Journal of
Mechanics of
Materials and Structures

EFFECT OF NONHOMOGENEITY ON THE CONTACT OF AN ISOTROPIC HALF-SPACE AND A RIGID BASE WITH AN AXIALLY SYMMETRIC RECESS

Sakti Pada Barik, Mridula Kanoria and Pranay Kumar Chaudhuri

Volume 3, № 1 January 2008

mathematical sciences publishers

JOURNAL OF MECHANICS OF MATERIALS AND STRUCTURES
Vol. 3, No. 1, 2008

EFFECT OF NONHOMOGENEITY ON THE CONTACT OF AN ISOTROPIC HALF-SPACE AND A RIGID BASE WITH AN AXIALLY SYMMETRIC RECESS

SAKTI PADA BARIK, MRIDULA KANORIA AND PRANAY KUMAR CHAUDHURI

We study an axially symmetric frictionless contact problem between a nonhomogeneous elastic half-space and a rigid base that has a small axisymmetric surface recess. We reduce the problem to solving Fredholm integral equations, solve these equations numerically, and establish a relationship between the applied pressure and the gap radius. We find and graph the effects of nonhomogeneity on the normal pressure, critical pressure and on the surface displacement.

1. Introduction

When two bodies are placed in contact, they touch either at a point, along a line, over a surface or in a combination thereof. While the initial contact is determined by the geometric features of the bodies, the extent generally changes when the bodies are deformed by applied forces, changes in temperature or other sources of stress. In the study of contact problems, a class of problems is considered when two bodies are in contact without a bond, so that the region of contact is not known. Here, determining the contact region, which depends on geometric features as well the load distribution, presents an additional task for finding the stress distribution. Contact problems have been studied extensively in the literature, but in most cases the study was confined to isotropic and homogeneous solids. With the increasing use of functionally graded materials or anisotropic materials in industry, the study needs to be extended to these materials also.

A comprehensive list of work by earlier investigators has been provided in [Sneddon and Lowengrub 1969] and also [Gladwell 1980]. Among the recent works on the contact problems, notable are the works of Civelek et al. [1978], Schmuesre et al. [1980], Gecit [1981], Selvanduri [1983], Loboda and Tauchert [1985], Martynyak [1985], Fabrikant [1986], Li and Dempsey [1988], Wu and Yen [1994], Shvets et al. [1996], Argatov [2000], Brock and Georiadis [2001], Kit and Monastyrsky [2001], Argatov [2004], and Barik et al. [2006].

The very important class of contact problems known as receding contact problems is the subject of study for many investigators. If the contact area diminishes as the load is applied, the contact is called receding. The analytical studies involving receding contact in homogeneous and graded media can be found in [Hussain et al. 1968; Noble and Hussain 1969; Weitsman 1969; Pu and Hussain 1970; Keer et al. 1972; Gecit 1986; Nowell and Hills 1988; Chaudhuri and Ray 1998; Birinci and Erdol 1999; Chaudhuri and Ray 2003; Comez et al. 2004; El-Borgi et al. 2006]. Numerical studies based either on the finite element method or on the boundary element method can be found in [Jing and Liao 1990; Garrido et al. 1991; Paris et al. 1995; Satish Kumar et al. 1996; Garrido and Lorenzana 1998].

Keywords: Hankel transform, dual integral equation, Fredholm integral equation, elastic nonhomogeneity.

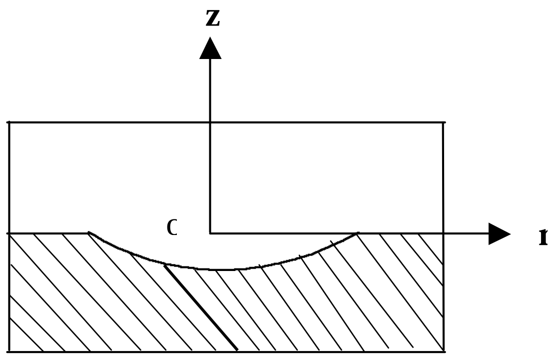

Figure 1. The semiinfinite solid with recess considered here.

In a recent paper, Kit and Monastyrsky [2001] discussed an axially symmetric problem of frictionless contact between an elastic half space and a rigid base with a small surface recess. The elastic material considered was isotropic and homogeneous. The present investigation aims at studying a similar problem, but in an nonhomogeneous elastic medium.

The nonhomogeneity arises because the rigidity modulus varies with distance in the medium. Although in the homogeneous case the displacement components are expressible in terms of potential functions, the solution of the governing equations is not so simple here, because of the nonhomogeneous modulus. Following the technique discussed in [Ozturk and Erdogan 1993], we have found the displacement components and, using the boundary conditions, have obtained a pair of dual integral equations Further manipulation yields Fredholm integral equations, which we solve numerically. We have done numerical computations to show the effect of the nonhomogeneity. Finally, we have checked that results of [Kit and Monastyrsky 2001] are recovered from ours by zeroing the nonhomogeneous parameter.

2. Formulating the problem

Let a semiinfinite solid of nonhomogeneous isotropic material with a flat surface lie on a semiinfinite solid of rigid material. The boundary of the solid of rigid material is everywhere planar except for a geometrical defect, which is shallow axisymmetric recess, as seen in Figure 1. We put the origin at the common boundary of the half spaces and point the z_1-axis of the cylindrical coordinate system (r_1, θ, z_1) into the elastic medium along the recess's axis of symmetry. We represent the bounding surface of the rigid base containing the recess by the equations

$$z_1 = f_1(r_1) = \begin{cases} -h_0\left(1 - \frac{r_1^2}{b^2}\right)^{3/2}, & 0 < r_1 \le b, \\ 0, & r_1 > b. \end{cases}$$

We shall assume that contact is smooth. In the absence of applied pressure on the solids, contact is made along the plane $z_1 = 0$, except for the central area $\mid r_1 \mid < b$. Applying the normal pressure p at infinity will cause the contact surface to increase. Let $a_1(p)$ be the radius of the gap, that is, the region in which there is no contact, as an as-yet unknown function of p, as shown in Figure 2. Before proceeding, it will be convenient to adopt dimensionless variables by rescaling all lengths by the problem's only length

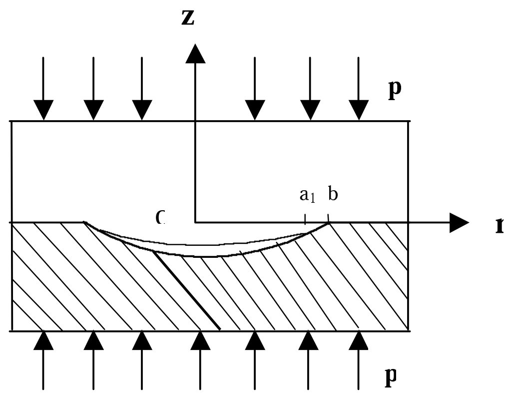

Figure 2. The first contact point a_1 moves inward as the pressure p increases.

scale b:

$$r = \frac{r_1}{b}, \qquad z = \frac{z_1}{b}, \qquad h = \frac{h_0}{b}, \qquad a = \frac{a_1}{b}, \qquad \hat{\sigma}_{i,j} = \frac{\sigma_{i,j}}{\mu_0}, \qquad \hat{u}_i = \frac{u_i}{b},$$

where here and in the following $i, j = r, \theta, z$. In the dimensionless variables, the surface with recess becomes

$$z = f(r) = \begin{cases} -h\left(1 - r^2\right)^{3/2}, & r \le 1, \\ 0, & r > 1. \end{cases} \tag{1}$$

In the analysis below, for notational convenience, we shall use only dimensionless variables and politely remove their hats $\hat{}$.

