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# A CRACKED BEAM FINITE ELEMENT FOR ROTATING SHAFT DYNAMICS AND STABILITY ANALYSIS

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In this paper, a method for the construction of a cracked beam finite element is presented. The additional flexibility due to the cracks is identified from three-dimensional finite element calculations taking into account the unilateral contact conditions between the crack lips. Based on this flexibility, which is distributed over the entire length of the element, a cracked beam finite element stiffness matrix is deduced. Considerable gain in computing efforts is reached compared to the nodal representation of the cracked section when dealing with the numerical integration of differential equations in structural dynamics. The stability analysis of a cracked shaft is carried out using the Floquet theory.

## 1. Introduction

Because of the increasing need for energy, the plants installed by electricity supply utilities throughout the world are becoming larger and more highly stressed. Thus, the risk of turbogenerator shaft cracking is increasing also. Rotating shafts are omnipresent in aeronautics, aerospace, automobile industries, and in particular in the energy sector which is vital for any economic development. Fatigue cracks is an important form of rotor damage which can lead to catastrophic failures unless detected early. They can have detrimental effects on the reliability of rotating shafts. According to [Bently and Muszynska 1986], in the 1970s and till the beginning of the 1980s, at least 28 shaft failures due to cracks were recorded in the US energy industry. Thus, since the 1980s, the interest of researchers to characterize structures containing cracks has grown remarkably. Between the beginning of the 1970s and the end of the 1990s, more than 500 articles concerning the cracked structures were published [Dimarogonas 1996; Bachschmid and Pennacchi 2008].

The problem of determining the behavior of cracked structures has been worked on for a long time. The fact that a crack presence or a local defect in a structural member introduces a local flexibility that affects its vibration response was known long ago. This local flexibility is related to the strain energy concentration in the vicinity of the crack.

The study of cracked turbine rotors began in the American energy industry with the works of Dimarogonas [1970; 1971]. In Europe, the first works appeared only some years later in papers by Gasch [1976], Mayes and Davies [1976], and Henry and Okah-Avae [1976]. These authors considered a simple model to account for the crack breathing mechanism to which we often refer to as the *switching crack model* or the *hinge crack model* where the crack is totally opened or totally closed.

The vibration analysis of cracked beams or shafts is a problem of great interest due to its practical importance. In fact, vibration measurements offer a nondestructive, inexpensive, and fast means to detect and locate cracks. Thus, vibration behavior analysis and monitoring of cracked rotors has received

*Keywords:* breathing crack, beam, unilateral contact, finite element, rotor, Floquet, stability.

considerable attention in the last three decades [Gudmundson 1982; Gudmundson 1983]. It has, perhaps, the greatest potential since it can be carried out without dismantling any part of the machine and usually done online thus avoiding the costly downtime of turbomachinery.

Zuo [1992] and Zuo and Curnier [1994] also used a bilinear model to characterize the vibrational response of a cracked shaft with the aim of developing an online cracks detection method. They first examined the one degree of freedom system and then studied the behavior of a system with two degrees of freedom. By extending the Rosenberg normal mode notion for smooth and symmetric nonlinear systems to bilinear systems, they defined the nonlinear modes of the bilinear system which were calculated numerically, and in certain simple cases, analytically. The application of this method to systems with high number of degrees of freedom is complicated and would lead to high computation costs.

Bachschmid and his coworkers [Bachschmid et al.1984; 2002; 2004a] examined the effects of the presence of a crack on the vibratory response of a rotor or a pump axis. Experimental and numerical models were proposed, the thermal effects on the crack breathing mechanism were taken into account [Bachschmid et al. 2004a]. It was reported that the temperature distribution is not influenced by the presence of the crack unlike those of the stresses and deformations.

A comparison of various cracked beams models was presented by Friswell and Penny [2002]. The authors showed that in the low frequencies domain, simple models of the crack breathing mechanism and beam type elements are adequate to monitor structures health. However, the approach of modeling based on a bilinear dynamic system, which still acts as a reference, remains a simplistic approach which leads to some reservations about the quality of the quantitative results stemming from its exploitation.

