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## STRESS ANALYSIS OF COMPOSITE CYLINDRICAL SHELLS WITH AN ELLIPTICAL CUTOUT

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A special-purpose, semianalytical solution method for determining the stress and deformation fields in a thin, laminated-composite cylindrical shell with an elliptical cutout is presented. The analysis includes the effects of cutout size, shape, and orientation; nonuniform wall thickness; oval cross-sectional eccentricity; and loading conditions. The loading conditions include uniform tension, uniform torsion, and pure bending. The analysis approach is based on the principle of stationary potential energy and uses Lagrange multipliers to relax the kinematic admissibility requirements on the displacement representations through the use of idealized elastic edge restraints. Specifying appropriate stiffness values for the elastic extensional and rotational edge restraints (springs) allows the imposition of the kinematic boundary conditions in an indirect manner, which enables the use of a broader set of functions for representing the displacement fields. Selected results of parametric studies are presented for several geometric parameters that demonstrate that this analysis approach is a powerful means for developing design criteria for laminated-composite shells.

### 1. Introduction

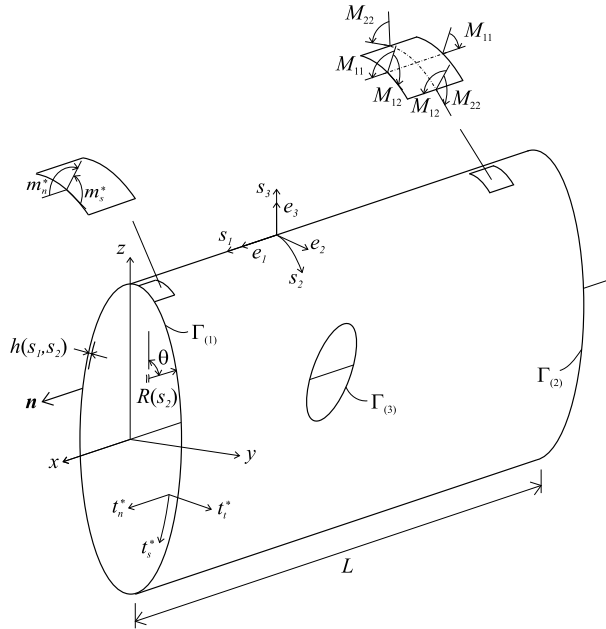
Cutouts in cylindrical shell-type components are unavoidable in the construction of aerospace structures. This fact is significant because the structural failure of these components usually begins near the cutout due to high stress concentrations that initiate the formation of cracks. Hence, a cutout can trigger a local failure at a load level lower than the global failure load of a corresponding shell without a cutout. As a result, preliminary design sizing of a cylindrical shell with a cutout is often based on the magnitude of the stress concentrations near the cutout. Therefore, an accurate assessment of the stress concentrations in a given shell subjected to various types of loading and support conditions is essential to the development of safe and reliable designs. Moreover, validated special-purpose analysis tools that enable rapid parametric studies would be very valuable to structural designers and for development of new design criteria and design concepts.

Several analytical, numerical, and experimental studies have been conducted during the past sixty years to determine stress distributions in cylindrical shells with a cutout and subjected to various types of loadings, such as axial tension and compression, torsion, and internal and external pressure. Pioneering analytical work was conducted by [Lurie 1946; 1947] to investigate the effects of axial tension, internal pressure, and shell curvature on the stress concentrations around a circular cutout. Many years later,

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*Keywords:* cylindrical, composite, cutout, stress.

The authors wish to dedicate this paper to the memory of Dr. James H. Starnes, Jr. of the NASA Langley Research Center. Dr. Starnes was an internationally recognized expert in aerospace structures technology and a proponent of the development of special-purpose, design-oriented analysis methods such as that presented herein.



**Figure 1.** Geometry, coordinates systems, and applied edge tractions for an oval cylindrical shell with an elliptical cutout and nonuniform wall thickness.

analytical studies were presented by [Lekkerkerker 1966; Van Dyke 1965; Ashmarin 1966; Murthy et al. 1974; Guz et al. 2001; Van Tooren et al. 2002] that further investigated the effects of various factors on the stress concentrations around a cutout in a cylindrical shell. Similarly, experimental investigations have been conducted by [Tennyson 1968; Starnes 1972; Pierce and Chou 1973; Zirka and Chernopiskii 2003; Bull 1982], and numerical studies have been conducted by [Liang et al. 1998; Shnerenko and Godzula 2003; Storozhuk and Chernyshenko 2005]. Hicks [1964] and Ebner and Jung [1972] summarized the results obtained from several of these studies and provided extensive lists of references related to this problem. Most of these previous studies are for isotropic cylindrical shells with a circular cutout. Only a few of these studies, such as those presented by Pierce and Chou [1973] and by Murthy et al. [1974], address the effects of cutout shape (elliptical cutouts) on the stress concentrations.

Mitigation of high stress concentrations by tailoring shell wall thickness, material orthotropy and anisotropy, and cutout reinforcement is also an important consideration in the design of aerospace structures made of lightweight composite materials. Likewise, the potential for using shells of noncircular cross section are relevant to fuselage-like structures. However, only a few studies have considered these effects. For example, the influence of variation in wall thickness on the stresses in axially loaded composite cylindrical shells without a cutout has been investigated by [Li et al. 1995]. Although numerous analyses exist in the literature on the analysis of shells with circular cross sections, only a few include noncircular cross sections. Sheinman and Firer [1994] provided an analytical investigation of stresses in laminated cylindrical shells with arbitrary noncircular cross sections. More recently, Hyer and Wolford [2002] and Hyer et al. [2003] studied the effect of noncircular cross sections on damage initiation and progressive failure in composite cylinders by employing the finite element method.

The objective of the present study is to present a special-purpose analysis for a laminated-composite cylindrical shell with an elliptical cutout that can be used to investigate rapidly and parametrically the effects of shell curvature, cutout size, shape, and orientation, and ply lay-up on stress-resultant concentrations near the cutout. The analysis is applicable to thin-walled cylindrical shells with nonuniform wall thickness, noncircular (for example, oval) shell cross-section, which are subjected to tension, torsion, and bending loads as illustrated in [Figure 1](#).

To accomplish this objective, an overview of the analysis is presented first. Next, the boundary value problem is defined along with the kinematics and stress-strain relations used in the analysis. Then, the derivation of the equations governing the response and numerical procedure are described. Finally, selected numerical results for oval and circular cylindrical shells with either circular or elliptical cutouts and subjected to either tension, torsion, or pure-bending loads are presented.

## 2. Analysis overview

The analytical approach used herein permits the determination of the pointwise variation of displacement and stress components. It is based on the principle of stationary potential energy, but utilizes local and global functions that are not required to satisfy the kinematic boundary conditions directly. Thus, the choice of local and global functions is not limited by a particular type of kinematic boundary condition. The kinematic boundary conditions are imposed by employing the Lagrange multiplier method. Both local and global functions are used, in contrast to the traditional approach, to enhance the robustness of the analysis method. In particular, the local functions are used to capture rapidly varying stress and strain gradients and local deformations near a cutout. Toward that goal, Laurent series are used for the local functions and are expressed in terms of the mapping functions introduced by [\[Lekhnitskii 1968\]](#). Fourier series are used for the global functions and are used to capture the overall deformation and stress fields. The kinematic admissibility requirements on the local and global functions are relaxed by defining the edges of the shell such that they are supported by extensional and rotational springs. Zero-valued displacement and rotation kinematic boundary conditions are enforced in an indirect manner by specifying values for the spring stiffnesses that are large compared to the corresponding shell stiffnesses. This approach effectively yields a prescribed kinematic boundary condition in the limit as the relative stiffness of the spring becomes much greater than the corresponding shell stiffness. Similarly, values for the spring stiffnesses can be selected that correspond to a given uniform elastic restraint along an edge, similar to that provided by an end ring. This capability is important and useful, because in some test fixtures or actual structures the edge supports may not be stiff enough to simulate a fully clamped boundary condition, or flexible enough to simulate a simply supported boundary condition.

As suggested by [\[Li et al. 1995\]](#) and [\[Sheinman and Firer 1994\]](#), nonuniform variations in wall thickness of a shell which lead to nonuniform laminate stiffnesses are represented using trigonometric series. Specifically, nonuniform shell wall thickness is represented in the present study by perturbing the ply thicknesses with a function that is periodic in either the longitudinal or the circumferential direction. The variation in wall thickness is accounted for by adjusting the lamina properties, resulting in nonhomogeneous in-plane and bending stiffness matrices. The nonuniform shell curvature associated with a noncircular cross section is represented by using trigonometric series for the coordinates of an oval cross-section shell reference surface (1958). The aspect ratio, or out-of-roundness, of the cross section is

represented in the analysis using an eccentricity parameter introduced by [Romano and Kempner 1962] and later used by [Culberson and Boyd 1971; Chen and Kempner 1976]. This parameter is defined in the subsequent section, and the aspect ratio, related to the eccentricity parameter, represents the ratio of the minor axis to the major axis.

In the derivation of the equations governing the response, the total potential energy consists of the elastic strain energy of the shell, the elastic edge restraints, and the potential energy of the applied loads. The conditions that may arise from the choice of displacement approximations without any kinematic restrictions are treated as constraint equations, and the potential energy arising from constraint reactions is invoked into the total potential energy through the use of Lagrange multipliers. The equations governing the shell response are obtained by enforcing the requirement that the first variation of the total potential energy vanish. The evaluation of the area integrals appearing in the potential energy are achieved numerically using a basic quadrature method in conjunction with standard triangulation of the entire domain, as described by [Shewchuk 1996]. Solutions to the equations governing the response are obtained using a standard Gaussian elimination procedure, which yields the generalized displacement coefficients and thus, the stress and strain fields. The accuracy of the analysis depends on the number of terms used for the functional representation of the displacement fields. As the number of terms increases, the results converge to the exact solution.

### 3. Representation of shell geometry

The geometry of a thin-walled, noncircular, cylindrical shell of length  $L$  and with an elliptical cutout located at the shell mid-length is shown in Figure 1. The origin of the global Cartesian coordinate system,  $(x, y, z)$ , is located at an end point of the longitudinal axis of the shell. As shown in Figure 1, the  $x$ -axis coincides with the longitudinal axis of the shell. The  $y$ - and  $z$ -coordinates span the cross-sectional plane. A curvilinear coordinate system is also attached to the mid-surface of the cylindrical shell. The coordinates of points in the longitudinal, circumferential (tangential), and normal-to-the-surface (transverse) directions of the shell are denoted by  $(s_1, s_2, s_3)$ , and the corresponding unit base vectors are  $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ .

Following [Romano and Kempner 1958], the noncircular cross section of the cylindrical shell is defined as an oval with the coordinates  $y$  and  $z$  expressed as

$$y = R_0 \sum_{\text{modd}} a_m(\xi) \sin \frac{ms_2}{R_0}, \quad z = R_0 \sum_{\text{modd}} b_m(\xi) \cos \frac{ms_2}{R_0}, \quad (1)$$

where  $\xi$  represents the eccentricity of the oval cross section and  $R_0$  is the equivalent radius of a circular cylindrical shell that has the same circumference as that of the oval cylindrical shell. The circumferential coordinate,  $s_2$ , varies between 0 and  $2\pi R_0$ . The derivation of Equation (1) and the explicit forms of the coefficients  $a_m(\xi)$  and  $b_m(\xi)$  are given by [Madenci and Barut 2003].

As derived by [Romano and Kempner 1958], the coordinates  $y$  and  $x$  in Equation (1) can be related to the radius of curvature of an oval-cross-section cylindrical shell  $R(s_2, \xi)$  by

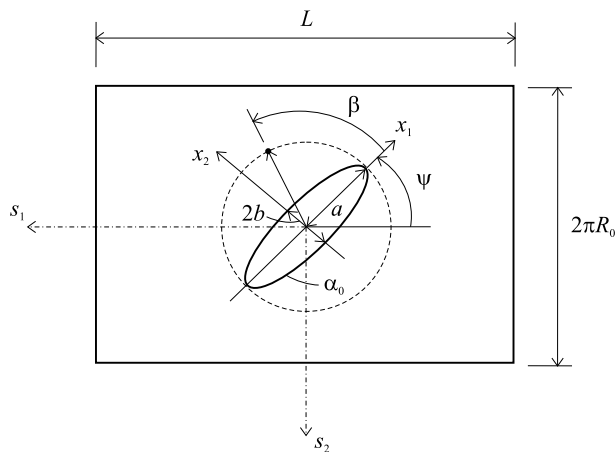
$$R(s_2, \xi) = \frac{R_0}{1 + \xi \cos(2s_2/R_0)}. \quad (2)$$

Therefore,  $\xi = 0$  implies no eccentricity and corresponds to a circular cross section with radius  $R_0$ . For positive values of the eccentricity parameter  $\xi$ , the  $z$ -coordinate becomes the major axis and the  $y$ -coordinate becomes the minor axis. For negative values of  $\xi$ , the major and minor axes switch to the  $y$ - and  $z$ -axes, respectively. The range of values of the eccentricity parameter  $\xi$  is bounded by  $-1 < \xi < 1$ .

As shown in Figure 1, the cylindrical shell contains a cutout. The shape of this cutout is defined such that if the shell is cut along a generator and flattened into a plane, the cutout becomes an ellipse, with major and minor axes denoted by  $a$  and  $b$ , respectively. For simplicity and convenience, the cutout is referred to herein as an *elliptical* cutout. Because the domain of the analysis shown in Figure 2 corresponds to a similar flat region, a subsequent mapping of the ellipse to a unit circle is possible, which enables the use of Laurent series expansions for the local functions. Note that the special case of a *circular* cutout is given by  $a = b$ .

In the flat analysis domain, the minor and major axes of the ellipse are aligned with a local coordinate system,  $(x_1, x_2)$ , whose origin is located at the center of the cutout and coincides with the origin of the parameter grid (given by constant values of  $s_1$  and  $s_2$ ) that forms the curvilinear coordinates  $(s_1, s_2)$  on the cylindrical shell mid-surface. The orientation of the elliptical cutout is arbitrary with respect to the longitudinal shell axis. Hence, the orientation of the local  $x_1$ -axis (major axis) of the cutout and the longitudinal  $s_1$ -axis of the cylindrical shell is denoted by the angle  $\psi$ . The elliptical coordinates  $\alpha$  and  $\beta$ , representing a family of confocal ellipses and hyperbolas, respectively, are utilized to obtain the stress-resultant distribution in the direction tangent to the cutout boundary. The coordinate  $\alpha$  is equal to  $\alpha_0 = \tanh^{-1}(b/a)$  on the particular ellipse that corresponds to the elliptical cutout. The other coordinate,  $\beta$ , varying from 0 to  $2\pi$ , is known as the eccentric angle and is related to the  $(x_1, x_2)$  coordinate system by  $x_1 = a \cos \beta$  and  $x_2 = b \sin \beta$ . The eccentric angle  $\beta$  is similar to the angle used for polar coordinates.