We suppose that the elastic material is nonhomogeneous by assuming the rigidity modulus μ varies along the z-axis as

$$\mu = \mu_0 e^{\alpha z}, \tag{2}$$

where α is the nonhomogeneity parameter. Because of axisymmetry, the field variables are independent of θ and the displacement vector $(u, 0, w)$ is a function of r and z only. Using the strain displacement relations

$$e_{rr} = \frac{\partial u}{\partial r}, \qquad e_{\theta\theta} = \frac{u}{r},$$
$$e_{zz} = \frac{\partial w}{\partial z}, \qquad 2e_{rz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r},$$

and Hooke's law

$$\sigma_{ij} = \frac{\lambda}{\mu_0} e_{kk}\delta_{ij} + 2\frac{\mu}{\mu_0} e_{ij},$$

where λ and μ are the Lame's constants and δ_{ij} is the Kronecker delta, the equations of equilibrium

$$\begin{aligned} 0 &= \frac{\partial \sigma_{rr}}{\partial r} + \frac{\partial \sigma_{rz}}{\partial z} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r}, \\ 0 &= \frac{\partial \sigma_{rz}}{\partial r} + \frac{\partial \sigma_{zz}}{\partial z} + \frac{\sigma_{rz}}{r}, \end{aligned} \tag{3}$$

may be expressed in terms of displacements components u and w as:

$$0=(\kappa+1)\left(\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}-\frac{u}{r^2}\right)+2\frac{\partial^2 w}{\partial r\partial z}+(\kappa-1)\frac{\partial^2 u}{\partial z^2}+(\kappa-1)\alpha\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial r}\right),$$

$$\begin{aligned}0=(\kappa+1)\frac{\partial^2 w}{\partial z^2}+\frac{2}{r}\frac{\partial u}{\partial z}+2\frac{\partial^2 u}{\partial r\partial z}+(\kappa-1)\left(\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\frac{\partial w}{\partial r}\right)\\ +(3-\kappa)\alpha\left(\frac{\partial u}{\partial r}+\frac{u}{r}\right)+(\kappa+1)\alpha\frac{\partial w}{\partial z},\end{aligned} \tag{4}$$

where $\kappa=3-4\nu$ and ν is Poisson's ratio.

The boundary conditions for the problem are

$$\begin{aligned}\lim_{z-\infty}\sigma_{zz}(r,z)=&-\frac{p}{\mu_0},\\ \lim_{z-\infty}\sigma_{rz}(r,z)=&\;\;0,\end{aligned} \tag{5}$$

$$\begin{aligned}\sigma_{zz}(r,0)&=0, && 0<r<a,\\ \sigma_{rz}(r,0)&=0, && 0<r<\infty,\\ w(r,0)&=f(r), && a<r<\infty,\end{aligned} \tag{6}$$

In addition to conditions Equation (6) we should have the condition

$$\sigma_{zz}(a,0)=0, \qquad 0\le a\le 1 \tag{7}$$

which follows from the smoothness of $f(r)\in[0,1)$.

The boundary condition of Equations (5) and (7) can also be written as

$$\begin{aligned}&\sigma_{zz}(r,0)[w(r,0)-f(r)]=0,\\ &\sigma_{zz}(r,0)\le 0, && w(r,0)\le f(r), 0\le r\le 1\\ &\sigma_{rz}(r,0)=0, && 0\le r<\infty,\\ &w(r,0)=f(r), && 1\le r<\infty.\end{aligned}$$

3. Solving the problem

Now we will describe the strained state of the elastic body and the relation between the gap geometry and ambient pressure. We will also find the critical value at which the gap disappears. To this end, we divide the solutions into two parts:

$$\sigma=\sigma^{(1)}+\sigma^{(2)}, \qquad u=u^{(1)}+u^{(2)}. \tag{8}$$

The first terms, $\sigma^{(1)}$ and $u^{(1)}$, are solutions of the problem of the contact of an elastic body and rigid base with flat surface. The second terms correspond to the perturbed stressed-strained state caused by geometric nonhomogeneity of the surface.

Because the perturbations are local, we can write

$$\lim_{(r,z)\to\infty} \sigma^{(2)}(r,z)=0, \qquad \lim_{(r,z)\to\infty} u^{(2)}(r,z)=0.$$

The first solution may be obtained in the form

$$\begin{aligned} &\sigma_{zz}^{(1)}=-\frac{p}{\mu_0}, \qquad \sigma_{rr}^{(1)}=0, \qquad \sigma_{rz}^{(1)}=0, \\ &u^{(1)}=\frac{\nu}{E}pr, \qquad w^{(1)}=\frac{\nu}{\alpha E}pz. \end{aligned} \tag{9}$$

Using the boundary conditions of Equations (5) and (6) and using Equations (8) and (9), the boundary conditions for the perturbed fields become

$$\lim_{z\to\infty} \sigma_{zz}^{(2)}(r,z)=0$$
$$\lim_{z\to\infty} \sigma_{rz}^{(2)}(r,z)=0,$$

$$\begin{aligned} &\sigma_{zz}^{(2)}(r,0)=\frac{p}{\mu_0}, && 0<r<a, \\ &\sigma_{rz}^{(2)}(r,0)=0, && 0<r<\infty, \\ &w^{(2)}(r,0)=f(r), && a<r<\infty \end{aligned} \tag{10}$$

To solve for the perturbed field, we assume the solution of Equation (3) in the form

$$\begin{aligned} u^{(2)}(r,z)&=\int_0^\infty d\rho\, F_1(z,\rho)\rho J_1(r\rho), \\ w^{(2)}(r,z)&=\int_0^\infty d\rho\, F_2(z,\rho) J_0(r\rho), \end{aligned} \tag{11}$$

where J_0 and J_1 are Bessel functions of first kind of orders zero and one, respectively. Substituting Equation (11) into Equation (4) and inverting the related Hankel transforms, we find

$$\begin{aligned} &\left[(\kappa-1)D^2+\alpha(\kappa-1)D-(\kappa+1)\rho^2\right]F_1-\left[2D+\alpha(\kappa-1)\right]F_2=0 \\ &\left[(\kappa+1)D^2+\alpha(\kappa+1)D-(\kappa-1)\rho^2\right]F_2+\rho^2\left[2D+\alpha(3-\kappa)\right]F_1=0, \end{aligned} \tag{12}$$

where $D=\frac{d}{dz}$. In deriving this, the following relationships have been used:

$$\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}-\frac{1}{r^2}\right)J_1(r\rho)=-\rho^2 J_1(r\rho),$$

$$\left(\frac{\partial}{\partial r}+\frac{1}{r}\right)J_1(r\rho)=\rho J_0(r\rho),$$

$$\frac{\partial^2}{\partial r^2}J_0(r\rho)=\frac{\rho}{r}J_1(r\rho)-\rho^2 J_0(r\rho).$$

