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A FINITE ELEMENT FOR DYNAMIC ANALYSIS OF A CYLINDRICAL ISOTROPIC HELICAL SPRING

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This paper presents a finite element for the dynamic analysis of the cylindrical isotropic helical spring. The hybrid-mixed formulation is used to compute the stiffness matrix. A simple approach is used to calculate the mass matrix. These matrices are used for solving the dynamic equation of the spring to calculate natural frequencies and the dynamic response of a simple or an assembled spring for different types of cross-section.

1. Introduction

The helical spring is one of fundamental mechanical elements used in various industrial applications such as balances, brakes, clutch, and valves. The investigation of its vibratory behavior in order to find its natural resonant frequencies permits a better conception of different dynamic conditions. The analysis of this type of element is complex due to the presence of bending, stretching, coupling, the effects of shear strain and the rotatory inertia, as well as the shape complexity of its structure. Neglecting one of these parameters to simplify the solution gives wrong results and erroneous frequencies. Since investigations in this area began in the 19th century with Michell [1890], researchers have predominantly investigated two aspects of springs.

The first field is the vibratory behavior of charged springs with purely axial compression or under compression and torsion [Haringx 1949; Pearson 1982; Becker and Cleghorn; 1992; 1993; 1994; Chassie et al. 1997; 2002]. Other investigations in this field are concentrated on the stability of this kind of structure after calculation of resonant frequencies. Mottershead [1982] and Pearson [1982] obtain governing equations by summing forces and moments on an element of the spring. Tabarrok and Xiong [1989; 1992] and Xiong and Tabarrok [1992] developed a finite element for the vibration and buckling of curved and twisted rods under loads, giving results which agree well with those given by Chassie et al. [1997].

The second field of research is the study of unloaded springs. In this study, many techniques are used to analyze the problem. The experimental method is used to determinate natural frequencies of the spring as in [Lin and Pisano 1987] and [Mottershead 1980], but these studies show the difficulty of finding these frequencies because they are close each to other, especially for higher modes. Other techniques are used to solve this problem, such as the analytical method. Wahl [1963] determined axial and torsional modes of cylindrical helical springs, but that approach is valid only for circular cross-sections with a small helix angle, and does not give realistic results, particularly for high frequencies. Pietra and Valle [1982] improve the last model by taking into account the effect of helix angle. Their model gives acceptable

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results. Other work also uses the analytical method [Kagawa 1968; Philips and Castello 1972; Castello 1975; Pietra 1976; Guido et al. 1978]. Recent research is now based on the transfer matrix and stiffness matrix methods to solve the free vibration problem [Haktanir 1994; Yildirim; 1995; 1996; 1997; 1999a; 1999b; Lee and Thompson 2001]. These methods take into account the axial and shear strain and rotary inertia and give good results with less error than other approaches.

The research presented above is limited to academic applicactions; results can not be used directly by engineers in the phase of spring design. In fact, in practical problems the spring is not alone, but is in assembly with other types of mechanisms. To get reasonably good results for those problems, a large number of terms will have to be used and getting them may not always be easy; thus analytical methods are restored for the study of the simple spring behavior. To solve this problem, the finite element method is often used [Mottershead 1982; 1980; Sawanbori and Fukushima 1983; 1983; Pearson and Wittrick 1986]. This method is an approximate technique and obtains a solution for specific problems and is characterized by its versatility and capacity to solve practical problems found in engineering. It can easily model the behavior of a spring in a complex mechanism with minimum of calculation. Examples of springs in different boundary cases (such as fixed-fixed and fixed-free) and for various numbers of parameters (such as number of coils and helix angle) are studied. Results given by this model are close to those determined by analytical and experimental methods. After treating a static study of the spring [Taktak et al. 2005b] and a stress analysis [Dammak et al. 2005], in which we presented the method of determination of the stiffness matrix of the structure, we present in this paper a method for computing the mass matrix and solve the dynamic equation. These methods were presented in previous communications [Abid et al. 2005; Taktak et al. 2005a] in the case of a single spring, the first of which presented the dynamic behavior of a single helical spring and the second was a parametric study of geometrical and mechanical proprieties effects on the natural frequencies of the helical spring. The aim of this paper is to present a finite element which permits the reduction of the number of elements needed to study the structure that can be used with other finite elements in cases where the spring is assembled with others structures. To validate the developed element, natural frequencies and the dynamic response of single and assembled springs calculated by this element are presented in comparison with results given by a three-dimensional elastic beam finite element.

2. Nomenclature

$[M_T]$	Total mass matrix	δe_t	Virtual membrane strain
$[K_T]$	Total stiffness matrix	$\delta\gamma_{tn,tb}$	Virtual shear strain
$[M_G]$	Element global mass matrix	$\delta\chi_{t,n,b}$	Virtual strains
$[K_G]$	Element global stiffness matrix	$\langle\delta\varepsilon\rangle$	Generalized virtual strains vector
$\{X\}$	Global displacements vector	N	Normal force
$\{U\}$	Eigen vector	T_n, T_b	Shearing forces
ω	Eigen pulsation	M_t	Torsional moment
t	Time	M_n, M_b	Bending moments
$[C_T]$	Damping matrix	$\langle R\rangle$	Resulting forces vector
$[F_T]$	External forces vector	$\sigma_{t,tb,tn}$	Stress tensor components
$[\Phi]$	Modal matrix	E	Young's module
n	Number of eigen modes	G	Shearing's module