The symmetrically laminated cylindrical shells considered herein are made of  $K$  specially orthotropic layers, and each layer has an orientation angle,  $\theta_k$ , that is defined with respect to the  $s_1$ -axis. Each layer also has elastic moduli  $E_L$  and  $E_T$ , shear modulus  $G_{LT}$ , and Poisson's ratio  $\nu_{LT}$ , where the subscripts  $L$  and  $T$  represent the longitudinal (fiber) and transverse principal material directions, respectively.



**Figure 2.** Computational domain of a cylindrical shell with an elliptical cutout.



As for the shell thickness variation, the nonuniform wall thickness of the shell is denoted by  $h(s_1, s_2)$ , and its variation is included by assuming that the thickness of each ply  $t_k$  varies as a function of the curvilinear coordinates of the form

$$t_k(s_1, s_2) = t_{k0} \left( 1 - \varepsilon_1 \cos \frac{2\pi m_1 s_1}{L} - \varepsilon_2 \cos \frac{m_2 s_2}{R_0} \right), \quad (3)$$

where  $t_{k0}$  denotes the nominal thickness of the  $k^{\text{th}}$  layer in the laminate, and the parameters  $(m_1, m_2)$  and  $(\varepsilon_1, \varepsilon_2)$  respectively denote the wave numbers and the amplitudes of the periodic thickness variation, in the longitudinal and circumferential directions. While the wall thickness of the shell is allowed to vary across the shell surface, the aspect ratio of the plies through the thickness is maintained. In other words, the thickness variation of each ply remains conformable to each other throughout the shell surface. A periodic thickness variation in the longitudinal direction is obtained by setting  $\varepsilon_1 \neq 0$  and  $\varepsilon_2 = 0$ , and in the circumferential direction by  $\varepsilon_1 = 0$  and  $\varepsilon_2 \neq 0$ . A shell with uniform thickness  $t_k = t_{k0}$  is obtained by setting  $\varepsilon_1 = 0$  and  $\varepsilon_2 = 0$ .

#### 4. Boundary conditions and external loads

To facilitate a general imposition of prescribed boundary tractions, displacements, or rotations; the edge boundary,  $\Gamma$ , of the shell is decomposed into  $\Gamma = \Gamma_{(1)} + \Gamma_{(2)} + \Gamma_{(3)}$ . As shown in [Figure 1](#),  $\Gamma_{(1)}$  and  $\Gamma_{(2)}$  denote the external edge boundary of the cylindrical shell and  $\Gamma_{(3)}$  represents the traction-free internal edge boundary around the cutout. The unit vector normal to an edge is represented by  $\mathbf{n}$ . Throughout this paper, a variable with the superscript  $*$  is treated as a known quantity, arising from the externally applied loads or from prescribed displacements and rotations. Also, the subscripts  $n$ ,  $s$ , and  $t$  denote the directions normal, tangent, and transverse (through-the-thickness) to the boundary, respectively. The details of how prescribed edge loads and displacements are imposed in the analysis are presented subsequently.

**4.1. Prescribed edge loads.** External loads are applied to a shell by specifying values for the positive-valued stress resultants shown in [Figure 1](#). More precisely, the membrane loads applied to the  $\ell^{\text{th}}$  boundary segment  $\Gamma_{(\ell)}$  are given by

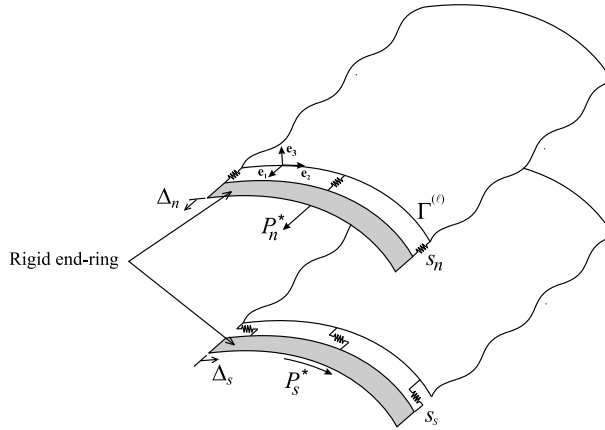
$$N_{11} = t_n^*, \quad N_{12} = t_s^*, \quad (4)$$

where  $N_{11}$  and  $N_{12}$  are the axial and shear stress resultants, respectively, defined in the cylindrical coordinate system. Likewise, shell-wall bending loads that are applied to the  $\ell^{\text{th}}$  boundary segment are given by

$$M_{11} = -m_n^*, \quad M_{11,1} + 2M_{12,2} = t_t^* - 2m_{s,2}^*, \quad (5)$$

where  $M_{11}$  and  $M_{12}$  are the pure bending and twisting stress resultants, respectively, defined in the cylindrical coordinate system. Moreover, the left hand side of the second equation in [Equation \(5\)](#) is the Kirchhoff shear stress resultant of classical shell theory.

As a matter of convenience, the analysis is formulated to permit in addition the specification of concentrated forces and moments that are transmitted to the ends of the shell as if through a rigid end ring, as shown in [Figure 3](#). At present, the concentrated force  $P_n^*$ , and the concentrated axial torque  $P_s^*$  are included in the analysis. The force  $P_n^*$  is simulated in the analysis by specifying a uniform distribution of the axial displacement, with the unknown magnitude  $\Delta_n$ , and the torque  $P_s^*$  is simulated by specifying



**Figure 3.** Application of a concentrated force through a rigid end-ring by using elastic springs along a shell edge.

a uniform distribution of the tangential displacement, with the unknown magnitude  $\Delta_s$  as

$$\int_{\Gamma(\ell)} N_{11}d\Gamma = P_n^*, \quad \int_{\Gamma(\ell)} N_{12}d\Gamma = P_s^*.$$

The analytical process used to ensure that the magnitudes of  $\Delta_n$  and  $\Delta_s$  correspond to the specified values of  $P_n^*$  and  $P_s^*$ , respectively, is described in the following section and in [Appendix A](#).

**4.2. Prescribed edge displacements and rotations.** Edge displacements and rotations are applied to a shell by specifying values for the displacements and rotations shown in [Figure 4](#) that correspond to the positive-valued stress resultants shown in [Figure 1](#). In particular, the axial and tangential displacements  $u_n^*$  and  $u_s^*$  that are applied to the  $\ell^{\text{th}}$  boundary segment  $\Gamma(\ell)$  are given by

$$u_1(\mathbf{n} \cdot \mathbf{e}_1) = u_n^*, \quad u_2[(\mathbf{e}_3 \times \mathbf{n}) \cdot \mathbf{e}_2] = u_s^*.$$

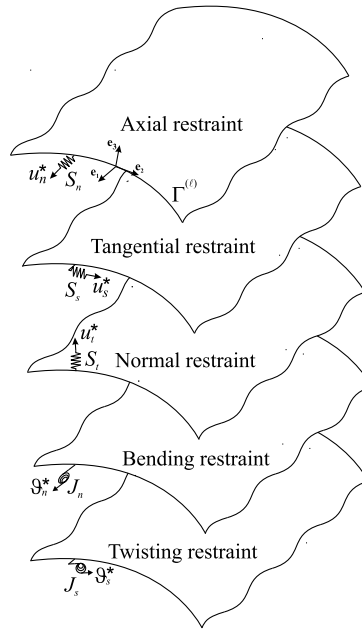
Similarly, the transverse displacement  $u_3^*$ , and the rotation about an axis tangent to an edge  $\vartheta_n^*$  which are applied to the  $\ell^{\text{th}}$  boundary segment are defined by

$$u_3 = u_t^*, \quad u_{3,1}(\mathbf{n} \cdot \mathbf{e}_1) = \vartheta_n^*.$$

As mentioned previously, these prescribed displacements are enforced through the use of elastic edge restraints (springs) to relax kinematic admissibility requirements on the functions that are used to represent the displacement fields. The uniformly distributed extensional and rotational springs that are attached to the shell edges in the normal, tangential, and transverse directions and are used to enforce the kinematic boundary conditions are depicted in [Figure 4](#).

Specifying appropriate stiffness values for the springs results in full or partial restraints along the shell edges. A zero value of the spring stiffness corresponds to a traction-free edge condition. In contrast, a value of the spring stiffness that is large compared to the corresponding shell stiffness effectively corresponds to a prescribed zero-valued boundary displacement or rotation. This approach effectively yields a prescribed kinematic boundary condition in the limit as the relative stiffness of the spring becomes much





**Figure 4.** Types of uniformly distributed elastic spring supports that can be prescribed along the edge of a cylindrical shell.

greater than the corresponding shell stiffness. Similarly, values for the spring stiffness can be selected that correspond to a specified uniform elastic restraint along an edge, similar to that provided by a rigid end ring. This capability is important and useful, because in some test fixtures or actual structures the edge supports may not be stiff enough to simulate a fully clamped boundary condition or flexible enough to simulate a simply supported boundary condition.

As depicted in [Figure 4](#), the membrane displacements  $u_n$  and  $u_s$ , and the transverse displacement  $u_t = u_3$  along the  $\ell^{\text{th}}$  boundary segment are restrained by extensional springs with stiffness values of  $S_n$ ,  $S_s$ , and  $S_t$  in the directions normal, tangential, and transverse to the boundary, respectively. In addition to the extensional springs, the edge rotations  $\vartheta_n$  and  $\vartheta_s$  along the  $\ell^{\text{th}}$  boundary segment are restrained by rotational springs with stiffness values of  $J_n$  and  $J_s$  that correspond to rotation about axes tangential and normal to the edge, respectively.

Extensional springs in the directions normal and tangent to the shell edge, with stiffness values of  $s_n$  and  $s_s$ , are also used to simulate load introduction through a rigid end ring, as shown in [Figure 3](#). Specifying values for the spring stiffnesses  $s_n$  and  $s_s$  that are relatively large compared to the corresponding shell stiffnesses causes the shell edge to behave as if a rigid end ring is attached, producing the uniformly distributed displacements with the corresponding magnitudes  $\Delta_n$  and  $\Delta_s$ . The values for  $\Delta_n$  and  $\Delta_s$  that correspond to the specified concentrated loads are determined by using a penalty parameter approach. This approach causes the difference between the edge displacements of the shell and the

unknown uniform rigid end ring displacements  $(u_n - \Delta_n)$  and  $(u_s - \Delta_s)$  to vanish while retaining the corresponding potential energy of the applied concentrated loads  $P_n^*$  and  $P_s^*$ .

### 5. Kinematics and stress-strain relations

The kinematic equations used in the present study are based to a large extent on the assumptions of the Love–Kirchhoff classical thin-shell theory. Specifically, the axial, circumferential (tangential), and normal (normal to the mid-surface) displacements of a generic point of the shell are denoted by  $U_1(s_1, s_2, s_3)$ ,  $U_2(s_1, s_2, s_3)$ , and  $U_3(s_1, s_2, s_3)$ , respectively. The corresponding displacements of a generic point of the shell mid-surface that share the same unit vector normal to the mid-surface are denoted by  $u_1(s_1, s_2)$ ,  $u_2(s_1, s_2)$ , and  $u_3(s_1, s_2)$ , respectively. In classical shell theory, these displacements are related by

$$U_1(s_i) = u_1(s_1, s_2) - s_3\beta_1(s_1, s_2), \quad U_2(s_i) = u_2(s_1, s_2) - s_3\beta_2(s_1, s_2), \quad U_3(s_i) = u_3(s_1, s_2),$$

where  $\beta_1(s_1, s_2)$  and  $\beta_2(s_1, s_2)$  are the mid-surface rotations about the  $s_2$  and  $s_1$  axes, respectively, as

$$\beta_1(s_1, s_2) = u_{3,1}(s_1, s_2), \quad \beta_2(s_1, s_2) = u_{3,2}(s_1, s_2) - \frac{1}{R(s_2)} u_2(s_1, s_2),$$

in which a subscript after a comma denotes partial differentiation. The corresponding linear membrane-strain displacement relations and the bending-strain displacement relations are given by

$$\boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{bmatrix} = \begin{bmatrix} u_{1,1} \\ \left(u_{2,2} + \frac{1}{R}u_3\right) \\ (u_{1,2} + u_{2,1}) \end{bmatrix}, \quad \boldsymbol{\kappa} = \begin{bmatrix} \kappa_{11} \\ \kappa_{22} \\ \kappa_{12} \end{bmatrix} = \begin{bmatrix} -u_{3,11} \\ -\left(u_{3,22} - \left(\frac{u_2}{R}\right)_{,2}\right) \\ -2\left(u_{3,12} - \frac{1}{R}u_{2,1}\right) \end{bmatrix}. \quad (6)$$

It is important to point out that the expression given for the change in surface twist due to deformation  $\kappa_{12}$  is that originally published by [Love 1888; Love 1944] for general shells, in terms of lines of principal-curvature coordinates, and derived in the book by [Timoshenko and Woinowsky-Krieger 1959] for circular cylindrical shells. As indicated by [Bushnell 1984], the expression for  $\kappa_{12}$  vanishes for rigid-body motions in contrast to the corresponding expression presented in Reissner’s version of Love’s first approximation shell theory [Reissner 1941; Kraus 1967; Naghdi 1962]. Equation (6) and the more general forms presented by [Bushnell 1984], are sometimes referred to as the Love–Timoshenko strain-displacement equations. Justification for this terminology is given by [Chaudhuri et al. 1986].

The stress-strain relations used in the present study are those of the classical theory of laminated plates and shells [Jones 1999], which are based on a linear through-the-thickness distribution of the strain fields. For a thin, symmetrically laminated cylindrical shell, with variable wall thickness, the relationship between the membrane and bending stress resultants and the membrane and bending strains is expressed conveniently in matrix notation by

$$\mathbf{N} = \mathbf{A}(s_1, s_2)\boldsymbol{\varepsilon}, \quad \mathbf{M} = \mathbf{D}(s_1, s_2)\boldsymbol{\kappa}. \quad (7)$$

The membrane and bending stress resultants in Equation (7) are defined as

$$\mathbf{N}^T = [N_{11} \ N_{22} \ N_{12}], \quad \mathbf{M}^T = [M_{11} \ M_{22} \ M_{12}]. \quad (8)$$

It is important to reiterate that when shell wall thickness variations are present, the membrane and bending stiffness matrices  $\mathbf{A}(s_1, s_2)$  and  $\mathbf{D}(s_1, s_2)$  are dependent on the curvilinear surface coordinates  $s_1$  and  $s_2$ .

