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A SURFACE CRACK IN A GRADED COATING BONDED TO A HOMOGENEOUS SUBSTRATE UNDER GENERAL LOADING CONDITIONS

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The elastostatic problem of a surface crack in a graded coating bonded to a homogeneous substrate under general loading conditions is considered. The coating is graded along the thickness direction and modeled as a nonhomogeneous medium with an isotropic stress-strain law. The problem is solved under the assumption of plane strain or generalized plane stress conditions. The crack surfaces are subjected to arbitrary loadings which give rise to mixed fracture modes which can be uncoupled due to the fact that the crack axis is parallel to the material gradient. Therefore, the opening and sliding mode problems may be formulated separately. For each problem, the solution of the composite medium may be determined by obtaining the solution of the homogeneous substrate and that of the graded layer. The latter solution may be expressed as the sum of two solutions, namely an infinite graded medium with a crack and a graded strip without a crack. The resulting mixed-boundary value problem is reduced to a set of two uncoupled singular integral equations which are solved numerically using Jacobi polynomials. The main objective of the paper is to study the effect of the layer thickness and nonhomogeneity parameter on the crack tip mixed-mode stress intensity factors for the purpose of gaining better understanding on the behavior of graded coatings.

1. Introduction

In high-temperature applications the potential of using homogeneous materials appears to be limited and in recent years the new trends in material design seem to be toward coating the main load-bearing component by a heat-resistive layer, generally a ceramic. Because of the relatively high mismatch in thermal expansion coefficients, the resulting bonded structure is generally subjected to very high residual and thermal stresses. As a result, the composite medium becomes vulnerable to cracking, debonding and spallation [Erdogan 1995].

The materials research community has recently been exploring the possibility of using new concepts in coating design, such as Functionally Graded Materials (FGMs), as an alternative to the conventional homogeneous coatings. These can be at least two-phase inhomogeneous particulate composites synthesized in such a way that the volume fractions of the constituent materials, such as ceramic and metal, vary continuously along a spatial direction to give a predetermined composition profile resulting in a relatively smooth variation of the mechanical properties. FGMs appear to promise attractive applications in a wide variety of wear coating and thermal shielding problems such as gears, cams, cutting tools, high temperature chambers, furnace liners, turbines, microelectronics and space structures [Holt et al. 1992]. FGM structures can also be made through physical vapor deposition techniques such as those described in [Chen et al. 2002c; 2002b], which produce nanolayers having properties varying with depth. In designing

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components involving FGMs, an important aspect of the problem is the question of mechanical failure, specifically the fracture failure [Lanutti 1994]. Fatigue and fracture characterization of materials and related analysis require the solution of certain standard crack problems.

Most of the crack problems solved over the past two decades on nonhomogeneous materials [Dhaliwal and Singh 1978; Delale and Erdogan 1983; Erdogan 1985; Ozturk and Erdogan 1993] provide the basis for the fracture mechanics research on FGMs which are essentially nonhomogeneous materials. In [Erdogan 1995] a brief discussion is given on the application of elementary concepts of fracture mechanics in nonhomogeneous materials and a number of typical problem areas are identified which relate to the fracture of FGMs. An important problem is the nature of stress singularities near the tip of a crack embedded in a nonhomogeneous medium. Konda and Erdogan [1994] and Jin and Noda [1994a] showed that such a crack has the standard square-root singularity in addition to others encountered in a homogeneous medium provided the material property model is continuous and piecewise differentiable.

A number of crack problems in FGMs were solved accounting only for mechanical loading or thermal loading or a combination of both. The crack can be either an internal crack parallel to the free surface or perpendicular to it. A particular case of a crack perpendicular to the free surface is the edge crack which is also called the surface crack. Noda and Jin [1993] studied the internal crack problem for an infinite FGM medium subjected to a steady-state heat flux over the crack surfaces by assuming continuously varying thermal properties. The same problem was extended by El-Borgi et al. [2004a] by considering a steady-state heat flux applied away from the crack region, by modeling the crack faces as partially insulated and by accounting for crack-closure effects. The case of an internal fully insulated crack parallel to the boundary of a semi-infinite graded medium subject to a steady-state heat flux applied at the free surface was studied in [Jin and Noda 1993]. This problem was later extended to the case of transient heat flux in [Jin and Noda 1994c]. Lee and Erdogan [1998a] studied the problem of interface cracking in FGM coatings under steady-state heat flow. Chen and Erdogan [1996] studied the problem of a graded coating on homogeneous substrate with an interface crack subjected to mechanically induced crack surface tractions. El-Borgi et al. [2003] extended this problem by considering both thermal and mechanical loads and accounting for crack-closure effects. In [El-Borgi et al. 2004b] we considered the problem of a graded coating bonded to a substrate subjected to a Hertzian contact pressure and with an internal crack embedded in the coating, and in [El-Borgi et al. 2000] the mixed-mode crack parallel to the boundary of an infinite strip, with the elastic modulus varying exponentially in an arbitrary direction. Long and Delale [2005] solved the more general problem of an arbitrarily oriented crack in a graded layer bonded to a homogeneous half-plane.

Jin and Noda [1994b] considered the problem of a surface crack in a semi-infinite nonhomogeneous medium subject to a steady-state heat flux. Erdogan and Wu [1997] considered a graded strip with a surface crack, perpendicular to the boundaries and parallel to the material gradient, subjected to mechanical crack surface tractions. The same problem was also solved in [Erdogan and Wu 1996] by considering thermal loads. Dag et al. [1999] and Kadioglu et al. [1998] studied a similar problem with the graded layer attached to an elastic foundation with a crack subjected, respectively, to thermal and mechanical loads. Yildirim and Erdogan [2004] considered an axisymmetric surface crack problem for thermal barrier coatings under a uniform temperature change. Guo et al. [2004a] investigated the mode I surface crack problem for an orthotropic graded strip. A similar problem was also solved in [Chen et al. 2002a] by considering a transient loading. Guo et al. [2004b] considered the mode I problem of a graded coating bonded to homogeneous substrate with a crack perpendicular to the coating's surface subjected to a transient load. Dag and Erdogan [2002] solved the problem of a surface crack in a graded semi-infinite medium under general loading conditions in which they proposed a method to uncouple the opening and sliding modes.

The practical use of FGMs is in the form of coating applications. On the other hand, the manufacturing of FGMs may lead to inherent surface flaws which may give rise to surface cracks that can eventually propagate to the component. Therefore, the present work consists of a surface crack located in an isotropic graded coating bonded to homogeneous substrate subjected to general loading conditions. This study is an extension of [Dag and Erdogan 2002] in the sense that problem geometry consists of a graded coating bonded to a substrate rather than a graded half-plane. The main objective of the paper is to study the effect of the coating thickness and material nonhomogeneity on the crack tip stress intensity factors for the purpose of further understanding the behavior and design of graded materials.

2. Problem description and governing equations

As shown in Figure 1, the problem under consideration consists of an infinitely long graded coating of thickness h, bonded to a homogeneous semi-infinite medium. The graded coating contains an edge crack of length d along the x-axis. For the graded coating, the material gradient is oriented along the

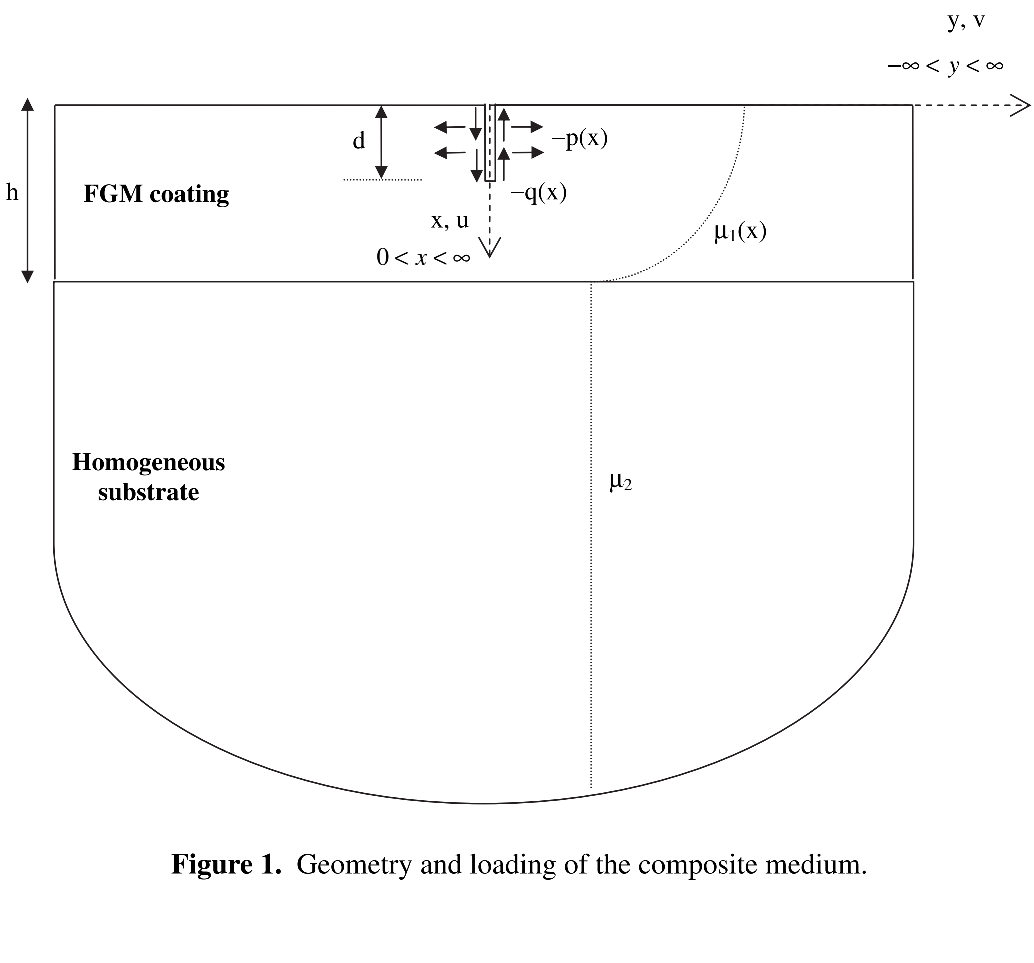

Figure 1. Geometry and loading of the composite medium.

x-direction. The Poisson's ratio ν is assumed to be a constant because the effect of its variation on the crack-tip stress intensity factors was shown in [Erdogan and Wu 1996; 1997] to be negligible and is equal to the same value as that of the homogeneous substrate. On the other hand, the shear modulus in the FGM layer μ_1 depends on the x-coordinate only and is modeled by an exponential function as expressed by

$$\mu_1 = \mu_0 \exp(\beta x), \tag{1a}$$

where μ_0 is the value of the shear modulus in the coating along the free surface $x = 0$ and β is the nonhomogeneity parameter controlling the variation of the shear modulus in the graded coating.

