Journal of
Mechanics of
Materials and Structures

THE EFFECT OF TAPER ON SECTION CONSTANTS FOR IN-PLANE DEFORMATION OF AN ISOTROPIC STRIP

Dewey H. Hodges, Jimmy C. Ho and Wenbin Yu

Volume 3, № 3

March 2008

mathematical sciences publishers

JOURNAL OF MECHANICS OF MATERIALS AND STRUCTURES
Vol. 3, No. 3, 2008

THE EFFECT OF TAPER ON SECTION CONSTANTS FOR IN-PLANE DEFORMATION OF AN ISOTROPIC STRIP

DEWEY H. HODGES, JIMMY C. HO AND WENBIN YU

The variational-asymptotic method is used to obtain an asymptotically-exact expression for the strain energy of a tapered strip-beam. The strip is assumed to be sufficiently thin to warrant the use of two-dimensional elasticity. The taper is represented by a nondimensional constant of the same order as the ratio of the maximum cross-sectional width to the wavelength of the deformation along the beam, and thus its cube is negligible compared to unity. The resulting asymptotically-exact section constants, being functions of the taper parameter, are then used to find section constants for a generalized Timoshenko beam theory. These generalized Timoshenko section constants are then used in the associated one-dimensional beam equations to obtain the solution for the deformation of a linearly tapered beam subject to pure axial, pure bending, and transverse shear forces. These beam solutions are then compared with plane stress elasticity solutions, developed for extension, bending, and flexure of a linearly tapered isotropic strip. The agreement is excellent, and the results show that correction of the section constants using the taper parameter is necessary in order for beam theory to yield accurate results for a tapered beam.

1. Introduction

According to Euler–Bernoulli beam theory for pure bending of a uniform beam made of isotropic material with Young's modulus E, the strain energy per unit length is given by

$$U = \tfrac{1}{2} E I \kappa^2,$$

where κ is the curvature of the beam neutral axis (the locus of cross-sectional area centroids) and I is the cross-sectional area moment of inertia. The bending stiffness according to Euler–Bernoulli theory is EI. For nonuniform beams it is typical that the bending stiffness is regarded as $EI(x)$, with x being the beam axial coordinate. For example, for a beam with narrow rectangular cross-section of width $2b$ and thickness t, EI is given by

$$EI = \frac{Et(2b)^3}{12} = \frac{2Etb^3}{3}.$$

Customarily, this expression remains the same regardless of whether or not the beam is uniform. For example, when $b = b(x)$, one just replaces b with $b(x)$; the local taper of the beam $b'(x) = -\tau(x)$ does not further influence the local bending stiffness. [Boley 1963] showed that the accuracy of predictions by beam theory, performed in the described manner, worsened as τ increased.

Keywords: beam theory, asymptotic methods, dimensional reduction, tapered beam.
This research was supported in part by the Army Vertical Lift Research Center of Excellence at Georgia Institute of Technology and its affiliate program through subcontract at Utah State University. The technical monitor is Dr. Michael J. Rutkowski.

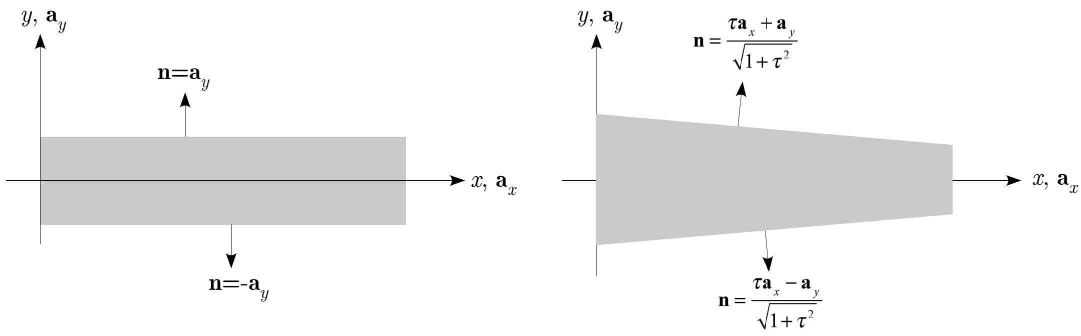

Figure 1. Contrast between a prismatic (left) and a tapered (right) beam.

In reality, taper introduces three-dimensional effects that cannot be accounted for by merely changing the sectional width in this formula. For example, stress at the boundaries of a solid continuum must conform to the traction-stress relationships from Cauchy's formula. Let us now introduce the y-axis as perpendicular to x along with unit vectors, $\boldsymbol{a}_x$ and $\boldsymbol{a}_y$, parallel to x and y respectively. In Figure 1 a prismatic beam is shown beside a linearly tapered beam. Comparing the two cases, one can easily see that neglecting the local taper parameter τ is equivalent to regarding the local upper and lower surface outward-directed normal vectors as parallel to $\boldsymbol{a}_y$, which is only true for beams with no taper. Instead, the true outward-directed normal vectors feature a component parallel to $\boldsymbol{a}_x$, omission of which means that the surface boundary conditions are erroneous. Cross-sectional analysis (for instance, solutions for the elastic constants used in a beam analysis) of tapered beams performed without consideration of taper would then be incorrect and thereby degrade results from one-dimensional beam analysis. The questions that must be answered are (a) how significant is this effect, and (b) is its inclusion tractable? Krahula [1975] obtained an exact plane stress elasticity solution for a tapered strip-beam undergoing flexure. This solution, along with others developed by Timoshenko and Goodier [1970] for pure extension and bending deformations, all three of which are included herein, provides a means to assess the error associated with omission of taper from the cross-sectional analysis of engineering beam theories for this simple configuration. It will also provide a means to assess the accuracy of more general cross-sectional analyses for spanwise nonuniform beams.

Andrade and Camotim [2005] considered this effect on the lateral-torsional buckling of I-beams. It was reported therein that taper can affect the local cross-sectional stiffnesses, expressions of which, if accurate, demonstrate that its effects are not, in general, negligible. For finite element analysis of linearly tapered I-beams, Vu-Quoc and Léger [1992] derived a flexibility matrix showing a dependence on τ; however, the basis of their derivation is, at best, only an approximation, in that they assumed that the bending stress equation for a prismatic beam, $\sigma_{xx} = My/I$, remains valid in the presence of taper. For tapered monosymmetric I-beams, Kitipornchai and Trahair [1975] introduced additional section constants to account for taper while leaving the traditional section constants unaltered. Ronagh et al. [2000] also employed this approach for tapered beams of arbitrary cross-sectional geometries. A drawback of introducing additional constants is that it inevitably adds complexity to the beam analysis. There is relatively little other information in the literature regarding the effect of taper on the local stiffness properties. On the other hand, beam analyses with stiffness (or flexibility) matrices formulated for numerical computations in [To 1981; Karabalis and Beskos 1983; Banerjee and Williams 1986; Cleghorn and Tabarrok 1992; Rajasekaran 1994; Tena-Colunga 1996] are only selected examples of the immense body of research performed on the analysis of tapered beams that does not consider the effect of taper on the cross-sectional constants and stress recovery relations.