It is convenient here to combine the relations given in Equation (7) into the matrix form

$$\mathbf{s} = \mathbf{C} \mathbf{e}, \tag{9}$$

where  $\mathbf{s}$ ,  $\mathbf{e}$ , and  $\mathbf{C}$  are defined as

$$\mathbf{s}^T = [\mathbf{N}^T \quad \mathbf{M}^T], \quad \mathbf{e}^T = [\boldsymbol{\epsilon}^T \quad \boldsymbol{\kappa}^T], \quad \mathbf{C} = \mathbf{C}(s_1, s_2) = \begin{bmatrix} \mathbf{A}(s_1, s_2) & \mathbf{0} \\ \mathbf{0} & \mathbf{D}(s_1, s_2) \end{bmatrix}. \tag{10}$$

### 6. Equations governing the response

A general analytical approach to obtain the exact solution of the equilibrium equations for a laminated-composite cylindrical shell with variable curvature is not mathematically tractable. Therefore, a semianalytic variational approach based on the principle of stationary potential energy is used in the present study to obtain numerical results. Because elastic edge restraints are used as a means to relax the kinematic admissibility conditions on the assumed displacement functions, and because a rigid end ring capability is used to impose shell-end force resultants, the potential energy consists of the elastic strain energy of the shell and the elastic edge restraints and the potential energy of the applied loads. In particular, the potential energy is expressed symbolically by  $\pi(\mathbf{q}, \Delta) = U(\mathbf{q}) + \Omega(\mathbf{q}, \Delta) + V(\mathbf{q}, \Delta)$ , in which  $U$  and  $\Omega$  represent the strain energy of the laminate and the elastic edge supports (springs), and  $V$  represents the potential energy due to external boundary loads. Their explicit forms are presented in Appendix A. The symbol  $\mathbf{q}$  is the vector of unknown, generalized displacement coefficients, which arises from the mathematical representation of the mid-surface displacement fields used in the variational solution process. In particular, the mid-surface displacement fields are given symbolically by  $u_1(\mathbf{q})$ ,  $u_2(\mathbf{q})$ , and  $u_3(\mathbf{q})$ . The symbol  $\Delta$  represents the vector of unknown edge displacements that arise from prescribing end loads.

Subjected to the constraint equations that arise from the use of Lagrange multipliers, the equations governing the shell response are obtained by enforcing the requirement that the first variation of the total potential energy vanish. As discussed by [McFarland et al. 1972], because the constraint equations are not functionally dependent on spatial coordinates  $s_1$  and  $s_2$ , the equations governing the response can be generated by modifying the total potential energy into the form

$$\pi^*(\mathbf{q}, \Delta, \boldsymbol{\lambda}) = \pi(\mathbf{q}, \Delta) + W(\mathbf{q}, \boldsymbol{\lambda}),$$

in which  $W$  is viewed as the potential energy arising from constraint reactions. In particular,

$$W(\mathbf{q}, \boldsymbol{\lambda}) = \boldsymbol{\lambda}^T \mathbf{G} \mathbf{q} = 0, \tag{11}$$

where  $\boldsymbol{\lambda}$  is the unknown vector of Lagrange multipliers and  $\mathbf{G}$  is the known constraint coefficient matrix.

Substituting the specific expressions for  $U(\mathbf{q})$ ,  $\Omega(\mathbf{q}, \Delta)$ ,  $V(\mathbf{q}, \Delta)$ , and  $W(\mathbf{q}, \boldsymbol{\lambda})$  that arise from approximation of the surface displacement field, and enforcing the requirement that the first variation of the modified form of the total potential energy to vanish leads to

$$\delta \pi^* = \delta \mathbf{q}^T [\mathbf{k}_{qq} \mathbf{q} + \mathbf{S}_{qq} \mathbf{q} - \mathbf{s}_{q\Delta} \Delta - \mathbf{f}^* - \mathbf{T}^* + \mathbf{G}^T \boldsymbol{\lambda}] + \delta \Delta^T [\mathbf{s}_{\Delta\Delta} \Delta - \mathbf{s}_{q\Delta}^T \mathbf{q} - \mathbf{P}^*] + \delta \boldsymbol{\lambda}^T \mathbf{G} \mathbf{q} = 0,$$

where the matrix  $\mathbf{k}_{qq}$  represents the stiffness matrix of the shell and requires evaluation of the corresponding integrand over a doubly connected region (see [Appendix A](#) for details). The spring stiffness matrices  $\mathbf{S}_{qq}$  and  $s_{\Delta\Delta}$  are associated with the deformation of the shell edges and displacement of the rigid end ring, respectively. The spring stiffness matrix  $s_{q\Delta}$  captures the coupling between the displacement of the shell edges and the rigid end ring. The vectors  $\mathbf{f}^*$ ,  $\mathbf{T}^*$ , and  $\mathbf{P}^*$  arise from the prescribed boundary displacements, external tractions and moments, and the concentrated forces applied to a rigid end ring, respectively. For the arbitrary variations ( $\delta\mathbf{q}$ ,  $\delta\Delta$ , and  $\delta\lambda$ ), the stationary condition requires that the following equations must be satisfied:

$$\left[ (\mathbf{k}_{qq} + \mathbf{S}_{qq})\mathbf{q} - s_{q\Delta}\Delta - \mathbf{f}^* - \mathbf{T}^* + \mathbf{G}^T\lambda \right] = \mathbf{0}, \quad \left[ s_{\Delta\Delta}\Delta - s_{q\Delta}^T\mathbf{q} - \mathbf{P}^* \right] = \mathbf{0}, \quad \mathbf{G}\mathbf{q} = \mathbf{0}. \quad (12)$$

It is convenient to express [Equation \(12\)](#) by the single matrix equation

$$\mathbf{K}\mathbf{Q} = \mathbf{F}, \quad (13)$$

where  $\mathbf{K}$  and  $\mathbf{F}$  represent the overall system stiffness matrix and the overall load vector, respectively. These matrices have the general expanded form

$$\mathbf{K} = \begin{bmatrix} \mathbf{K}_{qq} & -s_{q\Delta} & \mathbf{G}^T \\ -s_{q\Delta}^T & s_{\Delta\Delta} & \mathbf{0}^T \\ \mathbf{G} & \mathbf{0} & 0 \end{bmatrix}, \quad \mathbf{F} = \begin{bmatrix} \mathbf{F}^* \\ \mathbf{P}^* \\ 0 \end{bmatrix},$$

in which  $\mathbf{K}_{qq} = \mathbf{k}_{qq} + \mathbf{S}_{qq}$ , and  $\mathbf{F}^* = \mathbf{f}^* + \mathbf{T}^*$ . The vector of unknowns,  $\mathbf{Q}$ , that appears in [Equation \(13\)](#) is defined as  $\mathbf{Q}^T = [\mathbf{q} \ \Delta \ \lambda]$ . Solving for the vector of unknowns in [Equation \(13\)](#) yields all the information needed to obtain a complete variational solution to a specific problem. The accuracy of a solution depends on the number of terms included in the expressions for the local and global functions representing the displacement fields and converges to the corresponding exact solution as the number of terms increases.

**6.1. Displacement-field representation.** Representation of the mid-surface displacement field is a critical step in the variational solution to the problem. By relaxing the requirements for kinematic admissibility, the mid-surface displacement fields are represented in the present study by a combination of rigid-body modes,  $u_{Ri}$ , and global and local functions, denoted by  $\bar{u}_i$  and  $\bar{\bar{u}}_i$ , respectively, through  $u_i = u_{Ri} + \bar{u}_i + \bar{\bar{u}}_i$ , where the values of the index are given by  $i = 1, 2, 3$ . The explicit forms of the displacement fields can be found in [Appendix B](#). The rigid-body modes account for the overall or global translation and rotation of the shell, and are selected so that they produce neither membrane strain nor changes in shell curvature and twist. These terms are included for the completeness of the kinematics of the cylindrical shell. The presence of appropriate displacement boundary conditions inherently eliminates the rigid-body motion. However, for cases where an insufficient number of kinematic boundary conditions are imposed, these rigid-body terms need to be eliminated, as discussed in detail in [Appendix C](#). Following the complex variable solution techniques used in the theory of elasticity, the local functions are expressed in terms of robust, uniformly convergent Laurent series (used for doubly connected regions) to enhance capturing of steep stress gradients and deformations near the cutout. Complete sets of trigonometric expansions are used primarily to capture the overall global response of the shell.

Here, completeness means that all the fundamental waveforms needed to construct the typical overall deformations of a shell are included in the set.

For convenience, the displacement representations are rewritten in matrix form as

$$u_{i=1,2} = \mathbf{V}_{Ri}^T \boldsymbol{\alpha}_R + \bar{\mathbf{V}}_i^T \mathbf{c}_i + \bar{\bar{\mathbf{V}}}_i^T \boldsymbol{\alpha}, \quad u_3 = \mathbf{V}_{R3}^T \boldsymbol{\alpha}_R + \bar{\mathbf{V}}_3^T \mathbf{c}_3 + \bar{\bar{\mathbf{V}}}_3^T \boldsymbol{\beta}. \quad (14)$$

An even more useful, compact form is given by

$$u_i = \mathbf{V}_i^T \mathbf{q}, \quad (15)$$

where the vector of unknown displacement coefficients,  $\mathbf{q}$ , is defined by

$$\mathbf{q}^T = [\boldsymbol{\alpha}_R^T \ \mathbf{c}_1^T \ \mathbf{c}_2^T \ \mathbf{c}_3^T \ \boldsymbol{\alpha}^T \ \boldsymbol{\beta}^T]. \quad (16)$$

In Equation (16), the vector  $\boldsymbol{\alpha}_R$  contains the unknown coefficients for the rigid-body motion of the shell, and the vectors  $\boldsymbol{\alpha}$  and  $\boldsymbol{\beta}$  contain the real and imaginary parts of the unknown coefficients  $\alpha_{nm}$  and  $\beta_{nm}$ , respectively, that are associated with the local functions. The vectors  $\mathbf{c}_i$ , where  $i = 1, 2, 3$ , contain the real-valued unknown coefficients  $c_{i(mm)}$  associated with the global functions. The explicit forms used herein for the unknown coefficient vectors  $\boldsymbol{\alpha}_R$ ,  $\mathbf{c}_i$ ,  $\boldsymbol{\alpha}$ , and  $\boldsymbol{\beta}$  that appear in Equation (14) along with the vector functions  $\mathbf{V}_i$  (and the corresponding subvectors  $\mathbf{V}_{Ri}$ ,  $\bar{\mathbf{V}}_i$ , and  $\bar{\bar{\mathbf{V}}}_i$ ) are given in Appendix B.

In addition to the general representation of the shell surface-displacement fields, similar matrix expressions are needed for the displacements and rotations of points on the shell boundary. In the present study, the boundary displacement vector  $\mathbf{u}_\Gamma$  is introduced that consists of the mid-surface boundary displacements in the directions normal, tangent, and transverse to a shell edge, and the mid-surface rotations about axes that are normal and tangent to a shell edge. The boundary displacements in the directions normal, tangent, and transverse to a shell edge are denoted herein by  $u_n$ ,  $u_s$ , and  $u_t$ , respectively. Similarly, the mid-surface rotations about axes that are tangent and normal to a shell edge are denoted by  $\vartheta_n$  and  $\vartheta_s$ , respectively. In terms of the vector of unknowns defined by Equation (16), the boundary displacements  $\mathbf{u}_\Gamma$  and rotations are expressed in matrix form by

$$\mathbf{u}_\Gamma = \mathbf{B}\mathbf{q}, \quad \mathbf{u}_\Gamma^T = [u_n \ u_s \ u_t \ \vartheta_n]. \quad (17)$$

The matrix  $\mathbf{B}$  with known coefficients is defined as  $\mathbf{B}^T = [\mathbf{u}_n^T \ \mathbf{u}_s^T \ \mathbf{u}_t^T \ \boldsymbol{\theta}_n^T]$  in which the subvectors  $\mathbf{u}_n^T$ ,  $\mathbf{u}_s^T$ ,  $\mathbf{u}_t^T$ , and  $\boldsymbol{\theta}_n^T$  are known and defined by

$$\mathbf{u}_n^T = (\mathbf{n} \cdot \mathbf{e}_1) \mathbf{V}_1, \quad \mathbf{u}_s^T = [(\mathbf{e}_3 \times \mathbf{n}) \cdot \mathbf{e}_2] \mathbf{V}_2, \quad \mathbf{u}_t^T = \mathbf{V}_3^T, \quad \boldsymbol{\theta}_n^T = (\mathbf{n} \cdot \mathbf{e}_1) \mathbf{V}_{3,1}^T.$$

**6.2. Strain- and stress-resultant field representation.** After defining the shell mid-surface displacement field in terms of the generalized coordinate  $\mathbf{q}$ , the corresponding representation of the strains is obtained by substituting Equation (15) into the strain-displacement relations given in vector form by Equation (6). This substitution yields  $\boldsymbol{\varepsilon} = \mathbf{L}_\varepsilon \mathbf{q}$  and  $\boldsymbol{\kappa} = \mathbf{L}_\kappa \mathbf{q}$ , where the strain-coefficient matrices  $\mathbf{L}_\varepsilon$  and  $\mathbf{L}_\kappa$  are

$$\mathbf{L}_\varepsilon = \begin{bmatrix} \mathbf{V}_{1,1}^T \\ \mathbf{V}_{2,2}^T + \frac{1}{R} \mathbf{V}_3^T \\ \mathbf{V}_{1,2}^T + \mathbf{V}_{2,1}^T \end{bmatrix}, \quad \mathbf{L}_\kappa = \begin{bmatrix} -\mathbf{V}_{3,11}^T \\ -\mathbf{V}_{3,22}^T + \frac{1}{R} \mathbf{V}_{2,2}^T + \frac{R_{,2}}{R^2} \mathbf{V}_2^T \\ -2\mathbf{V}_{3,12}^T + \frac{2}{R} \mathbf{V}_{2,1}^T \end{bmatrix}.$$

Next, the representations for  $\boldsymbol{\varepsilon}$  and  $\boldsymbol{\kappa}$  are substituted into Equation (9) to obtain

$$\boldsymbol{e} = \boldsymbol{L}\boldsymbol{q}, \quad \boldsymbol{L}^T = [\boldsymbol{L}_\varepsilon^T \quad \boldsymbol{L}_\kappa^T], \quad (18)$$

where  $\boldsymbol{L}$  is the overall strain-coefficient matrix .