$\{a(t)\}$	Generalized displacements vector	A	Section area
$[M_m]$	Generalized mass matrix	J	Inertia of torsion
$[K_m]$	Generalized stiffness matrix	I_y, I_z	Central quadratic moments
$[C_m]$	Generalized damping matrix	k_y, k_z	Shears correction factors
$\{F_m\}$	Generalized forces vector	$[H]$	Elastic behavior Matrix
ω_i	Eigen pulsation	Π	Mixed functional of energy
ξ_i	Reduced damping coefficient	Π^e_{ext}	External energy element functional
X, Y, Z	Global coordinate system	Π^e_{int}	internal energy element functional
$\vec{I}, \vec{J}, \vec{K}$	Global basis	W^e_{int}	Internal work
s, y, z	Local coordinate system	e	Numbers of elements
$\vec{t}, \vec{n}, \vec{b}$	Local basis	$\{f\}$	Distributed forces Vector
s	Curvilinear axis	$\langle u\rangle$	Displacements vector
p	Point belonging to the axis ”s”	$\{F\}$	Concentrated forces vector
u, v, w	Local displacements	$\langle u_n\rangle$	Element degrees of freedom vector
$\theta_t, \theta_n, \theta_b$	Local rotations	$[P]$	Approximation matrix
U, V, W	Global displacements	$\{\alpha_n\}$	Discontinuous parameters vector
$\Theta_t, \Theta_n, \Theta_b$	Global rotations	$[A]$	Binodal approximation matrix
$\vec{X}_p$	Position vector	$[k]$	Local stiffness matrix
r	Helix radius	W^e_{inertial}	Element inertial work
P	Helix Pitch	ρ_0	Mass per unit volume
θ	Polar angle	L	Curvilinear length
N_s	Number of spires	$[m]$	Local mass matrix
R	Radius of curvature	ξ	Parametric variable
T	Radius of torsion	$\Delta\theta$	Difference of nodal angles
q	Point belonging to the beam	$[K_G]$	Global stiffness matrix
$\delta u,\ \delta v,\ \delta w$	Virtual displacements	$[M_G]$	Global mass matrix
$\delta\theta_{t,n,b}$	Virtual rotations	$[T]$	Transfer matrix
$\delta\varepsilon_t$	Axial virtual strain	$[Q_i]$	Rotation matrix
$\delta\gamma_n, b$	Virtual transverse shear strains		

3. Dynamic analysis

3.1. ***Modal analysis.*** The calculation of natural frequencies and modes is made by the resolution of the matrix system

$$[M_T]\left\{\ddot{X}\right\}+[K_T]\{X\}=\{0\}, \tag{3–1}$$

where $[M_T]$ is the total mass matrix, $[K_T]$ is the total stiffness matrix. These matrices are obtained by an assembly of element matrices in the global coordinate system. $[M_G]$ and $[K_G]$ are defined in the following sections. $\{X\}$ is the global nodal displacements vector. For a harmonic solution having the expression

$$\{X\}=\{U\}\exp\,(i\omega t), \tag{3–2}$$

$\{U\}$ is the eigenvector and ω is the eigen pulsation (rad s^{-1}). Equation (3–1) is reduced to the general eigenvalue problem

$$\big([K_T]-\omega^2[M_T]\big)\{U\}=\{0\}. \tag{3–3}$$

This eigen problem can be solved with one of many methods which exist in the literature, such as subspace iteration.

3.2. ***Dynamic response: method of modal superposition.*** The equation of movement of the system is written as

$$[M_T]\left\{\ddot{U}\right\}+[C_T]\left\{\dot{U}\right\}+[K_T]\{U\}=\{F_T\}, \tag{3–4}$$

where $[C_T]$ is the damping matrix and $\{F_T\}$ is the vector of external forces. The description of the movement of a system with several degrees of freedom can be made by its spatial coordinates or by its modal coordinates. The movement's equation of the structure without a second member admits a linear base of orthogonal real modes of the nondamped system. These eigen modes are characterized by the eigen pulsations ω and also by their eigenvectors $\{U_i\}$. We define the modal matrix $[\Phi]$ by

$$[\Phi]=[\{U_1\},\{U_2\},\{U_3\},....,\{U_n\}], \tag{3–5}$$

where n is the number of considered eigen modes to describe the movement of the system. The projection of the movement's equation on the modal basis leads to a system of n equations coupled only by the damping matrix. The equation of the movement according to the generalized parameters is written as

$$[M_m]\{\ddot{a}(t)\}+[C_m]\{\dot{a}(t)\}+[K_m]\{a(t)\}=\{F_m\}, \tag{3–6}$$

where $a(t)$ is the generalized displacements vector defined as $\{U(t)\}=[\Phi]\{a(t)\}$, $[M_m]$ is the generalized mass matrix $[M_m]=[\Phi]^t[M_T][\Phi]=\text{diag}\ (m_i)$, and $[K_m]$ is the generalized stiffness matrix $[K_m]=[\Phi]^t[K_T][\Phi]=\text{diag}\ (m_i\omega_i^2)$. ω_i are the eigen pulsations of each mode. The actions of damping are small. The matrix $[C_m]$ is obtained by adopting a reduced modal damping coefficient on each eigen mode [Dahtt and Touzout 1984], as in $[C_m]=\text{diag}\ (2m_i\omega_i\xi_i)$, where ξ_i is the reduced modal damping coefficient. $[F_m]$ is the vector of the generalized forces

$$\{F_m\}=[\Phi]^t\{F_T\}=\left\{\begin{array}{c} f_1\\ \vdots\\ f_i\\ \vdots\\ f_n\end{array}\right\}. \tag{3–7}$$

The matrices $[K_m]$, $[M_m]$ and $[C_m]$ are diagonal, so we obtain a system of uncoupled n oscillators with one degree of freedom for each. The equation of movement of each oscillator is written as

$$m_i\ddot{a}_i+2m_i\xi_i\omega_i\dot{a}_i+m_i\omega_i^2a_i=f_i, \qquad i=1,2,\ldots,n. \tag{3–8}$$

The advantage of this modal description is that it simplifies the resolution of the movement's equations, reducing them to a linear system of n completely uncoupled equations.

4. Finite element formulation

4.1. ***Geometric presentation.*** A spring's beam is a three-dimensional curved beam defined in the global coordinate system (O, X, Y, Z) of the basis $(\vec{I}, \vec{J}, \vec{K})$. This beam is generated by a succession of plane domains which are orthogonal to the middle fiber of the structure s. The dimensions of those domains are small in comparison to the beam's length. Two geometric hypotheses are taken. The first is that the orthogonal sections are identical along the curvilinear axis. The second is that studied beams have a full section. The position vector of any point p belonging to the middle fiber of the spring's beam is defined in the global coordinate system by

$$\vec{X}_p = \left\{ \begin{array}{c} r\cos\theta \\ r\sin\theta \\ \dfrac{P}{2\pi}\theta \end{array} \right\}_{(\vec{I},\vec{J},\vec{K})}, \tag{4–1}$$

where r and P are respectively the radius and pitch of the helix and θ is the polar angle. This angle is defined as

$$\theta = 2\pi N_s, \tag{4–2}$$

where N_s is the number of spires of the spring. The curvilinear coordinate s is related to this angle by the relation

$$ds = \rho\, d\theta \qquad \rho = \sqrt{r^2 + \left(\frac{P}{2\pi}\right)^2}. \tag{4–3}$$

Vectors of the local coordinate system, which are related to the point p, are defined as the tangential $\vec{t}$, the normal $\vec{n}$ and the binormal $\vec{b}$. The corresponding expression of each vector is

$$\vec{t} = \frac{1}{\rho}\left\{ \begin{array}{c} -r\sin\theta \\ r\cos\theta \\ \dfrac{P}{2\pi} \end{array} \right\}_{(\vec{I},\vec{J},\vec{K})}, \qquad \vec{n} = \left\{ \begin{array}{c} -\cos\theta \\ -\sin\theta \\ 0 \end{array} \right\}_{(\vec{I},\vec{J},\vec{K})}, \qquad \vec{b} = \frac{1}{\rho}\left\{ \begin{array}{c} \dfrac{P}{2\pi}\sin\theta \\ -\dfrac{P}{2\pi}\cos\theta \\ r \end{array} \right\}_{(\vec{I},\vec{J},\vec{K})},$$

where the radius of curvature R of this helical beam defined as $R = \frac{\rho^2}{r}$ and the radius of torsion T is $T = \rho^2 \frac{2\pi}{P}$. Figure 1 shows different parameters presented thus far.