Here the effect is examined from the point of view of an analytical treatment. In order to facilitate such a treatment, a tapered strip-beam is analyzed as a plane stress problem undergoing in-plane deformation.

The variational-asymptotic method (VAM) was developed by Berdichevsky and first used in construction of two-dimensional shell theory by dimensional reduction from three-dimensional elasticity theory (see [Berdichevsky 1979]). The VAM finds an asymptotically-exact approximation of the solution, to any desired order of accuracy, in terms of specified small parameters. The suitability of VAM for development of one-dimensional beam theory was shown by Berdichevsky [1981]. [Hodges 2006] and many references cited therein provide sample derivations for applying the VAM to the analysis of beams.

In this paper the VAM is used in Section 2 to analyze the in-plane deformation of a thin strip-beam and obtain its strain energy per unit length. In Section 3, this strain energy is repackaged into a generalized Timoshenko framework. In Section 4, the resulting generalized Timoshenko theory, with its modified bending and shear stiffnesses, is used to analyze the classical pure extension, pure bending, and flexure problems associated with the in-plane deformation of a linearly tapered beam. In Section 5, three elasticity solutions are presented for the pure extension, pure bending, and flexure problems. In Section 6, the elasticity results are compared with the beam solution. Finally, conclusions are drawn.

2. Beam strain energy per unit length

Beam theory requires an expression for the strain energy per unit length in terms of generalized strains that depend only on the axial coordinate. The process of finding this expression, to be rigorous, must begin with the strain energy for the accompanying elasticity problem on which dimensional reduction is being performed. As discussed in [Hodges 2006], and many papers cited therein (for example, [Cesnik and Hodges 1997; Yu et al. 2002]), dimensional reduction is based on the identification and exploitation of various small parameters, and may be rigorously carried out by asymptotic methods. The VAM of [Berdichevsky 1979] allows one to work directly in terms of energy functionals and still take advantage of small parameters.

In this section we develop the strain energy per unit length for a linearly tapered strip-beam such as the one shown on the right side of Figure 1. The undeformed state is described following the methods of [Hodges 2006], where the position vector to an arbitrary point in the undeformed beam is taken to be

$$\hat{\boldsymbol{r}} = x\boldsymbol{a}_x + y\boldsymbol{a}_y = \boldsymbol{r} + y\boldsymbol{a}_y,$$

where the x-axis is the reference line of the undeformed beam, taken for convenience as the locus of cross-sectional centroids. The position vector to an arbitrary point in the deformed beam can be written as

$$\hat{\boldsymbol{R}} = \boldsymbol{R} + y\boldsymbol{T}_y + w_x(x, y)\boldsymbol{T}_x + w_y(x, y)\boldsymbol{T}_y, \tag{1}$$

where $\boldsymbol{R} = (x + u)\boldsymbol{a}_x + v\boldsymbol{a}_y$, $\boldsymbol{T}_x$ is a unit vector tangent to the deformed reference line, and $\boldsymbol{T}_y$ is normal to $\boldsymbol{T}_x$ in the plane. If we only keep linear terms, then $\boldsymbol{T}_x = \boldsymbol{a}_x + v'\boldsymbol{a}_y$ and $\boldsymbol{T}_y = -v'\boldsymbol{a}_x + \boldsymbol{a}_y$. The displacement field is thus described in terms of beam variables $u(x)$, $v(x)$ along with warping functions $w_x(x, y)$ and $w_y(x, y)$. For the dimensional reduction, the warping functions are unknown at the outset but are solved for in the procedure. Two constraints on the warping are needed to make the displacement field unique. These constraints are not unique, so we choose to follow [Hodges 2006] in letting

$$\left\langle \hat{\boldsymbol{R}} \right\rangle = 2b\boldsymbol{R}, \tag{2}$$

where

$$\langle (\bullet) \rangle = \int_{-b(x)}^{b(x)} (\bullet) dy.$$

Equation (2) implies that

$$\langle w_x \rangle = \langle w_y \rangle = 0. \tag{3}$$

The beam is assumed to be homogeneous and isotropic, and the entire development is linear throughout. Under assumption of plane stress, appropriate for a thin body such as this one, twice the strain energy per unit length is given by

$$2U = \frac{Et}{1-\nu^2} \left\langle \Gamma_{xx}^2 + \Gamma_{yy}^2 + 2\nu \Gamma_{xx}\Gamma_{yy} + \frac{(1-\nu)}{2}\Gamma_{xy}^2 \right\rangle, \tag{4}$$

where ν is Poisson's ratio. According to the displacement field spelled out in Equation (1), the two-dimensional strain components are

$$\Gamma_{xx} = \bar{\epsilon} - y\bar{\kappa} + w_x', \qquad \Gamma_{yy} = w_{y,y}, \qquad \Gamma_{xy} = w_{x,y} + w_y', \tag{5}$$

where $(\,)'$ means the partial derivative with respect to x and $(\,)_{,y}$ means the partial derivative with respect to y. The one-dimensional generalized strains are $\bar{\epsilon}$ and $\bar{\kappa}$, both functions of x. Here $\bar{\kappa} = v''(x)$ is the usual curvature of the reference line of Euler–Bernoulli beam theory, and $\bar{\epsilon} = u'(x)$ is the stretching of the reference line. The one-dimensional generalized strains are taken as known in the dimensional reduction procedure.

There are three small parameters that can be identified. First, the strain is small compared to unity. It is straightforward to show that both $\bar{\epsilon}$ and $a\bar{\kappa}$ are $O(\varepsilon)$, where ε denotes the maximum strain, and $a = b(0)$ the maximum value taken on by y in the structure. The second small parameter is a/ℓ where ℓ is the wavelength of deformation along the beam, such that $\partial(\bullet)/\partial x = O(\bullet/\ell)$. Finally, in this study we select the nondimensional taper parameter τ as a small parameter. Because our problem is linear, the strain will only enter the strain energy quadratically, so the smallness of strain has no real effect on the formulation. For simplicity, we take a/ℓ and τ to be of the same order, $O(\delta)$, and will ultimately ignore δ^3 compared to unity.

The VAM procedure is summarized as follows:

(i) Identify and remove all terms $O(\delta)$ and higher in the strain.

(ii) Use this resulting zeroth-order approximation of the strain to form the zeroth-order approximation of the strain energy in terms of the warping.

(iii) Minimize the zeroth-order approximation of strain energy with respect to the warping to obtain the zeroth-order approximation of the warping.

(iv) Perturb the resulting zeroth-order warping by one order of δ and use the perturbed warping to express the strain components to a sufficiently high order approximation so that the energy contains all $O(\delta^2)$ terms and all higher-order terms are dropped.

(v) Minimize this second-order approximation of the energy with respect to the warping function perturbations.