A good review on the most relevant analytical, experimental and numerical works conducted in the last three decades and related to the cracked structures behavior were reported in [Dimarogonas and Paipetis 1983; Entwistle and Stone 1990; Dimarogonas 1996; Wauer 1990; Gasch 1993; El Arem 2006].

Today, most of the works on the cracked shafts vibration analysis are concerned with more detailed investigations of certain particular points such as the phase of acceleration or deceleration of the shaft and the passage through the critical speed or the coupling between diverse modes of vibrations to highlight parameters facilitating the online cracks detection when dealing with machines monitoring. These works such as those of Darpe et al. [2004], those of Jun et al. [1992] and those of Sinou and Lees [2005; 2007] where we find numerical and experimental results of their investigations on certain aforementioned points remain faithful to the theoretical principles formalized in the 70s in some reference papers. In the other hand, remarkable and continuous progress of the computational tools allows the realization of successful three-dimensional models. So it becomes possible to envisage an identification of the constitutive equations of a cracked shaft section (breathing mechanism description) which eliminates certain simplifying hypotheses and approximations made until now; this without degrading the costs of the calculations of the vibrational response.

The main objective of this work is the presentation of a method of construction of a cracked beam finite element. The stiffness variation of the element is deduced from three-dimensional finite element computations accounting for the unilateral contact between the crack lips. Based on an energy approach, this method could be applied to cracks of any shape. The validation of the approach on a case of a cracked rotating shaft is then presented and its stability analysis is carried using Floquet theory. The work of El Arem and Maitournam [2007] is used to show the general form of the stiffness matrix of the finite element and then facilitate the approximation of its terms.

## 2. Cracked rotors modeling: State of the art

The analysis of rotating machinery shafts behavior is a complex structural problem. For a relevant description, it requires a fine and precise modeling of the rotor and cracks in order to allow the identification and calculation of the parameters characterizing their presence.

Researchers dealing with the problem of a rotating cracked beam recognize its two main features: the determination of the local flexibility of the beam cracked section, and the consideration of the crack breathing mechanism responsible of the system nonlinearity (the stiffness of the system is dependent on the cracked section's position).

Most researchers agree with the application of the linear fracture mechanics theory to evaluate the local flexibility introduced by the crack [Gross and Srawley 1965; Anifantis and Dimarogonas 1983; Dimarogonas and Paipetis 1983; Dimarogonas 1996; Papadopoulos and Dimarogonas 1987a; 1987b; 1987c; Papadopoulos 2004]. Obviously, the first work was done in the early 1970s by Dimarogonas [1970; 1971] and Pafelias [1974] at the General Electric Company. There has been different attempts to quantify local effects introduced by cracks. The analysis of the local flexibility of a cracked region of a structural element was quantified in the 1950s by Irwin [1957a; 1957b], Bueckner [1958] , and Westmann and Yang [1967] by relating it to the stress intensity factors (SIF). Afterwards, the efforts to calculate the SIF for different cracked structures with simple geometry and loading was duplicated [Tada et al. 1973; Bui 1978].

For an elastic structure, the additional displacement $\boldsymbol{u}$ due to the presence of a straight crack of depth $a$ under the generalized loading $\boldsymbol{P}$ is given by the Castigliano theorem

$$\boldsymbol{u} = \frac{\partial}{\partial \boldsymbol{P}} \int_0^a \boldsymbol{G}(a)\,da.$$

$\boldsymbol{G}$ is the energy release rate defined in fracture mechanics and related to the SIF by the Irwin formula [Irwin 1957a]. Then, the local flexibility matrix coefficients are obtained by

$$c_{ij} = \frac{\partial^2}{\partial P_i \partial P_j} \int_0^a \boldsymbol{G}(a)\,da, \quad 1 \le i, j \le 6.$$

Extra diagonal terms of this matrix are responsible for longitudinal and lateral vibrations coupling that can be of great interest when dealing with cracks detection.

