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SUPPLEMENT TO

ASYMPTOTIC ANALYSIS AND REFLECTION PHOTOELASTICITY FOR THE STUDY OF TRANSIENT CRACK PROPAGATION IN GRADED MATERIALS

STRESS, STRAIN, AND DISPLACEMENT FIELDS

NITESH JAIN AND ARUN SHUKLA

The following expressions complement Section 2.4 of the article. Maple code is available from the authors upon request.

$$u=\left\{\begin{array}{l}\frac{3}{2}A_0r_l^{1/2}\cos\frac{1}{2}\theta_l+\frac{3}{2}C_0r_l^{1/2}\sin\frac{1}{2}\theta_l+2A_1r_l\cos\theta_l+2C_1r_l\sin\theta_l\\ \quad+\frac{5}{2}A_2r_l^{3/2}\cos\frac{3}{2}\theta_l+\frac{5}{2}C_2r_l^{3/2}\sin\frac{3}{2}\theta_l-r_l^{3/2}\cos\frac{1}{2}\theta_l\Big[\frac{3}{2}P_1+\frac{1}{4}B_l^A\Big]\\ \quad-r_l^{3/2}\cos\frac{3}{2}\theta_l\Big[P_1+\frac{5}{2}P_8\alpha_l\Big]+2B_1\alpha_sr_s\cos\theta_s\\ \quad+r_l^{3/2}\sin\frac{1}{2}\theta_l\Big[\frac{3}{2}P_2+\frac{1}{4}B_l^C\Big]-r_l^{3/2}\sin\frac{3}{2}\theta_l\Big[P_2+\frac{5}{2}P_7\alpha_l\Big]\\ \quad-r_s^{3/2}\cos\frac{1}{2}\theta_s\Big[\frac{3}{2}P_6\alpha_s-\frac{1}{4}B_s^B\Big]+r_s^{3/2}\cos\frac{3}{2}\theta_s\Big[\alpha_sP_6-\frac{5}{2}P_3\Big]\\ \quad-r_s^{3/2}\alpha_s\sin\frac{1}{2}\theta_s\Big[\frac{3}{2}P_5-\frac{1}{4}B_s^D\Big]-r_s^{3/2}\alpha_s\sin\frac{3}{2}\theta_s\Big[P_5-\frac{5}{2}P_4\Big]\\ \quad+\frac{3}{2}B_0r_s^{1/2}\alpha_s\cos\frac{1}{2}\theta_s+\frac{3}{2}D_0r_s^{1/2}\alpha_s\sin\frac{1}{2}\theta+\frac{5}{2}B_2\alpha_sr_s^{3/2}\cos\frac{3}{2}\theta_s\\ \quad-\frac{5}{2}D_2\alpha_sr_s^{3/2}\sin\frac{3}{2}\theta_{ss}+2D_1\alpha_sr_s\sin\theta_s\\ \quad-\frac{1}{16}r_l^{3/2}\Big[B_l^A\cos\frac{5}{2}\theta_l-B_l^C\sin\frac{5}{2}\theta_l\Big]\\ \quad-\frac{1}{16}r_s^{3/2}\Big[B_l^B\cos\frac{5}{2}\theta_s+B_l^D\sin\frac{5}{2}\theta_s\Big]\end{array}\right\}\tag{A-1a}$$