(vi) Substitute the result for the warping back into the original strain energy and discard all terms of orders higher than $O(\delta^2)$.

The result is the asymptotically-exact strain energy per unit length.

To begin we write twice the zeroth-order approximation of the energy, tantamount to ignoring δ altogether, as

$$2U_0 = \left\langle \frac{Et}{1-\nu^2}\left[(\bar{\epsilon} - y\bar{\kappa})^2 + 2\nu w_{y,y}(\bar{\epsilon} - y\bar{\kappa}) + \frac{(1-\nu)w_{x,y}^2}{2} + w_{y,y}^2\right] - 2\lambda_x w_x - 2\lambda_y w_y \right\rangle,$$

where Lagrange multipliers λ_x and λ_y are used to enforce constraints on the warping. The warping field that minimizes U_0 can be found as

$$w_x = 0, \qquad w_y = -\nu\bar{\epsilon}y + \frac{\nu\bar{\kappa}}{2}\left(y^2 - \frac{b^2}{3}\right).$$

Plugging this warping field back into the expression for $2U_0$, one obtains twice the zeroth-order energy as

$$2U_0 = 2Ebt\,\bar{\epsilon}^2 + \tfrac{2}{3}Etb^3\bar{\kappa}^2, \tag{6}$$

which is consistent with Euler–Bernoulli theory. Note that Equation (6) is derived without ad hoc assumptions such as assuming the cross section to be rigid in its own plane or assuming that $\nu = 0$. Such assumptions are sometimes used to derive classical beam theory, but they are neither necessary nor correct.

For the next approximation to the one-dimensional energy, we first perturb the above approximation of warping to arrive at

$$w_x = v_x, \qquad w_y = -\nu\bar{\epsilon}y + \frac{\nu\bar{\kappa}}{2}\left(y^2 - \frac{b^2}{3}\right) + v_y,$$

where v_x is the perturbation of w_x, and v_y is the perturbation of w_y; v_x and v_y are of one order higher in δ than w_x and w_y.

This new warping field is then substituted into the strain components from (5), at which point a new expression for the two-dimensional strain energy arises from (4) by virtue of the new strain components. Here one must be careful to retain all terms up through $O(\delta^2)$ and drop all terms of higher order in the energy, so now we find

$$2U_2 = \frac{Et}{1-\nu^2}\left\{\frac{1-\nu}{2}\left\langle\left[\frac{\nu\tau b\bar{\kappa}}{3} - y\nu\bar{\epsilon}' + \frac{\nu\left(3y^2 - b^2\right)\bar{\kappa}'}{6} + v_{x,y}\right]^2\right\rangle + \left\langle v_{y,y}^2 + 2v_x'(1-\nu^2)(\bar{\epsilon} - y\bar{\kappa})\right\rangle\right\}.$$

Expressions for the perturbation variables, v_x and v_y, that minimize U_2 subject to the constraints in (3) must be found; the constraints are again enforced by use of Lagrange multipliers, Λ_x and Λ_y, respectively. The stationary point of U_2 is found by setting its first variation equal to zero, which leads to the two Euler–Lagrange equations

$$\frac{\partial U_2}{\partial v_x} - \left(\frac{\partial U_2}{\partial v_x'}\right)' - \left(\frac{\partial U_2}{\partial v_{x,y}}\right)_{,y} = \Lambda_x, \qquad \frac{\partial U_2}{\partial v_y} - \left(\frac{\partial U_2}{\partial v_{y,y}}\right)_{,y} = \Lambda_y, \tag{7}$$

along with corresponding natural boundary conditions

$$\left.\frac{\partial U_2}{\partial v_{x,y}}\right|_{y=\pm b(x)} = 0, \qquad \left.\frac{\partial U_2}{\partial v_{y,y}}\right|_{y=\pm b(x)} = 0. \tag{8}$$

According to Saint-Venant's principle, boundary conditions, the warping at the beam ends (not shown) does not affect the behavior of the warping inside the beam and is not used in the solution of Equations (7) and (8).

Although both the Euler–Lagrange equations and boundary conditions for v_x and v_y look almost identical, the actual equations obtained are not. The Euler–Lagrange equation for v_y, the second equation of (7), reduces simply to $v_{y,yy} = 0$; from this and the second equation of (8), which requires $v_{y,y}$ to vanish at $y = \pm b$, one obtains by inspection that $\Lambda_x = v_y = 0$. On the other hand, the resulting Euler–Lagrange equation in v_x, the first part of Equation (7), can be simplified to

$$Et\left[(2+\nu)(\overline{\epsilon}' - y\,\overline{\kappa}') + v_{x,yy}\right] + 2(1+\nu)\Lambda_x = 0, \tag{9}$$

and the natural boundary conditions simplify to

$$\left.\frac{y\tau\,(\overline{\epsilon} - y\,\overline{\kappa})}{b} + \frac{2\nu\left(\tau b\,\overline{\kappa} - 3y\,\overline{\epsilon}'\right) + \nu\left(3y^2 - b^2\right)\overline{\kappa}' + 6v_{x,y}}{12(1+\nu)}\right|_{y=\pm b(x)} = 0. \tag{10}$$

Solving Equations (9) and (10) simultaneously gives the Lagrange multiplier as

$$\Lambda_x = Et\left(\frac{\tau\,\overline{\epsilon}}{b} - \overline{\epsilon}'\right),$$

and v_x as

$$v_x = \frac{\tau}{3b}\left[(1+\nu)\left(b^2 - 3y^2\right)\overline{\epsilon} + y(5\nu+6)b^2\,\overline{\kappa}\right] + \frac{1}{6}\left\{\nu\left(3y^2 - b^2\right)\overline{\epsilon}' + \left[y^3(\nu+2) - y(5\nu+6)b^2\right]\overline{\kappa}'\right\}.$$

Note that the first term is $O(\tau)$ and the second is $O(a/\ell)$, so that the perturbation is indeed $O(\delta)$. It can also be easily checked that the traction-free boundary conditions are satisfied asymptotically to the order of the perturbation variables, $O(\delta)$.

With both perturbation variables now known, the second-order energy is also known. The strain energy per unit length, asymptotically correct up to second order in δ, is then the sum of U_0 and U_2, and is equal to

$$U = Etb\left[1 - \frac{2}{3}(\nu+1)\tau^2\right]\overline{\epsilon}^2 + \frac{2Et\nu\tau b^2}{3}\overline{\epsilon}\,\overline{\epsilon}' + \frac{Etb^3}{9}\left[3 + 2(14\nu+15)\tau^2\right]\overline{\kappa}^2 - \frac{4Et\tau(8\nu+9)b^4}{9}\overline{\kappa}\,\overline{\kappa}' + \frac{4Et(1+\nu)b^5}{15}\overline{\kappa}'^2 + \frac{2Et(11\nu+12)b^5}{45}\overline{\kappa}\,\overline{\kappa}'', \tag{11}$$

which is of the same form as the refined beam theory presented in [Hodges 2006], namely

$$2U = S\,\overline{\epsilon}^2 + 2G\,\overline{\epsilon}\,\overline{\epsilon}' + A\,\overline{\kappa}^2 + 2B\,\overline{\kappa}\,\overline{\kappa}' + C\,\overline{\kappa}'^2 + 2D\,\overline{\kappa}\,\overline{\kappa}'', \tag{12}$$

with A, B, C, D, S, and G being scalars identified from Equation (11); they are implicit functions of x through the varying width $b(x)$ and explicit functions of τ. It is easy to see that terms A and S without τ correspond to those of Euler–Bernoulli theory. Terms with τ are the corrections from taper, and other terms from which τ is absent (C and D) pertain to shear deformation of prismatic beams.