For the homogeneous substrate, the shear modulus μ_2 is a constant and is equal to the value of the FGM coating shear modulus at the interface

$$\mu_2 = \mu_0 \exp(\beta h). \tag{1b}$$

The loading consists of arbitrary crack surface tractions (ie, normal and shear tractions) which can be expressed in terms of the external mechanical loads. The problem under consideration is similar to the graded half-plane surface crack problem studied in [Dag and Erdogan 2002]. In both problems, the material gradient in the graded medium and the crack orientation are both along the x-direction. As a result, a normal loading and a shear loading on the crack faces do not induce, respectively, mode II and mode I stress intensity factors. Therefore, the opening and sliding mode problems turn out to be uncoupled and may be formulated separately, as indicated in the work just cited.

The equations of the plane problem for nonhomogeneous isotropic elastic solids are the equilibrium equations

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} = 0, \qquad \frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} = 0, \tag{2}$$

(ignoring body forces), the strain-displacement relationships

$$\varepsilon_{xy} = \frac{1}{2}\Big(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\Big), \qquad \varepsilon_{xx} = \frac{\partial u}{\partial x}, \qquad \varepsilon_{yy} = \frac{\partial v}{\partial y}, \tag{3}$$

and the linear elastic stress-strain law

$$\sigma_{xx} = \frac{\mu_j}{\kappa - 1}\big((1+\kappa)\varepsilon_{xx} + (3-\kappa)\varepsilon_{yy}\big), \quad \sigma_{xy} = 2\mu_j \varepsilon_{xy}, \quad \sigma_{yy} = \frac{\mu_j}{\kappa - 1}\big((3-\kappa)\varepsilon_{xx} + (1+\kappa)\varepsilon_{yy}\big), \tag{4}$$

where $j = 1, 2$ and $\kappa = 3 - 4\nu$ for plane strain, $\kappa = (3-\nu)/(1+\nu)$ for generalized plane stress.

Substituting Equations (3) into (4), inserting the resulting expressions into (2) and using (1a) and (1b), we obtain the equations of plane elasticity

$$\begin{aligned}
&(\kappa+1)\frac{\partial^2 u}{\partial x^2} + (\kappa-1)\frac{\partial^2 u}{\partial y^2} + 2\frac{\partial^2 v}{\partial x \partial y} + \beta(\kappa+1)\frac{\partial u}{\partial x} + \beta(3-\kappa)\frac{\partial v}{\partial y} = 0, \quad 0 \le x < h, \\
&(\kappa-1)\frac{\partial^2 v}{\partial x^2} + (\kappa+1)\frac{\partial^2 v}{\partial y^2} + 2\frac{\partial^2 u}{\partial x \partial y} + \beta(\kappa-1)\frac{\partial v}{\partial x} + \beta(\kappa-1)\frac{\partial u}{\partial y} = 0, \quad 0 \le x < h,
\end{aligned} \tag{5}$$

together with

$$(\kappa+1)\frac{\partial^2 u}{\partial x^2} + (\kappa-1)\frac{\partial^2 u}{\partial y^2} + 2\frac{\partial^2 v}{\partial x \partial y} = 0 \quad \text{and} \quad (\kappa-1)\frac{\partial^2 v}{\partial x^2} + (\kappa+1)\frac{\partial^2 v}{\partial y^2} + 2\frac{\partial^2 u}{\partial x \partial y} = 0, \qquad x \ge h.$$

For both crack mode problems (I and II), these plane elasticity equations are subject to the appropriate boundary conditions, detailed in the next two sections. The main unknowns of interest in these mixed-boundary value problems are the so-called density functions chosen for convenience as the derivatives of the relative crack opening displacements. Because of symmetry, these density functions can be chosen as

$$f_1(x) = \frac{4\mu_0}{\kappa+1}\frac{\partial}{\partial x}v(x,0^+), \qquad f_2(x) = \frac{4\mu_0}{\kappa+1}\frac{\partial}{\partial x}u(x,0^+). \tag{6}$$

3. Solution of the opening mode problem

For the opening mode problem, the solution of the composite medium may be determined by obtaining the solution in the graded layer and that in the homogeneous substrate. The graded layer solution may be expressed as the sum of two solutions, namely an infinite graded medium with a crack along the x-direction and a graded strip without a crack [Erdogan and Wu 1996; Dag and Erdogan 2002; Bogy 1975]. For the cracked infinite graded medium, the plane elasticity Equations (5) are solved using standard Fourier transforms with respect to the x-coordinate. Furthermore, taking advantage of the problem symmetry and considering that the displacements need to be bounded as y goes to ∞, the solution for the half-plane $y \geq 0$ is given by

$$\begin{aligned} u_1^{i+}(x,y) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty}\sum_{k=3}^{4} C_k(\lambda)e^{m_k y}e^{ix\lambda}d\lambda, && y \geq 0,\\ v_1^{i+}(x,y) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty}\sum_{k=3}^{4} C_k(\lambda)s_k^i(\lambda)e^{m_k y}e^{ix\lambda}d\lambda, && y \geq 0, \end{aligned} \tag{7}$$

where the subscript 1 stands for the opening mode problem, the superscript i indicates the cracked infinite medium, the superscript $+$ stands for the half-plane $y \geq 0$, and $C_k(\lambda)$, $k = 3, 4$, are unknown functions. Moreover, $s_k^i(\lambda)$, $k = 3, 4$ are known functions given by

$$s_k^i(\lambda) = -\frac{(\kappa-1)m_k^2 + (\kappa+1)(i\lambda\beta - \lambda^2)}{(2i\lambda + \beta(3-\kappa))m_k} \quad (k = 3, 4), \tag{8}$$

and the m_k are the roots of the characteristic polynomial

$$m^4 - \big(2(\lambda^2 - i\lambda\beta) + \delta^2\big)m^2 + \big(\lambda^2 - i\lambda\beta\big)^2 = 0, \tag{9}$$

which are given by

$$m_{1,2} = \tfrac{1}{2}\big(\mp\delta + \sqrt{\delta^2 + 4\lambda^2 - 4i\lambda\beta}\big), \qquad \mathrm{Re}\, m_{1,2} \geq 0, \tag{10a}$$

$$m_{3,4} = \tfrac{1}{2}\big(\mp\delta - \sqrt{\delta^2 + 4\lambda^2 - 4i\lambda\beta}\big), \qquad \mathrm{Re}\, m_{3,4} \leq 0, \tag{10b}$$

where we have set $\delta = \sqrt{(3-\kappa)/(\kappa+1)}\,\beta$.

Substituting (7) into (3) and the result into (4) yields the stress field $\sigma_{xx1}^{i+}(x,y)$, $\sigma_{yy1}^{i+}(x,y)$ and $\sigma_{xy1}^{i+}(x,y)$ in the infinite graded medium:

$$\sigma_{xx1}^{i+}(x, y) = \frac{\mu_1}{2\pi(\kappa-1)} \int_{-\infty}^{+\infty} \sum_{k=3}^{4} p_{xx1,k}^{i}(\lambda) C_k(\lambda) e^{m_k y} e^{ix\lambda} d\lambda, \quad y \geq 0, \tag{11a}$$

$$\sigma_{yy1}^{i+}(x, y) = \frac{\mu_1}{2\pi(\kappa-1)} \int_{-\infty}^{+\infty} \sum_{k=3}^{4} p_{yy1,k}^{i}(\lambda) C_k(\lambda) e^{m_k y} e^{ix\lambda} d\lambda, \quad y \geq 0, \tag{11b}$$

$$\sigma_{xy1}^{i+}(x, y) = \frac{\mu_1}{2\pi} \int_{-\infty}^{+\infty} \sum_{k=3}^{4} p_{xy1,k}^{i}(\lambda) C_k(\lambda) e^{m_k y} e^{ix\lambda} d\lambda, \quad y \geq 0, \tag{11c}$$

where the $C_k(\lambda)$ $(k = 3, 4)$ are unknown functions, the roots m_k $(k = 3, 4)$ are given by (10b), and

$$\begin{aligned} p_{xx1,k}^{i}(\lambda) &= (i\lambda)(1+\kappa) + (3-\kappa) m_k s_k^i, \\ p_{yy1,k}^{i}(\lambda) &= (i\lambda)(3-\kappa) + (1+\kappa) m_k s_k^i \qquad (k = 3, 4), \\ p_{xy1,k}^{i}(\lambda) &= m_k + (i\lambda) s_k^i \quad (k = 1, \ldots, 4). \end{aligned}$$

We now consider the graded strip without a crack. By symmetry, the displacement along the x-axis is an even function, while the displacement along the y-axis is an odd function. Hence, the displacements u and v may be expressed using the cosine and sine transforms with respect to y as follows:

$$u_1^s(x, y) = \int_0^{+\infty} \sum_{k=1}^{4} E_k e^{n_k x} \cos(y\alpha)\, d\alpha, \qquad v_1^s(x, y) = \int_0^{+\infty} \sum_{k=1}^{4} E_k s_k^s e^{n_k x} \sin(y\alpha)\, d\alpha, \tag{12}$$

in which the superscript s indicates the uncracked graded strip, the $E_k(\lambda)$, $k = 1, 2, 3, 4$, are unknown functions, the $s_k^s(\lambda)$ are known functions given by

$$s_k^s(\lambda) = -\frac{(\kappa+1)n_k^2 + \beta(\kappa+1)n_k - (\kappa-1)\alpha^2}{2\alpha n_k + \beta(3-\kappa)\alpha} \quad k = 1, \ldots, 4 \tag{13}$$

and the n_k are the roots of the characteristic polynomial

$$n^4 + 2\beta n^3 + (\beta^2 - 2\alpha^2) n^2 - 2\beta\alpha^2 n + (\alpha^4 + \alpha^2\delta^2) = 0, \tag{14}$$

which are given by

$$\begin{aligned} n_{1,2} &= \tfrac{1}{2}\big(-\beta - \sqrt{\beta^2 + 4\alpha^2 \pm 4i\alpha\delta}\,\big), \qquad \operatorname{Re} n_{1,2} \leq 0, \\ n_{3,4} &= \tfrac{1}{2}\big(-\beta + \sqrt{\beta^2 + 4\alpha^2 \pm 4i\alpha\delta}\,\big), \qquad \operatorname{Re} n_{3,4} \geq 0. \end{aligned} \tag{15}$$