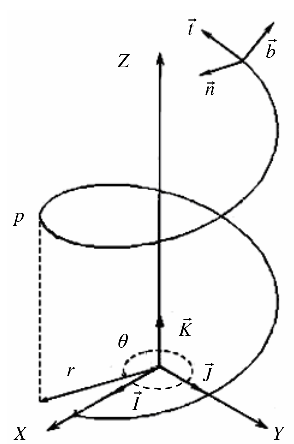

Figure 1. Geometric description of the helical spring and its two bases.

4.2. ***Kinematic presentation.*** To determine the displacement's field, two hypotheses are used. First, Timoshenko's hypothesis that the shear effect is not neglected and the strain of the helical beam takes place such that cross section remains planar. Second, the St. Venant hypothesis that the torsion is uniform along the beam and all sections undergo the same warping. Then the axial stress resultants and distortions due to the moment of torsion are null. The position vector of a point q, different from p in the section of the wire, is defined in local coordinate system as

$$\vec{X}_q = \vec{X}_p + \vec{h}, \qquad \vec{h} = \begin{Bmatrix} 0 \\ y \\ z \end{Bmatrix}_{(\vec{t},\vec{n},\vec{b})} . \tag{4–4}$$

For the point p, the vector of the virtual displacements in the local coordinate system is expressed as

$$\delta\vec{u}_p = \begin{Bmatrix} \delta u \\ \delta v \\ \delta w \end{Bmatrix}_{(\vec{t},\vec{n},\vec{b})} , \tag{4–5}$$

where δu, δv, and δw are the displacements parallel to the $\vec{t}$, $\vec{n}$ and $\vec{b}$ axes. The vector of the virtual displacements of the point q is given by the expression

$$\delta\vec{u}_q = \delta\vec{u}_p + \begin{Bmatrix} \delta\theta_t \\ \delta\theta_n \\ \delta\theta_b \end{Bmatrix}_{(\vec{t},\vec{n},\vec{b})} \wedge \vec{h}, \tag{4–6}$$

where $\delta\theta_t$, $\delta\theta_n$, and $\delta\theta_b$ are respectively the rotations of the point p around the $\vec{t}$, $\vec{n}$ and $\vec{b}$ axes. The expression of the vector of virtual displacements of the point q in the local coordinate system $(\vec{t}, \vec{n}, \vec{b})$ is expressed as

$$\delta\vec{u}_q = \begin{Bmatrix} \delta u + z\delta\theta_n - y\delta\theta_b \\ \delta v - z\delta\theta_t \\ \delta w + y\delta\theta_t \end{Bmatrix}_{(\vec{t},\vec{n},\vec{b})} . \tag{4–7}$$

The virtual displacement field in Equation (4–6) generates three components of virtual strain in any point of the beam, written as

$$\begin{aligned} \delta\varepsilon_t &= \delta\varepsilon_{tt} = \Big(1 - \frac{y}{R}\Big)^{-1}(\delta e_t - y\delta\chi_b + z\delta\chi_n), \\ \delta\gamma_n &= 2\delta\varepsilon_{nt} = \Big(1 - \frac{y}{R}\Big)^{-1}(\delta\gamma_{tn} - z\delta\chi_t), \\ \delta\gamma_b &= 2\delta\varepsilon_{bt} = \Big(1 - \frac{y}{R}\Big)^{-1}(\delta\gamma_{tb} + y\delta\chi_t), \end{aligned} \tag{4–8}$$

where $\delta\varepsilon_t$ is the axial virtual strain and $\delta\gamma_n$ and $\delta\gamma_b$ are the virtual transverse shear strains.

$$\delta e_t = \frac{d(\delta u)}{ds} - \frac{\delta v}{R}, \qquad \delta\gamma_{tn} = \frac{d(\delta v)}{ds} + \frac{\delta u}{R} - \frac{\delta w}{T} - \delta\theta_b, \qquad \delta\gamma_{tb} = \frac{d(\delta w)}{ds} - \frac{\delta v}{T} + \delta\theta_n,$$

$$\delta\chi_t = \frac{d(\delta\theta_t)}{ds} - \frac{\delta\theta_n}{R}, \qquad \delta\chi_n = \frac{d(\delta\theta_n)}{ds} + \frac{\delta\theta_t}{R} - \frac{\delta\theta_b}{T}, \qquad \delta\chi_b = \frac{d(\theta_b)}{ds} - \frac{\delta\theta_n}{R}.$$

δe_t is the virtual membrane strain, $\delta\gamma_{tn}$ and $\delta\gamma_{tb}$ are the virtual shear strains, $\delta\chi_t$ is the virtual strain due to the effect of torsion, and $\delta\chi_n$ and $\delta\chi_b$ are respectively the virtual curvatures in the planes (s, z) and (y, s). These strains constitute the components of the generalized virtual strains vector

$$\langle\delta\varepsilon\rangle = \langle\delta e_t \quad \delta\gamma_{tn} \quad \delta\gamma_{tb} \quad \delta\chi_t \quad \delta\chi_n \quad \delta\chi_b\rangle. \tag{4–9}$$