3. Transformation to generalized Timoshenko form

The strain energy function developed in the previous section is not suitable for use as an engineering beam theory because of the presence of derivatives of $\overline{\epsilon}$ and $\overline{\kappa}$. It is known, however, that the form of (11) can be transformed into a generalized Timoshenko theory, which is the main objective of this section. Thus, the strain energy will be put into the form

$$2U^* = W\kappa^2 + 2X\kappa\gamma + Y\gamma^2 + Z\epsilon^2, \tag{13}$$

where W, X, Y, and Z are scalars, and with W, X, and Z being functions of τ, while γ is the one-dimensional beam engineering transverse shear measure. The shear strain measure γ turns out to be one order higher in δ than the classical measures of strain; therefore the energy from (11), which is second-order accurate, is sufficient to construct a generalized Timoshenko model. Note that after being put in this form the energy will no longer be asymptotically correct, because information is lost in the conversion process. Also, because γ is $O(\delta)$, Y will not have corrections from the taper parameter in a second-order correct strain energy. By inspection of (12), extension $\overline{\epsilon}$ is coupled only with its own derivative, hence we expect it will not be coupled with any other strain measures in (13).

The major difference between classical and Timoshenko theories is that classical theory neglects transverse shear strain while the generalized Timoshenko theory includes it, so the relationship between the two theories is established here. (The term *generalized* is used to emphasize the fact that the theory is not Timoshenko theory, nor is it based on any of the myriad assumptions of that theory. Moreover, the present theory includes the bending-shear coupling effect embodied in X.) As depicted in Figure 2, $\boldsymbol{T}_x$

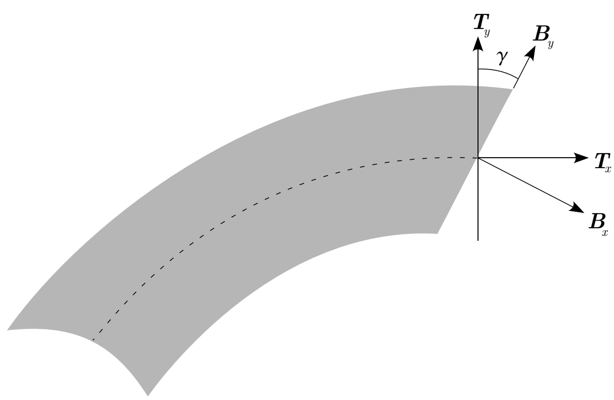

Figure 2. Coordinate systems used for transverse shear formulation.

and $\boldsymbol{T}_y$ collectively represent the dyad associated with classical theory, whereas $\boldsymbol{B}_x$ and $\boldsymbol{B}_y$ represent the dyad associated with generalized Timoshenko theory. $\boldsymbol{T}_x$ and $\boldsymbol{T}_y$ are aligned as parallel to and normal to the beam reference axis respectively. $\boldsymbol{B}_x$ and $\boldsymbol{B}_y$ are then rotated clockwise by an angle from $\boldsymbol{T}_x$ and $\boldsymbol{T}_y$ so that $\boldsymbol{B}_x$ is normal to the cross-sectional plane (which may be either defined as an average or at a point), so that

$$\boldsymbol{B}_x = \boldsymbol{T}_x - \gamma \boldsymbol{T}_y, \qquad \boldsymbol{B}_y = \gamma\, \boldsymbol{T}_x + \boldsymbol{T}_y.$$

Following the procedure of [Hodges 2006], which assumes that the strain components are small, the axial *force strain* measure is identical for the two theories so that $\bar{\epsilon} = \epsilon$ and the relationship of *moment strain* between the two theories is given by

$$\bar{\kappa} = \kappa + \gamma'. \tag{14}$$

Due to the presence of the derivatives in $\bar{\kappa}$ in Equation (12), we also mention that the derivatives are

$$\bar{\kappa}' = \kappa' + \gamma'', \qquad \bar{\kappa}'' = \kappa'' + \gamma''', \tag{15}$$

and that $v' = \theta + \gamma$, where θ is the total section rotation and $\kappa = \theta'$.

The derivatives of ϵ, κ, and γ must be written in terms of ϵ, κ, and γ, since the form of (13) contains no derivatives. The approach for *eliminating* the derivatives adopted here is to make use of the equilibrium equations. At each section the axial force (F), shear force (V), and bending moment (M) are, respectively,

$$F = \frac{\partial U^*}{\partial \epsilon} = Z\epsilon, \qquad V = \frac{\partial U^*}{\partial \gamma} = X\kappa + Y\gamma, \qquad M = \frac{\partial U^*}{\partial \kappa} = W\kappa + X\gamma.$$

In the absence of applied loading within the beam, the equilibrium equations are then

$$\begin{aligned} F' &= Z\epsilon' + Z'\epsilon = 0,\\ V' &= Y\gamma' + X\kappa' + Y'\gamma + X'\kappa = 0,\\ M' + V &= X\gamma' + W\kappa' + (X' + Y)\gamma + (W' + X)\kappa = 0. \end{aligned} \tag{16}$$

The above represents a system of equations which can be used to solve for ϵ', γ', and κ' in terms of ϵ, γ, and κ. The higher derivatives can then be obtained, in terms of ϵ, γ, and κ, by directly taking derivatives of (16). The resulting expressions are too lengthy to include here, but suffice it to say that the procedure is not at all challenging for symbolic computational tools such as Mathematica.

The desired strain energy of the beam, in the form of (13), can now be obtained by substituting Equations (14) and (15), along with the described approach for eliminating derivatives, into (12). Comparing the resultant second-order approximation to the strain energy with (13), one obtains the section constants as

$$W = \frac{2Etb^3}{3}\left[1 + \frac{(\nu - 48)\nu - 45}{45(\nu + 1)}\tau^2\right], \qquad X = \frac{Et(5\nu + 3)b^2\tau}{9(\nu + 1)},$$

$$Y = \frac{5Etb}{6(\nu + 1)}, \qquad Z = 2Etb\left(1 - \frac{2\tau^2}{3}\right).$$

The terms involving $\tau = -b'(x)$ are the corrections from our having included taper. From these expressions, we can observe that W is proportional to b^3 and is a quadratic polynomial in τ, X is proportional to b^2 and is linear in τ, Y is proportional to b and is independent of τ, and Z is proportional to b and is a quadratic polynomial in τ^2.