$$v = \left\{ \begin{array}{l} -\frac{3}{2}A_0\alpha_l r_l^{1/2}\sin\frac{1}{2}\theta_l + \frac{3}{2}C_0\alpha_l r_l^{1/2}\cos\frac{1}{2}\theta_l + 2A_1\alpha_l r_l\sin\theta_l + 2C_1\alpha_l r_l\cos\theta_l \\ -\frac{5}{2}A_2\alpha_l r_l^{3/2}\sin\frac{3}{2}\theta_l + \frac{5}{2}C_2\alpha_l r_l^{3/2}\cos\frac{3}{2}\theta_l - r_l^{3/2}\alpha_l\cos\frac{1}{2}\theta_l\Big[\frac{3}{2}P_2 + \frac{1}{4}B_l^C\Big] \\ -r_l^{3/2}\cos\frac{3}{2}\theta_l\Big[\alpha_l P_2 + \frac{5}{2}P_7\Big] - \alpha_l r_l^{3/2}\sin\frac{1}{2}\theta_l\Big[\frac{3}{2}P_1 + \frac{1}{4}B_l^A\Big] \\ -r_l^{3/2}\sin\frac{3}{2}\theta_l\Big[\alpha_l P_1 - \frac{5}{2}P_8\Big] + r_s^{3/2}\cos\frac{1}{2}\theta_s\Big[\frac{3}{2}P_6 + \frac{1}{4}B_s^D\Big] \\ +r_s^{3/2}\cos\frac{3}{2}\theta_s\Big[P_5 + \frac{5}{2}\alpha_s P_4\Big] - r_s^{3/2}\sin\frac{1}{2}\theta_s\Big[\frac{3}{2}P_6 + \frac{1}{4}B_s^B\Big] \\ +r_s^{3/2}\sin\frac{3}{2}\theta_s\Big[P_6 + \frac{5}{2}P_3\Big] - \frac{3}{2}B_0 r_s^{1/2}\sin\frac{1}{2}\theta_s - \frac{3}{2}D_0 r_s^{1/2}\cos\frac{1}{2}\theta_s \\ -2B_1 r_s\sin\theta_s - 2D_1 r_s\cos\theta_s - \frac{5}{2}B_2 r_s^{3/2}\sin\frac{3}{2}\theta_s - \frac{5}{2}D_2 r_s^{3/2}\cos\frac{3}{2}\theta_s \\ -\frac{1}{16}r_l^{3/2}\Big[B_l^A\sin\frac{5}{2}\theta_l + B_l^C\cos\frac{5}{2}\theta_l\Big] - \frac{1}{16}r_s^{3/2}\Big[B_l^B\sin\frac{5}{2}\theta_s - B_l^D\cos\frac{5}{2}\theta_s\Big] \end{array} \right\}, \qquad \text{(A-1b)}$$

where

$$\begin{aligned} P_1 &= -\frac{1}{4\alpha_l^2}(\alpha A_0(t) + \beta\alpha_l C_0(t)) + \frac{1}{6}\Big[D_l(A_0(t)) + \frac{1}{2}B_l^A(t)\Big], \\ P_2 &= \frac{1}{4\alpha_l^2}(\alpha C_0(t) - \beta\alpha_l A_0(t)) + \frac{1}{6}\Big[D_l(C_0(t)) + \frac{1}{2}B_l^C(t)\Big], \\ P_3 &= -\frac{2}{5}\frac{1}{(k+2)(\alpha_l^2-\alpha_s^2)}(\alpha\alpha_s B_0(t) - \beta D_0(t)), \\ P_4 &= -\frac{2}{5}\frac{1}{(k+2)(\alpha_l^2-\alpha_s^2)}(\alpha B_0(t) + \beta\alpha_s D_0(t)), \\ P_5 &= -\frac{1}{4\alpha_s^2}(\alpha D_0(t) + \beta\alpha_s B_0(t)) + \frac{1}{6}\Big[D_s(B_0(t)) + \frac{1}{2}B_s^B(t)\Big], \\ P_6 &= \frac{1}{4\alpha_s^2}(\alpha B_0(t) - \beta D_0(t)) + \frac{1}{6}\Big[D_s(D_0(t)) + \frac{1}{2}B_s^D(t)\Big], \\ P_7 &= \frac{2}{5}\frac{k}{(\alpha_l^2-\alpha_s^2)}(\alpha\alpha_l C_0(t) - \beta A_0(t)), \\ P_8 &= \frac{2}{5}\frac{k}{(\alpha_l^2-\alpha_s^2)}(\alpha\alpha_l A_0(t) + \beta C_0(t)). \end{aligned}$$