Substituting Equations (12) into (3) and the result into (4) yields the stress field in the uncracked graded strip:

$$\sigma_{xx1}^{s}(x, y) = \frac{\mu_1}{\kappa-1} \int_0^{+\infty} \sum_{k=1}^{4} p_{xx1,k}^{s}(\alpha) E_k e^{n_k x} \cos(y\alpha)\, d\alpha, \tag{16a}$$

$$\sigma_{yy1}^{s}(x, y) = \frac{\mu_1}{\kappa-1} \int_0^{+\infty} \sum_{k=1}^{4} p_{yy1,k}^{s}(\alpha) E_k e^{n_k x} \cos(y\alpha)\, d\alpha, \tag{16b}$$

$$\sigma^s_{xy1}(x, y) = \mu_1 \int_0^{+\infty} \sum_{k=1}^{4} p^s_{xy1,k}(\alpha) E_k e^{n_k x} \sin(y\alpha)\, d\alpha, \tag{16c}$$

where $E_k(\lambda)$ $(k = 1, 2, 3, 4)$ are unknown functions, the roots n_k, $k = 1, \dots, 4$, are given by (15) and

$$\begin{aligned} p^s_{xx1,k}(\alpha) &= (1+\kappa)n_k + (3-\kappa)\alpha s^s_k, \\ p^s_{xy1,k}(\alpha) &= -\alpha + s^s_k n_k, \quad (k = 1, \dots, 4), \\ p^s_{yy1,k}(\alpha) &= (3-\kappa)n_k + (1+\kappa)\alpha s^s_k, \quad (k = 1, \dots, 4). \end{aligned}$$

By superposing the two sets of solutions obtained from the cracked infinite graded medium and the uncracked graded strip, the solution of the cracked graded strip can be established as follows:

$$\begin{aligned} u^+_1(x, y) = u^{i+}_1(x, y) + u^s_1(x, y), \quad v^+_1(x, y) = v^{i+}_1(x, y) + v^s_1(x, y), \quad y \geq 0, \\ \sigma^+_{kl1}(x, y) = \sigma^{i+}_{kl1}(x, y) + \sigma^s_{kl1}(x, y), \quad k, l = x, y, \qquad\qquad y \geq 0, \end{aligned} \tag{17}$$

where $u^{i+}_1(x, y)$ and $v^{i+}_1(x, y)$ are given by (7), $u^s_1(x, y)$ and $v^s_1(x, y)$ are given by (12), $\sigma^{i+}_{kl1}(x, y)$ is given by (11) and $\sigma^s_{kl1}(x, y)$ is given by (16).

Considering that the displacements need to be bounded as x goes to ∞, the displacement solution of the homogeneous substrate can be obtained in a similar manner as the uncracked graded strip:

$$u^h_1(x, y) = \int_0^{+\infty} (C_5 + C_7 x) e^{-\alpha x} \cos(y\alpha)\, d\alpha, \tag{18a}$$

$$v^h_1(x, y) = \int_0^{+\infty} \Big(\Big(C_5 - C_7 \frac{\kappa}{\alpha}\Big) + C_7 x\Big) e^{-\alpha x} \sin(y\alpha)\, d\alpha, \tag{18b}$$

where C_5 and C_7 are unknown functions.

Substituting (18) into (3) and the result into (4) yields the stress field in the uncracked graded strip:

$$\sigma^h_{xx1}(x, y) = \frac{\mu_2}{\kappa-1} \int_0^{+\infty} \big(p^h_{xx1,1}(\alpha) C_5 + p^h_{xx1,2}(\alpha) C_7 + p^h_{xx1,1}(\alpha) C_7 x\big) e^{-\alpha x} \cos(y\alpha)\, d\alpha, \tag{19a}$$

$$\sigma^h_{yy1}(x, y) = \frac{\mu_2}{\kappa-1} \int_0^{+\infty} \big(p^h_{yy1,1}(\alpha) C_5 + p^h_{yy1,2}(\alpha) C_7 + p^h_{yy1,1}(\alpha) C_7 x\big) e^{-\alpha x} \cos(y\alpha)\, d\alpha, \tag{19b}$$

$$\sigma^h_{xy1}(x, y) = \mu_2 \int_0^{+\infty} \big(p^h_{xy1,1}(\alpha) C_5 + p^h_{xy1,2}(\alpha) C_7 + p^h_{xy1,1}(\alpha) C_7 x\big) e^{-\alpha x} \sin(y\alpha)\, d\alpha, \tag{19c}$$

where C_5 and C_7 are unknown functions and

$$\begin{aligned} p^h_{xx1,1}(\alpha) &= 2(1-\kappa)\alpha, & p^h_{xx1,2}(\alpha) &= (\kappa-1)^2, \\ p^h_{yy1,1}(\alpha) &= -2(1-\kappa)\alpha, & p^h_{yy1,2}(\alpha) &= 3 - 2\kappa - \kappa^2, \\ p^h_{xy1,1}(\alpha) &= -2\alpha, & p^h_{xy1,2}(\alpha) &= 1+\kappa. \end{aligned}$$

The solution is expressed in terms of the 9 unknown functions C_3, C_4 from (7), C_5, C_7 from (18), $E_1, \dots, E_4$ from (12), and the density function $f_1(x)$, to be determined from the 9 boundary conditions

$$\sigma_{xy1}(x, 0^+) = 0, \qquad x > 0, \tag{20}$$

$$\left.\begin{aligned}\sigma_{xx1}(x,y)=0,\\ \sigma_{xy1}(x,y)=0,\end{aligned}\right\}\qquad x=0,\ y<\infty, \tag{21}$$

$$\left.\begin{aligned}\sigma_{xx1}(h^+,y)=\sigma_{xx1}(h^-,y),\\ \sigma_{xy1}(h^+,y)=\sigma_{xy1}(h^-,y),\end{aligned}\right\}\qquad y<\infty, \tag{22}$$

$$\left.\begin{aligned}u_1(h^+,y)=u_1(h^-,y),\\ v_1(h^+,y)=v_1(h^-,y),\end{aligned}\right\}\qquad y<\infty, \tag{23}$$

$$v_1(x,0^+)=0,\qquad d<x<\infty, \tag{24}$$

$$\sigma_{yy1}(x,0^+)=-p(x),\qquad 0<x<d. \tag{25}$$

Equation (20) indicates that the shear stress is zero in the plane of symmetry. Equations (21) show that no tractions are applied at the top surface of the coating. Equations (22) and (23) describe the continuity conditions of the stress and displacement fields along the interface $x=h$. Equation (24) indicates the y-component of the displacement along the plane $y=0$ is zero outside the crack because of symmetry. Equation (25) describes the applied normal crack surface traction which can be expressed in terms of external general loads. It should be pointed out that the regularity conditions which state that the stresses and displacements need to be bounded at $x\to\infty$ were already incorporated in the solution given by Equations (18) and (19a)–(19c).

Applying the boundary conditions (20) and (24) allows expressing the unknown functions C_3 and C_4 in terms of the density function f_1:

$$C_3=\frac{\kappa+1}{4\mu_0}\frac{i}{\lambda}\frac{-p^i_{xy1,4}}{s^i_3p^i_{xy1,4}-s^i_4p^i_{xy1,3}}\int_0^d f_1(t)e^{-i\lambda t}dt,\qquad C_4=\frac{\kappa+1}{4\mu_0}\frac{i}{\lambda}\frac{p^i_{xy1,3}}{s^i_3p^i_{xy1,4}-s^i_4p^i_{xy1,3}}\int_0^d f_1(t)e^{-i\lambda t}dt.$$

Applying the boundary conditions (21)–(23) yields the linear system of equations

$$\begin{bmatrix} p^s_{xx1,1} & p^s_{xx1,2} & p^s_{xx1,3} & p^s_{xx1,4} & 0 & 0\\ -p^s_{xy1,1} & -p^s_{xy1,2} & -p^s_{xy1,3} & -p^s_{xy1,4} & 0 & 0\\ p^s_{xx1,1}e^{n_1h} & p^s_{xx1,2}e^{n_2h} & p^s_{xx1,3}e^{n_3h} & p^s_{xx1,4}e^{n_4h} & -p^h_{xx1,1}e^{-\alpha h} & -(p^h_{xx1,2}+hp^h_{xx1,1})e^{-\alpha h}\\ -p^s_{xy1,1}e^{n_1h} & -p^s_{xy1,2}e^{n_2h} & -p^s_{xy1,3}e^{n_3h} & -p^s_{xy1,4}e^{n_4h} & p^h_{xy1,1}e^{-\alpha h} & (p^h_{xy1,2}+hp^h_{xx1,1})e^{-\alpha h}\\ e^{n_1h} & e^{n_2h} & e^{n_3h} & e^{n_4h} & -e^{-\alpha h} & -he^{-\alpha h}\\ -s^s_1e^{n_1h} & -s^s_2e^{n_2h} & -s^s_3e^{n_3h} & -s^s_4e^{n_4h} & e^{-\alpha h} & (h-\frac{\kappa}{\alpha})e^{-\alpha h} \end{bmatrix}\begin{Bmatrix}E_1\\E_2\\E_3\\E_4\\C_5\\C_7\end{Bmatrix}=\begin{Bmatrix}F^R_1\\F^R_2\\F^R_3\\F^R_4\\F^R_5\\F^R_6\end{Bmatrix},$$

where we have defined

$$F^R_j=\frac{\kappa+1}{4\pi^2\mu_0}\int_0^d f_1(t)R_j(\alpha,t)\,dt\quad(j=1,\ldots,6),$$

the $R_j(\alpha, t)$ $(j = 1, \dots, 6)$ being integrals evaluated using the residue theorem [Kreyszig 1999]; their values are

$$R_j(\alpha, t) = L_j\left(H_{1j}\sin\frac{Bt}{2} + H_{2j}\cos\frac{Bt}{2}\right) \quad (j = 1, 2),$$
$$R_j(\alpha, t) = L_j\left(H_{1j}\sin\frac{B(h-t)}{2} + H_{2j}\cos\frac{B(h-t)}{2}\right) \quad (j = 3, \dots, 6),$$

where L_j, H_{ij}, and B are functions of α, β and κ.