The system of equilibrium equations for the helical beam is written [Batoz and Dhatt 1993] in the local coordinate system as

$$\begin{cases} \frac{dN}{d\theta} - aT_n = 0, \\ \frac{dT_n}{d\theta} + aN + bT_b = 0, \\ \frac{dT_b}{d\theta} - bT_n = 0, \end{cases} \qquad \begin{cases} \frac{dM_t}{d\theta} - aM_n = 0, \\ \frac{dM_n}{d\theta} + aM_t + bM_{bv} - \rho T_b = 0, \\ \frac{dM_b}{d\theta} - bM_n + \rho T_n = 0, \end{cases} \tag{4–10}$$

where $a = \frac{r}{\rho}$ and $b = -\frac{P}{2\pi\rho}$. N is the normal force, T_n and T_b the normal and binormal shearing forces, M_t the torsional moment, and M_n and M_b the bending moments around the normal and binormal axis. These resulting forces are defined as

$$N = \int_A \sigma_t \, dA, \qquad T_n = \int_A \sigma_{tn} \, dA, \qquad T_b = \int_A \sigma_{tb} \, dA$$

$$M_n = \int_A z\sigma_t \, dA, \qquad M_b = \int_A -y\sigma_t \, dA, \qquad M_t = \int_A (y\sigma_{tb} - z\sigma_{tn}) \, dA,$$

with $dA = dydz$. They constitute the components of the vector of generalized forces which correspond to the resulting forces

$$\langle R\rangle = \langle\, N \;\; T_n \;\; T_b \;\; M_t \;\; M_n \;\; M_b \,\rangle. \tag{4–11}$$

4.3. ***Constitutive relation.*** The spring has an isotropic material behavior, in which stresses are linearly related to the strains by the constitutive relations

$$\sigma_t = E\varepsilon_t, \qquad \sigma_{tn} = G\gamma_n, \qquad \sigma_{tb} = G\gamma_b, \tag{4–12}$$

where E is the Young's modulus and G is the shear modulus of the material. The principal axes of inertia are assumed to be coincident with the local coordinate system. So, the resulting force vector $\{R\}$ is written $\{R\} = [H]\{\varepsilon\}$, with

$$\langle\varepsilon\rangle = \langle e_t \quad \gamma_{tn} \quad \gamma_{tb} \quad \chi_t \quad \chi_n \quad \chi_b\rangle, \qquad [H] = \operatorname{diag}(H_m, H_{cn}, H_{cb}, H_t, H_{fn}, H_{fb}),$$

where $[H]$ is the matrix of the elastic comportment. Its components are given by

$$H_m \simeq EA, \qquad H_t \simeq GJ, \qquad H_{fn} \simeq EI_y,$$
$$H_{fb} \simeq EI_z, \qquad H_{cn} \simeq k_y GA, \qquad H_{cb} \simeq k_z GA, \tag{4–13}$$

where A is the area of the section, J the inertia of torsion, I_y and I_z the central quadratic moments regarding (p, y) and (p, z) axes, and k_y and k_z are the shear correction factors for y and z axis.

4.4. ***The mixed formulation.*** The mixed functional of energy associated to the equilibrium equations is expressed by Batoz and Dhatt [1993] as

$$\Pi = \sum_e (\Pi^e_{\text{int}} - \Pi^e_{ext}), \tag{4–14}$$

where

$$\Pi^e_{\text{int}} = \int_0^L (-\frac{1}{2}\langle R\rangle [H]^{-1}\{R\} + \langle\varepsilon\rangle\{R\})ds), \qquad \Pi^e_{ext} = \int_0^L \langle u\rangle\{f\}\,ds + (\langle u\rangle\{F\})_S. \tag{4–15}$$

Π^e_{int} is the element functional energy of internal forces, and Π^e_{ext} is the same for the external forces. e is the number of elements, $\langle u\rangle$ is the vector of the displacements in the local coordinate system, $\{f\}$ the vector of distributed forces, $\{F\}$ is the vector of concentrated forces, and S is the boundary of the wire.

4.5. ***Element stiffness matrix in the local coordinate system.*** The development of the element stiffness matrix in the local coordinate system was presented in [Taktak et al. 2005b]. In the case where an approximation of the generalized forces $\langle R\rangle$ verifies the equilibrium equations, the expression (4–15) becomes

$$\Pi^e_{\text{int}} = \int_0^L -\frac{1}{2}\langle R\rangle [H]^{-1}\{R\}\,ds + \langle u_n\rangle\{R_n\}, \tag{4–16}$$

where $\langle u_n\rangle$ and $\langle R_n\rangle$ represent respectively the vector of the degrees of freedom for the element and the vector of the nodal resulting forces

$$\begin{aligned} \langle u_n\rangle &= \langle u_1\ \ v_1\ \ w_1\ \ \theta_{t1}\ \ \theta_{n1}\ \ \theta_{b1}\ \ u_2\ \ v_2\ \ w_2\ \ \theta_{t2}\ \ \theta_{n2}\ \ \theta_{b2}\rangle, \\ \langle R_n\rangle &= \left\langle -N_1\ \ -T_{n1}\ \ -T_{b1}\ \ -M_{t1}\ \ -M_{n1}\ \ -M_{b1}\ \ N_2\ \ T_{n2}\ \ T_{b2}\ \ M_{t2}\ \ M_{n2}\ \ M_{b2}\right\rangle. \end{aligned} \tag{4–17}$$

The expression of internal virtual work is then written

$$W^e_{\text{int}} = \int_0^L - < \delta R_n > [H]^{-1}\{R_n\}\,ds + < \delta u_n > \{R_n\} + < \delta R_n > \{u_n\}. \tag{4–18}$$

The element stiffness matrix in the local coordinate system is determined using a mixed formulation where the equilibrium equations are enforced in the variational (4–18). The resolution of the equilibrium equations (4–10) permits choice for the resulting forces vector the approximation $\{R\} = [P]\{\alpha_n\}$, where $[P]$ is the approximation matrix of the resulting forces and $\{\alpha_n\}$ is the vector of the independent parameters, defined as

$$\{\alpha_n\}^T = \langle \alpha_1 \quad \alpha_2 \quad \alpha_3 \quad \alpha_4 \quad \alpha_5 \quad \alpha_6\rangle. \tag{4–19}$$