In-plane strains can be obtained using these displacements in (A-2) as

$$\varepsilon_{xx} = \frac{\partial u}{\partial x}, \qquad \varepsilon_{yy} = \frac{\partial v}{\partial y}, \qquad \varepsilon_{xy} = \frac{1}{2}\Big(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\Big). \qquad \text{(A-2)}$$

$$\varepsilon_{xx} = \left\{ \begin{aligned} & \frac{3}{4}A_0 r_l^{-1/2}\cos\frac{1}{2}\theta_l - \frac{3}{4}C_0 r_l^{-1/2}\sin\frac{1}{2}\theta_l + 2A_1 + \frac{15}{4}r_l^{1/2}A_2\cos\frac{1}{2}\theta_l \\ & \quad + \frac{15}{4}r_l^{1/2}C_2\sin\frac{1}{2}\theta_l - r_l^{1/2}\cos\frac{1}{2}\theta_l\left[3P_1 + \frac{15}{4}P_8 + \frac{1}{4}B_l^A\right] \\ & \quad - r_l^{1/2}\cos\frac{3}{2}\theta_l\left[\frac{3}{4}P_1 + \frac{1}{4}B_l^A\right] - \frac{3}{4}P_6\alpha_s r_s^{1/2}\cos\frac{3}{2}\theta_s \\ & \quad + r_l^{1/2}\sin\frac{3}{2}\theta_l\left[\frac{3}{4}P_2 + \frac{1}{4}B_l^C\right] - \alpha_s r_s^{1/2}\cos\frac{1}{2}\theta_s\left[\frac{15}{4}P_3 - \frac{1}{4}B_s^B\right] \\ & \quad + \alpha_s r_s^{1/2}\sin\frac{1}{2}\theta_s\left[\frac{15}{4}P_4 - \frac{1}{4}B_s^D\right] - \frac{3}{4}P_5\alpha_s r_s^{1/2}\sin\frac{3}{2}\theta_s \\ & \quad + \frac{3}{4}B_0\alpha_s r_s^{-1/2}\cos\frac{1}{2}\theta_s + \frac{3}{4}D_0\alpha_s r_s^{-1/2}\sin\frac{1}{2}\theta_s \\ & \quad + \frac{15}{4}\alpha_s r_s^{1/2}B_2\cos\frac{1}{2}\theta_s - r_l^{1/2}\sin\frac{1}{2}\theta_l\left[3P_2 + \frac{15}{4}P_7 + \frac{1}{4}B_l^C\right] \\ & \quad + 2\alpha_s B_1 + \frac{1}{32}r_l^{1/2}\left[B_l^A\cos\frac{7}{2}\theta_l - B_l^C\sin\frac{7}{2}\theta_l\right] \\ & \quad - \frac{15}{4}\alpha_s r_s^{1/2}D_2\sin\frac{1}{2}\theta_s + \frac{1}{32}\alpha_s r_s^{1/2}\left[B_s^B\cos\frac{7}{2}\theta_s + B_s^D\sin\frac{7}{2}\theta_l\right] \end{aligned} \right\} \tag{A-3a}$$