Finally, the corresponding matrix representation of the stress resultants in terms of the generalized coordinates is obtained by substituting Equation (18) into constitutive Equation (9). The resulting vector of stress resultants is given by

$$\boldsymbol{s} = \boldsymbol{C}\boldsymbol{L}\boldsymbol{q}. \quad (19)$$

**6.3. Constraint equations.** In the generalized coordinate representations for  $u_1$  and  $u_2$ , the coefficients  $c_{1(00)}$  and  $c_{2(00)}$  associated with the global functions  $\bar{u}_1$  and  $\bar{u}_2$  in Equation (B.2) also correspond to rigid-body translation in the  $s_1$  direction and rigid-body rotation about the  $s_1$  axis, respectively. These two redundant rigid-body modes are eliminated by introducing constraint conditions using Lagrange multipliers. In particular, the unknown Lagrange multipliers  $\lambda_{RRB(1)}$  and  $\lambda_{RRB(2)}$  are associated with the redundant rigid-body modes. Also, multivaluedness of the normal direction displacement  $u_3(s_1, s_2)$  that arises from the presence of logarithmic terms in the Laurent series expansion for the local function must be eliminated. The unknown Lagrange multipliers  $\lambda_{SV(r)}$  and  $\lambda_{SV(s)}$  are used herein to eliminate this multivaluedness. Likewise, the rigid-body modes of the cylindrical shell must be eliminated by the Lagrange multipliers  $\lambda_{RB(j)}$  for  $j = 1, \dots, 6$  if the specified kinematic boundary conditions are not sufficient enough to prevent them. In other words, the nonvanishing rigid-body modes must be eliminated by introducing constraint conditions prior to the stress analysis to allow the overall system stiffness matrix  $\boldsymbol{K}$ , given in Equation (13), to be nonsingular.

These requirements on the representation of the shell displacement field are enforced by using constraint equations that use Lagrange multipliers. These constraint equations are functionally independent, forming a set of linearly independent equations equal in number to the total number of Lagrange multipliers. The Lagrange multipliers can be viewed as the reactions needed to enforce the corresponding constraints. In the present study, all of these constraint conditions are included in the matrix equation given in Equation (11). The explicit forms of the vector of unknown Lagrange multipliers  $\boldsymbol{\lambda}$  containing  $\lambda_{RRB(1)}$  and  $\lambda_{RRB(2)}$ ,  $\lambda_{SV(r)}$ ,  $\lambda_{SV(s)}$  and  $\lambda_{RB(j)}$ , for  $j = 1, \dots, 6$ , and the known coefficient matrix,  $\boldsymbol{G}$ , in Equation (12) are given in Appendix C.

## 7. Overview of validation studies

A limited series of validation studies were conducted in the present study to determine the accuracy of results obtained using the analysis method presented herein. Specifically, the studies included circular and noncircular cylindrical shells with either a circular or an elliptical cutout under uniform tension. The stress resultants around the circular and elliptical cutout for varying aspect ratios and orientations in a circular cylinder, as well as the stress concentrations arising from a circular cutout in a noncircular cylindrical shell were computed. Comparisons of the stress-resultant distributions and magnitudes in the shells were made with corresponding results obtained using an in-house finite element program developed earlier by [Madenci and Barut 1994a]. This finite element program has been validated to a large extent against previously published experimental and numerical results for stress, buckling, and post-buckling of thin-shell structures [Madenci and Barut 1994b; 1994c]. Therefore, this finite element program is

expected to serve as a reliable indicator of the accuracy of the analysis methods and results presented herein. Overall, the comparisons indicate very good agreement (less than 1% difference) between the corresponding results produced by the two analysis methods. For shells with high aspect ratio cutouts, differences of approximately 5% were obtained and found to be the result of insufficient mesh refinement in the finite element models.

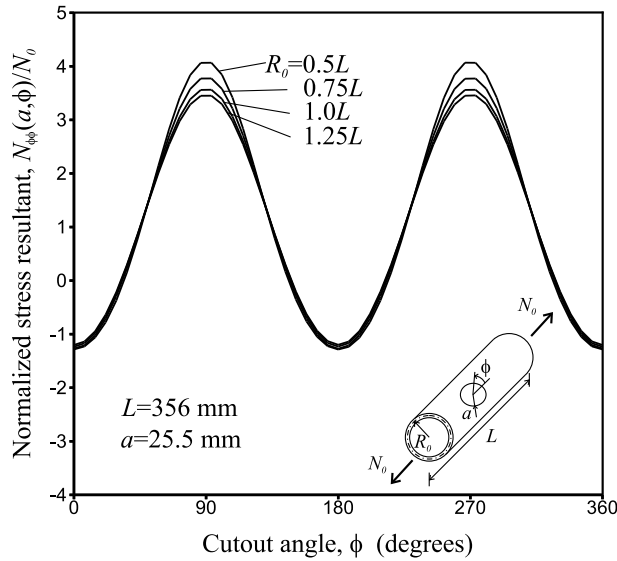
## 8. Selected numerical results

Selected numerical results are presented in this section to demonstrate the utility of the analysis method presented herein and the potential for its use in developing design technology. These results elucidate the effects of loading condition, noncircular cross-section geometry, variation in wall thickness, cutout shape, cutout size, and cutout orientation on the intensity of stress-resultant concentrations near a cutout. Specifically, tension, torsion, and pure-bending loads are considered for  $[45^\circ/-45^\circ/90^\circ/0_2^\circ/90^\circ/-45^\circ/45^\circ]_s$  quasiisotropic shells with length  $L = 356$  mm and made of graphite-epoxy plies. The nominal ply thickness is  $t_{k0} = 0.14$  mm, resulting in the total thickness of the shell given by  $h = 2.24$  mm. The ply orientation angles are measured with respect to the longitudinal shell axis. The Young's moduli of each ply in the longitudinal, fiber direction and in the direction transverse to the fibers are specified as  $E_L = 135.0$  GPa and  $E_T = 13.0$  GPa, respectively. The in-plane shear modulus and Poisson's ratio of each ply are given by  $G_{LT} = 6.4$  GPa and  $\nu_{LT} = 0.38$ .

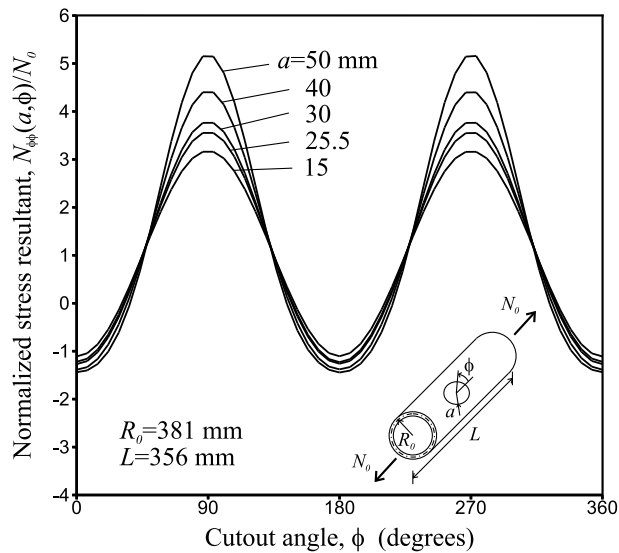
The effects of varying the radius of curvature  $R_0$  on the stress-resultant concentration along the contour of a circular cutout with radius  $a = 25.5$  mm are shown in [Figure 5](#) for a circular cylindrical shell subjected to a uniform axial tension load. Four curves that correspond to values of  $R_0/L = 0.5, 0.75, 1.0,$  and  $1.25$  are presented that show the tangential stress resultant  $N_{\phi\phi}$ , normalized by the far-field applied uniform stress resultant  $N_0$ , as a function of position around the cutout (indicated by the *cutout angle*,  $\phi$ ). As shown in [Figure 5](#), the stress-resultant concentration is a maximum at  $\phi = 90^\circ$  and  $270^\circ$  (at the net section of the shell) for each case and reduces from a maximum value of approximately 4.0 to a minimum value of 3.4 at the net section as the radius of curvature increases. In addition, the results show that the  $N_{\phi\phi}(a, 90^\circ)$  stress-resultant concentration approaches the well-known value of 3.0 for an isotropic plate as the shell radius increases. Away from the net section, changes in the radius of curvature have a relatively small effect on the stress-resultant concentration.

The effects of varying the circular cutout radius on the stress-resultant concentration along the contour of a circular cutout is shown in [Figure 6](#) for a circular cylindrical shell with radius  $R_0 = 381$  mm and subjected to a uniform axial tension load. Five curves that correspond to values of the cutout radius  $a = 15, 25.5, 30, 40,$  and  $50$  mm are presented that also show the tangential stress resultant  $N_{\phi\phi}(a, \phi)$ , normalized by the far-field applied uniform stress resultant  $N_0$ , as a function of the cutout angle  $\phi$ . The results in [Figure 6](#) show that the stress-resultant concentration is a maximum at the net section of the shell for each case, as expected, and changes significantly from a minimum value of approximately 3.1 to a maximum value of 5.1 at the net section as the cutout radius increases — an increase of approximately 65%. The results also show that the  $N_{\phi\phi}(a, 90^\circ)$  stress-resultant concentration approaches the well-known value of 3.0 for an isotropic plate as the cutout radius decreases. Away from the net section, changes in the cutout radius have a much smaller effect on the stress-resultant concentration.

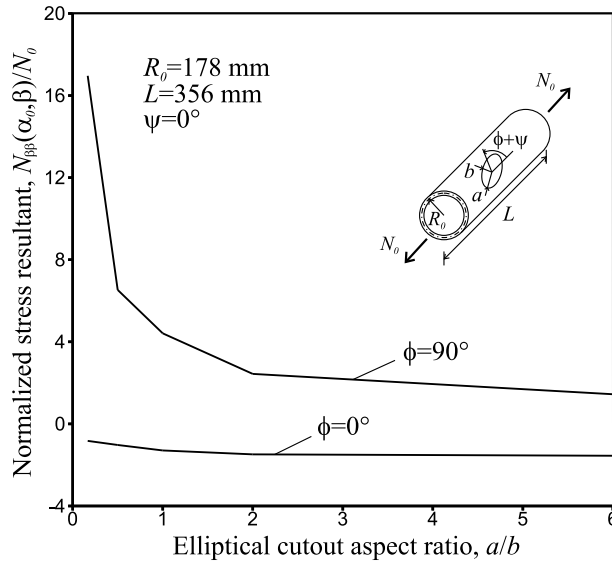




**Figure 5.** The effect of varying shell radius on the tangential stress resultants around a circular cutout in a quasiisotropic circular cylindrical shell subjected to a uniform tension load.



**Figure 6.** The effect of varying circular-cutout radius on the tangential stress resultants around a circular cutout in a quasiisotropic circular cylindrical shell subjected to a uniform tension load.

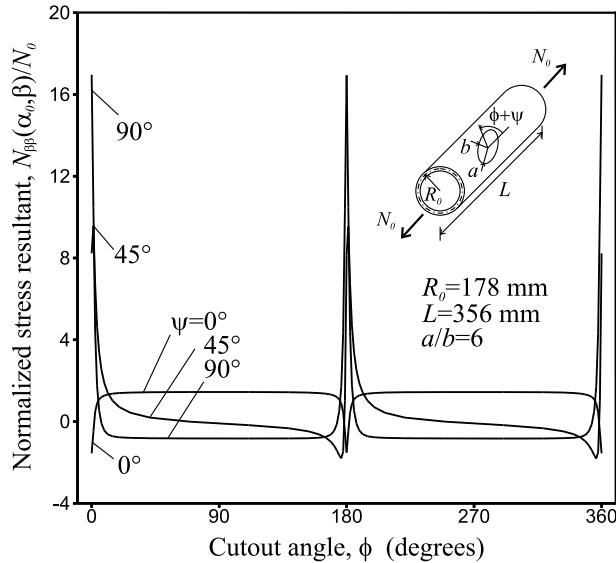


**Figure 7.** The effect of varying elliptical cutout aspect ratio on the tangential stress resultants around a cutout in a quasiisotropic cylindrical shell subjected to a uniform tension load.

The effect of varying the elliptical cutout aspect ratio  $a/b$  on the tangential stress-resultant distribution around the edge of a cutout in a cylindrical shell with radius  $R_0 = 178$  mm and subjected to uniform tension is presented in Figure 7. The orientation of the elliptical cutout is specified by  $\psi = 0^\circ$ . Two curves that correspond to the locations  $\phi = 0^\circ$  and  $90^\circ$  are presented that show the tangential stress resultant  $N_{\beta\beta}(\alpha_0, \beta)$ , normalized by the far-field applied uniform stress resultant  $N_0$ , as a function of the cutout aspect ratio. As expected, the normalized stress-resultant concentration,  $N_{\beta\beta}(\alpha_0, \beta)/N_0$ , remains negative for all aspect ratios at  $\phi = 0^\circ$ , consistent with the expected Poisson effect, and the magnitudes are relatively insignificant at this location. In contrast, large stress-resultant concentrations are indicated at the net section ( $\phi = 90^\circ$ ) that diminish from a maximum value of approximately 17.0 for a widthwise, slot-like cutout with  $a = 5$  mm and  $b = 30$  mm, or  $a/b = 1/6$ , to a minimum value of 1.4 for a lengthwise, slot-like cutout with  $a = 30$  mm and  $b = 5$  mm, or  $a/b = 6$ .

The effects of varying the orientation of a high aspect ratio, slot-like elliptical cutout on the stress-resultant concentration along the cutout contour is shown in Figure 8 for a circular cylindrical shell with radius  $R_0 = 178$  mm and subjected to a uniform axial tension load. The major and minor axes of the cutout are given by  $a = 30$  mm and  $b = 5$  mm, respectively. The orientation of the elliptical cutout, with respect to the longitudinal shell axis, is measured by the angle  $\psi$ . Three curves that correspond to values of  $\psi = 0^\circ$ ,  $45^\circ$ , and  $90^\circ$  are presented that show the tangential stress resultant at the cutout edge  $N_{\beta\beta}$ , normalized by the far-field applied uniform stress resultant  $N_0$ , as a function of the cutout angle  $\phi$ .