This system of equations can be analytically inverted, leading to the expressions of the unknown functions C_5, C_7, E_1, E_2, E_3, E_4 in terms of the density function f_1:

$$E_1 = Q\int_0^d S_1(\alpha, t) f_1(t)\, dt, \qquad E_3 = Q\int_0^d S_3(\alpha, t) f_1(t)\, dt, \qquad C_5 = Q\int_0^d S_5(\alpha, t) f_1(t)\, dt,$$
$$E_2 = Q\int_0^d S_2(\alpha, t) f_1(t)\, dt, \qquad E_4 = Q\int_0^d S_4(\alpha, t) f_1(t)\, dt, \qquad C_7 = Q\int_0^d S_6(\alpha, t) f_1(t)\, dt,$$

where $Q = (\kappa+1)/(4\pi^2\mu_0)$ and, for $j = 1, \dots, 6$,

$$S_j(\alpha, t) = \frac{D_{1j}}{D}R_1(\alpha, t) - \frac{D_{2j}}{D}R_2(\alpha, t) + \frac{D_{3j}}{D}R_3(\alpha, t) - \frac{D_{4j}}{D}R_4(\alpha, t) + \frac{D_{5j}}{D}R_5(\alpha, t) - \frac{D_{6j}}{D}R_6(\alpha, t).$$

4. Solution of the sliding mode problem

Similarly to the opening mode problem, the solution of the composite medium for the sliding mode problem may be determined by obtaining the solution in the graded layer and that in the homogeneous substrate with the exception that $y = 0$ is a plane of antisymmetry. The layer solution may be expressed as the sum of two solutions, namely an infinite graded medium with a crack and a graded strip without a crack [Erdogan and Wu 1996; Dag and Erdogan 2002; Bogy 1975]. By superposing the two sets of solutions, the displacement and stress field solutions of the $y \ge 0$ cracked graded strip can be established as follows:

$$\begin{aligned} &u_2^+(x, y) = u_2^{i+}(x, y) + u_2^s(x, y), \quad v_2^+(x, y) = v_2^{i+}(x, y) + v_2^s(x, y), \quad y \ge 0,\\ &\sigma_{kl2}^+(x, y) = \sigma_{kl2}^{i+}(x, y) + \sigma_{kl2}^s(x, y), \quad k, l = x, y, \qquad\qquad y \ge 0, \end{aligned} \tag{26}$$

where the subscript 2 stands for the sliding mode problem, the superscript i and s indicate, respectively, the cracked infinite medium and the strip, the superscript $+$ stands for the half-plane $y \ge 0$, $u_2^{i+}(x, y)$ and $v_2^{i+}(x, y)$ are given by Equations (A-1a), (A-1b) in the Appendix, $u_2^s(x, y)$ and $v_2^s(x, y)$ are given by (A-2a), (A-2b) and $\sigma_{kl2}^{i+}(x, y)$ and $\sigma_{kl2}^s(x, y)(k, l = x, y)$ are, respectively, given by (A-1c)–(A-1e) and (A-2c)–(A-2e).

The solution in the homogeneous half-plane can be obtained by considering the plane $y = 0$ as a plane of antisymmetry and that it needs to be bounded as x goes to ∞. The solution is given by (A-3). As in the opening mode problem, the sliding mode problem contains nine unknowns: D_3, D_4 from (A-1), D_5, D_7 from (A-3), $F_1, \dots, F_4$ from (A-2), and f_2, which can be determined from the nine boundary conditions

$$\sigma_{yy2}(x, 0^+) = 0, \qquad x > 0, \tag{27}$$

$$\left.\begin{aligned} \sigma_{xx2}(x, y) = 0, \\ \sigma_{xy2}(x, y) = 0, \end{aligned}\right\} \qquad x = 0, \ y < \infty, \tag{28}$$

$$\left.\begin{aligned} \sigma_{xx2}(h^+, y) = \sigma_{xx2}(h^-, y), \\ \sigma_{xy2}(h^+, y) = \sigma_{xy2}(h^-, y), \end{aligned}\right\} \qquad y < \infty, \tag{29}$$

$$\left.\begin{aligned} u_2(h^+, y) = u_2(h^-, y), \\ v_2(h^+, y) = v_2(h^-, y), \end{aligned}\right\} \qquad y < \infty, \tag{30}$$

$$\left.\begin{aligned} v_1(x, 0^+) = 0, \\ u_2(x, 0^+) = 0, \end{aligned}\right\} \qquad d < x < \infty, \tag{31}$$

$$\sigma_{xy2}(x, 0^+) = -q(x), \qquad 0 < x < d. \tag{32}$$

Applying the boundary conditions (27) and (31) allows us to express the unknown functions D_3 and D_4 in terms of the density function f_2, as shown in Equations (A-4a), (A-4b). Applying the boundary conditions (28)–(30) yields the linear system in A.5. This system of equations can be analytically inverted, leading to expressions for the unknown functions D_5, D_7, F_1, F_2, F_3, F_4 in terms of the density function f_2; these are given in A.6.

5. Singular integral equations

Based on the solutions obtained in Sections 3 and 4 for the opening and sliding mode problems, we can apply the boundary conditions (25) and (32), obtaining the dual uncoupled integral equations

$$\int_0^d (K_{11}(x, t) + K_{12}(x, t)) f_1(t)\, dt = -p(x), \quad 0 < x < d, \tag{33a}$$

$$\int_0^d (K_{21}(x, t) + K_{22}(x, t)) f_2(t)\, dt = -q(x), \quad 0 < x < d, \tag{33b}$$

where the kernels $K_{11}(x, t)$ and $K_{12}(x, t)$ representing the infinite graded medium and the graded strip for the opening mode problem are given by

$$K_{11}(x, t) = \frac{\kappa+1}{4\pi(\kappa-1)} e^{\beta x} \lim_{y\to 0^+} \left(\int_0^{+\infty} M_0(\lambda, y) \cos(\lambda(x-t))\, d\lambda + \int_0^{+\infty} N_0(\lambda, y) \sin(\lambda(x-t))\, d\lambda \right),$$

$$K_{12}(x, t) = \frac{\kappa+1}{4\pi^2(\kappa-1)} e^{\beta x} \lim_{y\to 0^+} \int_0^{+\infty} \sum_{k=1}^{4} p^s_{yy1,k}(\alpha) S_k(\alpha, t) e^{n_k x} \cos(y\alpha)\, d\alpha,$$

where $M_0(\lambda, y) = S_0(\lambda, y) + S_0(-\lambda, y)$ and $N_0(\lambda, y) = i\,(S_0(\lambda, y) - S_0(-\lambda, y))$ with

$$S_0(\lambda, y) = \frac{i}{2\lambda} \left(\frac{p^i_{yy1,4} p^i_{xy1,3} e^{m_4 y} - p^i_{yy1,3} p^i_{xy1,4} e^{m_3 y}}{s^i_3 p^i_{xy1,4} - s^i_4 p^i_{xy1,3}} \right),$$

and the corresponding kernels $K_{21}(x,t)$ and $K_{22}(x,t)$ for the sliding mode problem are given by

$$K_{21}(x,t)=\frac{\kappa+1}{4\pi}e^{\beta x}\lim_{y\to 0^+}\left(\int_0^{+\infty}M_0'(\lambda,y)\cos(\lambda(x-t))\,d\lambda+\int_0^{+\infty}N_0'(\lambda,y)\sin(\lambda(x-t))\,d\lambda\right)$$

$$K_{22}(x,t)=\frac{\kappa+1}{4\pi^2}e^{\beta x}\left(\lim_{y\to 0^+}\int_0^{+\infty}\sum_{k=1}^{4}q^s_{xy2,k}(\alpha)S_k'(\alpha,t)e^{n_k x}\cos(y\alpha)\,d\alpha\right),$$

where $M_0'(\lambda,y)$ and $N_0'(\lambda,y)$ are given analogously in terms of

$$S_0'(\lambda,y)=\frac{i}{2\lambda}\left(\frac{q^i_{xy2,4}q^i_{yy2,3}e^{m_4 y}-q^i_{xy2,3}q^i_{yy2,4}e^{m_3 y}}{q^i_{yy2,4}-q^i_{yy2,3}}\right).$$

The singular nature of the integral equations above and that of the solutions f_1 and f_2 may be determined by studying the asymptotic behavior of the integrands found in the expressions of $K_{11}(x,t)$, $K_{12}(x,t)$, $K_{21}(x,t)$ and $K_{22}(x,t)$. After a very lengthy analysis, the singular integral equations above become

$$\begin{aligned}&\int_0^d\left(\frac{1}{\pi(t-x)}+h_1^s(x,t)+h_1^f(x,t)\right)f_1(t)\,dt=-e^{-\beta x}p(x),\quad 0<x<d,\\&\int_0^d\left(\frac{1}{\pi(t-x)}+h_2^s(x,t)+h_2^f(x,t)\right)f_2(t)\,dt=-e^{-\beta x}q(x),\quad 0<x<d,\end{aligned}\tag{34}$$

where $h_1^f(x,t)$ and $h_2^f(x,t)$ are bounded Fredholm kernels and $h_1^s(x,t)$ and $h_2^s(x,t)$ are generalized Cauchy kernels of order $1/t$ that become unbounded as the arguments x and t tend to the end point zero simultaneously. The Cauchy kernels are of the form

$$h_1^s(x,t)=\frac{(\kappa+1)e^{\beta(t-x)/2}}{2(\kappa-1)}\left(\frac{2h_2^*}{(t+x)^3}+\frac{h_1^*}{(t+x)^2}+\frac{h_0}{(t+x)}+\frac{c_1^*}{(2h-t-x)^2}+\frac{c_0}{(2h-t-x)}\right),$$

$$h_2^s(x,t)=\frac{(\kappa+1)e^{\beta(t-x)/2}}{2}\left(\frac{2m_2^*}{(t+x)^3}+\frac{m_1^*}{(t+x)^2}+\frac{m_0}{(t+x)}+\frac{l_1^*}{(2h-t-x)^2}+\frac{l_0}{(2h-t-x)}\right),$$

where each of the coefficients h_0, h_1^*, h_2^*, c_0, c_1^*, m_0, m_1^*, m_2^*, l_0 and l_1^* is a lengthy function of κ, β, x and t.

