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A COMPARISON OF OPEN CELL AND CLOSED CELL PROPERTIES FOR LOW-DENSITY MATERIALS

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The stiffness and strength properties for open cell and closed cell low-density materials are collected and compared. These are the theoretical predictions for the Kelvin cell type and the oct-tet cell type of open cell forms and of the closed cell form from the generalized self-consistent method. The strength properties considered are those for plastic collapse and elastic instability, under both uniaxial stress and dilatational stress conditions.

1. Introduction and conditions of comparison

Low-density materials (LDM) have come into widespread usage. The properties and practice for materials with two-dimensional microstructures such as honeycomb forms, and other forms are well covered and well understood. [Gibson and Ashby 1997] is standard reading on the topic and there have been many review articles, such as that by Christensen [2000]. The situation with low-density materials having three-dimensional microstructures is more complex. They are widely used, but they are less well understood than the two-dimensional cases, partly because of the much more complex microscale geometry. The complications that arise in the three-dimensional case are immediately apparent when one considers whether to focus upon closed cell or open cell forms. This divergence does not even arise in the two-dimensional case. For this reason two-dimensional results are almost useless in projecting what three-dimensional behaviors may involve.

Until rather recently there did not appear to be enough information available to conduct an in-depth comparison of open cell and closed cell forms. Essentially the missing piece was that of the work of Deshpande et al. [2001] on a particular type of open cell form. To describe this further requires an understanding of the two basic types of microstructures for the various open cell forms. These are most easily illustrated by the corresponding two-dimensional cases. If a honeycomb form is deformed in any state except dilatation, its material members resist the deformation by means of a bending mechanism. Alternatively, if a two-dimensional LDM is composed of equilateral triangular cells its material members resist deformation by an axial or direct or stretching mechanism. The same situation exists in three dimensions, different microstructures resist deformation by either a direct or a bending mechanism. However, nearly all of the attention has been focused upon three-dimensional LDM's having only the bending mechanism. The work of Deshpande et al. [2001] convincingly detailed a microstructure which involves only the direct mechanism. In the work to be given here, two fundamental microstructures will be taken

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for the open cell case, one involving the bending mechanism and one involving the direct mechanism. These will be compared with each other as well as with a closed cell type microstructure.

Although one could in principle compare experimental data for open cell and closed cell forms, in practice this is difficult to do. Rarely are open cell and closed cell samples available at exactly the same volume fraction of voids and for the exact same constituent materials. Furthermore even when they are at least approximately the same, the experimental scatter and uncertainty renders comparisons difficult. Instead, theoretical predictions of properties will be given here for the three idealized microstructures, thus precise control will be exercised over the morphological forms being compared. It is well understood that imperfections are very important as they can reduce performance levels significantly. Nevertheless the ideal forms considered here will bring out strong differences in a relative sense between the microstructures.

The present considerations will be restricted to the very low-density range. The volume fraction of material will be taken to be much less than 0.1. In effect, this will involve retaining the first term in an expansion in terms of powers of the volume fraction of the material. For larger volume fractions, the differences between open cell and closed cell forms become blurred and they merely become ones of many different possible forms for porous materials wherein rather convoluted surface morphologies become possible.

2. Stiffness and strength properties

The three LDM cellular microstructures to be studied here are now prescribed. The two open cell forms are the Kelvin cell and the oct-tet cell. The closed cell form is that from the generalized self-consistent method (GSCM).

The Kelvin cell is obtained by starting with an octahedron, then cutting off the six corners such that the truncated octahedron is composed of six squares and eight hexagons. The resulting forms can then be perfectly packed into a space filling periodic structure having cubic symmetry. It was Kelvin who first understood the space-filling attribute of this simple cell. The resulting Kelvin cell is shown in Figure 1. For application to LDM's the edges of the cells are taken to be material members.

The oct-tet form was patented by Fuller [1961] and was extensively studied by Deshpande et al. [2001] as an LDM. The cellular form can be thought of as due to the three-dimensional packing of tetrahedrons with the octahedrons existing as the spaces left over from the necessarily imperfect packing of the tetrahedrons. The resulting microstructure is interpreted as an LDM by taking the edges of the tetrahedrons (and octahedrons) as material members. The resulting form has face centered cubic symmetry and is as shown in Figure 2. Material members connect the corners and face centers in the manner shown for the three front faces. Christensen [2004] described this three-dimensional cellular form as being the three-dimensional analog of the classical two-dimensional truss system.

For application to closed cell forms, the GSCM model is as shown in Figure 3. A thin spherical shell of the material is embedded in a continuous medium having the as yet unknown properties of the composite form. This model was developed by Christensen and Lo [1979] for applications to general composite material systems. The specialization of the general model to the case of the inclusion phase as voids and the material in the low volume fraction range gives the LDM model of interest here, for the effective stiffness properties.