In two technical notes from NASA, Gross and Srawley [1964; 1965] computed the local flexibility corresponding to tension and bending including their coupling terms. This coupling effect was observed by Rice and Levy [1972] in their study of cracked elastic plates for stress analysis.

Dimarogonas and his coworkers introduced the full $(6 \times 6)$ flexibility matrix of a cracked section [Dimarogonas 1982; 1987; 1988; Dimarogonas and Paipetis 1983; Anifantis and Dimarogonas 1983]. They noted the presence of extra diagonal terms which indicate the coupling between the longitudinal and lateral vibrations. Papadopoulos and Dimarogonas [1987a; 1987b; 1987c], and Ostachowicz and Krawwczuk [1992] computed all the $(6 \times 6)$ flexibility matrices of a Timoshenko beam cracked section for any loading case.

However, there are no results for the SIF for cracks on a cylindrical shaft. Thus, Dimarogonas and Paipetis [1983] have developed a procedure which is commonly used in FEM software: the shaft was considered as an assembly of elementary rectangular strips where an approximation of the SIF using fracture mechanics results remains possible. The SIF are obtained by integration of the energy release rate on the crack tip. Although it offers the advantage of being easily applicable in a numerical algorithm, this method remains an approximation whose convergence remains to be checked. In fact, some numerical problems were noted when the depth of the crack exceeds the section radius [Abraham et al. 1994; Dimarogonas 1994]. Moreover, generalization of this method to any geometry of cracked section is complex, even impossible in the case of nonconnate multiple cracks affecting the same transverse section.

An original method for deriving a lumped cracked section beam model was proposed by Varé and Andrieux [2000; 2002]. The procedure was designed by starting from three-dimensional computations and incorporating more realistic behavior on the cracks than the previous models, namely the unilateral contact conditions on the crack lips and the breathing mechanism of the cracks under variable loading. The approach was validated experimentally by Audebert and Voinis [2000] and applied for the study of real cracked structures especially turbines.

The cracked element of Figure 1 is submitted to an end moment $\boldsymbol{M}_{2L} = (M_x(2L), M_y(2L))$ at $z = 2L$. Andrieux [2000] has demonstrated certain properties of the problem elastic energy, $W^*$, leading to a considerable gain in three-dimensional calculus needed for the identification of the constitutive law. In particular, for a linear elastic material, under the assumption of small displacements and small deformations, and in the absence of friction on the crack lips, the energy function could be written by distinguishing the contribution of the cracked section from that of the uncracked element as

$$W^*(\boldsymbol{M}_{2L}) = W^*(\boldsymbol{M}) = W_s^*(\boldsymbol{M}) + w^*(\boldsymbol{M}) = \frac{L}{EI}\|\boldsymbol{M}\|^2(1+s(\Phi)), \tag{1}$$

where $\boldsymbol{M} = (M_x, M_y)$ is the resulting couple of flexural moments at the cracked section, $W_s^*(\boldsymbol{M})$ is the total elastic energy of the uncracked element submitted to the flexural moment $\boldsymbol{M}$, and $w^*(\boldsymbol{M})$ is the additional elastic energy due to the presence of the cracked section. The loading direction angle is defined by $\Phi = \arctan(M_y/M_x)$, $E$ is the Young modulus, and $I$ is the quadratic moment of inertia.

In this framework, the nonlinear constitutive law of the discrete element modeling of the cracked section is obtained by differentiating the function $w^*(\boldsymbol{M})$ with respect to $\boldsymbol{M}$. We obtain

$$[\boldsymbol{\theta}] = \begin{pmatrix} [\theta_x] \\ [\theta_y] \end{pmatrix} = \frac{2L}{EI}\begin{pmatrix} s(\Phi) & -\frac{1}{2}s'(\Phi) \\ \frac{1}{2}s'(\Phi) & s(\Phi) \end{pmatrix}\begin{pmatrix} M_x \\ M_y \end{pmatrix}, \quad \text{with } s'(\Phi) = \frac{ds(\Phi)}{d\Phi}.$$

However, for finite element computational codes in rotordynamics, the compliance function $s(\Phi)$ is of low interest and a nonlinear relation of the form $[\boldsymbol{\theta}] = f(\boldsymbol{M})$ is to be integrated in transient computations.