From these singular integral equations, we conclude there is no singularity at the crack mouth, while the standard square-root or Cauchy singularity, $1/(t-x)$, is retained at the crack tip, in addition to other singularities contained in the generalized Cauchy kernels. The solution of these equations is expressed as follows, where $\tilde{f}_1(t)$ and $\tilde{f}_2(t)$ are unknown bounded functions (see [Erdogan et al. 1973]):

$$f_1(t)=(d-t)^{-1/2}\tilde{f}_1(t),\quad f_2(t)=(d-t)^{-1/2}\tilde{f}_2(t),\quad 0<t<d,\tag{35}$$

The limits of the generalized Cauchy kernels are the same as for homogeneous materials [Erdogan and Wu 1997], namely

$$\lim_{\beta\to 0}h_1^s(x,t)=\lim_{\beta\to 0}h_2^s(x,t)=\frac{1}{\pi}\left(\frac{1}{t+x}+\frac{2t}{(t+x)^2}-\frac{4t^2}{(t+x)^3}\right),\quad 0<(t,x)<d.\tag{36}$$

6. Solution of the singular integral equations

We have the following normalizations for the singular integral equations (34):

$$
\begin{aligned}
t &= \tfrac{1}{2}d(r+1), & x &= \tfrac{1}{2}d(s+1), \quad 0<(t,x)<d,\ -1<(r,s)<1,\\
f_1(t) &= \bar{f}_1(r), & f_2(t) &= \bar{f}_2(r), \quad 0<t<d,\ -1<r<1,\\
h_1^s(x,t) &= \bar{h}_1^s(s,r), & h_2^s(x,t) &= \bar{h}_2^s(s,r), \quad 0<(x,t)<d,\ -1<(s,r)<1,\\
h_1^f(x,t) &= \bar{h}_1^f(s,r), & h_2^f(x,t) &= \bar{h}_2^f(s,r), \quad 0<(x,t)<d,\ -1<(s,r)<1,\\
p(x) &= \bar{p}(s), & q(x) &= \bar{q}(s), \quad 0<x<d,\ -1<s<1.
\end{aligned}
$$

Hence the equations become

$$
\begin{aligned}
&\int_{-1}^{1}\left(\frac{1}{\pi(r-s)}+\bar{h}_1^s(s,r)+\bar{h}_1^f(s,r)\right)\bar{f}_1(r)\,dr=-e^{-\beta(d/2)(s+1)}\bar{p}(s), \quad -1<s<1,\\
&\int_{-1}^{1}\left(\frac{1}{\pi(r-s)}+\bar{h}_2^s(s,r)+\bar{h}_2^f(s,r)\right)\bar{f}_2(r)\,dr=-e^{-\beta(d/2)(s+1)}\bar{q}(s), \quad -1<s<1.
\end{aligned}
\tag{37}
$$

It was shown in [Erdogan et al. 1973] that the solution of these equations may be expressed as $\bar{f}_i(r)=w(r)\psi_i(r)$, $i=1,2$, where $w(r)=1/\sqrt{1-r}$ is the weight function associated with the Jacobi polynomial $P_n^{(-1/2,0)}(r)$ and $\psi_i(r)$, $(i=1,2)$ are continuous and bounded function in the interval $[-1,1]$ which may be expressed as convergent series of Jacobi polynomials. Hence the solutions of (37) become

$$
\bar{f}_1(r)=\frac{1}{\sqrt{1-r}}\sum_{n=0}^{\infty}A_{1n}P_n^{(-1/2,0)}(r), \quad \bar{f}_2(r)=\frac{1}{\sqrt{1-r}}\sum_{n=0}^{\infty}A_{2n}P_n^{(-1/2,0)}(r), \quad -1<r<1. \tag{38}
$$

Substituting this into (37), truncating the series at N and regularizing the singular terms, the above integral equations reduce to a system of linear algebraic equations in the $2(N+1)$ unknowns A_{1n} and A_{2n}:

$$
\sum_{n=0}^{N}\left(-\frac{\Gamma(-1/2)\Gamma(n+1)}{\sqrt{2}\pi\Gamma(n+1/2)}F\big(n+1,-n+1/2;\,3/2;\,(1-s)/2\big)+M_1(s)\right)A_{1n}
= -e^{(-\beta d(1+s)/2)}p(d(1+s)/2), \quad -1<s<1, \tag{36a}
$$

$$
\sum_{n=0}^{N}\left(-\frac{\Gamma(-1/2)\Gamma(n+1)}{\sqrt{2}\pi\Gamma(n+1/2)}F\big(n+1,-n+1/2;\,3/2;\,(1-s)/2\big)+M_2(s)\right)A_{2n}
= -e^{(-\beta d(1+s)/2)}q(d(1+s)/2), \quad -1<s<1, \tag{36b}
$$

where Γ is the gamma function, F is the hypergeometric function and M_1, M_2 are given by

$$
M_k(s)=\int_{-1}^{1}(1-r)^{-1/2}H_k(s,r)P_n^{(-1/2,0)}(r)\,dr \quad (k=1,2),
$$

with

$$
H_k(s,r)=\frac{d}{2}\left(h_k^s\Big(\frac{d}{2}s+\frac{d}{2},\frac{d}{2}r+\frac{d}{2}\Big)+h_k^f\Big(\frac{d}{2}s+\frac{d}{2},\frac{d}{2}r+\frac{d}{2}\Big)\right).
$$

Equations (36) are solved numerically using a suitable collocation technique [Erdogan and Wu 1996]. Once the $2(N+1)$ unknowns coefficients A_{1n} and A_{2n} are obtained, the mixed-mode crack tip stress intensity factors may be estimated as follows:

$$\begin{aligned} k_1 &= \lim_{x\to d+0} \sqrt{2(x-d)}\sigma_{yy}(x,0) = -e^{\beta d}\sqrt{d}\sum_{n=0}^{N} A_{1n} P_n^{(-1/2,0)}(1), \\ k_2 &= \lim_{x\to d+0} \sqrt{2(x-d)}\sigma_{xy}(x,0) = -e^{\beta d}\sqrt{d}\sum_{n=0}^{N} A_{2n} P_n^{(-1/2,0)}(1). \end{aligned} \tag{40}$$

7. Results and discussion

Table 1 gives the thermomechanical properties of actual functionally graded materials studied in [Zhao et al. 2004; Jin and Paulino 2001; Shodja and Ghahremaninejad 2006; Shao 2005; Ma and Wang 2003; Ching and Yen 2005; Lee and Erdogan 1998b]. The tabulated FGMs are made of two constituents, such as ceramic Al_2O_3 and titanium carbide (TiC), or Rene 41 and zirconia. It is clear from this table that the thermomechanical properties, such as the elastic modulus, the thermal expansion coefficient and the heat conductivity, can be proportional (decreasing or increasing) or nonproportional from one face to the other face of the graded layer. For example, for a Rene 41 and zirconia FGM, the ratio of the elastic moduli,

components	reference	ν	E	E_1/E_2	α	α_1/α_2	k	k_1/k_2
ceramic Al_2O_3 titanium carbide	[Zhao et al. 2004]	0.26 0.19	380 450	<1	8.5 7.6	>1	40.37 24.28	>1
silicon carbide titanium carbide	[Jin and Paulino 2001]	0.20 0.20	400 400	$=1$	4.0 7.0	<1	60 20	>1
titanium nickel aluminum alloy	[Shodja and Ghahremaninejad 2006]	0.20 0.33	600 70	>1	8.08 23.05	<1	22 190	<1
titanium nickel carbon steel	[Shodja and Ghahremaninejad 2006]	0.20 0.30	600 207	>1	8.08 12.36	<1	22 40	<1
mullite molybednum	[Shao 2005]	0.27 0.30	225 330	<1	4.8 4.9	≈ 1	5.9 138	<1
zirconia aluminum	[Ma and Wang 2003]	0.3 0.3	151 70	>1	9.97 22.86	<1	2.09 204	<1
silicon carbide aluminum	[Ching and Yen 2005]	0.17 0.3	427 70	>1	4.3 23.4	<1	65 233	<1
Rene 41 zirconia	[Lee and Erdogan 1998b]	0.3 0.3	219.7 151	>1	16.7 10	>1	25.51 2.09	>1

Table 1. Thermomechanical properties of various types of functionally graded materials. Legend: ν, Poisson's ratio; E, elastic modulus, in GPa; α, thermal expansion coefficient, in $10^{-6}\mathrm{K}^{-1}$, k, thermal conductivity, in $10^3\mathrm{W}@(\mathrm{mK})^{-1}$. Subscript 1 corresponds to the top-row material in each pair and subscript 2 to bottom row.