The results in Figure 8 show that the stress-resultant concentration is the least pronounced for the case of  $\psi = 0^\circ$ . For this case, the cutout major axis is aligned lengthwise with the shell axis and the

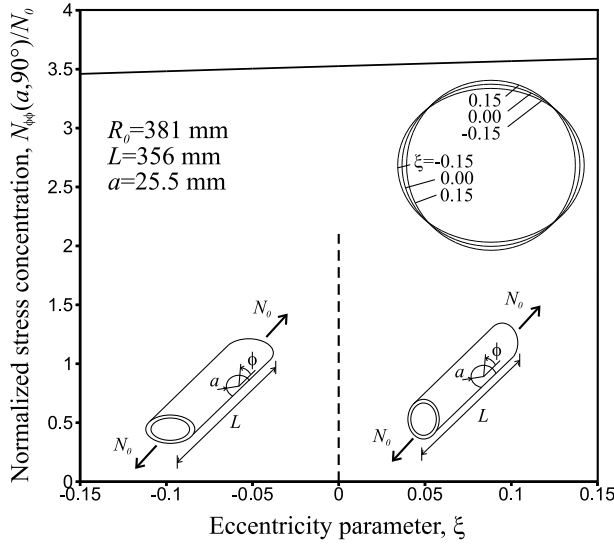


**Figure 8.** The effect of varying elliptical cutout orientation on the tangential stresses around the cutout in a quasiisotropic cylindrical shell subjected to a uniform tension load.

net section of the shell is the largest. The location on the cutout edge defined by  $\phi = 0^\circ$  corresponds to where the edge of the cutout intersects the major axis. At this location, the edge of the cutout is in tangential compression ( $N_{\beta\beta}/N_0 = -1.6$ ), consistent with a Poisson effect. The location defined by  $\phi = 90^\circ$  corresponds to where the edge of the cutout intersects the minor axis, that is, at the net section of the shell. At this location, the edge of the cutout is in tangential tension ( $N_{\beta\beta}/N_0 = 1.4$ ). Between approximately  $\phi = 0^\circ$  and  $170^\circ$  and between  $\phi = 190^\circ$  and  $350^\circ$ , the cutout width (and hence net section width) does not vary greatly. This attribute accounts for the corresponding flat regions in the  $\psi = 0^\circ$  curve shown in Figure 8.

For the case of  $\psi = 90^\circ$ , the cutout major axis is perpendicular to the shell axis and the net section of the shell is the smallest. As before, the locations defined by  $\phi = 0^\circ$  and  $180^\circ$  correspond to where the edge of the cutout intersects the major axis, that is, at the net section of the shell. The results in Figure 9 show that the edge of this high aspect ratio cutout has extremely high stress-resultant concentrations at these locations ( $N_{\beta\beta}/N_0 = 17.0$ ) that have very steep gradients. Between approximately  $\phi = 5^\circ$  and  $175^\circ$  and between  $\phi = 185^\circ$  and  $355^\circ$ , the analysis predicts relatively benign variations in the stress-resultant concentration. The case of  $\psi = 45^\circ$  exhibits stress-resultant concentrations that are, for the most part, bounded by the corresponding results for  $\psi = 0^\circ$  and  $90^\circ$ . The analysis also predicts very high stress-resultant concentrations where the cutout edge intersects the major principal cutout axis ( $N_{\beta\beta}/N_0 = 8.2$ ).

The effects of varying the cross-sectional eccentricity (2) of a tension loaded oval shell with a circular cutout are shown in Figure 9. The results in this figure correspond to the equivalent shell radius  $R_0 = 381$  mm and a circular cutout radius given by  $\alpha = 25.5$  mm. Moreover, the tangential stress-resultant concentration at the shell net section  $N_{\phi\phi}(a, 90^\circ)$ , normalized by the applied load  $N_0$ , is shown as a function of the eccentricity parameter for the range of  $-0.15 \leq \xi \leq 0.15$ . As indicated in the figure,



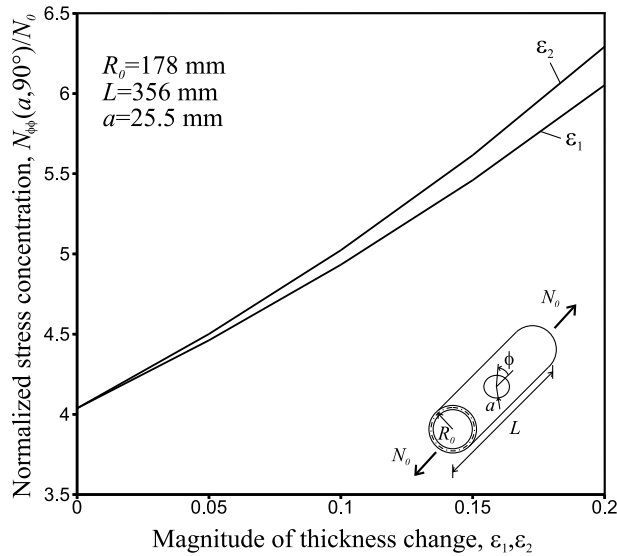
**Figure 9.** The effect of shell cross-sectional eccentricity on the stress-resultant concentration in an oval quasiisotropic cylindrical shell with a circular cutout and subjected to a uniform tension load.

negative and positive values of  $\xi$  correspond to cylindrical shells with the largest cross-sectional width, with parallel and perpendicular orientation, respectively. A value of  $\xi = 0$  corresponds to a circular cross section and a value of  $\xi = 0.15$  corresponds to a cross-sectional aspect ratio of 0.9.

The results presented in [Figure 9](#) show that the stress-resultant concentration is affected benignly by the cross-sectional eccentricity. In particular, the stress-resultant concentration increases almost linearly with increases in the eccentricity parameter from  $N_{\phi\phi}(a, 90^\circ)/N_0 = 3.5$  to 3.6, which is slightly less than a 3% variation. This trend is understood by noting that the shells that correspond to negative values of  $\xi$  are flatter near the cutout than those that correspond to positive values of  $\xi$  and, as indicated by the results in [Figure 5](#), are expected to have the lower values for the stress-resultant concentrations.

The effects of longitudinal and circumferential periodic variations in wall thickness on the stress-resultant concentration at the net section of the circular cylindrical shell with radius  $R_0 = 178$  mm, circular cutout radius  $a = 25.5$  mm, and subjected to uniform axial tension load are shown in [Figure 10](#). Two monotonically increasing curves that correspond to values of  $\varepsilon_1$  (with  $\varepsilon_2 = 0$ ) and  $\varepsilon_2$  (with  $\varepsilon_1 = 0$ ) are presented that show the tangential stress resultant  $N_{\phi\phi}(a, 90^\circ)$ , normalized by the far-field applied uniform stress resultant  $N_0$ , as a function of thickness-variation amplitudes (3) which range from 0 to 0.2. For the longitudinal thickness variation, the wave numbers used in [Equation \(3\)](#) are  $m_1 = 1$  and  $m_2 = 0$ . Similarly, for the circumferential thickness variation, the wave numbers used in [Equation \(3\)](#) are  $m_1 = 0$  and  $m_2 = 1$ .

The results shown in [Figure 10](#) indicate that the stress-resultant concentration at the shell net section increases as the magnitude of the thickness variation increases, for variations in either the longitudinal or circumferential direction. The maximum variation in the results is approximately 56%. Furthermore, the change in the stress-resultant concentration is slightly more pronounced for the circumferential thickness variation than for the longitudinal thickness variation. These increases are primarily due to a drastic loss

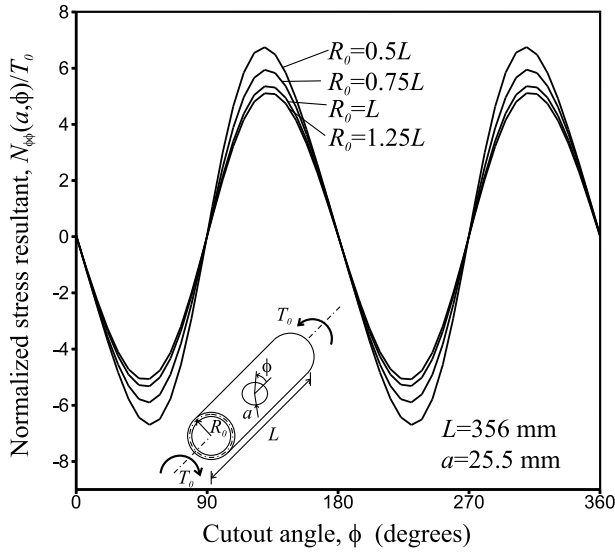


**Figure 10.** The effects of longitudinal ( $\epsilon_1 \neq 0$  and  $\epsilon_2 = 0$ ) and circumferential ( $\epsilon_1 \neq 0$  and  $\epsilon_2 = 0$ ) wall thickness variations on the tangential stress-resultant concentration around a circular cutout in a quasiisotropic circular cylindrical shell subjected to a uniform tension load.

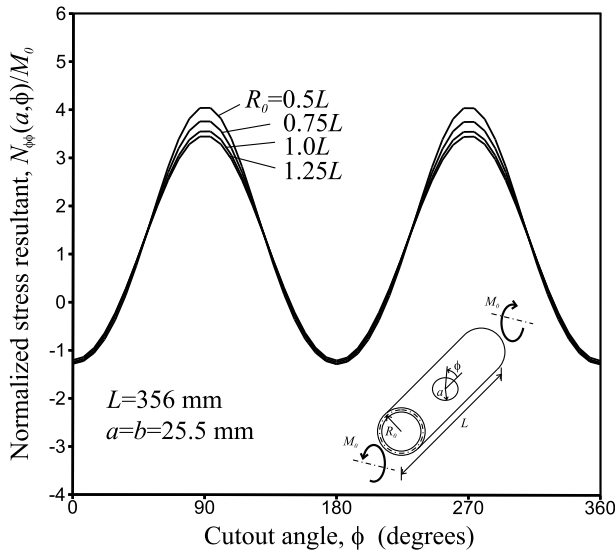
of bending stiffness near the net section of the shell, as indicated by the wave numbers  $m_1 = 0$  and  $m_2 = 1$ , where the thickness of the shell near the center of the cutout is smaller.

The effects of varying the radius of curvature  $R_0$  on the stress-resultant concentration along the contour of a circular cutout with radius  $a = 25.5$  mm is shown in Figures 11 and 12 for a circular cylindrical shell subjected to a uniform torsion load and a pure bending load, respectively. The pure-bending load corresponds to using  $t_n^* = M_0 \cos(\pi/s_2)$  in Equation (4). Four curves that correspond to values of  $R_0/L = 0.5, 0.75, 1.0,$  and  $1.25$  are presented that show the normalized values of the tangential stress resultant  $N_{\phi\phi}$  as a function of position around the cutout. In Figure 11,  $N_{\phi\phi}$  is normalized by the far-field applied uniform shear stress resultant,  $T_0$ . In Figure 12,  $N_{\phi\phi}$  is normalized by the far-field applied uniform bending stress resultant,  $M_0$ .

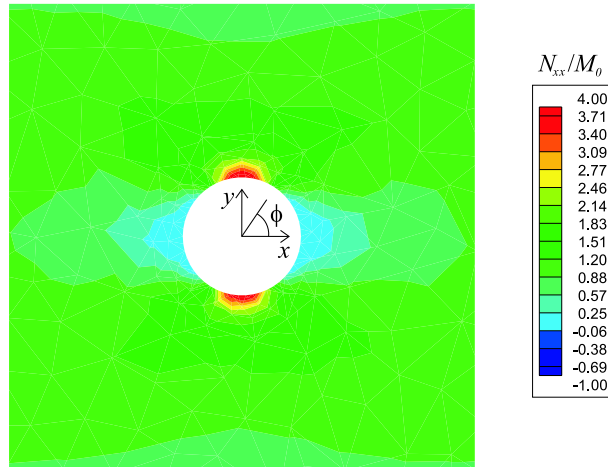
The results in Figure 11 indicate that the stress-resultant concentration has identical maximum magnitudes at  $\phi = 45^\circ, 135^\circ, 225^\circ,$  and  $315^\circ$  (at the net section of the shell) for each case, which corresponds to maximum diagonal tension and compression stress resultants associated with the shear stress resultants near the cutout. The magnitudes of the stress-resultant concentration for these four locations reduce from a maximum value of 6.8 to a minimum value of 5.1 as the radius of curvature increases (33% variation). Away from these four locations, changes in the radius of curvature have a smaller effect on the stress-resultant concentration. The results in Figure 12 indicate that the stress-resultant concentration for the shell subjected to the pure bending load is quite similar to that presented in Figure 5 for the corresponding tension loaded shell.



**Figure 11.** The effect of varying shell radius on the tangential stresses around a circular cutout in a quasiisotropic circular cylindrical shell subjected to a uniform torsion load.



**Figure 12.** The effect of varying shell radius on the tangential stresses around a circular cutout in a quasiisotropic circular cylindrical shell subjected to a pure-bending load.



**Figure 13.** Stress resultant distribution near the cutout in a quasiisotropic circular cylindrical shell subjected to a pure-bending load.

Specifically, the stress-resultant concentration is a maximum at  $\phi = 90^\circ$  and  $270^\circ$  (at the net section of the shell) for each case and reduces from a maximum value of 4.0 to a minimum value of 3.5 at the net section as the radius of curvature increases (14% variation). In addition,  $N_{\phi\phi}/M_0$  approaches the well-known value of three for an isotropic plate as the shell radius increases, and away from the net section, changes in the radius of curvature have a relatively small effect on the stress-resultant concentration. A contour plot of  $N_{\phi\phi}/M_0$  near the cutout is shown in Figure 13 for the case of  $R_0/L = 0.5$  shown in Figure 12. The extent of the stress concentration at the shell net section ( $\phi = 90^\circ$  and  $270^\circ$ ) is clearly captured by the analysis method presented herein. The highest stress-resultant concentration is  $N_{\phi\phi}(a, 90^\circ)/M_0 = 4$ , and it attenuates to the value of 1.01 at a radius of about 80 mm (approximately three times the cutout radius), measured from the center of the cutout.

## 9. Concluding remarks

A special-purpose, semianalytical approach based on complex potential functions has been presented that can be used to efficiently and parametrically investigate the behavior of thin, noncircular cross-section cylindrical shells made of laminated-composite materials and with a cutout. In particular, the effects of radius of curvature; elliptical cutout size, aspect ratio, and orientation; oval cross-section eccentricity; wall-thickness variations; and loading conditions on the stress-resultant concentration near the cutout have been presented for a quasiisotropic shell subjected to uniform tension, uniform torsion, and pure bending. In addition, studies that were conducted to validate the analysis method have been described.