$$\varepsilon_{yy} = \left\{ \begin{aligned} & -\frac{3}{4}A_0\alpha_l^2 r_l^{-1/2}\cos\frac{1}{2}\theta_l + \frac{3}{4}C_0\alpha_l^2 r_l^{-1/2}\sin\frac{1}{2}\theta_l - 2\alpha_l^2 A_1 \\ & -\frac{15}{4}r_l^{1/2}\alpha_l^2 A_2\cos\frac{1}{2}\theta_l - \frac{15}{4}r_l^{1/2}\alpha_l^2 C_2\sin\frac{1}{2}\theta_l \\ & - r_l^{1/2}\alpha_l\cos\frac{1}{2}\theta_l\left[3\alpha_l P_1 - \frac{15}{4}P_8 - \frac{1}{4}\alpha_l B_l^A\right] \\ & + r_l^{1/2}\alpha_l^2\cos\frac{3}{2}\theta_l\left[\frac{3}{4}P_1 - \frac{1}{4}B_l^A\right] - \frac{3}{4}P_5\alpha_s r_s^{1/2}\sin\frac{3}{2}\theta_s \\ & - r_l^{1/2}\alpha_l\sin\frac{1}{2}\theta_l\left[3\alpha_l P_2 - \frac{15}{4}P_7 - \frac{1}{4}\alpha_l B_l^C\right] \\ & + \alpha_s r_s^{1/2}\cos\frac{1}{2}\theta_s\left[\frac{15}{4}P_3\alpha_s - \frac{1}{4}B_s^B\right] - 2\alpha_s B_1 \\ & - \alpha_l^2 r_l^{1/2}\sin\frac{3}{2}\theta_l\left[\frac{3}{4}P_2 - \frac{1}{4}B_l^C\right] - \alpha_s r_s^{1/2}\sin\frac{1}{2}\theta_s\left[\frac{15}{4}P_4\alpha_s^- - \frac{1}{4}B_s^D\right] \\ & + \frac{3}{4}P_6\alpha_s r_s^{1/2}\cos\frac{3}{2}\theta_s - \frac{3}{4}B_0\alpha_s r_s^{-1/2}\cos\frac{1}{2}\theta_s - \frac{3}{4}D_0\alpha_s r_s^{-1/2}\sin\frac{1}{2}\theta_s \\ & - \frac{1}{32}\alpha_l^2 r_l^{1/2}\left[B_l^A\cos\frac{7}{2}\theta_l - B_l^C\sin\frac{7}{2}\theta_l\right] + \frac{15}{4}\alpha_s r_s^{1/2}D_2\sin\frac{1}{2}\theta_s \\ & - \frac{1}{32}r_s^{1/2}\left[B_s^B\cos\frac{7}{2}\theta_s + B_s^D\sin\frac{7}{2}\theta_l\right] - \frac{15}{4}\alpha_s r_s^{1/2}B_2\cos\frac{1}{2}\theta_s \end{aligned} \right\} \tag{A-3b}$$

$$\varepsilon_{xy}=\left\{\begin{aligned}&\frac{3}{4}A_0\alpha_l r_l^{-1/2}\sin\frac{1}{2}\theta_l+\frac{3}{4}C_0\alpha_l r_l^{-1/2}\cos\frac{1}{2}\theta_l+\frac{3}{8}B_0(1+\alpha_s^2)r_s^{-1/2}\sin\frac{1}{2}\theta_s\\&\quad-\frac{3}{8}D_0(1+\alpha_s^2)r_s^{-1/2}\cos\frac{1}{2}\theta_s-r_l^{1/2}\cos\frac{1}{2}\theta_l\Big[\frac{15}{8}P_7(1+\alpha_l^2)-\frac{1}{4}\alpha_l B_l^C\Big]\\&\quad-\frac{3}{4}P_2\alpha_l r_l^{1/2}\cos\frac{3}{2}\theta_l+r_l^{1/2}\sin\frac{1}{2}\theta_l\Big[\frac{15}{8}P_8(1+\alpha_l^2)-\frac{1}{4}\alpha_l B_l^A\Big]\\&\quad-\frac{3}{4}P_1\alpha_l r_l^{1/2}\sin\frac{3}{2}\theta_l\\&\quad+r_s^{1/2}\cos\frac{1}{2}\theta_s\Big[\frac{15}{4}P_4\alpha_s+\frac{3}{2}P_5(1-\alpha_s^2)+\frac{1}{8}(1+\alpha_s^2)B_s^D\Big]\\&\quad-\frac{15}{8}B_2(1+\alpha_s^2)r_s^{1/2}\cos\frac{1}{2}\theta_s+\frac{(1+\alpha_s^2)}{8}r_s^{1/2}\cos\frac{3}{2}\theta_s[3P_5+B_s^D]\\&\quad-\frac{(1+\alpha_s^2)}{8}r_s^{1/2}\sin\frac{3}{2}\theta_s[3P_6+B_s^B]\\&\quad-\frac{15}{8}D_2(1+\alpha_s^2)r_s^{1/2}\sin\frac{1}{2}\theta_s+2\alpha_l C_1\\&r_s^{1/2}\sin\frac{1}{2}\theta_s\Big[\frac{15}{4}P_3\alpha_s+\frac{3}{2}P_6(1-\alpha_s^2)+\frac{1}{8}(1+\alpha_s^2)B_s^B\Big]\\&\quad-\frac{15}{4}A_2\alpha_l r_l^{1/2}\cos\frac{1}{2}\theta_l+\frac{15}{4}C_2\alpha_l r_l^{1/2}\cos\frac{1}{2}\theta_l-D_1(1+\alpha_s^2)\\&\quad+\frac{1}{32}\alpha_l r_l^{1/2}\Big[B_l^A\sin\frac{7}{2}\theta_l+B_l^C\cos\frac{7}{2}\theta_l\Big]\\&\quad-\frac{1}{64}r_s^{1/2}(1+\alpha_s^2)\Big[B_s^B\sin\frac{7}{2}\theta_s+B_s^D\cos\frac{7}{2}\theta_l\Big]\end{aligned}\right\}. \tag{A-3c}$$