According to Renton [1991] there is no consensus on the precise definition of shear stiffness; thus, even though the expression for Y corresponds to results from [Washizu 1968; Young 1989; Renton 1991], it may not match those of other definitions.

4. Beam analysis of classical elasticity problems

4.1. *Pure extension.* In pure extension, a beam of length l is loaded at each end by equal and opposite axial tensile forces of magnitude T, depicted in Figure 3 for $Q = 0$. The potential of the applied loads is thus

$$\Phi = -T[u(l) - u(0)] = -T\int_0^l u' dx = -T\int_0^l \epsilon\, dx.$$

According to the principle of virtual work, the system is in equilibrium if and only if the variation of its total potential is zero. Upon setting the variation of the total potential equal to zero without imposing any geometric boundary conditions, one obtains

$$\int_0^l (W\kappa\delta\kappa + X\gamma\delta\kappa + X\kappa\delta\gamma + Y\gamma\delta\gamma + Z\epsilon\delta\epsilon - T\delta\epsilon)\, dx = 0.$$

The above equation requires the internal axial force, F, to be $F = Z\epsilon = T$. One can easily see that the elongation strain is

$$\epsilon = \frac{T}{Z}, \tag{17}$$

knowledge of which allows us to then integrate the kinematical differential equation $u' = \epsilon$ to obtain $u(x)$ for any given spanwise variation of the section constant Z. According to the model obtained from the VAM, the displacement $u(x)$ can be related directly to the elasticity solution in terms of the average axial displacement over the section.

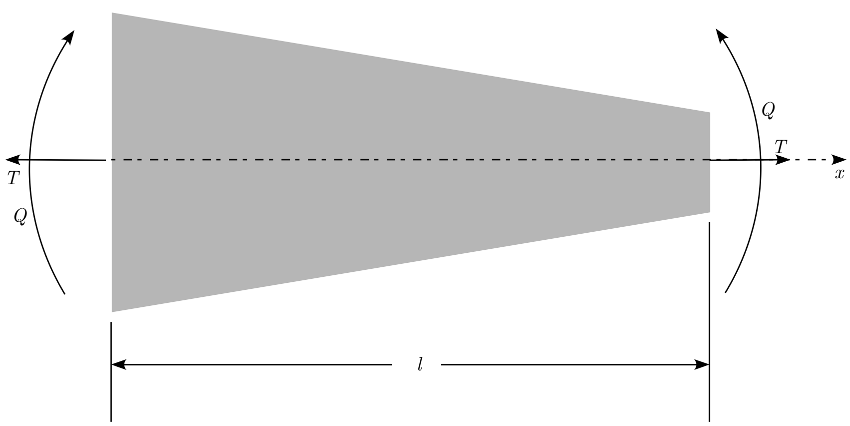

Figure 3. Schematic of beam loaded for either pure extension or pure bending.

4.2. *Pure bending.* To solve the pure bending problem, we use the kinematical differential equation $\kappa = \theta'$ and apply equal and opposite moments of magnitude Q on the ends of the beam. Figure 3, with $T = 0$, illustrates this case. This yields a potential of the applied loads of the form

$$\Phi = -Q[\theta(l) - \theta(0)] = -Q \int_0^l \theta' dx = -Q \int_0^l \kappa \, dx.$$

Equilibrium equations can then be found by minimizing the total potential subject to no geometric boundary conditions. The result is

$$\int_0^l (W\kappa\delta\kappa + X\gamma\delta\kappa + X\kappa\delta\gamma + Y\gamma\delta\gamma + Z\epsilon\delta\epsilon - Q\delta\kappa)\, dx = 0.$$

The resulting Euler–Lagrange equations require that the bending moment and shear force are, respectively,

$$M = W\kappa + X\gamma = Q, \qquad V = X\kappa + Y\gamma = 0.$$

Thus, eliminating $\gamma = -X\kappa/Y$, one obtains

$$\left(W - \frac{X^2}{Y}\right)\kappa = Q.$$

The solution can then be written as

$$\kappa = \frac{Q}{W - \frac{X^2}{Y}}, \tag{18}$$

which allows one to integrate the kinematical differential equation, $\theta' = \kappa$, to obtain $\theta(x)$ for any given spanwise variation of W, X, and Y. Unlike the prismatic case, even though Q is constant, κ is not. Moreover, the transverse displacement $v(x)$ can then be obtained by integration of another kinematical differential equation, $v' = \theta + \gamma = \theta - X\kappa/Y$. It is clear that loading by pure bending produces transverse shear deformation in a tapered beam.

4.3. *Flexure.* For the flexure problem, we load the beam with an equal and opposite transverse force P at each end, and a moment Pl at the left end to counteract the moment of the force at the right end (see Figure 4). For this loading the potential of the applied loads takes the form

$$\Phi = -P\,[v(l) - v(0)] + Pl\theta(0) = -P \int_0^l \left[(v' - \theta) + (l - x)\theta'\right] dx = -P \int_0^l [\gamma + (l - x)\kappa]\, dx.$$

Equilibrium equations can then be found by minimizing the total potential subject to no geometric boundary conditions. The result is

$$\int_0^l \{W\kappa\delta\kappa + X\gamma\delta\kappa + X\kappa\delta\gamma + Y\gamma\delta\gamma + Z\epsilon\delta\epsilon - P[\delta\gamma + (l - x)\delta\kappa]\}\, dx = 0.$$

The resulting Euler–Lagrange equations and boundary conditions require that the bending moment and shear force are, respectively,

$$M = W\kappa + X\gamma = P(l - x), \qquad V = X\kappa + Y\gamma = P.$$

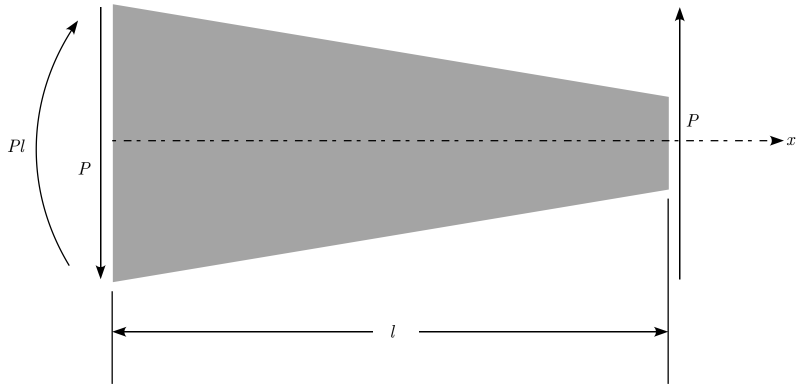

Figure 4. Schematic of beam loaded for flexure.