The solutions of the system of differential equations (12) are found to be

$$F_1(z,\rho)=\sum_{k=1}^{4}A_k(\rho)e^{m_k z},$$
$$F_2(z,\rho)=\sum_{k=1}^{4}a_k(\rho)A_k(\rho)e^{m_k z}, \tag{13}$$

where the functions A_k, $k=1,\dots,4$, are unknowns, and m_k, $k=1,\dots,4$, are the roots of the characteristic equation

$$m^4+2\alpha m^3+(\alpha^2-2\rho^2)m^2-2\alpha\rho^2 m+\alpha^2\rho^2\frac{3-\kappa}{1+\kappa}+\rho^4=0, \tag{14}$$

and the coefficients a_k, $k=1,\dots,4$, are given by

$$a_k(\rho)=-\rho^2\frac{2m_k+\alpha(3-\kappa)}{(1+\kappa)m_k^2+(1+\kappa)\alpha m_k-(\kappa-1)\rho^2}. \tag{15}$$

The characteristic Equation (14) may easily be rewritten as

$$\left(m^2+\alpha m-\rho^2\right)^2+\alpha^2\rho^2\frac{3-\kappa}{1+\kappa}=0,$$

from which it follows that

$$m_3=\overline{m_1}=\frac{-\alpha+\beta}{2},\qquad m_4=\overline{m_2}=-\frac{\alpha+\beta}{2}$$

where

$$\beta=\sqrt{\alpha^2+4\rho^2+i4\alpha\rho\sqrt{\frac{3-\kappa}{1+\kappa}}},$$

After solving for m_k, $k=1,\dots,4$, the expressions for the coefficients a_k in Equation (15) may be simplified as follows:

$$a_k(\rho)=-\rho\frac{2m_k+\alpha(3-\kappa)}{2\rho+i\alpha\sqrt{(3-\kappa)(1+\kappa)}},$$
$$a_{k+2}=\overline{a_k},$$

for $k=1,2$. We observe that $\Re m_1$, $\Re m_3>0$ and $\Re m_2$, $\Re m_4<0$, so, to satisfy the regularity condition at $z=\infty$ in the solution given by Equation (13), we must put $A_1=A_3=0$ for $z>0$. Thus the problem may be considered as that of an elastic upper half space with rigid base involving only two unknowns A_2 and A_4, and, for $z>0$, we have

$$F_1(z,\rho)=A_2(\rho)e^{m_2 z}+A_4(\rho)e^{m_4 z},$$
$$F_2(z,\rho)=a_2(\rho)A_2(\rho)e^{m_2 z}+a_4(\rho)A_4(\rho)e^{m_4 z}.$$

The unknowns A_2 and A_4 may be determined from the mixed boundary conditions of Equation (6). Hence, the dimensionless displacement and stress components can be written in terms of A_2 and A_4 as

$$\begin{aligned} u^{(2)}(r,z) &= \int_0^\infty d\rho \, \left(A_2 e^{m_2 z} + A_4 e^{m_4 z}\right) \rho J_1(r\rho) \\ w^{(2)}(r,z) &= \int_0^\infty d\rho \, \left(a_2 A_2 e^{m_2 z} + a_4 A_4 e^{m_4 z}\right) J_0(r\rho) \\ \sigma_{rz}^{(2)}(r,z) &= \frac{\mu}{\mu_0} \int_0^\infty d\rho \, \left((m_2 - a_2) A_2 e^{m_2 z} + (m_4 - a_4) A_4 e^{m_4 z}\right) \rho J_1(r\rho) \\ \sigma_{zz}^{(2)}(r,z) &= \frac{\mu}{\mu_0} \int_0^\infty d\rho \, \left[\frac{\kappa+1}{\kappa-1} \left(m_2 a_2 A_2 e^{m_2 z} + m_4 a_4 A_4 e^{m_4 z}\right) \right. \\ &\qquad \left. + \frac{3-\kappa}{\kappa+1} \left(A_2 e^{m_2 z} + A_4 e^{m_4 z}\right) \rho^2 \right] J_0(r\rho) \end{aligned} \tag{16}$$

Using the second boundary condition of Equation (10) yields

$$A_4(\rho) = -\frac{m_2 - a_2}{m_4 - a_4} A_2(\rho), \tag{17}$$

and the remaining two mixed boundary conditions, with the help of the last two of Equation (16) and Equation (17), give a pair of dual integral equations

$$\begin{aligned} \int_0^\infty d\rho \, A_2^*(\rho) J_0(r\rho) &= f(r), \qquad a < r < \infty, \\ \int_0^\infty d\rho \, G_1(\rho) A_2^*(\rho) J_0(r\rho) &= \frac{\kappa - 1}{\mu_0} p, \qquad 0 < r < a, \end{aligned} \tag{18}$$

where

$$\begin{aligned} A_2^*(\rho) &= \frac{a_2 m_4 - a_4 m_2}{m_4 - a_4} A_2(\rho), \\ G_1(\rho) &= \frac{1}{a_2 m_4 - a_4 m_2} \left[(\kappa+1) \left(|m_2|^2 (a_2 - a_4) - |a_2|^2 (m_2 - m_4)\right) \right. \\ &\qquad \left. + (3-\kappa)(a_2 - a_4 + m_4 - m_2)\rho^2\right]. \end{aligned} \tag{19}$$

Denoting

$$\begin{aligned} a_2(\rho) &= A(\rho) + i B(\rho), \\ m_2(\rho) &= -\frac{\alpha + c(\rho)}{2} - i \frac{d(\rho)}{2}, \end{aligned}$$

we get $G_1(\rho)$ in the form

$$G_1(\rho) = \frac{(\kappa+1)\left(2|m_2|^2 B(\rho) - |a_2|^2 d(\rho)\right) + (3-\kappa)\rho^2 \left(2B(\rho) + d(\rho)\right)}{A(\rho) d(\rho) - (\alpha + c(\rho)) B(\rho)}.$$

If we write

$$A_2^*(\rho) = C^*(\rho) + D^*(\rho),$$

where

$$D^*(\rho) = \rho \int_0^\infty dr\, J_0(r\rho) r f(r), \tag{20}$$

then the dual integral equations of Equation (18) transform into

$$\int_0^\infty d\rho\, C^*(\rho) J_0(r\rho) = 0, \qquad a < r < \infty,$$
$$\int_0^\infty d\rho\, G_1(\rho) C^*(\rho) J_0(r\rho) = g(r), \qquad 0 < r < a, \tag{21}$$

where

$$g(r) = \frac{\kappa - 1}{\mu_0} p - \int_0^\infty d\rho\, G_1(\rho) D^*(\rho) J_0(r\rho). \tag{22}$$