By substituting (6) into (5), in-plane stress component can be written in terms of displacement potentials Φ and Ψ as,

$$\begin{aligned}\frac{\sigma_{xx}}{\mu_c}&=\Big[\Big(\frac{p+1}{p-1}\Big)\frac{\partial^2\Phi}{\partial^2X}+\Big(\frac{3-p}{p-1}\Big)\frac{\partial^2\Phi}{\partial^2Y}+2\frac{\partial^2\Psi}{\partial X\,\partial Y}\Big]\exp(\alpha x)\\\frac{\sigma_{yy}}{\mu_c}&=\Big[\Big(\frac{3-p}{p-1}\Big)\frac{\partial^2\Phi}{\partial^2X}+\Big(\frac{p+1}{p-1}\Big)\frac{\partial^2\Phi}{\partial^2Y}-2\frac{\partial^2\Psi}{\partial X\,\partial Y}\Big]\exp(\alpha x)\\\frac{\sigma_{xy}}{\mu_c}&=\Big[-\frac{\partial^2\Psi}{\partial^2X}+\frac{\partial^2\Psi}{\partial^2Y}+2\frac{\partial^2\Phi}{\partial X\,\partial Y}\Big]\exp(\alpha x),\end{aligned} \tag{A-4}$$

where $p=3-4\nu$ for plane strain and $p=\frac{3-\nu}{1+\nu}$ for plane stress

Substituting for Φ and Ψ in (A-4) gives

$$\frac{\sigma_{xx}}{\mu_c \exp(\alpha x + \beta y)} =$$

$$\left\{\begin{array}{l}
R_1 \left\{\begin{array}{l}
\frac{3}{4}A_0 r_l^{-1/2}\cos\left(\frac{\theta_l}{2}\right) - \frac{3}{4}C_0 r_l^{-1/2}\sin\left(\frac{\theta_l}{2}\right) + 2A_1 \\
\quad + \frac{15}{4}r_l^{1/2}\cos\left(\frac{\theta_l}{2}\right)A_2 + \frac{15}{4}r_l^{1/2}\sin\left(\frac{\theta_l}{2}\right)C_2 \\
\quad + \frac{15}{4}P_4 r_s^{1/2}\sin\left(\frac{\theta_s}{2}\right) - r_l^{1/2}\cos\left(\frac{3\theta_l}{2}\right)\left[\frac{3}{4}P_1 + \frac{1}{4}B_l^C\right] \\
\quad - r_l^{1/2}\cos\left(\frac{\theta_l}{2}\right)\left[3P_1 + \frac{1}{4}B_l^A\right] - r_l^{1/2}\sin\left(\frac{\theta_l}{2}\right)\left[3P_2 + \frac{1}{4}B_l^C\right] \\
\quad - \frac{15}{4}P_3 r_s^{1/2}\cos\left(\frac{\theta_s}{2}\right) r_l^{1/2}\sin\left(\frac{3\theta_l}{2}\right)\left[\frac{3}{4}P_2 + \frac{1}{4}B_l^A\right] \\