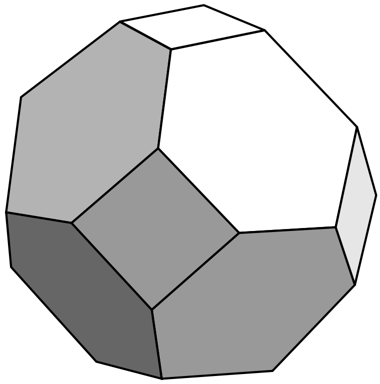

Figure 1. Open cell, Kelvin form, truncated octahedron.

The Kelvin cell has four material members meeting at a node, whereas the oct-tet cell has twelve material members meeting at a node. For the oct-tet cell the resistance to deformation is supplied by the direct mechanism in all possible deformation states. In the Kelvin cell a dilatational deformation is resisted by the direct mechanism of the material members, but all other deformation modes are resisted by the bending mechanism. In these cases of the bending mechanism being operative, the torsion of the material members also must be included. The two open cell forms thus employ fundamentally different mechanisms of material resistance. Also, these two cellular forms are of cubic symmetry and the properties will be stated in cubic symmetry form. The GSCM closed cell form is isotropic. Many

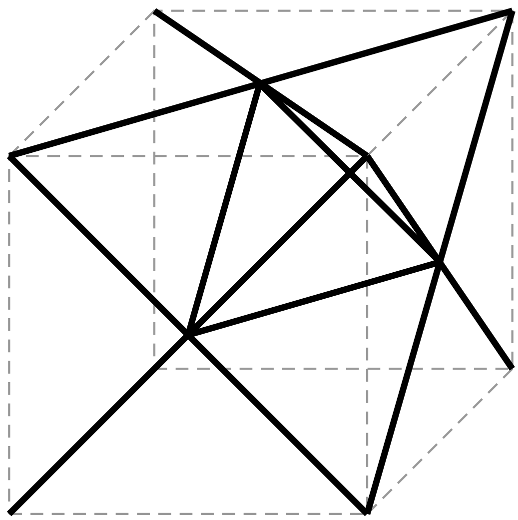

Figure 2. Open cell, oct-tet form.

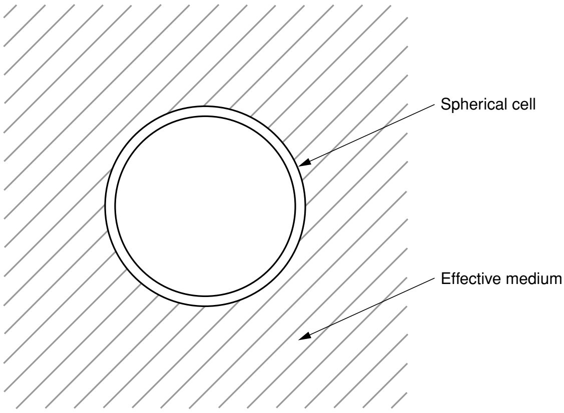

Figure 3. Closed cell, GSCM model.

other cellular forms have been studied (see [Wicks and Hutchinson 2001; Gong et al. 2005; Gong and Kyriakides 2005; Roberts and Garboczi 2001]) but the three forms considered here suffice to cover most of the basic effects needed to compare open cell and closed cell forms.

For the two open cell forms the properties are sometimes stated in terms of (r/ℓ) where r is the radius of a circular material member and ℓ is its length. However, for present purposes of comparing different microstructures it is necessary to use the volume fraction of material, c_m, as the basis of comparison. Various powers of c_m will be found to enter the properties, such as c_m, c_m^2, etc. Since $c_m \ll 0.1$ in the LDM range the different powers of c_m will project very different properties, sometimes different by orders of magnitude. All results to be given here will be for material members having cross sections of solid, circular form in the open cell cases.

A table has been constructed of all the major stiffness and strength properties for all three forms of LDM's. The properties information is in Table 1, and is obtained directly from the references as noted. It should be recognized that many of these results required major expenditures of effort in their original derivations, and as such represent a considerable resource. The first four entries in Table 1 are for the usual stiffness properties and Poisson's ratio of the effective medium. E_m and υ_m are the properties of the composing material. The two plastic collapse entries are for yielding where the material has a uniaxial stress at yield of σ_y. Plastic collapse dilatational refers to the imposition of a hydrostatic stress state which in the open cell cases imposes compressive stress states in the material members. Collapse occurs when this stress reaches the yield value. In the closed cell case, the biaxial stress state reaches its yield value. In Table 1, the term plastic collapse uniaxial stress is similarly defined. In the oct-tet cell and the closed cell cases, these results follow from simple geometry and equilibrium. It is only in the case of the Kelvin cell that the plastic collapse under uniaxial stress is far from obvious and requires a somewhat lengthy derivation. This derivation is given in Appendix A. All of these results will be further described below.