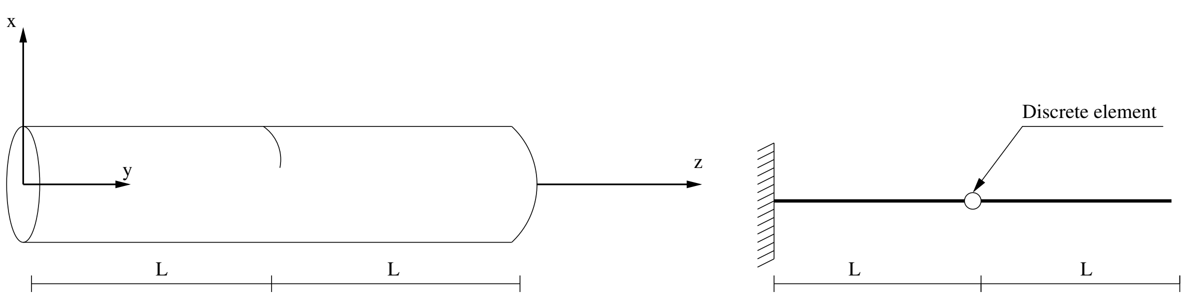


**Figure 1.** The cracked beam element modeling. Left: Three-dimensional model; right: beam model.

Andrieux and Varé [2002] introduced some properties of the additional strain energy due to the cracked section, $w([\boldsymbol{\theta}])$. Thus, $w$ could be written as a quadratic function of the rotations jumps as

$$w([\theta_x],[\theta_y]) = \frac{EI}{4L}k(\varphi)\|[\boldsymbol{\theta}]\|^2, \qquad \text{with } \varphi = \arctan\frac{[\theta_y]}{[\theta_x]}.$$

By using the Légendre–Fenchel transform to establish the relation between the two energy functions $w^*(\boldsymbol{M})$ and $w([\boldsymbol{\theta}])$, the stiffness function $k(\varphi)$ is obtained from the compliance function $s(\Phi)$ identified from three-dimensional calculus. As described in [Andrieux and Varé 2002], the behavior law is finally deduced from differentiation of $w([\boldsymbol{\theta}])$ with respect to $[\theta]$ as

$$\begin{pmatrix} M_x \\ M_y \end{pmatrix} = \frac{EI}{2L}\begin{pmatrix} k(\varphi) & -\frac{1}{2}k'(\varphi) \\ \frac{1}{2}k'(\varphi) & k(\varphi) \end{pmatrix}\begin{pmatrix} [\theta_x] \\ [\theta_y] \end{pmatrix}, \qquad \text{with } k'(\varphi) = \frac{dk(\varphi)}{d\varphi}.$$

## 3. A cracked beam finite element construction

There are two procedures to introduce the local flexibility generated by a cracked section when dealing with the numerical integration of differential equations in dynamics. The first technique considers the construction of a stiffness matrix exclusively for the cracked section by computing the inverse of the flexibility matrix. However, for small cracks, the additional flexibility is very small and consequently the corresponding stiffness coefficient is extremely large leading to high numerical integration costs and convergence problems [El Arem 2006].

The second procedure, adopted here, consists of constructing a cracked finite element stiffness matrix which is later assembled with the other uncracked elements of the system. Thus, the elastic energy due to the cracks is distributed over the entire length of the cracked element. This method has been used in [Bachschmid et al. 2004b] and [Saavedra and Cuitino 2001].

The studies of Verrier and El Arem [2003], El Arem et al. [2003], Varé and Andrieux [2005], and El Arem [2006] showed that the shear effects on the breathing mechanism of the cracks is insignificant and will be neglected in this study.