The matrix $[P]$ is obtained by resolving the equilibrium equations while expressing the resulting forces in any point p of the beam, according to the forces exerted on one of extremities expressed by $\{\alpha_n\}$. This matrix is given by

$$[P] = \begin{bmatrix} P_{1,1} & P_{1,2} & P_{1,3} & 0 & 0 & 0 \\ -P_{1,2} & P_{2,2} & P_{2,3} & 0 & 0 & 0 \\ -P_{1,3} & P_{2,3} & P_{3,3} & 0 & 0 & 0 \\ P_{4,1} & P_{4,2} & P_{4,3} & -P_{1,1} & -P_{1,2} & -P_{1,3} \\ -P_{4,2} & P_{5,2} & P_{5,3} & P_{1,2} & -P_{2,2} & -P_{2,3} \\ -P_{4,3} & P_{5,3} & P_{6,3} & P_{1,3} & -P_{3,2} & -P_{3,3} \end{bmatrix}, \tag{4–20}$$

$$\begin{aligned}
&P_{1,1} = a^2C + b, && P_{1,2} = aS,\\
&P_{1,3} = ab(1-C), && P_{2,2} = C,\\
&P_{2,3} = bS, && P_{3,3} = -b^2C - a^2,\\
&P_{4,1} = -2\rho a^2 b(1-C) + a^2 b\rho\theta S, &&\\
&P_{4,2} = -ab(\rho\theta C - \rho S), &&\\
&P_{4,3} = -ab^2\rho\theta S - r(a^2+b^2)(1-C), &&\\
&P_{5,2} = b\rho\theta S, &&\\
&P_{5,3} = -b^2\rho\theta C - raS, &&\\
&P_{6,3} = -b^3\rho\theta S - 2\rho b a^2(1-C), &&
\end{aligned} \tag{4–21}$$

where $C = \cos\,\theta$ and $S = \sin\,\theta$. We define the matrix $[A]$ joining the nodal resulting forces to the independent parameters α_n $\langle R_n\rangle = \langle\alpha_n\rangle\,[A]$, expressed by $[A] = [-\,[P_1]^T\;[P_2]^T]$, where $[P_1]$ and $[P_2]$ are the approximation matrices of the resulting forces at nodes 1 and 2 of the element. We also define the matrix $[B]$ as

$$[B] = \int_0^L [P]^T\,[H]^{-1}\,[P]\,ds. \tag{4–22}$$

We express Equation (4–18) as

$$W^e_{\text{int}} = -< \delta\alpha_n > [B]\,\{\alpha_n\} + < \delta u_n > [A]^T\left\{\alpha_n\right\} + < \delta\alpha_n > [A]\,\{u_n\} \tag{4–23}$$

This leads to

$$[A]\,\{u_n\} - [B]\,\{\alpha_n\} = 0, \tag{4–24}$$

which permits expression of the independent variables as

$$\{\alpha_n\} = [B]^{-1}\,[A]\,\{u_n\} \tag{4–25}$$

and the virtual internal work's expression becomes

$$W^e_{\text{int}} =< \delta u_n > [k]\,\{u_n\}\,, \tag{4–26}$$

with the element stiffness matrix $[k]$ defined in the local coordinate system by

$$[k] = [A]^T\,[B]^{-1}\,[A]\,. \tag{4–27}$$

4.6. ***The element mass matrix in the local coordinate system.*** The term of element inertial work is written as

$$W^e_{\text{inertial}} = \int_V \rho_0 < \delta u_q > \left\{\ddot{u}_q\right\} dV, \tag{4–28}$$

where ρ_0 is the mass per unit volume of the spring's material, $\langle\delta u_q\rangle$ is the vector of virtual displacement in a point q of the section, $\{\ddot{u}_q\}$ is the vector of acceleration at this point, and V is the volume of the corresponding portion of the spring modelled by the element.

By taking account of the field of displacement (4–7), the expression (4–28) becomes

$$W^e_{inertial} = \int_V \begin{bmatrix} (\delta u + z\delta\theta_n - y\delta\theta_b)\ \rho_0\ (\ddot{u} + z\ddot{\theta}_n - y\ddot{\theta}_b) \\ +(\delta v - z\delta\theta_t)\ \rho_0\ (\ddot{v} - z\ddot{\theta}_t) \\ +(\delta w + y\delta\theta_t)\ \rho_0\ (\ddot{w} + y\ddot{\theta}_t) \end{bmatrix} dV. \tag{4–29}$$

We define the homogenized inertias as

$$\begin{cases} \rho_m = \int_A \rho_0(1-\frac{y}{R})dA, \\ \rho_1 = \int_A \rho_0 z(1-\frac{y}{R})dA, \\ \rho_2 = \int_A \rho_0 y(1-\frac{y}{R})dA, \end{cases} \qquad \begin{cases} \rho_3 = \int_A \rho_0 z^2(1-\frac{y}{R})dA, \\ \rho_4 = \int_A \rho_0 y^2(1-\frac{y}{R})dA, \\ \rho_5 = \int_A \rho_0 yz(1-\frac{y}{R})dA, \end{cases} \tag{4–30}$$

and the expression of W^e_{inertial} becomes

$$\begin{aligned} W^e_{\text{inertial}} = \int_s (\delta u(\rho_m\ddot{u} + \rho_1\ddot{\theta}_n - \rho_2\ddot{\theta}_b) + \delta v(\rho_m\ddot{v} - \rho_1\ddot{\theta}_t) + \delta w(\rho_m\ddot{w} + \rho_2\ddot{\theta}_t) \\ + \delta\theta_t(-\rho_1\ddot{v} + \rho_3\ddot{\theta}_t + \rho_2\ddot{w} + \rho_4\ddot{\theta}_t) + \delta\theta_n(\rho_1\ddot{u} + \rho_3\ddot{\theta}_n - \rho_5\ddot{\theta}_b) \\ + \delta\theta_b(-\rho_2\ddot{u} - \rho_5\ddot{\theta}_n + \rho_4\ddot{\theta}_b))\ ds. \end{aligned} \tag{4–31}$$

The element mass matrix in the local coordinate system is obtained by discretization of the expression (4–31). We choose a linear interpolation for virtual displacements ($\delta u,\ \delta v,\ \delta w,\ \delta\theta_t,\ \delta\theta_n\ and\ \delta\theta_b$) and accelerations($\ddot{u},\ \ddot{v},\ \ddot{w},\ \ddot{\theta}_t,\ \ddot{\theta}_n\ and\ \ddot{\theta}_b$) and we follow the geometry of the spring during the integration. The expression (4–28) in the discretized form is written as

$$\begin{aligned} & W^e_{\text{inertial}} = <\delta u_n> [m] \{\ddot{u}_n\} \\ & <\delta u_n> = <\delta u_1\ \delta v_1\ \delta w_1\ \delta\theta_{t1}\ \delta\theta_{n1}\ \delta\theta_{b1}\ \delta u_2\ \delta v_2\ \delta w_2\ \delta\theta_{t2}\ \delta\theta_{n2}\ \delta\theta_{b2}> \\ & <\ddot{u}_n> = <\ddot{u}_1\ \ddot{v}_1\ \ddot{w}_1\ \ddot{\theta}_{t1}\ \ddot{\theta}_{n1}\ \ddot{\theta}_{b1}\ \ddot{u}_2\ \ddot{v}_2\ \ddot{w}_2\ \ddot{\theta}_{t2}\ \ddot{\theta}_{n2}\ \ddot{\theta}_{b2}> \end{aligned} \tag{4–32}$$

where $[m]$ is the element mass matrix in the local coordinate system.