The last two entries in Table 1 are for the elastic stability under compressive dilatational and uniaxial stress. The elastic stability dilatational refers to stress levels in the open cell material members that buckle under the imposition of hydrostatic stress to the cell. In the closed cell case, the result arises

Property	Open cell (bending/Kelvin)	Open cell (direct/oct-tet)	Closed cell GSCM
$\frac{E_{11}}{E_m}$	$\frac{4\sqrt{2}}{3\pi} c_m^2$	$\frac{1}{9} c_m$	$\frac{2}{5+3\upsilon_m} c_m$
υ_{12}	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1+3\upsilon_m}{5+3\upsilon_m}$
μ_{12}/E_m	$\frac{2\sqrt{2}}{3\pi}\left(\frac{6+5\upsilon_m}{9+8\upsilon_m}\right) c_m^2$	$\frac{1}{12} c_m$	$\frac{1}{6(1+\upsilon_m)} c_m$
k/E_m	$\frac{1}{9} c_m$	$\frac{1}{9} c_m$	$\frac{2}{9(1-\upsilon_m)} c_m$
Plastic collapse dilatational σ	$\frac{\sigma^y}{3} c_m$	$\frac{\sigma^y}{3} c_m$	$\frac{2}{3} \sigma^y c_m$
Plastic collapse uniaxial σ_{11}	$\frac{8(2)^{3/4}}{9\sqrt{3}\pi^{3/2}} \sigma^y c_m^{3/2}$	$\frac{\sigma^y}{3} c_m$	$\frac{2}{3} \sigma^y c_m$
Elastic stability dilatational σ	$\frac{\pi E_m}{4\sqrt{2}} c_m^2$	$\frac{\pi E_m}{72\sqrt{2}} c_m^2$	$\left[\frac{2}{9}\frac{E_m}{\sqrt{3(1-\upsilon_m{}^2)}}\right] c_m^2$
Elastic stability uniaxial σ_{11}	not applicable	$\frac{\pi E_m}{72\sqrt{2}} c_m^2$	unknown

Table 1. LDM properties.

directly from the buckling of a thin spherical shell. For the Kelvin cell case, the stiffness properties are from [Warren and Kraynik 1997] and [Zhu et al. 1997]. The strength properties for the Kelvin cell were constructed for this work. For the oct-tet cell the stiffness and strength properties were given by Deshpande et al. [2001]. Some of the properties were put into slightly different forms for the present purposes. The closed-cell stiffness properties were given by Christensen [1998]. The closed cell strength properties were worked out here and are extremely simple to derive.

The method of deducing the strength properties for the oct-tet cell case, given by Deshpande et al. [2001], also forms the basis for deducing the strength properties for the other two cell types. For plastic collapse, plastic hinges form in the Kelvin cell material members, see Appendix A, and plastic yield stress is reached in the closed cell case. The 3/2 power in the Kelvin cell, uniaxial stress plastic collapse term of Table 1 is due to the formation of the plastic hinges in the collapse mechanism. Similarly for elastic stability under compressive stress, the open cell material members reach the limits of elastic stability. These limiting stresses are found under both hydrostatic stress and uniaxial stress states.

In the closed cell case it was necessary to construct the elastic stability under dilatational and uniaxial stress. The dilatational case was taken as that for a free spherical shell under uniform pressure. The effect of the constraining effective medium attached to the spherical shell, Figure 3, was not included. Thus the constraint of the effective medium would increase somewhat the result shown in Table 1. This

Property	Open cell (bending/Kelvin)	Open cell (direct/oct-tet)	Closed cell GSCM
$\frac{E_{11}}{E_m}$	$0.600c_m^2$	$\frac{1}{9}c_m$	$\frac{1}{3}c_m$
υ_{12}	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{3}$
μ_{12}/E_m	$0.197c_m^2$	$\frac{1}{12}c_m$	$\frac{1}{8}c_m$
k/E_m	$\frac{1}{9}c_m$	$\frac{1}{9}c_m$	$\frac{1}{3}c_m$
Plastic collapse dilatational σ	$\frac{1}{3}\sigma^y c_m$	$\frac{1}{3}\sigma^y c_m$	$\frac{2}{3}\sigma^y c_m$
Plastic collapse uniaxial σ_{11}	$0.155\sigma^y c_m^{3/2}$	$\frac{1}{3}\sigma^y c_m$	$\frac{2}{3}\sigma^y c_m$
Elastic stability dilatational σ	$0.555E_m c_m^2$	$0.03E_m c_m^2$	$0.136E_m c_m^2$
Elastic stability uniaxial σ_{11}	not applicable	$0.03E_m c_m^2$	unknown

Table 2. Properties for $\upsilon_m = 1/3$.

result in the hydrostatic case follows directly from [Timoshenko 1961]. The case of the elastic stability under uniaxial stress is expected to have a critical value larger than that in the dilational case, but an exact result is unknown.