Consider the cracked finite element of length $2L_e$, circular transverse section of diameter $D$ and quadratic moment of inertia $I$; see Figure 2. First, we clamp all the displacements of node $A$ and establish a relation of the form $\boldsymbol{u} = \mathcal{S}(\boldsymbol{f})\cdot\boldsymbol{f}$, where $\boldsymbol{f} = \{T_{x_2}, T_{y_2}, M_{x_2}, M_{y_2}\}^t$ and $\boldsymbol{u} = \{u_{x_2}, u_{y_2}, \theta_{x_2}, \theta_{y_2}\}^t$ denote, respectively, the loading and displacements vectors at the end section ($z = 2L_e$), and $\mathcal{S}(\boldsymbol{f})$ represents

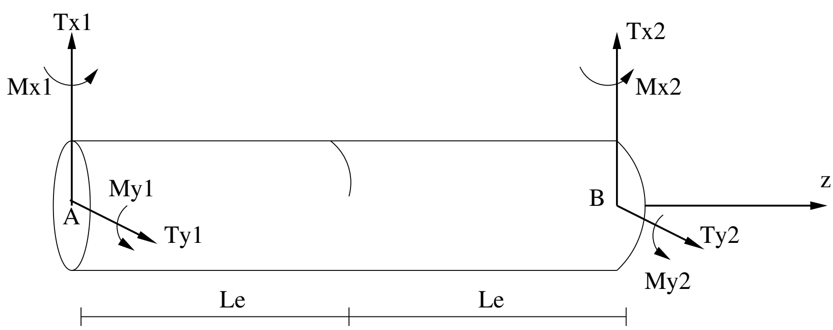


**Figure 2.** The cracked beam finite element.

the compliance matrix of the structure. At the cracked section ($z = L_e$), the internal forces are given by

$$T_x = T_{x_2}, \quad T_y = T_{y_2}, \quad M_x = M_{x_2} - L_e T_y, \quad M_y = M_{y_2} + L_e T_x. \tag{2}$$

The breathing mechanism of the cracks is governed by the flexural moment direction $\Phi = \arctan(M_y/M_x)$ at the cracked section. The elastic energy of the cracked element could be written as

$$W^*(\boldsymbol{f}) = W_s^*(\boldsymbol{f}) + w^*(\boldsymbol{M}) = W_s^*(\boldsymbol{f}) + \frac{L}{EI}\|\boldsymbol{M}\|^2 s(\Phi), \tag{3}$$

where $W_s^*(\boldsymbol{f})$ denotes the elastic energy of the uncracked finite element of the same geometry and submitted to the same loading conditions. By using Equation (1), the additional elastic energy due to the cracked section is given by

$$w^*(\boldsymbol{M}) = \frac{L}{EI}\|\boldsymbol{M}\|^2 s(\Phi), \tag{4}$$

where $L$ is the half length of the three-dimensional element used to identify the compliance function $s(\Phi)$ as described below. The nonlinear relation between the applied forces and the resulting displacements at the end section ($z = 2L_e$) are obtained by differentiating $W^*$ with respect to $\boldsymbol{f}$. Using (2), we get

$$\boldsymbol{u} = \mathcal{S}(\boldsymbol{f}) \cdot \boldsymbol{f} = \mathcal{S}(\Phi) \cdot \boldsymbol{f}, \tag{5}$$

where

$$\mathcal{S}(\Phi) = \mathcal{S}_{\mathbf{0}} + \frac{2L}{EI}\begin{pmatrix} L_e^2 s(\Phi) & -\frac{1}{2}L_e^2 s'(\Phi) & \frac{1}{2}L_e s'(\Phi) & L_e s(\Phi) \\ \frac{1}{2}L_e^2 s'(\Phi) & L_e^2 s(\Phi) & -L_e s(\Phi) & \frac{1}{2}L_e s'(\Phi) \\ -\frac{1}{2}L_e s'(\Phi) & -L_e s(\Phi) & s(\Phi) & -\frac{1}{2}s'(\Phi) \\ L_e s(\Phi) & -\frac{1}{2}L_e s'(\Phi) & \frac{1}{2}s'(\Phi) & s(\Phi) \end{pmatrix}.$$