The choice of the linear interpolation for the nodal variables is expressed as

$$\begin{cases} \delta u = \delta u_1 N_1 + \delta u_2 N_2, \\ \delta v = \delta v_1 N_1 + \delta v_2 N_2, \\ \delta w = \delta w_1 N_1 + \delta w_2 N_2, \\ \delta\theta_t = \delta\theta_{t1} N_1 + \delta\theta_{t2} N_2, \\ \delta\theta_n = \delta\theta_{n1} N_1 + \delta\theta_{n2} N_2, \\ \delta\theta_b = \delta\theta_{b1} N_1 + \delta\theta_{b2} N_2, \end{cases} \qquad \begin{cases} \ddot{u} = \ddot{u}_1 N_1 + \ddot{u}_2 N_2, \\ \ddot{v} = \ddot{v}_1 N_1 + \ddot{v}_2 N_2, \\ \ddot{w} = \ddot{w}_1 N_1 + \ddot{w}_2 N_2, \\ \ddot{\theta}_t = \ddot{\theta}_{t1} N_1 + \ddot{\theta}_{t2} N_2, \\ \ddot{\theta}_n = \ddot{\theta}_{n1} N_1 + \ddot{\theta}_{n2} N_2, \\ \ddot{\theta}_b = \ddot{\theta}_{b1} N_1 + \ddot{\theta}_{b2} N_2, \end{cases} \tag{4–33}$$

where $N_1 = \frac{1-\xi}{2}$ and $N_2 = \frac{1+\xi}{2}$ are the interpolation functions.

The transformation from the curvilinear variable s to the parametric variable ξ is done by analogy between the reference element and the real element. We suppose that ξ follows a linear law according to θ, as in

$$\xi = \frac{2}{\Delta\theta}\theta - 1, \tag{4–34}$$

where $\Delta\theta$ is the difference between the corresponding angles of each node, defined as

$$\Delta\theta = \theta_2 - \theta_1. \tag{4–35}$$

The expression of element inertial work becomes

$$W^e_{\text{inertial}} = \int_0^L < \delta u_n > [N]\{\ddot{u}_n\}\, ds = \frac{\rho\Delta\theta}{2}\int_{-1}^{1} < \delta u_n > [N]\{\ddot{u}_n\}\, d\xi, \tag{4–36}$$

where L is the curvilinear length of the corresponding portion of the spring modelled by the element.

$$[N] = \begin{bmatrix} [N_{11}] & [N_{12}] \\ [N_{12}] & [N_{22}] \end{bmatrix}, \tag{4–37}$$

$$[N_{11}] = \begin{bmatrix} N_1^2\rho_m & 0 & 0 & 0 & N_1^2\rho_1 & -N_1^2\rho_2 \\ & N_1^2\rho_m & 0 & -N_1^2\rho_1 & 0 & 0 \\ & & N_1^2\rho_m & N_1^2\rho_2 & 0 & 0 \\ & & & N_1^2(\rho_3+\rho_4) & 0 & 0 \\ & & & & N_1^2\rho_3 & -N_1^2\rho_5 \\ Sym & & & & & N_1^2\rho_4 \end{bmatrix}, \tag{4–38}$$

$$[N_{22}] = \begin{bmatrix} N_2^2\rho_m & 0 & 0 & 0 & N_2^2\rho_1 & -N_2^2\rho_2 \\ & N_2^2\rho_m & 0 & -N_2^2\rho_1 & 0 & 0 \\ & & N_2^2\rho_m & N_2^2\rho_2 & 0 & 0 \\ & & & N_2^2(\rho_3+\rho_4) & 0 & 0 \\ & & & & N_2^2\rho_3 & -N_2^2\rho_5 \\ Sym & & & & & N_2^2\rho_4 \end{bmatrix}, \tag{4–39}$$

$$[N_{21}] = [N_{12}] = \begin{bmatrix} N_1N_2\rho_m & 0 & 0 & 0 & N_1N_2\rho_1 & -N_1N_2\rho_2 \\ & N_1N_2\rho_m & 0 & -N_1N_2\rho_1 & 0 & 0 \\ & & N_1N_2\rho_m & N_1N_2\rho_2 & 0 & 0 \\ & & & N_1N_2(\rho_3+\rho_4) & 0 & 0 \\ & & & & N_1N_2\rho_3 & -N_1N_2\rho_5 \\ Sym & & & & & N_1N_2\rho_4 \end{bmatrix}, \tag{4–40}$$

The analogy between expressions (4–32) and (4–36) defines the element mass matrix in the local coordinate system as

$$[m] = \frac{\rho\Delta\theta}{2}\int_{-1}^{1} [N]\, d\xi. \tag{4–41}$$

Then the element stiffness matrix $[K_G]$ and mass matrix $[M_G]$ in the global coordinate system are defined by

$$[K_G] = [T]^T\,[k]\,[T] \tag{4–42}$$

go to line

$$[M_G] = [T]^T\,[m]\,[T], \tag{4–43}$$

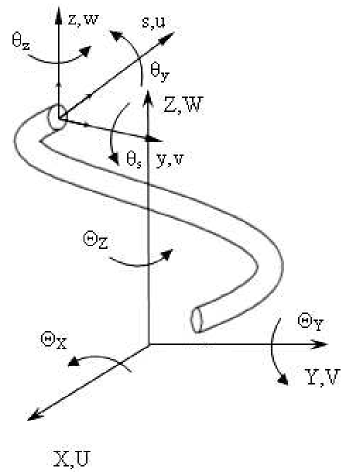

Figure 2. Local and global coordinate systems.

where $[T]$ is the transfer matrix from the local coordinate system to the global coordinate system, presented in Figure 2 and defined as

$$[T]^T = [T]^{-1} = \begin{bmatrix} [Q_1] & 0 & 0 & 0 \\ 0 & [Q_1] & 0 & 0 \\ 0 & 0 & [Q_2] & 0 \\ 0 & 0 & 0 & [Q_2] \end{bmatrix}. \tag{4–44}$$

$[Q_1]$ and $[Q_2]$ are the rotation matrixes defined by the leading cosines of local axes x, y and z in each of the elements

$$[Q_i] = \begin{bmatrix} -\frac{r}{\rho}\sin(\theta_i) & -\cos(\theta_i) & \frac{P}{2\pi\rho}\sin(\theta_i) \\ \frac{r}{\rho}\cos(\theta_i) & -\sin(\theta_i) & -\frac{P}{2\pi\rho}\cos(\theta_i) \\ \frac{P}{2\pi\rho} & 0 & \frac{r}{\rho} \end{bmatrix}. \tag{4–45}$$