Thus, one obtains

$$\kappa = \frac{P}{WY - X^2}\left[Y(l-x) - X\right], \qquad \gamma = \frac{P}{WY - X^2}\left[W - X(l-x)\right], \tag{19}$$

which allows one to integrate the same kinematical differential equations as in the pure bending case to obtain the total section rotation $\theta(x)$ and the displacement of the neutral axis $v(x)$ for any given spanwise variation of W, X, and Y. Unlike the prismatic case, although the bending moment is linear, κ is not. Also, although the shear force is constant, γ is not constant.

5. Solutions for classical elasticity problems

This section presents exact solutions for the purpose of comparing with the above beam solutions based on a refined beam theory. These solutions are appropriately based on linear, plane stress elasticity theory for a linearly tapered strip for problems of pure extension, pure bending, and flexure. For all three cases, the components of the stress tensor are presented (σ_{xx}, σ_{xy}, and σ_{yy}). Components of the strain tensor may then be obtained from the plane stress form of Hooke's law. Lastly, the strains can be integrated to obtain displacements, $u_x(x, y)$ and $u_y(x, y)$. In the formulae that ensue, the y-coordinate varies between $\pm b(x)$, where $b = a - x\tau$, a is the half-width of the strip at $x = 0$, $h = a - l\tau > 0$ is the half-width of the strip at $x = l$, t is the thickness of the strip, l is its length (not to be confused with the wavelength ℓ that was previously used), and $s = l - x$.

We now set forth a way to extract information from the elasticity solutions so that the results can be compared with those from the beam solutions. Let us denote the displacement fields from elasticity by $u_x(x, y)$ and $u_y(x, y)$. These can be related to those from beam theory by making use of Equation (1), yielding

$$u_x = u - yv' + w_x, \qquad u_y = v + w_y, \tag{20}$$

where we have earlier approximated the warping displacements. Integrating both sides of Equation (20) over y and using the constraints on the warping, one obtains

$$u = \frac{1}{2b}\langle u_x\rangle, \qquad v = \frac{1}{2b}\langle u_y\rangle. \tag{21}$$

Multiplying both sides of the first part of Equation (20) by y and integrating allows us to identify

$$\theta = \frac{3}{2b^3}\langle -yu_x\rangle, \qquad \gamma = v' - \theta = \frac{3}{2b^3}\langle yw_x\rangle. \tag{22}$$

Finally, the stretching and bending strain measures, $\epsilon = u'$ and $\kappa = \theta'$, along with the shear strain measure γ, can now be compared directly with results from applying the beam theory to a specific problem such as pure extension, pure bending or flexure.

5.1. ***Pure extension.*** The solution for the deformation of a wedge described by polar coordinates r and ϕ, presented in [Timoshenko and Goodier 1970, p. 110], is quite simple. The stresses for this case are

$$\sigma_\phi = \sigma_{r\phi} = 0, \qquad \sigma_r = \frac{T\cos\phi}{rt\,(\alpha + \cos\alpha\sin\alpha)},$$

where, referring back to Figure 3, $Q = 0$, T is nonzero, and

$$\alpha = \tan^{-1}\tau, \qquad r = \sqrt{y^2 + \frac{b^2}{\tau^2}}, \qquad \phi = \tan^{-1}\left(\frac{y\tau}{b}\right).$$

The stresses in the Cartesian system can be found as

$$\sigma_{xx} = \sigma_r\cos^2\phi - \sigma_{r\phi}\sin 2\phi, \qquad \sigma_{xy} = -\sigma_{r\phi}\cos 2\phi - \tfrac{1}{2}\sigma_r\sin 2\phi, \qquad \sigma_{yy} = \sigma_r\sin^2\phi + \sigma_{r\phi}\cos 2\phi.$$

In terms of the geometric parameters and loads, the stresses finally become

$$\sigma_{xx} = \frac{T\tau b^3\left(\tau^2+1\right)}{t\left(b^2+y^2\tau^2\right)^2\left[\tau + \left(\tau^2+1\right)\tan^{-1}(\tau)\right]}, \qquad \sigma_{xy} = -\frac{Ty\tau^2 b^2\left(\tau^2+1\right)}{t\left(b^2+y^2\tau^2\right)^2\left[\tau + \left(\tau^2+1\right)\tan^{-1}(\tau)\right]},$$
$$\sigma_{yy} = \frac{Ty^2\tau^3 b\left(\tau^2+1\right)}{t\left(b^2+y^2\tau^2\right)^2\left[\tau + \left(\tau^2+1\right)\tan^{-1}(\tau)\right]}.$$

5.2. ***Pure bending.*** This case is also shown in Figure 3, here with $T = 0$ and Q nonzero. The stresses in polar coordinates are given by Timoshenko and Goodier [1970, pp. 112–13], as $\sigma_\phi = 0$ and

$$\sigma_r = \frac{2Q\sin 2\phi}{r^2 t\,(2\alpha\cos 2\alpha - \sin 2\alpha)}, \qquad \sigma_{r\phi} = -\frac{Q(\cos 2\phi - \cos 2\alpha)}{r^2 t\,(2\alpha\cos 2\alpha - \sin 2\alpha)}.$$

Making the above transformation to Cartesian coordinates, one may obtain the stresses as

$$\sigma_{xx} = -\frac{2bQy\tau^3\left[b^2\left(2\tau^2+1\right) - y^2\tau^2\right]}{t\left(b^2+y^2\tau^2\right)^3\left[\tau + \left(\tau^2-1\right)\tan^{-1}(\tau)\right]}, \qquad \sigma_{xy} = -\frac{Q\tau^4\left[b^4 - 3y^2\left(\tau^2+1\right)b^2 + y^4\tau^2\right]}{t\left(b^2+y^2\tau^2\right)^3\left[\tau + \left(\tau^2-1\right)\tan^{-1}(\tau)\right]},$$
$$\sigma_{yy} = \frac{2bQy\tau^5\left[b^2 - y^2\left(\tau^2+2\right)\right]}{t\left(b^2+y^2\tau^2\right)^3\left[\tau + \left(\tau^2-1\right)\tan^{-1}(\tau)\right]}.$$

To visualize the deformed shape, finite element analysis was performed using ABAQUS. The deformed shape of the structure is shown in Figure 5. To eliminate rigid body motion, the geometric boundary conditions were set as $u_x = u_y = 0$ at the point $(x = 0,\, y = 0)$ and $u_y = 0$ at the point $(x = l,\, y = 0)$. Modeling in ABAQUS was done using its CPS8R elements, and its results were validated with the

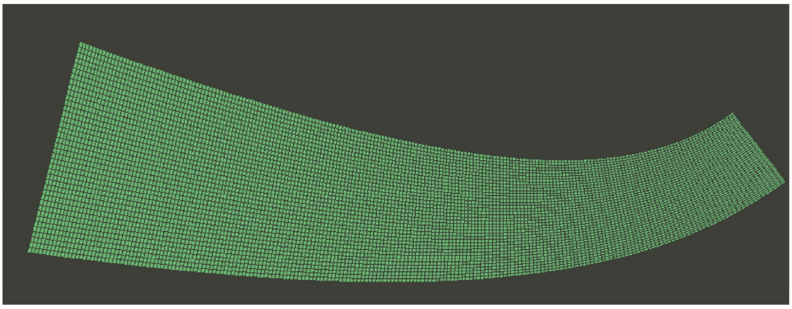

Figure 5. Deformed shape of the tapered strip under pure bending.

l (m)	a (m)	τ	t (m)	E (GPa)	ν	Q (N-m)
20	3	0.1	0.1875	200	0.3	1

Table 1. Dimensions, material properties, and loading for the tapered strip evaluated for ABAQUS calculations.

elasticity solution. The specific dimensions, material properties, and loading chosen are given in Table 1. It is clear that κ increases as the width of the structure decreases.