Here $\mathcal{S}_{\mathbf{0}}$ denotes the compliance matrix of an uncracked beam element of length $2L_e$. Let us consider $\{u_{B/A}\}$ the relative displacement of node $B$ in relation to node $A$. It satisfies the relation

$$\{T_{x_2}, T_{y_2}, M_{x_2}, M_{y_2}\}^t = (\mathcal{S}(\Phi))^{-1}\{u_{B/A}\}. \tag{6}$$

From equilibrium conditions of the element of Figure 2, the internal forces in $B$ can be expressed in terms of those in $A$ as $T_{x_1} = -T_{x_2}$, $T_{y_1} = -T_{y_2}$, $M_{x_1} = -M_{x_2} + 2L_e T_{y_2}$, $M_{y_1} = -M_{y_2} - 2L_e T_{x_2}$, or, written in matrix form,

$$\{T_{x_1}, T_{y_1}, M_{x_1}, M_{y_1}, T_{x_2}, T_{y_2}, M_{x_2}, M_{y_2}\}^t = \mathbf{\Pi_1}\{T_{x_2}, T_{y_2}, M_{x_2}, M_{y_2}\}^t \tag{7}$$

with

$$\mathbf{\Pi_1} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 2L_e & -1 & 0 \\ -2L_e & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{8}$$

In addition, writing $\{u_{B/A}\}$ as $\{u_{B/A}\} = \{u^1_{B/A}, u^2_{B/A}, u^3_{B/A}, u^4_{B/A}\}$, we obtain $u_{x_2} = u^1_{B/A} + u_{x_1} + 2L_e\theta_{y_1}$, $u_{y_2} = u^2_{B/A} + u_{y_1} - 2L_e\theta_{x_1}$, $\theta_{x_2} = u^3_{B/A} + \theta_{x_1}$, $\theta_{y_2} = u^4_{B/A} + \theta_{y_1}$, or, in a matrix form

$$\{u_{B/A}\} = \boldsymbol{\Pi_2}\{u_{x_1}, u_{y_1}, \theta_{x_1}, \theta_{y_1}, u_{x_2}, u_{y_2}, \theta_{x_2}, \theta_{y_2}\}^t, \tag{9}$$

where

$$\boldsymbol{\Pi_2} = \begin{pmatrix} -1 & 0 & 0 & -2L_e & 1 & 0 & 0 & 0 \\ 0 & -1 & 2L_e & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{10}$$

Comparing with (8), we see that $\boldsymbol{\Pi_2} = \boldsymbol{\Pi_1^t}$. Moreover, the cracked beam finite element stiffness matrix $\boldsymbol{K_{ef}}$ satisfies

$$\{T_{x_1}, T_{y_1}, M_{x_1}, M_{y_1}, T_{x_2}, T_{y_2}, M_{x_2}, M_{y_2}\}^t = \boldsymbol{K_{ef}}\{u_{x_1}, u_{y_1}, \theta_{x_1}, \theta_{y_1}, u_{x_2}, u_{y_2}, \theta_{x_2}, \theta_{y_2}\}^t. \tag{11}$$

Substituting (7) into (11) gives

$$\boldsymbol{\Pi_1}\{T_{x_2}, T_{y_2}, M_{x_2}, M_{y_2}\}^t = \boldsymbol{K_{ef}}\{u_{x_1}, u_{y_1}, \theta_{x_1}, \theta_{y_1}, u_{x_2}, u_{y_2}, \theta_{x_2}, \theta_{y_2}\}^t. \tag{12}$$