3. Discussion

The results shown in Table 1 are specialized to the case of Poisson's ratio $\upsilon_m = 1/3$ for ease of interpretation, Table 2. Some particular comparisons between the three cellular forms will be noted here but the main purpose is to collect and codify the significant results shown in Tables 1 and 2. There could be a huge variety of possible applications, most of which cannot even be anticipated here.

In the case of the two open cell forms, the two forms are both cubic and the properties are stated relative to the symmetry axes designated in the references. It is interesting however to note that in the case of the Kelvin cell and for the members of solid circular cross section, the cubic stiffness properties are very nearly isotropic. The properties are within 1.5% of being isotropic at $\upsilon_m = 1/2$ and they are exactly isotropic at $\upsilon_m = 0$. The oct-tet cell form is cubic with the maximum to minimum value of modulus E being in the ratio of 9/5 [Christensen 2004].

It is also interesting to note that the dilatational strength properties are not significantly larger than the uniaxial ones, for two of the three cellular forms. This contrasts somewhat with the behavior of homogeneous materials.

Observe again that properties dependent on c_m^2 are at least an order of magnitude less than those dependent on c_m. Thus the Kelvin cell case has elastic moduli properties as much less than those of the oct-tet cell. On the other hand, it is seen that the oct-tet form has serious shortcomings because of the extremely low value of the elastic stability tolerance due to the relatively large number of material members in the oct-tet cell. However, this result is for solid material members; for applications where hollow members could be used, this deficiency perhaps could be overcome.

In the elastic stability results, the material members in the open cell case could be taken either with pinned or fixed-end conditions. The results shown are for the pinned case. The other case would be a factor of four times larger.

As an overall observation, it can be noted that except for the dilatational elastic stability case, the closed cell properties are always greater, sometimes much greater, than are the open cell properties. The closed cell geometry is simply a more efficient use of the material. However, in particular applications, other requirements could supercede the properties considerations shown here. Finally, it is again cautioned that these results are for the corresponding idealized (perfect) morphologies of microstructure. Imperfections always have a strongly degrading effect.

Appendix A

The uniaxial stress σ_{11} for plastic collapse in the Kelvin cell form is derived here. Referring to Figure 1, take the stress σ_{11} to be in the horizontal direction, normal to the square face on the right hand side of the cell. The material member connecting this face and the top square face in Figure 1 has the horizontal component of force P at its ends given

$$P = \frac{2M}{\ell \sin\phi}, \tag{A.1}$$

where M is the maximum moment in the member, ℓ is the members length, and $\phi = 45°$. Then

$$P = \frac{2\sqrt{2}M}{\ell}. \tag{A.2}$$

Take the Kelvin cell in Figure 1 as being inside a cube of length a with the square faces in the Kelvin cell being on the faces of the cube. Dimension a is related to ℓ through

$$a = 2\sqrt{2}\ell. \tag{A.3}$$

The uniaxial stress σ_{11} is related to P through

$$\sigma_{11} = \frac{4P}{a^2}. \tag{A.4}$$

Combining (A.2), (A.3) and (A.4) gives

$$\sigma_{11} = \sqrt{2}\frac{M}{\ell^3}. \tag{A.5}$$

For a plastic hinge to form in a fully plastic, circular member the bending moment must attain the value

$$M = \frac{2}{3}\sigma^y r^3, \tag{A.6}$$

where the yield stress is σ^y and r is the radius of the cross section. Combining (A.5) and (A.6) gives

$$\sigma_{11} = \frac{2\sqrt{2}}{3}\sigma^y \left(\frac{r}{\ell}\right)^3. \tag{A.7}$$

The volume fraction c_m of the material members is given by

$$c_m = \frac{3\pi}{2\sqrt{2}}\left(\frac{r}{\ell}\right)^2. \tag{A.8}$$

Combining (A.7) and (A.8) gives the final result

$$\sigma_{11} = \left(\frac{8(2)^{3/4}}{9\sqrt{3}\pi^{3/2}}\right)\sigma^y c_m^{3/2}. \tag{A.9}$$

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