Substituting the expression for $f(r)$ from Equation (1) into Equation (20) we get

$$D^*(\rho) = -3h\sqrt{\frac{\pi}{2}}\rho^{-3/2} J_{5/2}(\rho)$$

If we write

$$C^*(\rho) = \int_0^a dt\, \phi(t) \sin(t\rho), \tag{23}$$

where $\phi(t)$ is an unknown integrable function, then the first of Equation (21) is automatically satisfied. Substitution of Equation (23) in the second of Equation (21) leads to the equation

$$r \int_0^a dt\, \phi(t) \int_0^\infty d\rho\, \frac{G_1(\rho)}{\rho} J_1(r\rho) \sin(t\rho) = \int_0^r ds\, s g(s) \equiv g_1(r), \tag{24}$$

for determining the unknown function ϕ. We note that for large ρ,

$$\frac{G_1(\rho)}{\rho} = \chi_1 + O(\rho^{-1}), \qquad \text{where} \qquad \chi_1 = 4\frac{1-\kappa}{1+\kappa}.$$

Equation (24) may be reduced to a standard integral equation by rewriting it in the form

$$r \int_0^a dt\, \phi(t) \int_0^\infty d\rho\, J_1(r\rho) \sin(t\rho) \left[\chi_1 + \left(\frac{G_1(\rho)}{\rho} - \chi_1\right)\right] = g_1(r). \tag{25}$$

Using the properties of Bessel function, this equation is finally reduced to a Fredholm integral equation of the second kind;

$$\chi_1 \phi(r) + \int_0^a dt\, K(r,t)\phi(t) = \frac{2}{\pi}\int_0^r dy\, \frac{y\, g(y)}{\sqrt{r^2 - y^2}}, \quad 0 < r < a, \tag{26}$$

where the kernel $K(r,t)$ is given by

$$K(r,t) = \frac{2}{\pi}\int_0^\infty d\rho\, G_2(\rho) \sin(r\rho) \sin(t\rho), \tag{27}$$

and

$$G_2(\rho) = \frac{G_1(\rho)}{\rho} - \chi_1.$$

We shall discuss the integral equation (26) and its kernel $K(r,t)$ in the next section.

Equations (16), (17), (19), (23), and (26) determine that the deformed state of the elastic half-space, the normal displacement $w^{(2)}$, and the pressure on its surface are given by

$$w^{(2)}(r,0)=\begin{cases}\int_0^a dt\,\phi(t)H_2(t,r)+f(r), & 0<r<a,\\ f(r), & r>a,\end{cases} \tag{28}$$

where

$$H_2(t,r)=\int_0^\infty d\rho\ \sin(t\rho)J_0(r\rho)$$

and

$$\sigma_{zz}^{(2)}(r,0)=\begin{cases}\frac{p}{\mu_0}, & 0<r<a,\\ \frac{1}{\kappa-1}\left[H_1(r)-\frac{3h\pi\chi_1}{4}\left(1-\frac{3}{2}r^2]\right)\right], & a<r\le 1,\\ \frac{1}{\kappa-1}\left[H_1(r)+\frac{h\chi_1}{5r^3}\,{}_2F_1\left(\frac{3}{2},\frac{3}{2};\frac{7}{2},\frac{1}{r^2}\right)\right], & r>1,\end{cases} \tag{29}$$

where

$$H_1(r)=\int_0^a dt\,\phi(t)\int_0^\infty d\rho\,G_2(\rho)J_0(\rho)\sin(t\rho)-3h\sqrt{\frac{\pi}{2}}\int_0^\infty d\rho\,\rho^{-1/2}G_2(\rho)J_0(\rho)J_{5/2}(\rho)$$
$$-\chi_1\frac{\phi(a)}{\sqrt{r^2-a^2}}+\chi_1\frac{\phi(0)}{r}+\chi_1\phi'(a)\sin^{-1}(a/r)-\chi_1\int_0^a dt\,\phi''(t)\sin^{-1}(t/r)\,. \tag{30}$$

Finally, putting Equations (9), (28), and (29) into Equation (8), we get the expressions for $w(r,0)$ and $\sigma_{zz}(r,0)$:

$$w(r,0)=\begin{cases}\int_0^a dt\,\phi(t)H_2(t,r)+f(r), & 0<r<a,\\ f(r), & r>a,\end{cases} \tag{31}$$

$$\sigma_{zz}(r,0)=\begin{cases}\frac{p}{\mu_0} & 0<r<a\\ \frac{1}{\kappa-1}\left[H_1(r)-\frac{3h\pi\chi_1}{4}(1-\frac{3}{2}r^2)\right], & a<r\le 1\\ \frac{1}{\kappa-1}\left[H_1(r)+\frac{h\chi_1}{5r^3}\,{}_2F_1\left(\frac{3}{2},\frac{3}{2};\frac{7}{2},\frac{1}{r^2}\right)\right], & r>1.\end{cases} \tag{32}$$

The boundary condition Equation (7) demands that

$$\phi(a)=0. \tag{33}$$

Putting this and Equation (22) into Equation (26) yields

$$\frac{p}{\mu_0}=\frac{1}{a(\kappa-1)}\left[\frac{\pi}{2}\int_0^a dt\,K(a,t)\phi(t)+\int_0^\infty d\rho\,\frac{G_1(\rho)}{\rho}D^*(\rho)\sin(a\rho)\right]. \tag{34}$$

This equation gives a relationship between the dimensionless applied pressure p/μ_0 and the dimensionless radius a of the gap. For a given p/μ_0, the equation determining a is nonlinear. From this equation, it is easy to obtain the critical pressure $p^\star$ at which gap disappears, that is, at which $a = 0$. The value of p^* is obtained as

$$\frac{p^\star}{\mu_0} = \frac{1}{\kappa - 1}\int_0^\infty d\rho G_1(\rho) D^*(\rho). \tag{35}$$

It should be noted as an important check on our results that if we take $\alpha = 0$—that is, if we assume the upper half space is homogeneous and isotropic—then the expressions for p and $p^\star$ reduce to those obtained by Kit and Monastyrsky [2001].

4. Determining $\phi(r)$

The infinite integral $K(r, t)$ in Equation (27) is smooth except possibly at $\rho = 0$ and $\rho = \infty$. We may check that

$$G_2(\rho) = \frac{\psi(\rho)}{\rho^2} - \chi_1 \qquad \text{as} \qquad \rho \to 0, \tag{36}$$

where $\psi(\rho)$ is regular in the neighborhood of $\rho = 0$. This implies that

$$\lim_{\rho\to 0} G_2(\rho)\sin(t\rho)\sin(r\rho) = \text{a finite quantity}, \tag{37}$$

so $\rho = 0$ is not a singularity for that integral.