βd	$k_1/(\sigma_n d^{1/2})$							
	$p(x)=\sigma_0$		$p(x)=\sigma_1(x/d)$		$p(x)=\sigma_2(x/d)^2$		$p(x)=\sigma_3(x/d)^3$	
	lit.	cur.	lit.	cur.	lit.	cur.	lit.	cur.
-3.0	4.4345	4.4481	1.9324	1.9375	1.2148	1.2176	0.8897	0.8916
-2.0	3.1238	3.1254	1.4495	1.4501	0.9525	0.9528	0.7209	0.7211
-1.0	1.9846	1.9853	1.0196	1.0199	0.7152	0.7154	0.5663	0.5664
-0.5	1.4988	1.4991	0.8317	0.8318	0.6099	0.6099	0.4970	0.4970
10^{-4}	1.1215	1.1215	0.6828	0.6828	0.5255	0.5255	0.4410	0.4410
0.5	1.0225	1.0225	0.6439	0.6439	0.5035	0.5035	0.4264	0.4264
1.0	0.9930	0.9931	0.6328	0.6328	0.4947	0.4974	0.4225	0.4225
2.0	0.9807	0.9806	0.6289	0.6289	0.4956	0.4956	0.4215	0.4215
3.0	09884	0.9885	0.6329	0.6329	0.4981	0.4982	0.4233	0.4234

βd	$k_2/(\tau_n d^{1/2})$							
	$p(x)=\tau_0$		$p(x)=\tau_1(x/d)$		$p(x)=\tau_2(x/d)^2$		$p(x)=\tau_3(x/d)^3$	
	lit.	cur.	lit.	cur.	lit.	cur.	lit.	cur.
-3.0	1.6704	1.6708	0.9273	0.9276	0.6738	0.6739	0.5437	0.5438
-2.0	1.4765	1.4767	0.8398	0.8399	0.6202	0.6202	0.5063	0.5063
-1.0	1.2825	1.2824	0.7534	0.7534	0.5678	0.5677	0.4700	0.4700
-0.5	1.1940	1.1939	0.7144	0.7144	0.5443	0.5443	0.4539	0.4539
10^{-4}	1.1215	1.1216	0.6829	0.6829	0.5255	0.5255	0.4410	0.4410
0.5	1.0727	1.0727	0.6620	0.6620	0.5132	0.5132	0.4327	0.4327
1.0	1.0429	1.0428	0.6497	0.6497	0.5062	0.5062	0.4280	0.4280
2.0	1.0164	1.0165	0.6397	0.6397	0.5008	0.5008	0.4245	0.4246
3.0	1.0128	1.0127	0.6394	0.6394	0.5011	0.5011	0.4249	0.4250

Table 2. Normalized mode I (top) and mode II (bottom) stress intensity factors for a graded half-plane: comparison of results from [Dag and Erdogan 2002] (columns labeled "lit.") with those obtained in the current study.

the ratio of the thermal expansion coefficients and the ratio of the heat conductivity are all greater than one. For a titanium nickel (TiNi) and aluminum alloy FGM, the ratio of the elastic moduli is greater than one while the ratio of the thermal expansion coefficients and the ratio of the heat conductivity are both less than one. For a ceramic Al_2O_3 and titanium carbide FGM, the ratio of the elastic moduli is less than one while the ratio of the thermal expansion coefficients and the ratio of the heat conductivity are both greater than one. Since this study is limited only to variations of the elastic modulus, we, therefore, consider two cases of FGM coatings, a stiff graded layer with $\beta d < 0$ and a compliant graded layer with $\beta d > 0$, which correspond, respectively, to a ratio of the elastic moduli greater than and less than 1.

Furthermore, for the problem considered, if the coating thickness tends to infinity ($h \to \infty$), the configuration of the problem tends toward a graded half-plane, with the effect of the homogeneous

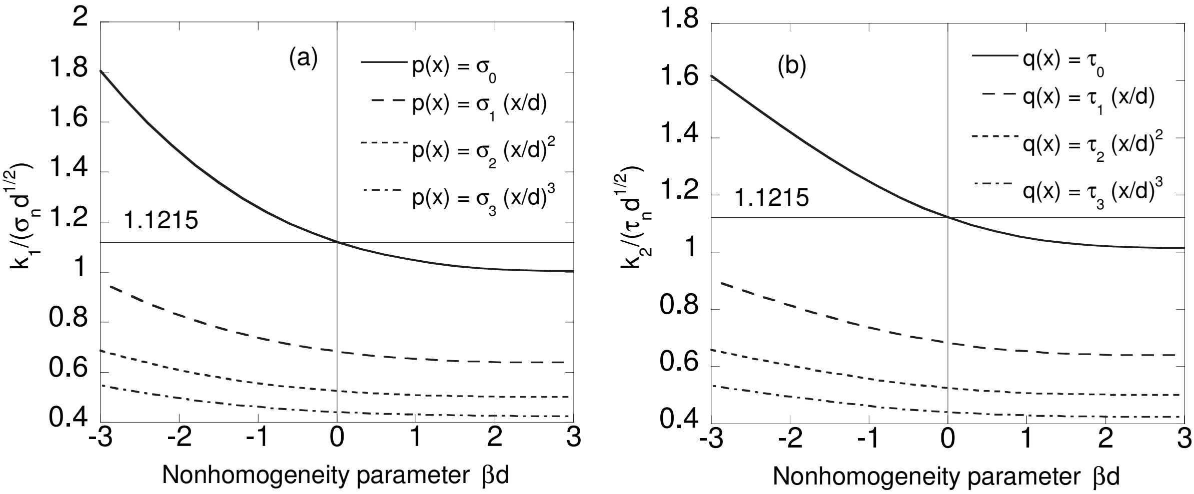

Figure 2. Effect of the nonhomogeneity parameter βd on the normalized mode I (left) and mode (II) stress intensity factors under plane strain and various distributions of crack surface tractions σ_n or τ_n, $n = 0, 1, 2, 3$. In each case, $\nu = 0.25$ and $h/d = 2$.

substrate almost negligible. As mentioned in the introduction, this crack problem was solved in [Dag and Erdogan 2002]. We see in Table 2 that the results obtained here are in agreement with those published in that work; the tables give the normalized mode I and II stress intensity factors for an edge crack in a graded half-plane, assuming different values of the nonhomogeneity parameter βd, various polynomial forms of crack surface tractions, plane strain conditions, a Poisson's ratio $\nu = 0.25$ and a large value of the coating thickness h ($h/d = 50$).

When $|\beta d|$ approaches zero, the composite medium becomes a homogeneous elastic half-plane. The closed-form solution for the stress intensity factor for a surface crack subject to a uniform normal traction $p(x) = \sigma_0$ was obtained in [Koiter 1965] in terms of an infinite integral as

$$\frac{k_1}{\sigma_0\sqrt{d}} = \sqrt{\frac{2(B+1)}{\sqrt{\pi}A}}, \quad \log A = -\frac{1}{\pi}\int_0^\infty \frac{1}{1+\alpha^2}\log\left(\frac{\alpha\sinh(\pi\alpha)}{\sqrt{B^2+\alpha^2}\,(\cosh(\pi\alpha)-2\alpha^2-1)}\right)d\alpha, \tag{41}$$

where B is an arbitrary real constant greater than 1 and the result is independent of the choice of B.

The numerical evaluation of (41), performed in [Kaya and Erdogan 1987], shows that $k_1/(\sigma_0\sqrt{d}) = 1.12152226$, which is the result obtained in Table 2 (top) for $\beta d = 0.0001$.

Figure 2 illustrates the effect of the nonhomogeneity parameter βd [in $\mu_1 = \mu_0\exp(\beta x)$] on the mode I and II crack tip stress intensity factors (SIF) for the case of constant $p(x) = \sigma_0$, linear $p(x) = \sigma_1(x/d)$, quadratic $p(x) = \sigma_2(x/d)^2$ and cubic $p(x) = \sigma_3(x/d)^3$ crack surface normal tractions. The results were calculated for plane strain conditions, a Poisson's ratio $\nu = 0.25$ and a coating thickness $h/d = 2$. Increasing the value of the nonhomogeneity parameter βd from -3 to 3 results in a decrease in both mode I and mode II stress intensity factors. In addition, k_1 and k_2 are much more sensitive to the variations in βd for $\beta d < 0$ (that is, compliant graded coating) than for $\beta d > 0$ (stiff graded coating).

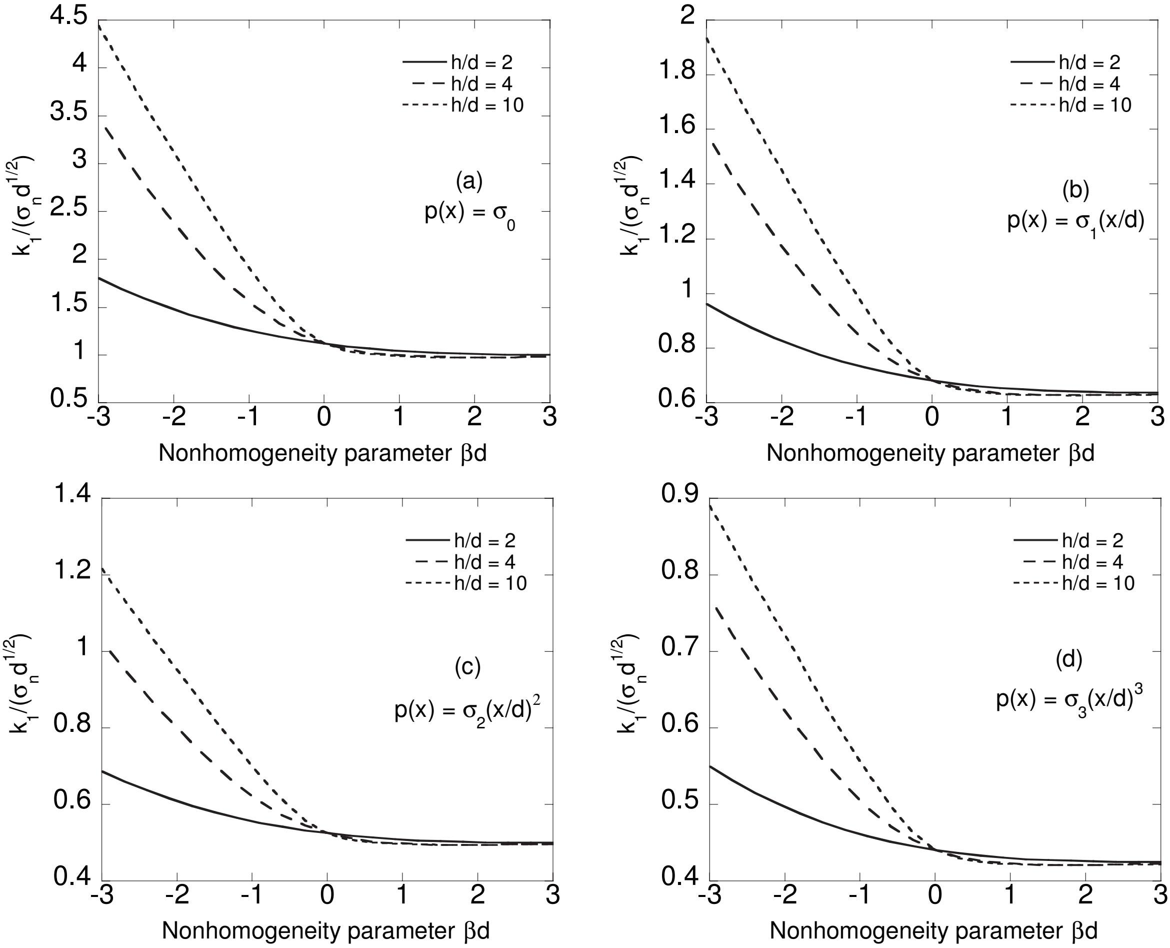

Figure 3. Effect of coating thickness h/d and stiffness parameter βd on the normalized mode I stress intensity factor under various distributions of crack surface normal tractions, $\nu = 0.25$, plane strain, (a) constant normal traction $p(x) = \sigma_0$, (b) linear normal traction $p(x) = \sigma_1(x/d)$, (c) quadratic normal traction $p(x) = \sigma_2(x/d)^2$, (d) cubic normal traction $p(x) = \sigma_3(x/d)^3$; σ_n $(n = 0, 1, 2, 3)$, τ_n $(n = 0, 1, 2, 3)$.