5.3. ***Flexure.*** The stresses of this case, shown in Figure 4, are given in polar coordinates by Krahula [1975], with $\sigma_\phi = 0$ and

$$\sigma_r = \frac{2P}{r^2 t}\left[\frac{r\sin\phi}{\sin 2\alpha - 2\alpha} + \frac{(l - a\cot\alpha)\sin 2\phi}{2\alpha\cos 2\alpha - \sin 2\alpha}\right], \qquad \sigma_{r\phi} = -\frac{P(\cos 2\phi - \cos 2\alpha)\cot\alpha\,(a - l\tan\alpha)}{r^2 t(\sin 2\alpha - 2\alpha\cos 2\alpha)}.$$

(It is noted that several small printing mistakes in the paper had to be corrected in order to obtain this result.) Making the transformation to Cartesian coordinates, one finds the stresses to be

$$\sigma_{xx} = \frac{bPy\tau^2}{t\left(b^2+y^2\tau^2\right)^3}\left\{\frac{2h\left[b^2+\left(2b^2-y^2\right)\tau^2\right]}{\tau+\left(\tau^2-1\right)\tan^{-1}(\tau)} - \frac{b\left(\tau^2+1\right)\left(b^2+y^2\tau^2\right)}{\left(\tau^2+1\right)\tan^{-1}(\tau)-\tau}\right\},$$

$$\sigma_{yy} = -\frac{Py\tau^4}{t\left(b^2+y^2\tau^2\right)^3}\left\{\frac{\left(\tau^2+1\right)\left(b^2+y^2\tau^2\right)y^2}{\left(\tau^2+1\right)\tan^{-1}(\tau)-\tau} + \frac{2bh\left[b^2-y^2\left(\tau^2+2\right)\right]}{\tau+\left(\tau^2-1\right)\tan^{-1}(\tau)}\right\},$$

$$\sigma_{xy} = -\frac{P\tau^4\left[b^5 - s\tau b^4 - 4y^2\left(\tau^2+1\right)b^3 + 3sy^2\tau\left(\tau^2+1\right)b^2 - y^4\tau^4 b - sy^4\tau^3\right]}{t\left(b^2+y^2\tau^2\right)^3\left\{2\tau\tan^{-1}(\tau)+\left(\tau^4-1\right)\left[\tan^{-1}(\tau)\right]^2-\tau^2\right\}}$$
$$+\frac{P\tau^3\left(\tau^2+1\right)\left[b^5 - s\tau b^4 - 2y^2\left(\tau^2+2\right)b^3 + 3sy^2\tau\left(\tau^2+1\right)b^2 + y^4\tau^4 b - sy^4\tau^3\right]\tan^{-1}(\tau)}{t\left(b^2+y^2\tau^2\right)^3\left\{2\tau\tan^{-1}(\tau)+\left(\tau^4-1\right)\left[\tan^{-1}(\tau)\right]^2-\tau^2\right\}}.$$

6. Comparison

In this section we wish to compare the beam section constants obtained by the VAM with results for the same quantities extracted from our elasticity solutions. To do so, the one-dimensional displacement and rotation variables u, v, and θ are extracted from the elasticity solutions above by averaging two-dimensional displacements over y in accordance with Equations (21) and (22). Then these quantities are differentiated with respect to x, leading to the values of one-dimensional generalized strains ϵ, γ, and κ. Finally, effective stiffnesses are found by dividing appropriate applied loads by corresponding one-dimensional generalized strains. These effective stiffnesses are then compared directly with values of the section constants determined from the VAM.

6.1. ***Pure extension.*** For pure extension, it is appropriate to compare the quantity T/ϵ using the stiffness constants obtained from the VAM with an expansion of the elasticity solution in τ. The beam solution, from Equation (17), and the second-order asymptotic expansion of the elasticity solution both agree that this quantity is

$$\frac{T}{\epsilon} = Z = 2Etb\left(1 - \frac{2\tau^2}{3}\right).$$

The term involving τ^2 represents the correction to taper. The perfect agreement of these two solutions reflects that the strain energy from the classical model is asymptotically exact for this problem, which is expected because shearing deformations are not involved in pure extension. For a section with a linear taper of $\tau = 0.1763$, which corresponds to 10° taper, and is not uncommon as local taper on rotor blades, the axial stiffness is overpredicted by 2.12% if the taper effect is neglected.

6.2. ***Pure bending.*** The quantity to be compared for this problem is Q/κ. The beam solution, from Equation (18) is

$$\frac{Q}{\kappa} = W - \frac{X^2}{Y} = \frac{2Etb^3}{3} - \frac{4Etb^3(4\nu+9)\tau^2}{45},$$

whereas the second-order asymptotic expansion of the elasticity solution yields

$$\frac{Q}{\kappa} = \frac{2Etb^3}{3} - \frac{4Etb^3(\nu+3)\tau^2}{15}.$$

For a linear taper of $\alpha = 10^\circ$ and $\nu = 0.3$, the taper effect reduces the bending stiffness by 4.28% and 4.42% from the elasticity and beam solutions respectively. The relative difference between the beam solution and the elasticity solution is $2\nu\tau^2/15$, with the beam solution being softer. This small difference between the asymptotic expansion of the exact solution versus the beam results can be attributed to our having approximated the asymptotically-exact energy, Equation (11), by forcing it into the mold of the generalized Timoshenko model, Equation (13). Obviously, the correction due to taper is itself much larger than the difference between the elasticity and beam solutions.

6.3. ***Flexure.*** For the flexure problem we compare the quantities P/κ and P/γ at $x = l$. The beam solution, Equation (19), yields

$$\frac{P}{\kappa} = X - \frac{WY}{X} = -\frac{5Etb^2}{(3+5\nu)\tau} + O(\tau), \qquad \frac{P}{\gamma} = Y - \frac{X^2}{W} = \frac{5Etb}{6(1+\nu)} + O(\tau^2) = \frac{5Gtb}{3} + O(\tau^2).$$

An order of magnitude analysis shows that we cannot trust either of the correction terms to these results, because we do not have sufficient data to ensure that we have all the contributions to them. That is to say, the VAM solution would have to be extended to include terms of higher order in τ than we needed to construct the beam model; in particular corrections of third-order to X and second-order to Y would be needed. As expected, the elasticity solution is in agreement with the above P/γ result since it does not involve taper. It should be noted, however, that there is more than one possible result from this exercise. The method of [Yu and Hodges 2004] was used here. The result for P/κ does involve taper and is given by

$$\frac{P}{\kappa} = -\frac{10Etb^2}{3(2+3\nu)\tau} + O(\tau).$$

The beam solution differs from the elasticity solution by less than 4% for practical values of ν. Note that this term tends to infinity as taper decreases and the beam approaches being prismatic.