Now, for large ρ, $G_2(\rho) = O(\rho^{-1})$. From Gradshteyn and Ryzhik [1963],

$$\int_0^\infty d\rho\, \frac{\sin(t\rho)\sin(r\rho)}{\rho} = \frac{1}{2}\log\left|\frac{r+t}{r-t}\right|, \qquad r \neq t, \tag{38}$$

and it follows that the behavior of the integral as $\rho \to \infty$ is not smooth for $r = t$, so determination of $\phi(r)$ from Equation (26) could be a problem. However, this difficulty may be easily overcome by writing Equation (25) in the form

$$\int_0^a dt\, \left[\frac{1}{t-r} + L(r, t)\right]\phi(t) = g_1(r) \tag{39}$$

where

$$\begin{aligned} L(r, t) &= \chi_1 L_0(r, t) + L_1(r, t) - \frac{1}{t-r}, \\ L_0(r, t) &= r\int_0^\infty d\rho\, J_1(r\rho)\sin(t\rho) = \begin{cases} \dfrac{t}{\sqrt{r^2 - t^2}}, & t < r, \\ 0, & t \geq r, \end{cases} \\ L_1(r, t) &= r\int_0^\infty d\rho\, \left(\frac{G_1(\rho)}{\rho} - \chi_1\right) J_1(r\rho)\sin(t\rho), \end{aligned} \tag{40}$$

Using the rescalings

$$t(u) = \frac{a}{2}(1+u), \qquad r(x) = \frac{a}{2}(1+x), \tag{41}$$

and the notations

$$\phi^*(u) = \phi\,(t(u)), \qquad g^*(x) = g\,(r(x)), \tag{42}$$

Equation (39) is transformed into a singular integral equation

$$\int_{-1}^{1} du\,\phi^*(u)\left[\frac{1}{u-x}+\frac{a}{2}L^*(u,x)\right]du = g_1^*(x), \quad \text{where } L^*(u,x) = L\left(\frac{a}{2}(1+u), \frac{a}{2}(1+x)\right). \quad (43)$$

This integral equation can be evaluated by using Gauss–Chebyshev method. It may be noted that $u = -1$ corresponds to $t = 0$ and $u = 1$ corresponds to $t = a$. Since the contact is smooth, neither the point $t = 0$ (that is, $u = -1$) nor the point $t = a$ (that is, $u = 1$) is a singularity of the function ϕ. Following [Erdogan and Gupta 1972], we take the solution $\phi^*(u)$ of integral equation Equation (43) in the form

$$\phi^*(u) = \sqrt{1-u^2}\,G(u), \qquad -1 \le u \le 1,$$

where $G(u)$ is a bounded unknown function. Now, using Gauss–Chebyshev formula, we express the integral equation (43) in discretized form as

$$\sum_{k=1}^{N} \frac{\pi\sqrt{1-u_k^2}}{N+1} G(u_k)\left[\frac{1}{u_k - x_j} + \frac{a}{2}\,L^*(u_k, x_j)\right] = g_1(x_j), \tag{44}$$

where $j = 1, \ldots, N+1$, and u_k and x_j are given by

$$u_k = \cos\left(\frac{k\pi}{N+1}\right) \qquad \text{and} \qquad x_j = \cos\left(\pi\frac{j-1/2}{N+1}\right).$$

Equation (44) gives a set of $(N+1)$ linear equations—one for each collocation point x_j—in the N unknowns $G(u_1), \ldots, G(u_N)$. To determine the $G(u_k)$, we may ignore one collocation point, say the midpoint $x_{N/2+1}$ when N is even, and let the remaining N equations determine $G(u_k)$. This linear algebraic system can be solved by gaussian eliminations.

We note here that although Equation (26) is not quite suitable for numerical evaluation of $\phi(r)$, it is very much useful for determining the applied pressures p and p^* in relatively simple forms (see Equations (34) and (35)). Also, evaluating the integral

$$\int_0^a dt\,K(a,t)\phi(t) \tag{45}$$

in Equation (34) will not pose any problem here, since $\phi(a) = 0$.

5. Numerical results and discussions

We have noted that zero is not a singularity for the infinite integrals in Equations (27) and (30), and, because the integrals are regular for finite ρ, we will, in numerically evaluating these integrals, study only their behavior at infinity. We find that $G_2(\rho)$ can be asymptotically represented for large ρ as

$$G_2(\rho) = \frac{\chi_2}{\rho} + \frac{\chi_3}{\rho^2} + \frac{\chi_4}{\rho^3} + \cdots. \tag{46}$$

The coefficients χ_2, χ_3, and χ_4 are given in Appendix A.

To examine the convergence of the improper integrals in Equations (27) and (30), let us take any one, say, the improper integral

$$\int_0^\infty d\rho\, G_2(\rho)\ J_0(\rho)\sin(t\rho) \tag{47}$$

in Equation (30). We write

$$\int_0^\infty d\rho\, G_2(\rho)\ J_0(\rho)\sin(t\rho) = \lim_{A\to\infty}\int_0^A d\rho\, G_2(\rho)\ J_0(\rho)\sin(t\rho). \tag{48}$$

For large ρ, we write the integral

$$\int_0^A d\rho\, G_2(\rho)\ J_0(\rho)\sin(t\rho) \tag{49}$$

as the sum of two integrals I_1 and $I_2(A)$ such that

$$\begin{aligned} I_1 &= \int_0^\zeta d\rho\, G_2(\rho)\ J_0(\rho)\sin(t\rho), \\ I_2(A) &= \int_\zeta^A d\rho\, G_2(\rho)\ J_0(\rho)\sin(t\rho), \end{aligned} \tag{50}$$

where ζ is a suitably chosen cutoff point. The first integral I_1 is numerically evaluated using gaussian quadratures. Noting the asymptotic behavior of $G_2(\rho)$ from Equation (46) and shown in Figure 3, it is expected that for sufficiently large A, the integral I_2 should converge. This is numerically checked and shown in Table 1.

Thus, for a given tolerance ϵ, the upper limit A can be determined numerically, and the value of the integral $I_2(\infty)$ can then be approximated as $I_2(A)$ at tolerance level ϵ. For example, from Table 1, for $\zeta = 25$ and $\epsilon = 10^{-5}$, we have $A = 205$.

We can then determine ϕ numerically by the method described in Section 4. Finally, we find $w(r, 0)$, $\sigma_{zz}(r, 0)$, p/μ_0, and p^*/μ_0 from Equations (31), (32), (34), and (35).

The dimensionless gap radius a depends on p/μ_0 as well as on the nonhomogeneity parameter α. From Equation (2), it is clear that as α increases, the material becomes harder. This implies in turn that if α is larger, a larger pressure p/μ_0 will be required to produce same change in a. Also, it is expected that

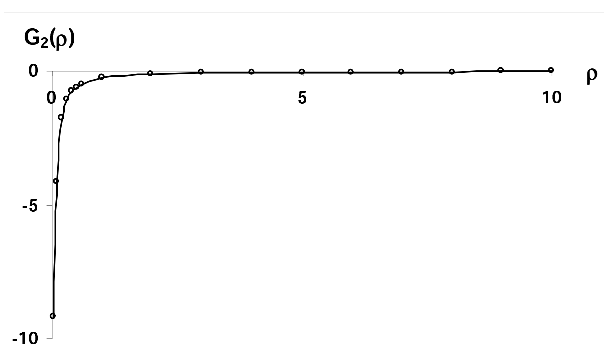

Figure 3. A plot of $G_2(\rho)$.