\quad + \frac{1}{32}r_l^{1/2}\left[B_l^A\sin\left(\frac{7\theta_l}{2}\right) - B_l^C\cos\left(\frac{7\theta_l}{2}\right)\right]
\end{array}\right\} + 4\alpha_s B_1 \\
R_2 \left\{\alpha_l^2 \left[\begin{array}{l}
\frac{3}{4}C_0 r_l^{-1/2}\sin\left(\frac{\theta_l}{2}\right) - \frac{3}{4}A_0 r_l^{-1/2}\cos\left(\frac{\theta_l}{2}\right) - 2A_1 \\
\quad - \frac{15}{4}r_l^{1/2}\cos\left(\frac{\theta_l}{2}\right)A_2 - \frac{15}{4}r_l^{1/2}\sin\left(\frac{\theta_l}{2}\right)C_2 \\
\quad + r_l^{1/2}\cos\left(\frac{3\theta_l}{2}\right)\left[\frac{3}{4}P_1 - \frac{1}{4}B_l^A\right] - r_l^{1/2}\cos\left(\frac{\theta_l}{2}\right)\left[3P_1 - \frac{1}{4}B_l^A\right] \\
- \frac{15}{4}P_4\frac{\alpha_s^2}{\alpha_l^2}r_s^{1/2}\sin\left(\frac{\theta_s}{2}\right) - r_l^{1/2}\sin\left(\frac{3\theta_l}{2}\right)\left[\frac{3}{4}P_2 - \frac{1}{4}B_l^C\right] \\
\quad - \frac{15}{4}P_3\frac{\alpha_s^2}{\alpha_l^2}r_s^{1/2}\cos\left(\frac{\theta_s}{2}\right) - r_l^{1/2}\sin\left(\frac{\theta_l}{2}\right)\left[3P_2 - \frac{1}{4}B_l^C\right] \\
- \frac{1}{32}r_l^{1/2}\left[B_l^A\cos\left(\frac{7\theta_l}{2}\right) - B_l^C\sin\left(\frac{7\theta_l}{2}\right)\right]
\end{array}\right]\right\} + \\
\quad - \frac{3}{2}B_0\alpha_s r_s^{-1/2}\cos\left(\frac{\theta_s}{2}\right) + \frac{3}{2}D_0\alpha_s r_s^{-1/2}\sin\left(\frac{\theta_s}{2}\right) + \frac{15}{2}B_2\alpha_s r_s^{1/2}\cos\left(\frac{\theta_s}{2}\right) \\
\quad - \frac{15}{2}P_7\alpha_l r_l^{1/2}\sin\left(\frac{\theta_l}{2}\right) - \frac{15}{2}D_2\alpha_s r_s^{1/2}\sin\left(\frac{\theta_s}{2}\right) - \frac{3}{2}P_5\alpha_s r_s^{1/2}\sin\left(\frac{3\theta_s}{2}\right) \\
\quad - \frac{3}{2}P_6\alpha_s r_s^{1/2}\cos\left(\frac{3\theta_s}{2}\right) - \frac{15}{2}P_8\alpha_l r_l^{1/2}\cos\left(\frac{\theta_l}{2}\right)
\end{array}\right\} \tag{A-5a}$$