Figures 3 and 4 illustrate the effect of varying the coating thickness h/d $(= 2, 4, 10)$ and the stiffness parameter βd $(= -3, \ldots, 3)$ on the normalized mode I and mode II stress intensity factor under various distributions of crack surface normal tractions. The results were calculated for plane strain conditions, a Poisson's ratio $\nu = 0.25$. It can be seen from Figure 3 that k_1 is sensitive to the variations of the coating thickness h/d for $\beta d < 0$ and tends to be insensitive for $\beta d > 0$. Furthermore, for negative values of βd, the value of k_1 tends to increase when the coating thickness is increased. In addition, the rate of change of k_1 with respect to βd for $\beta d < 0$ becomes more important for increasing values of the coating thickness. Moreover, increasing h/d from 2 to 10 indicates that the crack-tip is located more and more in the stiffer side of the coating, and, therefore, the increase of k_1 can be compensated somewhat by a larger fracture toughness which decreases for a compliant graded coating ($\beta d < 0$). On the other hand,

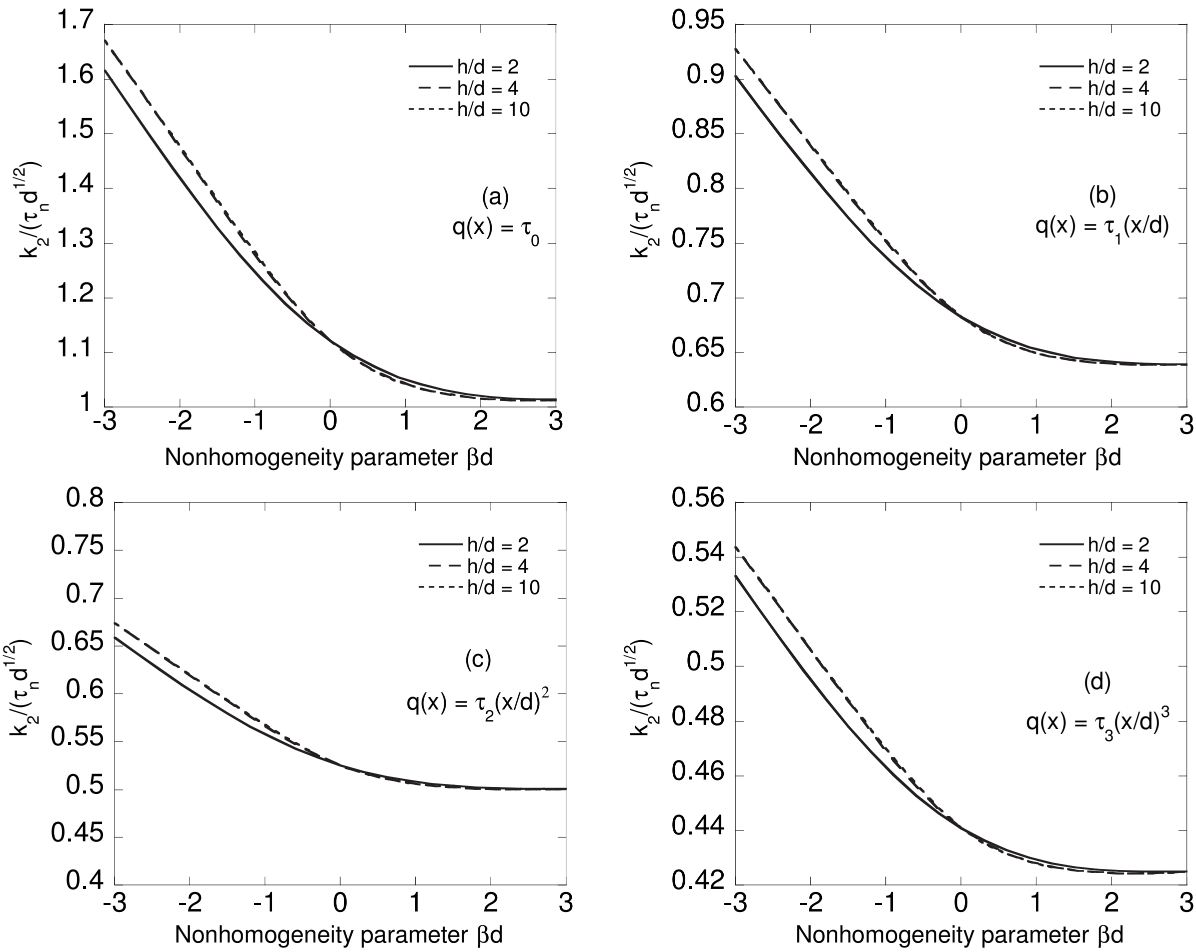

Figure 4. Effect of coating thickness h/d and stiffness parameterβd on the normalized mode II stress intensity factor under various distributions of crack surface shear tractions, $\nu = 0.25$, plane strain, (a) constant shear traction $q(x) = \tau_0$, (b) linear shear traction $q(x) = \tau_1(x/d)$, (c) quadratic shear traction $q(x) = \tau_2(x/d)^2$, (d) cubic shear traction $q(x) = \tau_3(x/d)^3$.

Figure 4 indicates that the mode II stress intensity factor is relatively insensitive to the variations of the coating thickness h/d. In other words, further increase of h/d from 4 to 10 does not seem to affect the values of k_2.

Figure 5 shows some sample results for the normalized normal and tangential crack opening displacements, $v^*(x) = v(x, 0^+) - v(x, 0^-)$ and $u^*(x) = u(x, 0^+) - u(x, 0^-)$, obtained by applying, respectively, constant normal and shear tractions. The results were calculated for plane strain conditions, a Poisson's ratio $\nu = 0.25$ and a coating thickness $h/d = 2$. It may be observed that as βd increases, the crack opening displacements decrease. Furthermore, the influence of βd on the crack opening displacement is more significant for a compliant graded layer ($\beta d < 0$) than for a stiff graded layer ($\beta d > 0$). The crack opening displacements for the homogeneous medium ($\beta d = 0.0001$) is bracketed by the results obtained

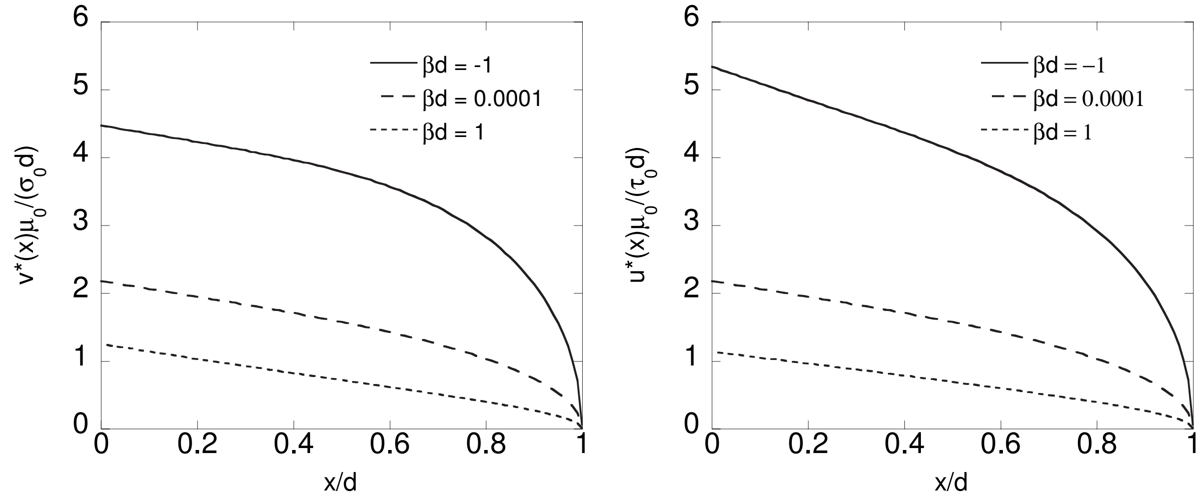

Figure 5. Normal (left) and tangential (right) crack opening displacement under constant normal traction $p(x) = \sigma_0$ (left) or constant shear traction $q(x) = \tau_0$ (right). In both cases, $\nu = 0.25$ and $h/d = 2$.

for $\beta d < 0$ and $\beta d > 0$. In addition, for $\beta d < 0$, crack opening displacements under mode I loading (that is, normal crack surface tractions) is greater than that under mode II loading (shear crack surface tractions).