A	$I_2(A)$	A	$I_2(A)$	A	$I_2(A)$	A	$I_2(A)$
30	–0.00077027	75	–0.00162461	120	–0.00093715	165	–0.00105215
35	–0.00056793	80	–0.00095082	125	–0.00103405	170	–0.00096191
40	–0.00162461	85	–0.00084971	130	–0.00098509	175	–0.00101668
45	–0.00070269	90	–0.00118899	135	–0.00090544	180	–0.00099818
50	–0.00114482	95	–0.00090935	140	–0.00107440	185	–0.00093061
55	–0.00090488	100	–0.00105012	145	–0.00095260	190	–0.00103696
60	–0.00077959	105	–0.00097268	150	–0.00102384	195	–0.00096784
65	–0.00129027	110	–0.00088441	155	–0.00099292	200	–0.00101127
70	–0.00085225	115	–0.00110913	160	–0.00091986	205	-0.00100183

Table 1. $I_2(A)$ for $t = 0.5$ and cutoff point ζ=25.

as p/μ_0 increases, a should decrease. To check the expected behavior we have computed from Equation (34), we compute p/μ_0 for different a and α. The results are graphed in Figure 4. In Figure 5, we show how dimensionless critical pressure p^*/μ_0 varies with nonhomogeneity parameter α. As expected, the critical pressure increases with α.

From Equation (1), we find that the function $f(r)$ is not differentiable at $r = 1$. Because the geometrical structure of the recess is expected to contribute to the behavior of surface stress distribution, we have evaluated the surface stress $\sigma_{zz}(r, 0)$ as $r \to 1$. We find that there is a finite discontinuity in the normal stress $\sigma_{zz}(r, 0)$ in the neighborhood of $r = 1$. The values of $\sigma_{zz}(r, 0)$ have been plotted against the nonhomogeneity parameter α and the gap radius a in Figures 6 and 7, respectively. In Figures 8 and 9, we show how $w(r, 0)$ varies with a and α.

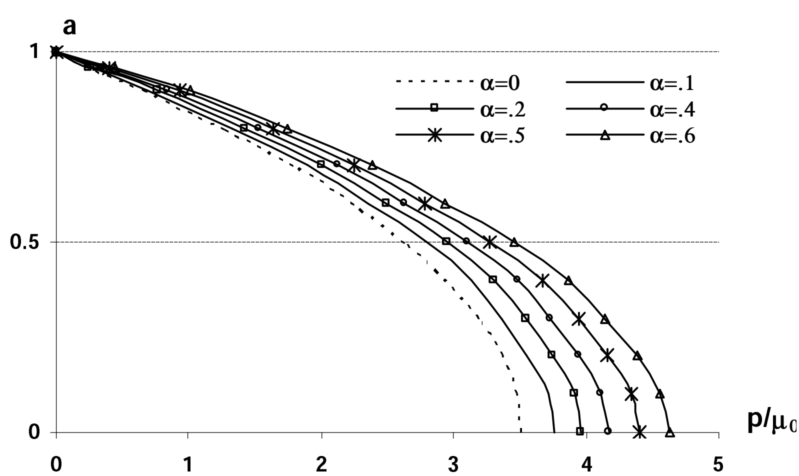

Figure 4. Variation of ambient pressure p/μ_0 with the first contact radius a for different values of the nonhomogeneity parameter α.

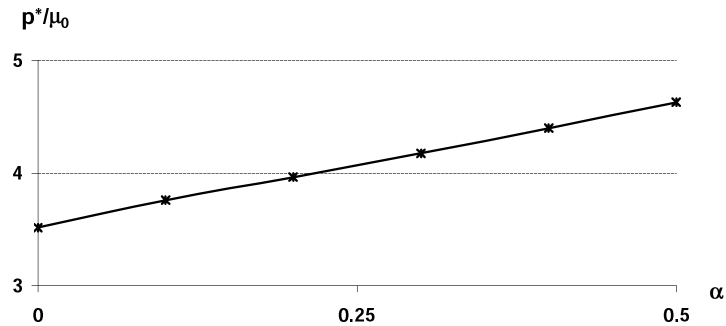

Figure 5. Variation of dimensionless critical pressure p^*/μ_0 for different values of the nonhomogeneity parameter α.

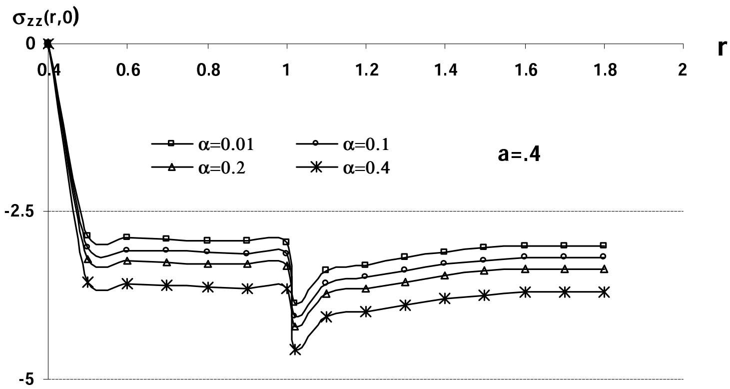

Figure 6. Variation of dimensionless normal stress $\sigma_{zz}(r, 0)$ with r $(> a)$ for various values of the nonhomogeneity parameter α.

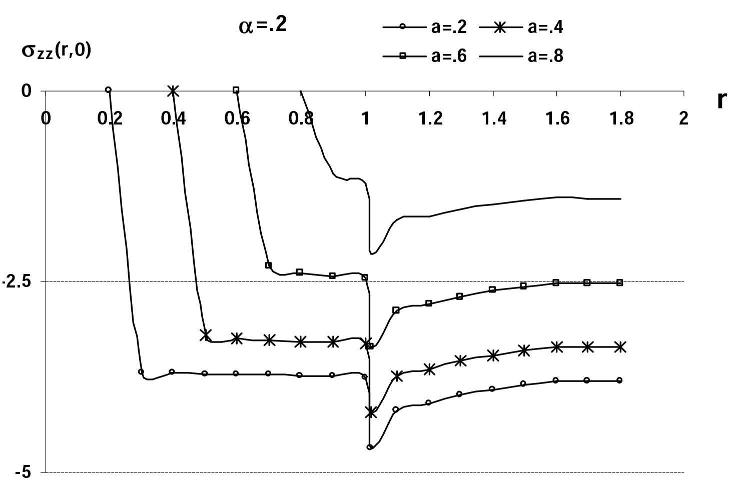

Figure 7. Variation of dimensionless normal stress $\sigma_{zz}(r, 0)$ with r $(> a)$ for different values of a.