$$
\frac{\sigma_{yy}}{\mu_c \exp(\alpha x + \beta y)} =
$$

$$
\left\{
\begin{aligned}
& R_1 \left\{ \alpha_l^2 \left[
\begin{aligned}
& -\frac{3}{4}A_0 r_l^{-1/2}\cos\left(\frac{\theta_l}{2}\right) + \frac{3}{4}C_0 r_l^{-1/2}\sin\left(\frac{\theta_l}{2}\right) - 2A_1 \\
& -\frac{15}{4}r_l^{1/2}\cos\left(\frac{\theta_l}{2}\right)A_2 - \frac{15}{4}r_l^{1/2}\sin\left(\frac{\theta_l}{2}\right)C_2 \\
& -r_l^{1/2}\sin\left(\frac{3\theta_l}{2}\right)\left[\frac{3}{4}P_2 - \frac{1}{4}B_l^C\right] + r_l^{1/2}\cos\left(\frac{3\theta_l}{2}\right)\left[\frac{3}{4}P_1 - \frac{1}{4}B_l^A\right] \\
& -r_l^{1/2}\sin\left(\frac{\theta_l}{2}\right)\left[3P_2 - \frac{1}{4}B_l^C\right] + \frac{15}{4}P_3\frac{\alpha_s^2}{\alpha_l^2}r_s^{1/2}\cos\left(\frac{\theta_s}{2}\right) \\
& -\frac{15}{4}P_4\frac{\alpha_s^2}{\alpha_l^2}r_s^{1/2}\sin\left(\frac{\theta_s}{2}\right) - \frac{1}{32}r_l^{1/2}\left[B_l^A\cos\left(\frac{7\theta_l}{2}\right) - B_l^C\sin\left(\frac{7\theta_l}{2}\right)\right] \\
& -r_l^{1/2}\cos\left(\frac{\theta_l}{2}\right)\left[3P_1 - \frac{1}{4}B_l^A\right]
\end{aligned}
\right]\right\} \\
& R_2 \left\{ \left[
\begin{aligned}
& -\frac{3}{4}C_0 r_l^{-1/2}\sin\left(\frac{\theta_l}{2}\right) + \frac{3}{4}A_0 r_l^{-1/2}\cos\left(\frac{\theta_l}{2}\right) + 2A_1 \\
& +\frac{15}{4}r_l^{1/2}\cos\left(\frac{\theta_l}{2}\right)A_2 + \frac{15}{4}r_l^{1/2}\sin\left(\frac{\theta_l}{2}\right)C_2 \\
& -r_l^{1/2}\cos\left(\frac{3\theta_l}{2}\right)\left[\frac{3}{4}P_1 + \frac{1}{4}B_l^A\right] - r_l^{1/2}\cos\left(\frac{\theta_l}{2}\right)\left[3P_1 + \frac{1}{4}B_l^A\right] \\
& +r_l^{1/2}\sin\left(\frac{\theta_l}{2}\right)\left[3P_2 - \frac{1}{4}B_l^C\right] - r_l^{1/2}\sin\left(\frac{3\theta_l}{2}\right)\left[\frac{3}{4}P_2 - \frac{1}{4}B_l^C\right] \\
& -\frac{15}{4}P_3\frac{\alpha_s^2}{\alpha_l^2}r_s^{1/2}\cos\left(\frac{\theta_s}{2}\right) + \frac{15}{4}P_4\frac{\alpha_s^2}{\alpha_l^2}r_s^{1/2}\sin\left(\frac{\theta_s}{2}\right) \\
& +\frac{1}{32}r_l^{1/2}\left[B_l^A\cos\left(\frac{7\theta_l}{2}\right) - B_l^C\sin\left(\frac{7\theta_l}{2}\right)\right]
\end{aligned}
\right]\right\} \\
& +\frac{3}{2}B_0\alpha_s r_s^{-1/2}\cos\left(\frac{\theta_s}{2}\right) + \frac{3}{2}D_0\alpha_s r_s^{-1/2}\sin(\frac{\theta_s}{2}) + 4\alpha_s B_1 \\
& -\alpha_s r_s^{1/2}\cos\left(\frac{\theta_s}{2}\right)\left[\frac{15}{2}P_8 - \frac{1}{2}B_s^B\right] - \alpha_s r_s^{1/2}\sin\left(\frac{\theta_s}{2}\right)\left[\frac{15}{2}P_7 - \frac{1}{2}B_s^D\right] \\
& +\frac{15}{2}B_2\alpha_s r_s^{1/2}\cos\left(\frac{\theta_s}{2}\right) - \frac{15}{2}D_2\alpha_s r_s^{1/2}\sin\left(\frac{\theta_s}{2}\right) \\
& +\frac{1}{16}\alpha_s r_s^{1/2}\left[B_s^B\cos\left(\frac{7\theta_s}{2}\right) + B_s^D\sin\left(\frac{7\theta_s}{2}\right)\right] - \frac{3}{2}P_5\alpha_s r_s^{1/2}\sin\left(\frac{3\theta_s}{2}\right) \\
& -\frac{3}{2}P_6\alpha_s r_s^{1/2}\cos\left(\frac{3\theta_s}{2}\right)
\end{aligned}
\right\} \tag{A-5b}
$$