A key finding of the results obtained with this analysis method is that the maximum tangential stress-resultant concentration near a circular cutout in a tension loaded, circular, quasiisotropic shell increases by approximately 18% as the shell radius-to-length ratio decreases from 1.25 to 0.5. Likewise, increases in the maximum tangential stress-resultant concentration as large as 65% have been found to occur with a five-fold increase in cutout radius. Results have also been presented showing that extremely high



tangential stress-resultant concentrations can occur for high aspect ratio elliptical cutouts whose principal axes are not aligned with the longitudinal axis of a tension loaded shell.

Additionally, results have been presented showing that tension loaded oval shells with a circular cutout on one of the flatter sides exhibit slightly lower tangential stress-resultant concentrations than the corresponding shell with the cutout on one of the more highly curved sides. Results have also been presented that show that variations in wall thickness in either the longitudinal or circumferential direction significantly affect the stress concentration, with respect to that for the corresponding shell with a nominal thickness. The analysis also predicts that a quasiisotropic shell with a circular cutout and subjected to pure bending that yields the maximum tensile stress resultant at the longitudinal axis of the cutout behaves similarly to the corresponding tension loaded shell. The corresponding shell subjected to torsion was found to exhibit the maximum tangential stress-resultant concentrations at locations consistent with the maximum diagonal tension and compression near the cutout. Overall, the results demonstrate that the analysis approach is a powerful means for developing design criteria for laminated composite shells.

### Appendix A

**Strain energy of the shell.** Based on classical laminated shell theory, the strain energy of the shell can be expressed as

$$U = \frac{1}{2} \int_A \mathbf{s}^T \mathbf{e} dA,$$

where  $A$  is the planform area of the shell mid-surface. Substituting the expressions for the resultant stress and strains, given in terms of the vector of unknown displacement coefficients,  $\mathbf{q}$ , by Equations (18) and (19), leads to

$$U(\mathbf{q}) = \frac{1}{2} \int_A \mathbf{q}^T (\mathbf{L}^T \mathbf{C} \mathbf{L}) \mathbf{q} dA.$$

The matrix  $\mathbf{L}$  involves the derivatives of the assumed functional displacement representations, and  $\mathbf{C}$  is the overall constitutive matrix defined by Equation (10). The expression for the strain energy is rewritten into the final form used herein as

$$U(\mathbf{q}) = \frac{1}{2} \mathbf{q}^T \mathbf{k}_{qq} \mathbf{q}, \quad \mathbf{k}_{qq} = \int_A (\mathbf{L}^T \mathbf{C} \mathbf{L}) dA.$$

The evaluation of this area integral is performed numerically by employing basic quadrature techniques. In this analysis, the quadrature points are predetermined by employing standard triangulation of the entire domain as described by [Shewchuk 1996].

**Strain energy of elastic restraints.** The strain energy of the elastic edge restraints (springs),  $\Omega$ , is expressed as

$$\Omega = \sum_{\ell=1}^2 \left[ \frac{1}{2} \sum_{\alpha=n,s,t} \int_{\Gamma(\ell)} S_{\alpha} (u_{\alpha} - u_{\alpha}^*)^2 d\Gamma + \frac{1}{2} \sum_{\alpha=n,s} \int_{\Gamma(\ell)} J_{\alpha} (\vartheta_{\alpha} - \vartheta_{\alpha}^*)^2 d\Gamma + \frac{1}{2} \sum_{\alpha=n,s} \int_{\Gamma(\ell)} s_{\alpha} (u_{\alpha} - \Delta_{\alpha})^2 d\Gamma \right]. \quad (\text{A.1})$$

As depicted in Figure 4, the boundary displacements  $u_n$ ,  $u_s$ , and  $u_t$  along the  $\ell^{\text{th}}$  boundary segment are restrained by extensional springs with the stiffness values  $S_n$ ,  $S_s$ , and  $S_t$ , respectively. Likewise, the

boundary rotations  $\vartheta_n$  and  $\vartheta_s$  are restrained by rotational springs with the stiffness values  $J_n$  and  $J_s$ , respectively.

To apply concentrated forces along the edge of a shell and introduce edge displacements similar to those introduced by a rigid end ring or by the loading platens of a testing machine, additional springs are used to simulate the load introduction effects of a rigid end ring. In particular, rigid end ring loads are introduced into the shell using extensional springs in the directions normal and tangent to the boundary with corresponding stiffness values of  $s_n$  and  $s_s$ , as shown in [Figure 3](#). By specifying relatively large values for the spring stiffnesses  $s_n$  and  $s_s$ , the laminate edge behaves as if a rigid end ring is attached that produces the uniform displacements  $\Delta_n$  and  $\Delta_s$ . In contrast, a relatively small spring stiffness between the shell edge and the rigid end ring eliminates the presence of a rigid end ring.

The desired form of the elastic-restraint strain energy is obtained in terms of the unknown vector  $\mathbf{q}$  by substituting expressions for the boundary displacements and rotations, given collectively by [Equation \(17\)](#), into [Equation \(A.1\)](#). This step yields

$$\Omega = \sum_{\ell=1}^2 \left[ \frac{1}{2} \sum_{\alpha=n,s,t} (\mathbf{q}^T \mathbf{S}_{\alpha\alpha}^{(\ell)} \mathbf{q} + \Omega_{u(\alpha)}^{(\ell)*} - 2\mathbf{q}^T \mathbf{f}_{\alpha}^{(\ell)*}) + \frac{1}{2} \sum_{\alpha=n,s} (\mathbf{q}^T \mathbf{J}_{\alpha\alpha}^{(\ell)} \mathbf{q} + \Omega_{\vartheta(\alpha)}^{(\ell)*} - 2\mathbf{q}^T \mathbf{r}_{\alpha}^{(\ell)*}) + \frac{1}{2} \sum_{\alpha=n,s} \left( \mathbf{q}^T \mathbf{s}_{\alpha\alpha}^{(\ell)} \mathbf{q} + \int_{\Gamma(\ell)} s_{\alpha} \Delta_{\alpha}^2 d\Gamma - 2\mathbf{q}^T \mathbf{s}_{\alpha}^{(\ell)} \Delta_{\alpha} \right) \right],$$

where the matrices  $\mathbf{S}_{\alpha\alpha}^{(\ell)}$  and  $\mathbf{J}_{\alpha\alpha}^{(\ell)}$  represent the stiffness contribution of the extensional and rotational springs attached to the  $\ell^{th}$  segment of the boundary. These matrices are defined as

$$\mathbf{S}_{\alpha\alpha}^{(\ell)} = \int_{\Gamma(\ell)} S_{\alpha} \mathbf{u}_{\alpha} \mathbf{u}_{\alpha}^T d\Gamma, \quad \mathbf{J}_{\beta\beta}^{(\ell)} = \int_{\Gamma(\ell)} J_{\beta} \boldsymbol{\theta}_{\beta} \boldsymbol{\theta}_{\beta}^T d\Gamma, \quad (\alpha = n, s, t; \beta = n, s).$$

The matrix  $\mathbf{s}_{\alpha\alpha}^{(\ell)}$ , representing the stiffness of the springs attached to the rigid end ring, is defined as

$$\mathbf{s}_{\alpha\alpha}^{(\ell)} = \int_{\Gamma(\ell)} s_{\alpha} \mathbf{u}_{\alpha} \mathbf{u}_{\alpha}^T d\Gamma, \quad (\alpha = n, s).$$

The load vectors  $\mathbf{f}_{\alpha}^{(\ell)*}$  and  $\mathbf{r}_{\alpha}^{(\ell)*}$  are associated with the prescribed boundary displacements and rotations

$$\mathbf{f}_{\alpha}^{(\ell)*} = \int_{\Gamma(\ell)} S_{\alpha} u_{\alpha}^* \mathbf{u}_{\alpha} d\Gamma, \quad \mathbf{r}_{\beta}^{(\ell)*} = \int_{\Gamma(\ell)} J_{\beta} \vartheta_{\beta}^* \boldsymbol{\theta}_{\beta} d\Gamma, \quad (\alpha = n, s, t; \beta = n, s).$$

The vector  $\mathbf{s}_{\alpha}^{(\ell)}$  is associated with the unknown end displacements that correspond to a given concentrated load, and is defined as

$$\mathbf{s}_{\alpha}^{(\ell)} = \int_{\Gamma(\ell)} s_{\alpha} \mathbf{u}_{\alpha} d\Gamma, \quad (\alpha = n, s).$$

The strain energies in the springs that arise from the known prescribed displacements ( $u_n^*$ ,  $u_s^*$ ,  $u_t^*$ ) and rotations ( $\vartheta_n^*$ ,  $\vartheta_s^*$ ) are defined as

$$\Omega_{u(\alpha)}^{(\ell)*} = \int_{\Gamma(\ell)} S_{\alpha} u_{\alpha}^{*2} d\Gamma, \quad \Omega_{\vartheta(\beta)}^{(\ell)*} = \int_{\Gamma(\ell)} J_{\beta} \vartheta_{\beta}^{*2} d\Gamma, \quad (\alpha = n, s, t; \beta = n, s).$$

For convenience, the expression for the strain energy in the springs is recast in matrix form as

$$\Omega(\mathbf{q}, \Delta) = \frac{1}{2} \mathbf{q}^T \mathbf{S}_{qq} \mathbf{q} + \frac{1}{2} \Delta^T \mathbf{s}_{\Delta\Delta} \Delta - \mathbf{q}^T \mathbf{s}_{q\Delta} \Delta - \mathbf{q}^T \mathbf{f}^* + \Omega^*,$$

in which the matrices  $\mathbf{S}_{qq}$ ,  $\mathbf{s}_{\Delta\Delta}$ , and  $\mathbf{s}_{q\Delta}$  represent the stiffness of the springs associated with the deformation of the laminate, the end displacements, and their coupling, respectively. They are

$$\begin{aligned} \mathbf{S}_{qq} &= \sum_{\ell=1}^2 \sum_{\alpha=n,s,t} \mathbf{S}_{\alpha\alpha}^{(\ell)} + \sum_{\ell=1}^2 \sum_{\alpha=n,s} \mathbf{J}_{\alpha\alpha}^{(\ell)} + \sum_{\ell=1}^2 \sum_{\alpha=n,s} \mathbf{s}_{\alpha\alpha}^{(\ell)}, \\ \mathbf{s}_{\Delta\Delta} &= \text{Diag}[s_n^{(1)}, s_n^{(2)}, s_s^{(1)}, s_s^{(2)}] \times 2\pi R_0, \\ \mathbf{s}_{q\Delta} &= \begin{bmatrix} \mathbf{s}_n^{(1)} & \mathbf{s}_n^{(2)} & \mathbf{s}_s^{(1)} & \mathbf{s}_s^{(2)} \end{bmatrix}. \end{aligned}$$

The vector of unknown end displacements  $\Delta$  is defined by  $\Delta^T = [\Delta_n^{(1)} \ \Delta_n^{(2)} \ \Delta_s^{(1)} \ \Delta_s^{(2)}]$ . The load vector arising from all prescribed boundary displacements and rotations  $\mathbf{f}^*$ , and the strain energy of all the springs due to prescribed displacements and rotations are defined as

$$\mathbf{f}^* = \frac{1}{2} \sum_{\ell=1}^2 \sum_{\alpha=n,s,t} \mathbf{f}_{\alpha}^{*(\ell)} + \frac{1}{2} \sum_{\ell=1}^2 \sum_{\alpha=n,s} \mathbf{r}_{\alpha}^{*(\ell)}, \quad \Omega^* = \frac{1}{2} \sum_{\ell=1}^2 \sum_{\alpha=n,s,t} \Omega_{u(\alpha)}^{(\ell)*} + \frac{1}{2} \sum_{\ell=1}^2 \sum_{\alpha=n,s} \Omega_{\vartheta(\alpha)}^{(\ell)*}.$$

**Potential of external loads.** The potential energy of the external tractions ( $t_n^*$ ,  $t_s^*$ ,  $t_t^*$ ) and moments ( $m_n^*$ ,  $m_s^*$ ) acting along the  $\ell^{\text{th}}$  boundary segment, and the concentrated loads ( $P_n^*$ ,  $P_s^*$ ) acting on the rigid end rings, is given in terms of the corresponding boundary displacements and rotations by

$$V = - \sum_{\ell=1}^2 \sum_{\alpha=n,s,t} \int_{\Gamma(\ell)} t_{\alpha}^* u_{\alpha} d\Gamma - \sum_{\ell=1}^2 \sum_{\alpha=n,s} \int_{\Gamma(\ell)} m_{\alpha}^* \theta_{\alpha} d\Gamma - \sum_{\ell=1}^2 \sum_{\alpha=n,s} P_{\alpha}^* \Delta_{\alpha}.$$

Substituting the expressions for the boundary displacements and rotations, given in terms of the vector  $\mathbf{q}$ , and combining terms yields  $V(\mathbf{q}, \Delta) = -\mathbf{q}^T \mathbf{T}^* - \Delta^T \mathbf{P}^*$ , where the vector  $\Delta$ , containing the uniform end-displacements  $\Delta_n$  and  $\Delta_s$  of the  $\ell^{\text{th}}$  boundary segment, is defined by  $\Delta^T = [\Delta_n^{(1)} \ \Delta_n^{(2)} \ \Delta_s^{(1)} \ \Delta_s^{(2)}]$ . The load vectors  $\mathbf{T}^*$  and  $\mathbf{P}^*$  are defined by

$$\mathbf{T}^{*T} = \sum_{\ell=1}^2 \sum_{\alpha=n,s,t} \int_{\Gamma(\ell)} t_{\alpha}^* \mathbf{u}_{\alpha}^T d\Gamma + \sum_{\ell=1}^2 \sum_{\alpha=n,s} \int_{\Gamma(\ell)} \theta_{\alpha}^* \boldsymbol{\theta}_{\alpha}^T d\Gamma, \quad \mathbf{P}^{*T} = \begin{bmatrix} P_n^{*(1)} & P_n^{*(2)} & P_s^{*(1)} & P_s^{*(2)} \end{bmatrix},$$

in which  $P_{\alpha}^{*(\ell)}$ , with  $\alpha = n, s$ , represents the membrane forces applied on the  $\ell^{\text{th}}$  boundary segment through a rigid end ring.

## Appendix B

**Rigid-body modes.** Following [Madenci and Barut 2003], the rigid-body displacements ( $u_{R1}$ ,  $u_{R2}$ ,  $u_{R3}$ ) of a cylindrical shell, defined with respect to the curvilinear coordinates, ( $s_1$ ,  $s_2$ ,  $s_3$ ), are

$$\begin{aligned} u_{R1} &= \alpha_1 - \alpha_6 y + \alpha_5 z, \\ u_{R2} &= \alpha_2 \cos \theta - \alpha_3 \sin \theta - \alpha_4 (y \sin \theta + z \cos \theta) + \alpha_5 x \sin \theta + \alpha_6 x \cos \theta, \\ u_{R3} &= \alpha_2 \sin \theta + \alpha_3 \cos \theta + \alpha_4 (y \cos \theta - z \sin \theta) - \alpha_5 x \cos \theta + \alpha_6 x \sin \theta, \end{aligned} \quad (\text{B.1})$$

where  $\theta$  denotes the angle between the radius of curvature at a point on the shell surface and  $z$ -axis as shown in Figure 1.