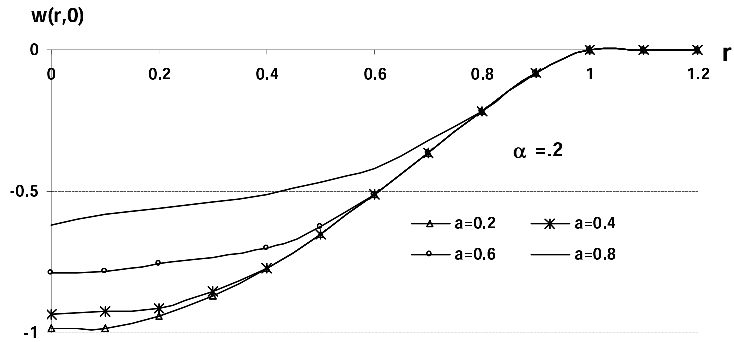

Figure 8. Variation of dimensionless normal displacement $w(r, 0)$ with r for different values of a.

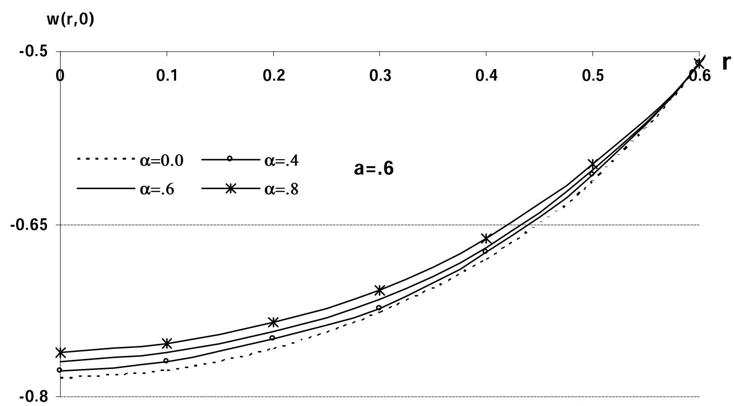

Figure 9. Variation of dimensionless normal displacement $w(r, 0)$ with $r < a$ for various values of the nonhomogeneity parameter α.

Appendix A

$$
\begin{aligned}
\chi_2 &= 2\frac{b_1\alpha\left(\chi_1+2(\kappa+1)\right)+2b_2\left(\chi_1+2(\kappa-1)\right)+A_1d_1\left(2(\kappa+1)-\chi_1\right)}{\alpha\sqrt{(3-\kappa)(\kappa+1)}},\\
\chi_3 &= \frac{1}{p_2}\Big[\frac{4d_1\alpha^2(1-\kappa-\kappa^2)}{\kappa+1}-4A_2\alpha\sqrt{(3-\kappa)(\kappa+1)}-2\alpha(\kappa+1)b_2\\
&\qquad\qquad +(\kappa+1)d_1(A_1^2+2A_2+b_1^2)\Big]+\frac{1}{p_2^2}\left(p_1p_2\chi_2-p_1^2\chi_1+p_2p_3\chi_1\right),\\
\chi_4 &= \frac{1}{p_2^2}\left[(\kappa+1)\left(p_2(u_4+w_4)-p_1(u_1+w_1)\right)+(3-\kappa)\left(2p_2b_4-p_1(2b_3-d_2)\right)\right],
\end{aligned}
\tag{A.1}
$$

where

$$2p_1/d_1 = \kappa^2+\kappa-4)\alpha, \qquad A_1 = \frac{\alpha(\kappa-2)}{2},$$
$$p_2/d_1 = \kappa+1, \qquad A_2 = \frac{\alpha^2}{4(\kappa+1)}\left(2-\kappa(\kappa+1)(3-\kappa)\right),$$
$$p_3/d_1 = \frac{(\kappa^3-\kappa-2)\alpha}{4(\kappa+1)} + (\kappa+2)A_2, \qquad A_3 = \frac{\alpha^3}{8}(2-\kappa)(\kappa+1)(3-\kappa),$$

$$u_1 = \frac{4b_1}{\kappa+1}\alpha^2 + 2\alpha b_2 + 2b_3, \qquad w_1/d_1 = (A_1^2+2A_2+b_1^2) - \frac{\alpha^2}{2(\kappa+1)},$$
$$u_4 = \frac{4b_2}{\kappa+1}\alpha^2 + 2\alpha b_3 + 2b_4 + \frac{\alpha^3 b_1}{\kappa+1}, \qquad w_4/d_1 = (2A_1A_2+2A_3+2b_1b_2) - 2A_1\frac{\alpha^2}{2(\kappa+1)},$$

$$b_1 = -\frac{\kappa d_1}{2}, \qquad b_3 = -\frac{\alpha}{2}A_2\sqrt{(3-\kappa)(\kappa+1)} + \frac{\alpha^2}{4(\kappa+1)}d_1, \qquad d_1 = \alpha\sqrt{\frac{3-\kappa}{\kappa+1}},$$
$$b_2 = -\frac{(\kappa+1)A_1d_1}{2}, \qquad b_4 = \frac{\alpha^4}{16}(\kappa-2)\sqrt[3]{(\kappa+1)(3-\kappa)}, \qquad d_2 = -\frac{\alpha^2}{2(\kappa+1)}d_1.$$

Acknowledgement

The authors express sincere thanks to the reviewers for valuable comments.

References

[Argatov 2000] I. I. Argatov, "Asymptotic modelling of the contact interaction between a system of rigidly connected punches and an elastic base", *Sib. Zh. Ind. Math.* **3**:2 (2000), 10–22.

[Argatov 2004] I. I. Argatov, "The method of averaging in a contact problem for a system of punches", *J. Appl. Math. Mech.* **68**:1 (2004), 93–104.

[Barik et al. 2006] S. P. Barik, M. K. Kanoria, and P. K. Chaudhuri, "Contact problem for an anisotropic elastic layer lying on an anisotropic elastic foundation under gravity", *J. Indian Acad. Math.* **28**:1 (2006), 205–223.

[Birinci and Erdol 1999] A. Birinci and R. Erdol, "Frictionless contact between a rigid stamp and an elastic layered composite resting on simple supports", *Math. Comput. Appl.* **4** (1999), 261–272.

[Brock and Georiadis 2001] L. M. Brock and H. G. Georiadis, "An illustration of sliding contact at any constant speed on highly elastic half-spaces", *J. Appl. Math.* **66** (2001), 551–566.

[Chaudhuri and Ray 1998] P. K. Chaudhuri and S. Ray, "Receding contact between an orthotropic layer and an orthotropic half-space", *Arch. Mech.* **50** (1998), 743–755.

[Chaudhuri and Ray 2003] P. K. Chaudhuri and S. Ray, "Receding axisymmetric contact between a transversely isotropic layer and a transversely isotropic half-space", *Bull. Am. Math. Soc.* **95** (2003), 151–164.