$$
\frac{\sigma_{xy}}{\mu_c \exp(\alpha x + \beta y)} =
$$

$$
\left\{
\begin{aligned}
&\frac{3}{2}A_0\alpha_l r_l^{-1/2}\sin\Big(\frac{\theta_l}{2}\Big) + \frac{3}{2}C_0\alpha_l r_l^{-1/2}\cos\Big(\frac{\theta_l}{2}\Big) + \frac{3}{4}(1+\alpha_s^2)B_0 r_s^{-1/2}\sin\Big(\frac{\theta_s}{2}\Big)\\
&\quad -\frac{3}{4}(1+\alpha_s^2)D_0 r_s^{-1/2}\sin\Big(\frac{\theta_s}{2}\Big) + 4\alpha_l C_1 + 2(1+\alpha_s^2)D_1\\
&\quad -\frac{15}{2}A_2\alpha_l r_l^{1/2}\sin\Big(\frac{\theta_l}{2}\Big) + \frac{15}{2}C_2\alpha_l r_l^{1/2}\cos\Big(\frac{\theta_l}{2}\Big) - \frac{15}{4}B_2 r_s^{1/2}\sin\Big(\frac{\theta_s}{2}\Big)\\
&\quad -\frac{15}{4}D_2(1+\alpha_s^2)r_s^{1/2}\sin\Big(\frac{\theta_s}{2}\Big) - \frac{15}{4}P_7 r_l^{1/2}(1+\alpha_l^2)\cos\Big(\frac{\theta_l}{2}\Big)\\
&\quad -\frac{3}{2}P_2 r_l^{1/2}\alpha_l\cos\Big(\frac{3\theta_l}{2}\Big) + \frac{15}{4}P_8 r_l^{1/2}(1+\alpha_l^2)\sin\Big(\frac{\theta_l}{2}\Big) - \frac{3}{2}P_1 r_l^{1/2}\alpha_l\sin\Big(\frac{3\theta_l}{2}\Big)\\
&\quad + r_s^{1/2}\cos\Big(\frac{\theta_s}{2}\Big)\Big(3P_5[1-\alpha_s^2] + \frac{15}{2}P_4\alpha_s + \frac{(1+\alpha_s^2)}{4}B_s^D\Big)\\
&\quad + \frac{1}{16}\alpha_l r_l^{1/2}\Big[B_l^A\sin\Big(\frac{7\theta_l}{2}\Big) + B_l^C\cos\Big(\frac{7\theta_l}{2}\Big)\Big]\\
&\quad - r_s^{1/2}\frac{(1+\alpha_s^2)}{4}\sin\Big(\frac{3\theta_s}{2}\Big)[3P_6 - B_s^B]\\
&\quad - \frac{1}{2}\alpha_l r_l^{1/2}\Big[B_l^A\sin\Big(\frac{\theta_l}{2}\Big) - B_l^C\cos\Big(\frac{\theta_l}{2}\Big)\Big]\\
&\quad + r_s^{1/2}\frac{(1+\alpha_s^2)}{4}\cos\Big(\frac{3\theta_s}{2}\Big)[3P_5 + B_s^D]\\
&\quad + r_s^{1/2}\sin\Big(\frac{\theta_s}{2}\Big)\Big(3P_6[1-\alpha_s^2] + \frac{15}{2}P_3\alpha_s + \frac{(1+\alpha_s^2)}{4}B_s^B\Big)\\
&\quad + \frac{1}{32}(1+\alpha_s^2)r_s^{1/2}\Big[B_s^B\sin\Big(\frac{7\theta_s}{2}\Big) - B_s^D\cos\Big(\frac{7\theta_s}{2}\Big)\Big]
\end{aligned}
\right\}, \tag{A-5c}
$$

where

$$
R_1 = \frac{p+1}{p-1} \qquad \text{and} \qquad R_1 = \frac{3-p}{p-1}.
$$

NITESH JAIN: nitesh_jain@goodyear.com

ARUN SHUKLA: shuklaa@egr.uri.edu