5. Numerical examples

5.1. ***Natural frequencies of clamped-free spring.*** To verify the developed model, the natural frequencies are determined for three types of clamped-free springs (S_1, S_2, and S_3) that differ in the nature of their cross-sections. The common proprieties of these springs are the follows: number of coils N_s = 10 coils; mean diameter of the spring D= 113 mm; pitch P = 26 mm; Young's modulus $E = 2.124810^{11}$ N/m^2; Poisson ratio ν = 0.28 and mass per unit volume of wire ρ_0 = 8000 Kg/m^3. The specific properties of each spring are:

spring S_1: Circular cross-section with a wire diameter $d = 15$ mm and a shear correction ratio $k = 0.886$ [Batoz and Dhatt 1993];

spring S_2: Square cross-section with a thickness $h = 15$ mm and a shear correction ratio $k = 0.833$ [Batoz and Dhatt 1993];

Frequency	Three-dimensional beam element			Presented element		
	e = 60	e = 240	e = 600	e = 2	e = 5	e = 10
1	8.059	8.071	9.987	9.306	9.386	9.391
2	9.461	9.297	10.001	9.465	9.598	9.611
3	17.043	20.007	21.118	21.949	21.712	21.656

Frequency	Three-dimensional beam element			Presented element		
	e = 60	e = 240	e = 600	e = 2	e = 5	e = 10
1	8.837	10.181	10.950	10.243	10.328	10.333
2	8.868	10.411	10.989	10.507	10.661	10.676
3	18.123	20.881	22.457	23.544	23.313	23.258

Frequency	Three-dimensional beam element			Presented element		
	e = 60	e = 240	e = 600	e = 2	e = 5	e = 10
1	11.199	12.903	13.877	12.748	12.885	12.897
2	11.249	12.961	13.939	12.998	13.100	13.107
3	22.407	25.8167	27.765	29.097	28.732	28.657

Table 1. Natural frequencies (Hz) of the spring S_1 (top), S_1 (middle), and S_3 (bottom).

spring S_3: Rectangular Cross-section with a thickness $e_1 = 15$ mm and width $e_2 = 20$ mm and a shear correction ratio $k = 0.833$ [Batoz and Dhatt 1993].

Natural frequencies are determined using two types of finite elements: The first is a three-dimensional elastic beam finite element [Cosmos 1990]. The second is the finite element developed in this paper.

Table 1 presents the first three natural frequencies, given by the two elements, for each spring and for a different number of elements used. In fact, according to Abid et al. [2005], these frequencies are dangerous natural frequencies of the structure because they present high vibration amplitudes which can cause failure of the structure. Natural frequencies presented in Table 1 correspond to the first three simple modes of the spring which are

mode 1: bending mode around Y axis;

mode 2: bending mode around X axis;

mode 3: compression mode.

These modes are presented in Figure 3. The similarity between the two results is clear for each spring and mode. These results confirm the efficiency of the finite element developed.

5.2. ***Dynamic response of a helical spring.*** In this study, we are interested in determining the dynamic response of the free extremity of a clamped-free spring subjected at its free end to a harmonic compressive excitation.

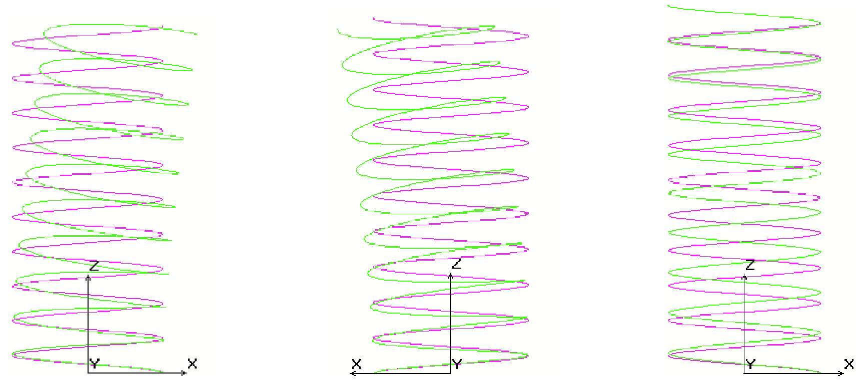

Figure 3. The three first spring's modes: Bending mode around Y axis (left), bending mode around X axis (middle) and compression mode (right). Pink represents the state case and green the vibrating case.

By varying the frequency of the excitation, the dynamic response of the free end of the spring is determined. The studied spring S_3 is subjected to a harmonic force with maximum amplitude $F_{\max} = 100$ N. We suppose that the material of the spring has a modal damping coefficient $\xi = 0.01$.

Results given by the present model are compared with those given by the three-dimensional elastic beam finite element [Cosmos 1990]. Figure 4 show this comparison respectively for the resultant displacement and rotation:

$$RD = \sqrt{(U^2 + V^2 + W^2)}, \qquad RR = \sqrt{(\Theta_X^2 + \Theta_Y^2 + \Theta_Z^2)}$$

of the top end of the spring expressed in the global coordinate system. The chosen frequency band is between 0 and 100 Hz.

Dynamic responses given by the two methods are close to each other. The developed finite element gives the right natural frequencies with a number of elements less than used in the first model (only 10 of the present elements for 600 of the three-dimensional elastic finite element). This means less calculation and programming needed for realistic results.

These figures indicate that the dangerous zone is especially located in the frequency band containing the three first natural frequencies. The difference between the vibration amplitude given by the two finite elements is due to the linear form of the developed finite element, but doesn't reduce the efficiency of the element. The helical spring is sensitive to its three first modes (two modes of bending and one of compression): If the spring is excited with these frequencies, resonance is produced and vibrations will have great amplitude that may damage the structure. At the other natural frequencies, the resonance

Mode	1	2	3	4	5	6	7	8
Model 1 ($e = 600$)	6.01424	6.61148	8.8772	9.41936	12.9729	14.0427	30.4497	31.1247
Model 2 ($e = 10$)	5.8720	6.1350	9.0642	9.6332	13.997	14.729	28.094	29.209

Table 2. The first eight natural frequencies of the studied system (Hz).