**Global functions.** The global functions  $\bar{u}_i$ , that are used to capture the overall deformations away from the cutout are expressed in terms of a series expansion of orthogonal functions of the form

$$\bar{u}_i(s_1, s_2) = \sum_{m=0}^M \sum_{n=0}^m c_{i(mn)} T_m(s_1) W_n(s_2). \quad (\text{B.2})$$

The symbols  $c_{i(mn)}$  are the unknown real-valued coefficients, and  $T_m(s_1)$  and  $W_n(s_2)$  are defined as

$$T_m(s_1) = \begin{cases} 1, & m = 0, \\ \zeta, & m = 1, \\ \sin\left[\frac{(m-1)}{2}(\zeta + 1)\right], & m > 1, \end{cases} \quad W_n(s_2(\theta)) = \begin{cases} \cos \frac{n\theta}{2}, & n \text{ even}, \\ \sin \frac{(n+1)\theta}{2}, & n \text{ odd}, \end{cases} \quad (\text{B.3})$$

in which  $-1 \leq \zeta \leq 1$  and  $s_1$  is related to  $\zeta$  as  $s_1 = \zeta L/2$ , with  $L$  being the length of the cylinder. Note that  $W_n$  is periodical. These particular functions were chosen because they form a complete set of functions when used with Equation (B.2). Hence, they are desirable for employing in energy based semianalytic solution techniques such as the total potential energy principle that is used in this study.

**Local functions.** The local functions are expressed in terms of mapping functions that transform the contour of an elliptical cutout to a unit circle. These mapping functions are used to permit the use of Laurent series expansions as local functions, which is desirable because Laurent series are analytic and uniformly convergent in domains with a circular hole. As a result, the use of mapping functions significantly reduces the number of terms in the Laurent series that are needed to adequately capture steep stress and strain gradients and local deformations near a cutout. In accordance with the principle of minimum potential energy, the local functions are not required to satisfy the traction boundary conditions at the cutout boundary. Thus, the local functions  $\bar{\bar{u}}_i$  are expressed in the form of Laurent series in terms

of complex functions, as

$$\bar{u}_{i=1,2} = 2 \operatorname{Re} \left[ \sum_{m=1}^2 u_m^{(i=1,2)} \sum_{\substack{n=-N \\ n \neq 0}}^N \alpha_{nm} \Phi_{nm}^*(z_{\varepsilon m}) \right] H(\rho), \tag{B.4}$$

$$\bar{u}_3 = 2 \operatorname{Re} \left[ \sum_{m=1}^2 \sum_{\substack{n=-N \\ n \neq 0}}^N \beta_{nm} F_{nm}^*(z_{\varepsilon m}) \right] H(\rho), \tag{B.5}$$

with  $\rho = (x_1^2 + x_2^2)^{1/2}$ , where the parameter  $N$  defines the extent of the complex series. In these series,  $\alpha_{nm}$  and  $\beta_{nm}$  are the unknown complex coefficients that appear in Equations (14)–(16). The auxiliary function  $H(\rho)$  that defines the domain of influence of the local functions is expressed as a polynomial

$$H(\rho) = \begin{cases} 1 - 10\left(\frac{\rho}{\rho_o}\right)^3 + 15\left(\frac{\rho}{\rho_o}\right)^4 - 6\left(\frac{\rho}{\rho_o}\right)^5, & 0 \leq \rho \leq \rho_o, \\ 0, & \rho > \rho_o, \end{cases}$$

such that  $H(\rho_o) = H'(\rho_o) = H''(\rho_o) = 0$ , where prime marks denote differentiation with respect to the variable  $\rho$ , and the parameter  $\rho_o$  denotes the radius of the region in which the local functions are effective. The purpose of choosing the auxiliary function is to prevent any possible linear dependency between the local and global functions and to restrict the influence of the local functions to a limited domain around the cutout.

The complex functions  $u_m^{(1)}(z_{\varepsilon m})$  and  $u_m^{(2)}(z_{\varepsilon m})$  that appear in Equation (B.4) are defined as

$$u_m^{(1)}(z_{\varepsilon m}) = \cos \psi p_m(z_{\varepsilon m}) - \sin \psi q_m(z_{\varepsilon m}), \quad u_m^{(2)}(z_{\varepsilon m}) = \sin \psi p_m(z_{\varepsilon m}) + \cos \psi q_m(z_{\varepsilon m}), \tag{B.6}$$

where the complex constants  $p_m$  and  $q_m$  are given by

$$p_m = a_{11}\mu_{\varepsilon m}^2 + a_{12} - a_{16}\mu_{\varepsilon m}, \quad q_m = a_{12}\mu_{\varepsilon m} + a_{22}/\mu_{\varepsilon m} - a_{26}.$$

In Equation (B.6), the unknown complex constants  $\mu_{\varepsilon m}$  are the roots of the characteristic equation associated with membrane deformation, that is,

$$a_{11}\mu_{\varepsilon m}^4 - 2a_{16}\mu_{\varepsilon m}^3 + (2a_{26} + a_{66})\mu_{\varepsilon m}^2 - 2a_{26}\mu_{\varepsilon m} + a_{22} = 0, \tag{B.7}$$

in which the coefficients  $a_{ij}$  are the coefficients of the flexibility matrix  $\mathbf{a}$ , which is the inverse of the stiffness matrix  $\mathbf{A}$  defined by Equation (7). Both the flexibility and the stiffness matrices,  $\mathbf{a}$  and  $\mathbf{A}$ , are measured with respect to the local coordinate system  $(x_1, x_2)$ . The angle  $\psi$  represents the orientation of the local coordinate system with respect to the global coordinate system,  $(s_1, s_2)$ .

The complex potential function  $\Phi_{nm}^*(z_{\varepsilon m})$ , appearing in Equation (B.4) is defined as  $\Phi_{nm}^*(z_{\varepsilon m}) = \xi_{\varepsilon m}^n$ , in which the mapping functions  $\xi_{\varepsilon m}$  map a cutout onto a unit circle. The mapping functions for an elliptical cutout, introduced by [Lekhnitskii 1968], are given by

$$\xi_{\varepsilon m} = \frac{z_{\varepsilon m} \pm \sqrt{z_{\varepsilon m}^2 - a^2 - \mu_{\varepsilon m}^2 b^2}}{a - i\mu_{\varepsilon m}^2 b}, \quad m = 1, 2, \tag{B.8}$$

where  $z_{\varepsilon m} = x_1 + \mu_{\varepsilon m}x_2$ ,  $a$  and  $b$  are the major and minor axes of the elliptical cutout, and  $i = \sqrt{-1}$ . The sign of the square root term is chosen so that  $|\xi_{\varepsilon m}| \geq 1$  (that is, the mapped point is guaranteed to be on or outside the unit circle).

Inverting the mapping function provides  $\omega_{\varepsilon m}(\xi_{\varepsilon m})$  as

$$\begin{aligned} z_{\varepsilon m} &= \omega_{\varepsilon m}(\xi_{\varepsilon m}) = r_{\varepsilon m}\xi_{\varepsilon m} - \frac{s_{\varepsilon m}}{\xi_{\varepsilon m}}, \\ r_{\varepsilon m} &= \frac{1}{2}(a - i\mu_{\varepsilon m}b), \\ s_{\varepsilon m} &= \frac{1}{2}(a + i\mu_{\varepsilon m}b). \end{aligned} \quad (\text{B.9})$$

The unknown complex constants  $\mu_{\varepsilon 1}$  and  $\mu_{\varepsilon 2}$ , and their complex conjugates, that is,  $\mu_{\varepsilon 3} = \bar{\mu}_{\varepsilon 1}$  and  $\mu_{\varepsilon 4} = \bar{\mu}_{\varepsilon 2}$ , are the roots obtained from the characteristic equation associated with membrane deformation. The complex potential functions,  $F_{nm}^*(z_{\kappa m})$  in Equation (B.5) are defined as

$$F_{nm}^*(z_{\kappa m}) = \begin{cases} \frac{r_{\kappa m}}{n+1}\xi_{\kappa m}^n - \frac{s_{\kappa m}}{n-1}\xi_{\kappa m}^{n-2}, & |n| > 1, \\ \frac{r_{\kappa m}}{2}\xi_{\kappa m}^2 - s_{\kappa m} \ln \xi_{\kappa m}, & n = 1, \\ r_{\kappa m} \ln \xi_{\kappa m} + \frac{s_{\kappa m}}{2}\xi_{\kappa m}^{-2}, & n = -1, \end{cases} \quad (\text{B.10})$$

in which the expressions for the mapping function  $\xi_{\kappa m}$  and the constants  $r_{\kappa m}$  and  $s_{\kappa m}$  have the same form as the corresponding expressions for  $\xi_{\varepsilon m}$ ,  $r_{\varepsilon m}$ , and  $s_{\varepsilon m}$  given by Equations (B.8) and (B.9), except that the subscript  $\varepsilon$  is replaced by  $\kappa$ . The complex variables  $z_{\kappa m}$  are defined by  $z_{\kappa m} = x_1 + \mu_{\kappa m}x_2$ , in which the unknown complex constants  $\mu_{\kappa 1}$  and  $\mu_{\kappa 2}$  and their conjugates, that is,  $\mu_{\kappa 3} = \bar{\mu}_{\kappa 1}$  and  $\mu_{\kappa 4} = \bar{\mu}_{\kappa 2}$ , are the roots obtained from the characteristic equation associated with the bending equilibrium equation

$$D_{22}\mu_{\kappa m}^4 + 4D_{26}\mu_{\kappa m}^3 + (2D_{12} + 4D_{66})\mu_{\kappa m}^2 + 4D_{16}\mu_{\kappa m} + D_{11} = 0, \quad (\text{B.11})$$

where  $D_{ij}$  are the components of the bending stiffness matrix  $\mathbf{D}$  (Equation (7)), which is defined with respect to the local coordinate system,  $(x_1, x_2)$ .

It is important to note that the local functions in Equations (B.4) and (B.5) satisfy the in-plane and bending equilibrium equations of a homogeneous, flat laminate of uniform thickness, not a cylindrical shell. Therefore, the roots of the characteristic equations, Equations (B.7) and (B.11), serve as approximations of their exact values, which are not mathematically tractable. Because the solution procedure is based on the principle of minimum potential, their exact values are not necessarily required. However, they capture the stress concentration and local deformation near the cutout in cylindrical shells because these functions possess the inherent solution characteristics. They satisfy the equilibrium equations exactly as the radius of curvature approaches infinity and are uniformly convergent in a doubly connected region.

In the displacement representations defined by Equation (14), the vectors,  $\alpha_R$ ,  $c_i$ ,  $\alpha$ ,  $\beta$  are defined as

$$\begin{aligned}\alpha_R^T &= [\alpha_{R1} \quad \alpha_{R2} \quad \alpha_{R3} \quad \alpha_{R4} \quad \alpha_{R5} \quad \alpha_{R6}], \\ c_i^T &= [c_{i(00)} \quad c_{i(10)} \quad c_{i(01)} \quad \dots \quad c_{i(M0)} \quad c_{i((M-1)1)} \quad \dots \quad c_{i(1(M-1))} \quad c_{i(0M)}], \\ \alpha^T &= [\alpha_{-N}^T \quad \alpha_{-N+1}^T \quad \dots \quad \alpha_{-1}^T \quad \alpha_1^T \quad \dots \quad \alpha_{N-1}^T \quad \alpha_N^T], \\ \beta^T &= [\beta_{-N}^T \quad \beta_{-N+1}^T \quad \dots \quad \beta_{-1}^T \quad \beta_1^T \quad \dots \quad \beta_{N-1}^T \quad \beta_N^T],\end{aligned}$$

in which  $\gamma_n^T = [\gamma_{n1}^T \quad \gamma_{n2}^T]$ , with  $\gamma_{nj}^T = [\text{Re}(\gamma_{nj}), \text{Im}(\gamma_{nj})]$  for  $\gamma = \alpha, \beta$ .