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Appendix: Mode II problem expressions

A.1. ***Expressions of displacements and stresses in the infinite graded medium (half-plane $y \geq 0$).***

$$u_2^{i+}(x, y) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \sum_{k=3}^{4} D_k e^{m_k y} e^{ix\lambda} d\lambda, \quad y \geq 0, \tag{A-1a}$$

$$v_2^{i+}(x, y) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \sum_{k=3}^{4} D_k s_k^i e^{m_k y} e^{ix\lambda} d\lambda, \quad y \geq 0, \tag{A-1b}$$

$$\sigma_{xx2}^{i+}(x, y) = \frac{\mu_1}{2\pi(\kappa-1)} \int_{-\infty}^{+\infty} \sum_{k=3}^{4} p_{xx1,k}^i(\lambda) D_k e^{m_k y} e^{ix\lambda} d\lambda, \quad y \geq 0, \tag{A-1c}$$

$$\sigma^{i+}_{yy2}(x,y) = \frac{\mu_1}{2\pi(\kappa-1)}\int_{-\infty}^{+\infty}\sum_{k=3}^{4} p^i_{yy1,k}(\lambda) D_k e^{m_k y} e^{ix\lambda} d\lambda, \quad y \geq 0, \tag{A-1d}$$

$$\sigma^{i+}_{xy2}(x,y) = \frac{\mu_1}{2\pi}\int_{-\infty}^{+\infty}\sum_{k=3}^{4} p^i_{xy1,k}(\lambda) D_k e^{m_k y} e^{ix\lambda} d\lambda, \quad y \geq 0, \tag{A-1e}$$

where $D_k(\lambda)$, $(k = 3, 4)$ are unknowns functions, the roots m_k $(k = 3, 4)$ are given by (10b), the known function $s^i_k(\lambda)$ is given by (8) and $p^i_{xx1,k}(\lambda)$, $p^i_{yy1,k}(\lambda)$ and $p^i_{xy1,k}(\lambda)$ are known functions given by the equations following (11).

A.2. ***Expressions of displacements and stresses in the graded strip.***

$$u^s_2(x,y) = \int_0^{+\infty}\sum_{k=1}^{4} F_k e^{n_k x}\sin(y\alpha)\,d\alpha, \tag{A-2a}$$

$$v^s_2(x,y) = \int_0^{+\infty}\sum_{k=1}^{4} F_k r^s_k e^{n_k x}\cos(y\alpha)\,d\alpha, \tag{A-2b}$$

$$\sigma^s_{xx2}(x,y) = \frac{\mu_1}{\kappa-1}\int_0^{+\infty}\sum_{k=1}^{4} q^s_{xx2,k}(\alpha) F_k e^{n_k x}\sin(y\alpha)\,d\alpha, \tag{A-2c}$$

$$\sigma^s_{yy2}(x,y) = \frac{\mu_1}{\kappa-1}\int_0^{+\infty}\sum_{k=1}^{4} q^s_{yy2,k}(\alpha) F_k e^{n_k x}\sin(y\alpha)\,d\alpha, \tag{A-2d}$$

$$\sigma^s_{xy2}(x,y) = \mu_1\int_0^{+\infty}\sum_{k=1}^{4} q^s_{xy2,k}(\alpha) F_k e^{n_k x}\cos(y\alpha)\,d\alpha, \tag{A-2e}$$

where $F_k(\lambda)$ $(k = 1, 2, 3, 4)$, are unknown functions, the roots n_k, $k = 1, \ldots, 4$, are given by (15), the known functions $r^s_k(\lambda)$, $(k = 1, 2, 3, 4)$, are such that $r^s_k(\lambda) = -s^s_k(\lambda)$ in which $s^s_k(\lambda)$ is given by (13) and $q^s_{xx2,k}(\alpha)$, $q^s_{yy2,k}(\alpha)$ and $q^s_{xy2,k}(\alpha)$ are known functions given by

$$\begin{aligned}
q^s_{xx2,k}(\alpha) &= (1+\kappa)n_k - (3-\kappa)\alpha r^s_k,\\
q^s_{xy2,k}(\alpha) &= \alpha + r^s_k n_k, \quad (k = 1, \ldots, 4),\\
q^s_{yy2,k}(\alpha) &= (3-\kappa)n_k - (1+\kappa)\alpha r^s_k, \quad (k = 1, \ldots, 4).
\end{aligned}$$

A.3. ***Expressions of displacements and stresses in the homogeneous substrate.***

$$u^h_2(x,y) = \int_0^{+\infty}(D_5 + D_7 x)e^{-\alpha x}\sin(y\alpha)\,d\alpha, \tag{A-3a}$$

$$v^h_2(x,y) = \int_0^{+\infty}\Big(D_5 + D_7\frac{\kappa}{\alpha} - D_7 x\Big)e^{-\alpha x}\cos(y\alpha)\,d\alpha, \tag{A-3b}$$

$$\sigma^h_{xx2}(x,y) = \frac{\mu_2}{\kappa-1}\int_0^{+\infty}\big(q^h_{xx2,1}D_5 + q^h_{xx2,2}D_7 + q^h_{xx2,1}D_7 x\big)e^{-\alpha x}\sin(y\alpha)\,d\alpha, \tag{A-3c}$$

$$\sigma^h_{yy2}(x,y) = \frac{\mu_2}{\kappa-1}\int_0^{+\infty}\big(q^h_{yy2,1}D_5 + q^h_{yy2,2}D_7 + q^h_{yy2,1}D_7 x\big)e^{-\alpha x}\sin(y\alpha)\,d\alpha, \tag{A-3d}$$

$$\sigma^h_{xy2}(x,y) = \mu_2\int_0^{+\infty}\big(q^h_{xy2,1}D_5 + q^h_{xy2,2}D_7 + q^h_{xy2,1}D_7 x\big)e^{-\alpha x}\cos(y\alpha)\,d\alpha, \tag{A-3e}$$

where D_5 and D_7 are unknown functions and $q^h_{xx2,1}$, $q^h_{xx2,2}$, $q^h_{yy2,1}$, $q^h_{yy2,2}$, $q^h_{xy2,1}$ and $q^h_{xy2,2}$ are known functions given by

$$\begin{aligned} q^h_{xx2,1}(\lambda) &= 2(1-\kappa)\alpha, & q^h_{xx2,2}(\lambda) &= (\kappa-1)^2,\\ q^h_{yy2,1}(\lambda) &= -2(1-\kappa)\alpha, & q^h_{yy2,2}(\lambda) &= 3-2\kappa-\kappa^2,\\ q^h_{xy2,1}(\lambda) &= 2\alpha, & q^h_{xy2,2}(\lambda) &= -(1+\kappa). \end{aligned}$$

A.4. ***Expressions of D_3 and D_4 in terms of the density function f_2.***

$$D_3 = \frac{\kappa+1}{4\mu_0}\frac{i}{\lambda}\frac{(-q^i_{yy2,4})}{q^i_{yy2,4}-q^i_{yy2,3}}\int_0^d f_2(t)e^{-i\lambda t}dt, \tag{A-4a}$$

$$D_4 = \frac{\kappa+1}{4\mu_0}\frac{i}{\lambda}\frac{q^i_{yy2,3}}{q^i_{yy2,4}-q^i_{yy2,3}}\int_0^d f_2(t)e^{-i\lambda t}dt. \tag{A-4b}$$

A.5. ***Linear system of equations obtained after applying boundary conditions*** (28)–(30).

$$\begin{bmatrix} -q^s_{xx2,1} & -q^s_{xx2,2} & -q^s_{xx2,3} & -q^s_{xx2,4} & 0 & 0\\ q^s_{xy2,1} & q^s_{xy2,2} & q^s_{xy2,3} & q^s_{xy2,4} & 0 & 0\\ -q^s_{xx2,1}e^{n_1h} & -q^s_{xx2,2}e^{n_2h} & -q^s_{xx2,3}e^{n_3h} & -q^s_{xx2,4}e^{n_4h} & q^h_{xx2,1}e^{-\alpha h} & (q^h_{xx2,2}+hq^h_{xx2,1})e^{-\alpha h}\\ q^s_{xy2,1}e^{n_1h} & q^s_{xy2,2}e^{n_2h} & q^s_{xy2,3}e^{n_3h} & q^s_{xy2,4}e^{n_4h} & -q^h_{xy2,1}e^{-\alpha h} & -(q^h_{xy2,2}+hq^h_{xx2,1})e^{-\alpha h}\\ -e^{n_1h} & -e^{n_2h} & -e^{n_3h} & -e^{n_4h} & e^{-\alpha h} & he^{-\alpha h}\\ r^s_1e^{n_1h} & r^s_2e^{n_2h} & r^s_3e^{n_3h} & r^s_4e^{n_4h} & e^{-\alpha h} & (h-\frac{\kappa}{\alpha})e^{-\alpha h} \end{bmatrix}\begin{Bmatrix} F_1\\ F_2\\ F_3\\ F_4\\ D_5\\ D_7 \end{Bmatrix} = \begin{Bmatrix} G^R_1\\ G^R_2\\ G^R_3\\ G^R_4\\ G^R_5\\ G^R_6 \end{Bmatrix},$$

where we have defined

$$G^R_j = \frac{\kappa+1}{4\pi^2\mu_0}\int_0^d f_2(t)R'_j(\alpha,t)\,dt \quad (j=1,\ldots,6),$$

the $R'_j(\alpha,t)$ $(j=1,\ldots,6)$ being integrals evaluated using the residue theorem in a similar manner as the integrals R_j for the mode I problem.

A.6. ***Expressions of D_5, D_7, F_1, F_2, F_3 and F_4 in terms of the density function f_2.***

$$F_1 = Q\int_0^d S'_1(\alpha,t)f_2(t)\,dt,\quad F_3 = Q\int_0^d S'_3(\alpha,t)f_2(t)\,dt,\quad D_5 = Q\int_0^d S'_5(\alpha,t)f_2(t)\,dt,$$

$$F_2 = Q\int_0^d S'_2(\alpha,t)f_2(t)\,dt,\quad F_4 = Q\int_0^d S'_4(\alpha,t)f_2(t)\,dt,\quad D_7 = Q\int_0^d S'_6(\alpha,t)f_2(t)\,dt,$$

in which $Q = (\kappa+1)/(4\pi^2\mu_0)$ and, for $j = 1, \ldots, 6$,

$$S'_j(\alpha, t) = \frac{H_{1j}}{H} R'_1(\alpha, t) - \frac{H_{2j}}{H} R'_2(\alpha, t) + \frac{H_{3j}}{H} R'_3(\alpha, t) - \frac{H_{4j}}{H} R'_4(\alpha, t) + \frac{H_{5j}}{H} R'_5(\alpha, t) - \frac{H_{6j}}{H} R'_6(\alpha, t).$$

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