[Civelek et al. 1978] M. B. Civelek, F. Erdogan, and A. O. Cakiroglu, "Interface separation for an elastic layer loaded by a rigid stamp", *Int. J. Eng. Sci.* **16**:9 (1978), 669–679.

[Comez et al. 2004] I. Comez, A. Birinci, and R. Erdol, "Double receding contact problem for a rigid stamp and two elastic layers", *Eur. J. Mech. A Solid.* **23**:2 (2004), 301–309.

[El-Borgi et al. 2006] S. El-Borgi, R. Abdelmoula, and L. Keer, "A receding contact plane problem between a functionally graded layer and a homogeneous substrate", *Int. J. Solids Struct.* **43**:3-4 (2006), 658–674.

[Erdogan and Gupta 1972] F. Erdogan and G. D. Gupta, "On the numerical solution of singular integral equations", *J. Appl. Math.* **29** (1972), 525–534.

[Fabrikant 1986] V. I. Fabrikant, "Flat punch of arbitrary shape on an elastic half-space", *Int. J. Eng. Sci.* **24**:11 (1986), 1731–1740.

[Garrido and Lorenzana 1998] J. A. Garrido and A. Lorenzana, "Receding contact problem involving large displacements using the BEM", *Eng. Anal. Bound. Elem.* **21**:4 (1998), 295–303.

[Garrido et al. 1991] J. A. Garrido, A. Foces, and F. Paris, "B.E.M. applied to receding contact problems with friction", *Math. Comput. Model.* **15**:3-5 (1991), 143–153.

[Gecit 1981] M. R. Gecit, "Axisymmetric contact problem for an elastic layer and an elastic foundation", *Int. J. Eng. Sci.* **19**:6 (1981), 747–755.

[Gecit 1986] M. R. Gecit, "Axisymmetric contact problem for a semiinfinite cylinder and a half space", *Int. J. Eng. Sci.* **24**:8 (1986), 1245–1256.

[Gladwell 1980] G. M. L. Gladwell, *Contact problems in the classical theory of elasticity*, Sijthoff and Noordhoff, Alphen aan den Rijn, The Netherlands, 1980.

[Gradshteyn and Ryzhik 1963] I. S. Gradshteyn and I. M. Ryzhik, *Table of integrals, series, and products*, Academic Press, New York, 1963.

[Hussain et al. 1968] M. A. Hussain, S. L. Pu, and M. A. Sadowsky, "Cavitation at the ends of an elliptic inclusion inside a plane under tension", *J. Appl. Mech.* (*Trans. ASME*) **35** (1968), 505–509.

[Jing and Liao 1990] H.-S. Jing and M.-L. Liao, "An improved finite element scheme for elastic contact problems with friction", *Comput. Struct.* **35**:5 (1990), 571–578.

[Keer et al. 1972] L. M. Keer, J. Dundurs, and K. C. Tasi, "Problem involving receding contact between a layer and a half-space", *J. Appl. Mech.* (*Trans. ASME*) **39** (1972), 1115–1120.

[Kit and Monastyrsky 2001] G. S. Kit and B. E. Monastyrsky, "A contact problem for a half-space and a rigid base with an axially symmetric recess", *J. Math. Sci.* **107**:1 (2001), 3545–3549.

[Li and Dempsey 1988] H. Li and J. Dempsey, "Unbonded contact of a square plate on an elastic half-space or a winkler foundation", *J. Appl. Mech.* (*Trans. ASME*) **55** (1988), 430–435.

[Loboda and Tauchert 1985] V. V. Loboda and T. R. Tauchert, "The elastic contact problem for dissimilar orthotropic semi-infinite and infinite strips", *Int. J. Eng. Sci.* **23**:12 (1985), 1337–1349.

[Martynyak 1985] R. M. Martynyak, "The interaction of elastic half-planes under an incomplete mechanical contact", *Mat. Met. Fiz.-Mekh. Polya.* **22** (1985), 89–92.

[Noble and Hussain 1969] B. Noble and M. A. Hussain, "Exact solution of certain dual series for indentation and inclusion problems", *Int. J. Eng. Sci.* **7**:11 (1969), 1149–1161.

[Nowell and Hills 1988] D. Nowell and D. A. Hills, "Contact problems incorporating elastic layers", *Int. J. Solids Struct.* **24**:1 (1988), 105–115.

[Ozturk and Erdogan 1993] M. Ozturk and F. Erdogan, "The axisymmetric crack problen in a non-homogeneous medium", *J. Appl. Mech.* (*Trans. ASME*) **60** (1993), 406–413.

[Paris et al. 1995] F. Paris, A. Blazquez, and J. Canas, "Contact problems with nonconforming discretizations using boundary element method", *Comput. Struct.* **57**:5 (1995), 829–839.

[Pu and Hussain 1970] S. L. Pu and M. A. Hussain, "Note on the unbonded contact between plates and an elastic half-space", *J. Appl. Mech.* (*Trans. ASME*) **37** (1970), 859–861.

[Satish Kumar et al. 1996] K. Satish Kumar, B. Dattaguru, T. S. Ramamurthy, and K. N. Raju, "Elasto-plastic contact stress analysis of joints subjected to cyclic loading", *Comput. Struct.* **60**:6 (1996), 1067–1077.

[Schmuesre et al. 1980] D. Schmuesre, M. Comminou, and J. Dundrus, "Separation and slip between a layer and subtrate caused by tensile load", *Int. J. Eng. Sci.* **18**:9 (1980), 1149–1155.

[Selvanduri 1983] A. P. S. Selvanduri, "The body force inducing separation at a frictionless precompressed transversely isotropic interface", *T. Can. Soc. Mech. Eng.* **7** (1983), 154–157.

[Shvets et al. 1996] R. M. Shvets, R. M. Martynyak, and A. A. Kryshtafovych, "Discontinuous contact of an anisotropic half-plane and a rigid base with disturbed surface", *Int. J. Eng. Sci.* **34**:2 (1996), 183–200.

[Sneddon and Lowengrub 1969] I. N. Sneddon and M. Lowengrub, *Crack problems in the classical theory of elasticity*, Wiley, New York, 1969.

[Weitsman 1969] Y. Weitsman, "On the unbonded contact between plates and an elastic half-space", *J. Appl. Mech.* (*Trans. ASME*) **36** (1969), 198–202.

[Wu and Yen 1994] E. Wu and C. S. Yen, "The contact behavior between laminated composite plates and rigid spheres", *J. Appl. Mech.* (*Trans. ASME*) **61** (1994), 60.

Received 7 Mar 2007. Accepted 7 Mar 2007.

SAKTI PADA BARIK: spbarik@yahoo.com
Department of Applied Mathematics, University Calcutta, 92, A.P.C. Road, Kolkata 700 009, India

MRIDULA KANORIA: k_mri@yahoo.com
Department of Applied Mathematics, University Calcutta, 92, A.P.C. Road, Kolkata 700 009, India

PRANAY KUMAR CHAUDHURI: pranay_chaudhuri@yahoo.co.in
Department of Applied Mathematics, University Calcutta, 92, A.P.C. Road, Kolkata 700 009, India