The vector functions,  $V_{i=1,2,3}$  associated with the unknown generalized coordinates  $q$  appearing in Equation (15) are defined as

$$\begin{aligned}V_1^T &= [V_{R1}^T \quad \bar{V}_1^T \quad \bar{\mathbf{0}}^T \quad \bar{\mathbf{0}}^T \quad \bar{\bar{V}}_1^T \quad \bar{\bar{\mathbf{0}}}^T], \\ V_2^T &= [V_{R2}^T \quad \bar{\mathbf{0}}^T \quad \bar{V}_2^T \quad \bar{\mathbf{0}}^T \quad \bar{\bar{V}}_2^T \quad \bar{\bar{\mathbf{0}}}^T], \\ V_3^T &= [V_{R3}^T \quad \bar{\mathbf{0}}^T \quad \bar{\mathbf{0}}^T \quad \bar{V}_3^T \quad \bar{\mathbf{0}}^T \quad \bar{\bar{V}}_3^T],\end{aligned}$$

where  $\bar{\mathbf{0}}^T = [0 \dots 0]$  of order  $[(M+1)(M+2)/2]$  and  $\bar{\bar{\mathbf{0}}}^T = [0 \dots 0]$  of order  $8N$ . The vectors associated with rigid-body motion are

$$\begin{aligned}V_{R1}^T &= [1 \quad 0 \quad 0 \quad 0 \quad z \quad -y], \\ V_{R2}^T &= \left[ 0 \quad \frac{dy}{ds_2} \quad \frac{dz}{ds_2} \quad \left( y \frac{dz}{ds_2} - z \frac{dy}{ds_2} \right) \quad -x \frac{dz}{ds_2} \quad -x \frac{dy}{ds_2} \right], \\ V_{R3}^T &= \left[ 0 \quad -\frac{dz}{ds_2} \quad \frac{dy}{ds_2} \quad \left( y \frac{dy}{ds_2} + z \frac{dz}{ds_2} \right) \quad -x \frac{dy}{ds_2} \quad -x \frac{dz}{ds_2} \right].\end{aligned}$$

Similarly, the vectors associated with the global functions are

$$\bar{V}_i^T = [T_0 W_0 \quad T_1 W_0 \quad T_0 W_1 \quad T_2 W_0 \quad T_1 W_1 \quad T_0 W_2 \quad \dots \quad T_M W_0 \quad T_{M-1} W_1 \quad \dots \quad T_1 W_{M-1} \quad T_0 W_M],$$

in which the expressions for  $T_i(s_1)$  and  $W_i(s_2)$  are given by Equation (B.3), and

$$\begin{aligned}\bar{\bar{V}}_i^T &= [\bar{\bar{V}}_{i(-N)}^T \quad \bar{\bar{V}}_{i(-N+1)}^T \quad \dots \quad \bar{\bar{V}}_{i(-1)}^T \quad \bar{\bar{V}}_{i(1)}^T \quad \dots \quad \bar{\bar{V}}_{i(N-1)}^T \quad \bar{\bar{V}}_{i(N)}^T], \quad i = 1, 2, \\ \bar{\bar{V}}_3^T &= [\bar{\bar{V}}_{3(-N)}^T \quad \bar{\bar{V}}_{3(-N+1)}^T \quad \dots \quad \bar{\bar{V}}_{3(-1)}^T \quad \bar{\bar{V}}_{3(1)}^T \quad \dots \quad \bar{\bar{V}}_{3(N-1)}^T \quad \bar{\bar{V}}_{3(N)}^T],\end{aligned}$$

with

$$\begin{aligned}\bar{\bar{V}}_{i(n)}^T &= [\bar{\bar{V}}_{i(n1)}^T \quad \bar{\bar{V}}_{i(n2)}^T], \quad \bar{\bar{V}}_{i(nj)}^T = [2 \text{Re}[u_j^{(i)} \Phi_{nj}^*] \quad -2 \text{Im}[u_j^{(i)} \Phi_{nj}^*]], \quad (i, j = 1, 2), \\ \bar{\bar{V}}_{3(n)}^T &= [\bar{\bar{V}}_{3(n1)}^T \quad \bar{\bar{V}}_{3(n2)}^T], \quad \bar{\bar{V}}_{3(nj)}^T = [2 \text{Re}[F_{nj}^*] \quad -2 \text{Im}[F_{nj}^*]], \quad (j = 1, 2).\end{aligned}$$



**Appendix C**

**Constraint equations.** The unknown vector of Lagrange multipliers  $\lambda$ , and the known coefficient matrix  $G$ , in Equation (11) are defined by  $\lambda^T = [\lambda_{RRB(1)} \lambda_{RRB(2)} \lambda_{SV(r)} \lambda_{SV(s)} \lambda_{RB(1)} \dots \lambda_{RB(6)}]$ , and

$$G = \begin{bmatrix} \mathbf{0}_R^T & \mathbf{g}_{RRB(1)}^T & \bar{\mathbf{0}}^T & \bar{\mathbf{0}}^T & \bar{\bar{\mathbf{0}}}^T & \bar{\bar{\mathbf{0}}}^T \\ \mathbf{0}_R^T & \bar{\mathbf{0}}^T & \mathbf{g}_{RRB(2)}^T & \bar{\mathbf{0}}^T & \bar{\mathbf{0}}^T & \bar{\bar{\mathbf{0}}}^T \\ \mathbf{0}_R^T & \bar{\mathbf{0}}^T & \bar{\mathbf{0}}^T & \bar{\mathbf{0}}^T & \bar{\mathbf{0}}^T & \mathbf{g}_{SV(s)}^T \\ \mathbf{0}_R^T & \mathbf{0}_L^T & \bar{\mathbf{0}}^T & \bar{\mathbf{0}}^T & \bar{\mathbf{0}}^T & \mathbf{g}_{SV(r)}^T \\ \mathbf{g}_{RB(1)}^T & \bar{\mathbf{0}}^T & \bar{\mathbf{0}}^T & \bar{\mathbf{0}}^T & \bar{\mathbf{0}}^T & \bar{\bar{\mathbf{0}}}^T \\ \mathbf{g}_{RB(2)}^T & \bar{\mathbf{0}}^T & \bar{\mathbf{0}}^T & \bar{\mathbf{0}}^T & \bar{\mathbf{0}}^T & \bar{\bar{\mathbf{0}}}^T \\ \mathbf{g}_{RB(3)}^T & \bar{\mathbf{0}}^T & \bar{\mathbf{0}}^T & \bar{\mathbf{0}}^T & \bar{\mathbf{0}}^T & \bar{\bar{\mathbf{0}}}^T \\ \mathbf{g}_{RB(4)}^T & \bar{\mathbf{0}}^T & \bar{\mathbf{0}}^T & \bar{\mathbf{0}}^T & \bar{\mathbf{0}}^T & \bar{\bar{\mathbf{0}}}^T \\ \mathbf{g}_{RB(5)}^T & \bar{\mathbf{0}}^T & \bar{\mathbf{0}}^T & \bar{\mathbf{0}}^T & \bar{\mathbf{0}}^T & \bar{\bar{\mathbf{0}}}^T \\ \mathbf{g}_{RB(6)}^T & \bar{\mathbf{0}}^T & \bar{\mathbf{0}}^T & \bar{\mathbf{0}}^T & \bar{\mathbf{0}}^T & \bar{\bar{\mathbf{0}}}^T \end{bmatrix},$$

in which the constant coefficient vectors  $\mathbf{g}_{RRB(1)}$  and  $\mathbf{g}_{RRB(2)}$  are associated with redundant rigid-body modes  $\mathbf{g}_{SV(r)}$  and  $\mathbf{g}_{SV(s)}$ , with single-valuedness of the radial displacement component, and  $\mathbf{g}_{RB(j)}$  with the rigid-body modes introduced by the global functions defined in Equation (B.2). The rigid-body modes must be eliminated in the absence of a sufficient number of specified kinematic boundary conditions. These terms, as well as the vectors with zeros, are defined in the following subsections.

**Redundant rigid-body modes.** The coefficients  $c_{1(00)}$  and  $c_{2(00)}$  in Equation (B.2) for the global displacement functions produce additional rigid-body translation in the  $s_1$  direction and rigid body rotation about the  $s_1$  axis, respectively. Because these rigid-body modes are already represented by  $\alpha_{R1}$  and  $\alpha_{R6}$  in Equation (B.1), the redundant rigid-body motion arising from the presence of  $c_{1(00)}$  and  $c_{2(00)}$  must be eliminated in order to obtain a unique representation of the displacements. These redundant rigid-body modes are eliminated by using the constraint conditions  $\lambda_{RRB(1)}c_{1(00)} = 0, \lambda_{RRB(2)}c_{2(00)} = 0$  in which the unknown Lagrange multipliers are denoted by  $\lambda_{RRB(1)}$  and  $\lambda_{RRB(2)}$ . In terms of the vector of unknowns  $\mathbf{q}$ , these constraints are rewritten in vector form as

$$\lambda_{RB(1)} \begin{bmatrix} \mathbf{0}_R^T & \mathbf{g}_{RRB}^T & \bar{\mathbf{0}}^T & \bar{\mathbf{0}}^T & \bar{\bar{\mathbf{0}}}^T & \bar{\bar{\mathbf{0}}}^T \end{bmatrix} \begin{bmatrix} \alpha_R \\ \mathbf{c}_1 \\ \mathbf{c}_2 \\ \mathbf{c}_3 \\ \alpha \\ \beta \end{bmatrix} = 0, \quad \lambda_{RB(2)} \begin{bmatrix} \mathbf{0}_R^T & \bar{\mathbf{0}}^T & \mathbf{g}_{RRB}^T & \bar{\mathbf{0}}^T & \bar{\bar{\mathbf{0}}}^T & \bar{\bar{\mathbf{0}}}^T \end{bmatrix} \begin{bmatrix} \alpha_R \\ \mathbf{c}_1 \\ \mathbf{c}_2 \\ \mathbf{c}_3 \\ \alpha \\ \beta \end{bmatrix} = 0,$$

in which the vectors of zeros,  $\mathbf{0}_R^T, \bar{\mathbf{0}}^T$ , and  $\bar{\bar{\mathbf{0}}}^T$ , are defined by  $\mathbf{0}_R^T = [0\ 0\ 0\ 0\ 0\ 0]$ ,  $\bar{\mathbf{0}}^T = [0\ 0\ \dots\ 0]$  of order  $[(M + 1)(M + 2)/2]$ , and  $\bar{\bar{\mathbf{0}}}^T = [0\ 0\ \dots\ 0]$  of order  $8N$ , while the constant coefficient vector,  $\mathbf{g}_{RRB}$ , is defined as  $\mathbf{g}_{RRB}^T = [1\ 0\ 0\ \dots\ 0]$  of order  $[(M + 1)(M + 2)/2]$ .

**Single-valuedness of the radial displacement component.** The multivaluedness of the normal displacement component that arises from the presence of logarithmic terms in the local expression for the radial displacement component in Equation (B.5) must be rendered single-valued in order to obtain a unique solution. The logarithmic terms associated with complex constants  $\beta_{-1m}$  and  $\beta_{1m}$ , with  $m = 1, 2$ , in Equation (B.10) result in two real constants. Representing the complex variable of the Laurent series  $\xi_m = \rho_m e^{i\theta_m}$ , the single-valuedness requirement is enforced as

$$u_z(\xi_m = \rho_m e^{i\theta_m}) - u_z(\xi_m = \rho_m e^{i(\theta_m+2\pi)}) = 0.$$

Associated with the complex constants  $\beta_{\mp 1m}$ , with  $m = 1, 2$ , in Equation (B.10), this condition yields

$$2 \operatorname{Re} \left\{ \sum_{m=1}^2 \left\{ F_{-1m}^*(\xi_m = \rho_m e^{i\theta_m}) - F_{-1m}^*(\xi_m = \rho_m e^{i(\theta_m+2\pi)}) \right\} \beta_{-1m} \right\} = 0 \quad \text{or} \quad \operatorname{Im} \sum_{m=1}^2 [r_m \beta_{-1m}] = 0,$$

$$2 \operatorname{Re} \left\{ \sum_{m=1}^2 \left\{ F_{1m}^*(\xi_m = \rho_m e^{i\theta_m}) - F_{1m}^*(\xi_m = \rho_m e^{i(\theta_m+2\pi)}) \right\} \beta_{1m} \right\} = 0 \quad \text{or} \quad \operatorname{Im} \sum_{m=1}^2 [s_{\kappa m} \beta_{1m}] = 0.$$

In order to ensure single-valuedness, these constraints are enforced as

$$\lambda_{SV(r)} \operatorname{Im} \sum_{m=1}^2 [r_{\kappa m} \beta_{-1m}] = 0, \quad \lambda_{SV(s)} \operatorname{Im} \sum_{m=1}^2 [s_{\kappa m} \beta_{1m}] = 0,$$

in which the unknown Lagrange multipliers are denoted by  $\lambda_{SV(r)}$  and  $\lambda_{SV(s)}$ .

In terms of the vector of unknowns  $\mathbf{q}$ , these constraint conditions can be recast in matrix form as

$$\lambda_{SV(r)} \begin{bmatrix} \mathbf{0}_R^T & \bar{\mathbf{0}}^T & \bar{\mathbf{0}}^T & \bar{\mathbf{0}}^T & \bar{\mathbf{0}}^T & \mathbf{g}_{SV(r)}^T \end{bmatrix} \begin{bmatrix} \alpha_R \\ \mathbf{c}_1 \\ \mathbf{c}_2 \\ \mathbf{c}_3 \\ \alpha \\ \beta \end{bmatrix} = 0, \quad \lambda_{SV(s)} \begin{bmatrix} \mathbf{0}_R^T & \bar{\mathbf{0}}^T & \bar{\mathbf{0}}^T & \bar{\mathbf{0}}^T & \bar{\mathbf{0}}^T & \mathbf{g}_{SV(s)}^T \end{bmatrix} \begin{bmatrix} \alpha_R \\ \mathbf{c}_1 \\ \mathbf{c}_2 \\ \mathbf{c}_3 \\ \alpha \\ \beta \end{bmatrix} = 0,$$

where the constant coefficient vectors  $\mathbf{g}_{SV(r)}$  and  $\mathbf{g}_{SV(s)}$  are given by

$$\mathbf{g}_{SV(j)}^T = \left[ \mathbf{g}_{SV(j)(-N)}^T \quad \mathbf{g}_{SV(j)(-N+1)}^T \quad \cdots \quad \mathbf{g}_{SV(j)(-1)}^T \quad \mathbf{g}_{SV(j)(1)}^T \quad \cdots \quad \mathbf{g}_{SV(j)(N-1)}^T \quad \mathbf{g}_{SV(j)(N)}^T \right],$$

$$\mathbf{g}_{SV(r)(1)}^T = [\operatorname{Im}[r_1] \quad \operatorname{Re}[r_1] \quad \operatorname{Im}[r_2] \quad \operatorname{Re}[r_2]], \quad \mathbf{g}_{SV(r)(n \neq 1)}^T = [0 \quad 0 \quad 0 \quad 0],$$

$$\mathbf{g}_{SV(s)(1)}^T = [\operatorname{Im}[s_1] \quad \operatorname{Re}[s_1] \quad \operatorname{Im}[s_2] \quad \operatorname{Re}[s_2]], \quad \mathbf{g}_{SV(s)(n \neq 1)}^T = [0 \quad 0 \quad 0 \quad 0],$$

with  $-N \leq n \leq N$ .

**Rigid-body modes.** In the absence of kinematic boundary conditions, the rigid-body modes of the displacement field are eliminated by enforcing the constraint conditions in the form  $\lambda_{RB(j)} \alpha_{R(j)} = 0$ , where  $j = 1, \dots, 6$  and the unknown Lagrange multipliers are denoted by  $\lambda_{RB(j)}$ . In terms of the vector of

unknowns  $\mathbf{q}$ , these constraints are rewritten in vector form as

$$\lambda_{RB(j)} \begin{bmatrix} \mathbf{g}_{RB(j)}^T & \bar{\mathbf{0}}^T & \bar{\mathbf{0}}^T & \bar{\mathbf{0}}^T & \bar{\mathbf{0}}^T & \bar{\mathbf{0}}^T \end{bmatrix} \begin{bmatrix} \alpha_R \\ c_1 \\ c_2 \\ c_3 \\ \alpha \\ \beta \end{bmatrix} = 0,$$

where the constant coefficient vectors  $\mathbf{g}_{RB(j)}$  are defined as  $\mathbf{g}_{RB(j)}^T = [\delta_{1j} \delta_{2j} \delta_{3j} \delta_{4j} \delta_{5j} \delta_{6j}]$ , in which  $\delta_{ij}$  is the Kronecker delta.

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