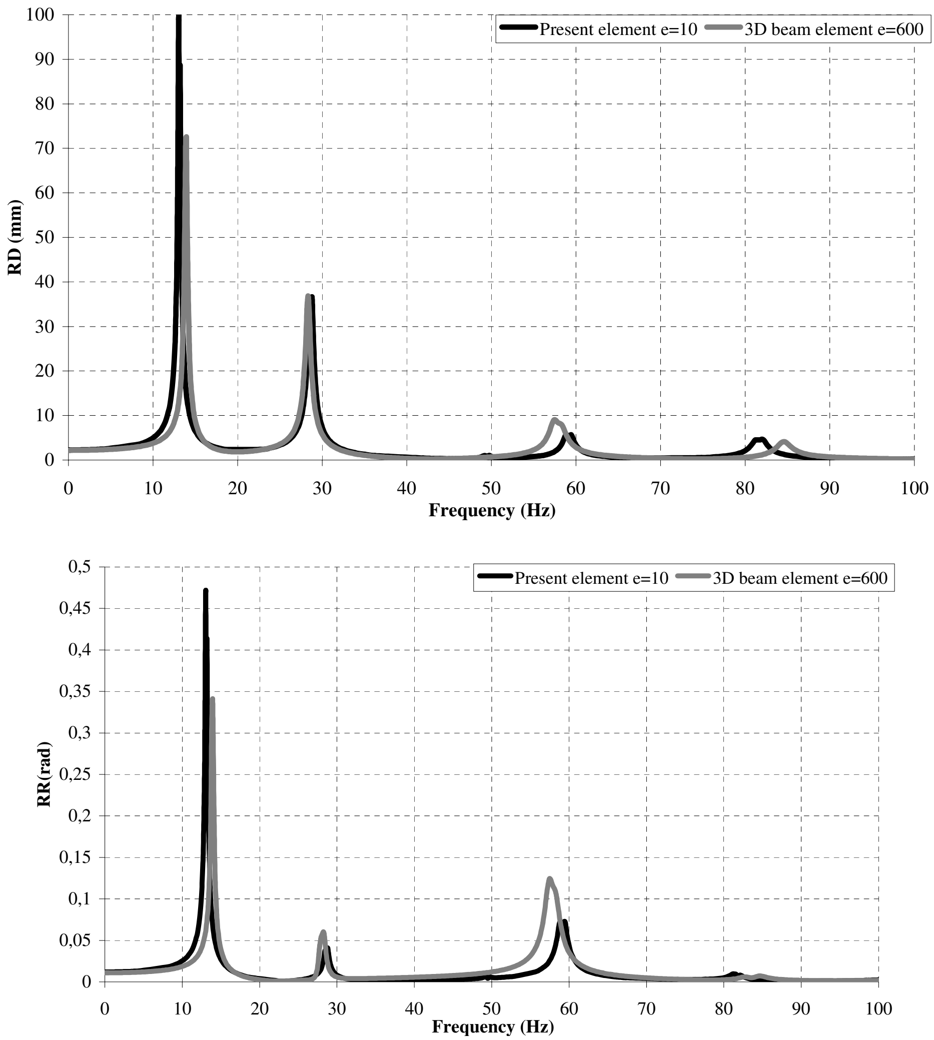

Figure 4. Resultant displacement of the free end of the studied spring (top) and resultant rotation of the free end of the studied spring (bottom).

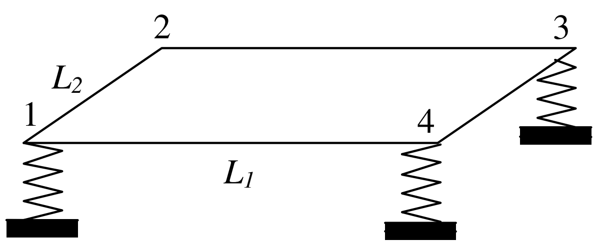

Figure 5. Studied system.

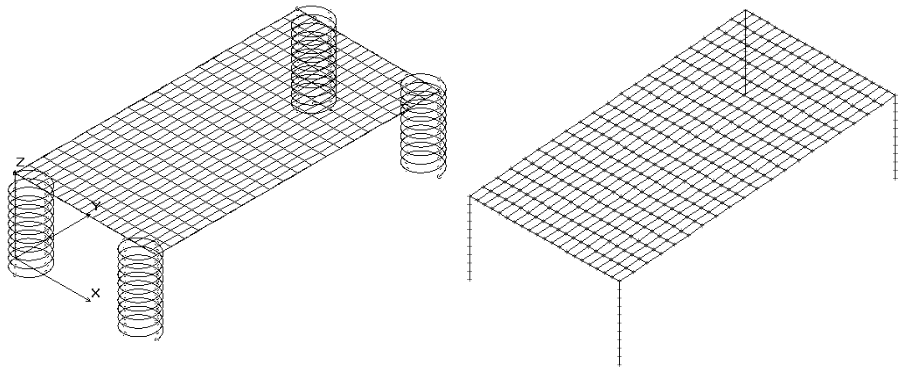

Figure 6. Meshing of the studied system. three-dimensional beam elements spring modeling (left); Present element modeling (right).

phenomena also manifests, but with smaller amplitudes. Thus, it is important to avoid the three first natural frequencies in practical applications of the spring and develop technological solutions to pass this frequency band without structural damage.

5.3. ***Vibrating plate.*** We next apply the developed and validated model to the practical example of a vibrating plate. This system consists of a thin plate supported at each extremity by a helical spring, as shown in Figure 5.

The thin plate has the characteristics:

length:	$L_1 = 1m$,	width:	L_2 = 0.5m,
thickness:	$c = 7mm$,	Young's modulus:	$E = 2.110^{11} N/m^2$,
Poisson ratio:	$\upsilon = 0.28$,	mass per unit of volume	$\rho_0 = 8000 Kg/m^3$.

With this plate we use four springs of type S_1.

The modeling of this plate is done by four-node quadrilateral thin shell elements with six degrees of freedom per node [Cosmos 1990]. The springs are modeled by the same two elements presented in Section 5.2 (proposed element + three-dimensional element). Figure 6 presents the meshing of the system in each method. The first eight natural frequencies of this system, given by the two modeling methods, are presented in Table 2.

Results given by the two methods are similar. This demonstrates the efficiency of the developed element in assembly with other types of structures. The element is not only capable of accurately the simple spring, but also the spring in practical applications.

6. Conclusion

In the present study, we develop a finite element for the dynamic analysis of a helical spring. The mixed-hybrid formulation is established from geometric and cinematic hypothesis and takes into account the effect of shear strain to calculate the stiffness matrix. A simple approach is used for calculating the mass matrix. Comparison with an other types of finite elements shows the efficiency of the element to model simple and assembled springs. The study shows the sensitivity of the spring to its first three natural frequencies where the phenomena of resonance appears with large displacements and cause the damage to the structure.

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