Then, substituting (6) into (12) results in

$$\boldsymbol{\Pi_1}(\mathcal{S}(\Phi))^{-1}\{u_{B/A}\} = \boldsymbol{K_{ef}}\{u_{x_1}, u_{y_1}, \theta_{x_1}, \theta_{y_1}, u_{x_2}, u_{y_2}, \theta_{x_2}, \theta_{y_2}\}^t$$

Finally, using (9) leads to

$$\boldsymbol{K_{ef}} = \boldsymbol{\Pi_1}(\mathcal{S}(\Phi))^{-1}\boldsymbol{\Pi_1^t}. \tag{13}$$

In this relation, the stiffness matrix appears as depending on the applied moments represented by angle $\Phi$. However, in a finite element code, it is preferable to express relation (13) as a function of the problem's unknowns, that is, the nodal displacements. By writing

$$(\mathcal{S}(\Phi))^{-1} = \mathcal{K}(\varphi_e) = \mathcal{K}_{\boldsymbol{0}} - \boldsymbol{K_e}(\varphi_e) \tag{14}$$

and using (13), we distinguish the stifness matrix of an uncracked element, $\boldsymbol{\Pi_1}\mathcal{K}_{\boldsymbol{0}}\boldsymbol{\Pi_1^t}$, from the matrix modeling the cracked section, $\boldsymbol{\Pi_1}\boldsymbol{K_e}(\varphi_e)\boldsymbol{\Pi_1^t}$. Here $\varphi_e$ is the angle given by

$$\varphi_e = \arctan\Big(\frac{\theta_{y_2} - \theta_{y_1}}{\theta_{x_2} - \theta_{x_1}}\Big) \qquad \text{and} \qquad \mathcal{K}_{\boldsymbol{0}} = \mathcal{S}_{\boldsymbol{0}}^{-1} = \frac{EI}{2L_e(1+a)} \begin{pmatrix} \frac{3}{L_e^2} & 0 & 0 & -\frac{3}{L_e} \\ 0 & \frac{3}{L_e^2} & \frac{3}{L_e} & 0 \\ 0 & \frac{3}{L_e} & 4+a & 0 \\ -\frac{3}{L_e} & 0 & 0 & 4+a \end{pmatrix},$$

(see Lalanne and Ferraris [1990]) where $a = 12EI/(4\mu kSL_e^2)$ is the shearing effect coefficient. For an

Euler–Bernoulli beam element, $a$ is zero. Equation (14) leads to

$$\mathbf{K_e}(\varphi_e) = \mathcal{K}_\mathbf{0} - (\mathcal{S}(\Phi))^{-1} = \frac{EI}{2L}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & k_{xx}(\varphi_e) & k_{xy}(\varphi_e) \\ 0 & 0 & k_{yx}(\varphi_e) & k_{yy}(\varphi_e) \end{pmatrix}, \tag{15}$$

where

$$k_{xx}(\varphi_e) = k_{yy}(\varphi_e) = \frac{L^2(4L_e s(\Phi) + 4Ls^2(\Phi) + Ls'^2(\Phi))}{L_e(4L_e^2 + 8LL_e s(\Phi) + 4L^2 s^2(\Phi) + L^2 s'^2(\Phi))},$$

$$k_{xy}(\varphi_e) = -k_{yx}(\varphi_e) = -\frac{2L^2 s'(\Phi)}{4L_e^2 + 8LL_e s(\Phi) + 4L^2 s^2(\Phi) + L^2 s'^2(\Phi)}.$$

When $L = L_e$, we obtain

$$k_{xx}(\varphi_e) = k_{yy}(\varphi_e) = \frac{4s(\Phi) + 4s^2(\Phi) + s'^2(\Phi)}{4 + 8s(\Phi) + 4s^2(\Phi) + s'^2(\Phi)}, \quad k_{xy}(\varphi_e) = -k_{yx}(\varphi_e) = -\frac{2s'(\Phi)}{4 + 8s(\Phi) + 4s^2(\Phi) + s'^2(\Phi)}.$$

Using the three-dimensional calculus conducted on the cracked element of Figure 1, we identify the compliance function $s(\Phi)$ as described previously. Then, we determine the stiffness matrix $\mathbf{K_e}(\varphi_e)$ terms by using relation (15).