7. Conclusion

A beam model is constructed using the variational-asymptotic method that is capable of handling extension, in-plane bending, and in-plane shear for a homogeneous, isotropic strip-beam, the width of which is linearly tapered along the span. The resulting beam model reveals that (a) section constants are influenced by the local taper such that $b'(x) = -\tau$ appears explicitly, and (b) bending and shear deformation are coupled by τ in the resulting model. To validate the theory, solutions for the corresponding plane stress elasticity problems for pure extension, pure bending, and flexure are presented, and the corrections caused by $\tau \neq 0$ are found. Excellent agreement is demonstrated between the elasticity solutions and the beam solutions based on the constructed model.

Examples of this influence include a decrease in both axial and bending stiffnesses, the latter being large enough that its neglect cannot be justified for tapered beams. To avoid errors, the taper effect must be accounted for in the cross-sectional analysis prior to performing the beam analysis. The present results will be of practical use in validating general cross-sectional analyses when they are eventually extended to include the influence of taper. In particular, additional work is needed to account for this effect in the cross-sectional analysis of spanwise nonuniform composite beams with arbitrary cross-sectional geometries and to determine asymptotically-exact strain/stress recovery relations.

References

[Andrade and Camotim 2005] A. Andrade and D. Camotim, "Lateral-torsional buckling of singly symmetric tapered beams: theory and applications", *J. Eng. Mech.* **131**:6 (2005), 586–597.

[Banerjee and Williams 1986] J. Banerjee and F. Williams, "Exact Bernoulli-Euler static stiffness matrix for a range of tapered beam columns", *Int. J. Numer. Meth. Eng.* **23**:9 (1986), 1615–1628.

[Berdichevsky 1979] V. L. Berdichevsky, "Variational-asymptotic method of constructing a theory of shells", *PMM* **43**:4 (1979), 664–687.

[Berdichevsky 1981] V. L. Berdichevsky, "On the energy of an elastic rod", *PMM* **45** (1981), 518–529.

[Boley 1963] B. A. Boley, "On the accuracy of the Bernoulli-Euler theory for beams of variable section", *J. Appl. Mech.* (*Trans. ASME*) **30** (1963), 374–378.

[Cesnik and Hodges 1997] C. E. S. Cesnik and D. H. Hodges, "VABS: a new concept for composite rotor blade cross-sectional modeling", *J. Am. Helicopter Soc.* **42**:1 (1997), 27–38.

[Cleghorn and Tabarrok 1992] W. Cleghorn and B. Tabarrok, "Finite element formulation of a tapered Timoshenko beam for free lateral vibration analysis", *J. Sound Vib.* **152**:3 (1992), 461–470.

[Hodges 2006] D. H. Hodges, *Nonlinear composite beam theory*, vol. 213, Progress in astronautics and aeronautics, American Institute of Aeronautics and Astronautics, Reston, VA, 2006.

[Karabalis and Beskos 1983] D. Karabalis and D. Beskos, "Static, dynamic and stability analysis of structures composed of tapered beams", *Comput. Struct.* **16**:6 (1983), 731–748.

[Kitipornchai and Trahair 1975] S. Kitipornchai and N. Trahair, "Elastic behaviour of tapered monosymmetric, I: beams", *J. Struct. Div. ASCE* **101** (1975), 1661–1678.

[Krahula 1975] J. L. Krahula, "Shear formula for beams of variable cross section", *AIAA J.* **13**:10 (1975), 1390–1391.

[Rajasekaran 1994] S. Rajasekaran, "Equations for tapered thin-walled beams of generic open section", *J. Eng. Mech.* **120**:8 (1994), 1607–1629.

[Renton 1991] J. D. Renton, "Generalized beam theory applied to shear stiffness", *Int. J. Solids Struct.* **27**:15 (1991), 1955–1967.

[Ronagh et al. 2000] H. Ronagh, M. Bradford, and M. Attard, "Nonlinear analysis of thin-walled members of variable cross-section, I: theory", *Comput. Struct.* **77**:3 (2000), 285–299.

[Tena-Colunga 1996] A. Tena-Colunga, "Stiffness formulation for nonprismatic beam elements", *J. Struct. Eng.* **122**:12 (1996), 1484–1489.

[Timoshenko and Goodier 1970] S. P. Timoshenko and J. N. Goodier, *Theory of elasticity*, 3rd ed., McGraw-Hill, New York, 1970.

[To 1981] C. To, "A linearly tapered beam finite element incorporating shear deformation and rotary inertia for vibration analysis", *J. Sound Vib.* **78**:4 (1981), 475–484.

[Vu-Quoc and Léger 1992] L. Vu-Quoc and P. Léger, "Efficient evaluation of the flexibility of tapered I-beams accounting for shear deformations", *Int. J. Numer. Meth. Eng.* **33**:3 (1992), 553–566.

[Washizu 1968] K. Washizu, *Variational methods in elasticity and plasticity*, Pergamon Press, Oxford, UK, 1968.

[Young 1989] W. C. Young, *Roark's formulas for stress and strain*, 6th ed., McGraw-Hill, New York, 1989.

[Yu and Hodges 2004] W. Yu and D. H. Hodges, "Elasticity solutions versus asymptotic sectional analysis of homogeneous, isotropic, prismatic beams", *J. Appl. Mech.* (*Trans. ASME*) **71**:1 (2004), 15–23.

[Yu et al. 2002] W. Yu, D. H. Hodges, V. V. Volovoi, and C. E. S. Cesnik, "On Timoshenko-like modeling of initially curved and twisted composite beams", *Int. J. Solids Struct.* **39**:19 (2002), 5101–5121.

Received 6 Jul 2007. Revised 3 Oct 2007. Accepted 8 Oct 2007.

DEWEY H. HODGES: dhodges@gatech.edu
Daniel Guggenheim School of Aerospace Engineering, Georgia Institute of Technology, Atlanta, GA 30332-0150, United States
http://www.ae.gatech.edu/~dhodges/

JIMMY C. HO: jho@gatech.edu
Daniel Guggenheim School of Aerospace Engineering, Georgia Institute of Technology, Atlanta, GA 30332-0150, United States

WENBIN YU: wenbin.yu@usu.edu
Wenbin Yu, Mechanical & Aerospace Engineering Department, Utah State University, Logan, UT 84322-4130, United States