We have noticed that $k_{xy}(\varphi_e) = -\frac{1}{2}k'_{xx}(\varphi_e)$. Thus, $\mathbf{K_e}(\varphi_e)$ can be written in the form

$$\mathbf{K_e}(\varphi_e) = \frac{EI}{2L}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & k_{xx}(\varphi_e) & -\frac{1}{2}k'_{xx}(\varphi_e) \\ 0 & 0 & \frac{1}{2}k'_{xx}(\varphi_e) & k_{xx}(\varphi_e) \end{pmatrix}.$$

For a straight crack of depth $a = D/2$, these terms are shown by Figure 3. We notice small variations in the interval $[0, \pi/2]$. However, in $[\pi/2, 2\pi]$, these variations becomes important but remain regular. The crack is totally closed when $\varphi_e = \pi/2$, image of $\Phi = \pi/2$. It opens totally at $\varphi_e = 3\pi/2$, the image of $\Phi = 3\pi/2$. Here $\Phi$ is the angle defined in Section 2 by $\Phi = \arctan(M_y/M_x)$.

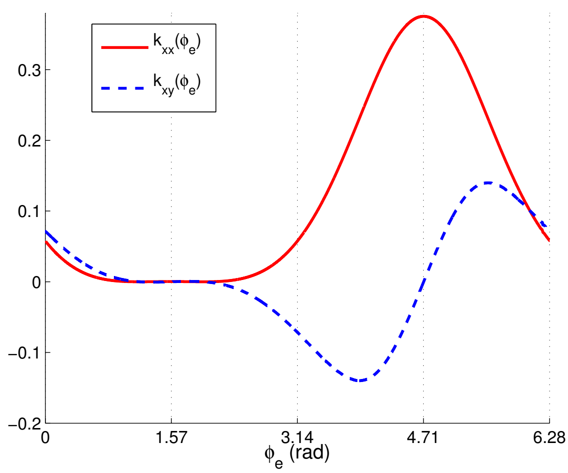


**Figure 3.** $\mathbf{K_e}(\varphi_e)$ terms for a straight crack of depth $a = D/2$, $L_e = L = 2D$.

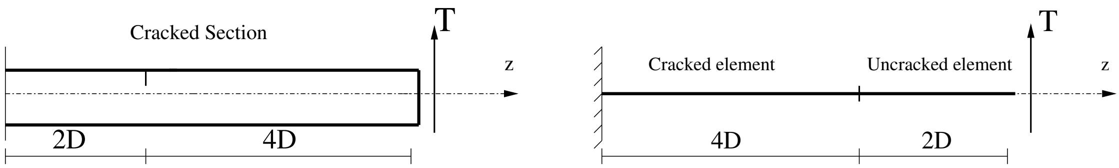


**Figure 4.** Three-dimensional and beam modeling of the system. Left: three-dimensional model; right: beam model.

## 4. Validation of the approach

In order to validate the stiffness beam finite element matrix construction method presented above, we propose to compare the three-dimensional modeling calculus results to those obtained using a beam modeling of the cracked structure of Figure 4. The cylinder element, of axis $0z$, diameter $D = 0.5$ m, and total length $L_t = 3$ m, is clamped at its end $z = 0$ and submitted at the other end to the force vector $\mathbf{T}$ $(T_x = \cos\alpha,\ T_y = \sin\alpha)$, with $\alpha \in [0, 2\pi[$. The cracked section is located at $z = 1$ m. The crack is straight with depth $a = 0.5D$; see Figure 4, left. The three-dimensional finite element calculus take into account the unilateral contact conditions between the crack lips. The beam model consists of two beam finite elements: the first is a cracked beam finite element of length 2 m, and the second is a classic beam finite element of length 1 m (Figure 4, right). Figure 5 shows excellent agreement for the results.

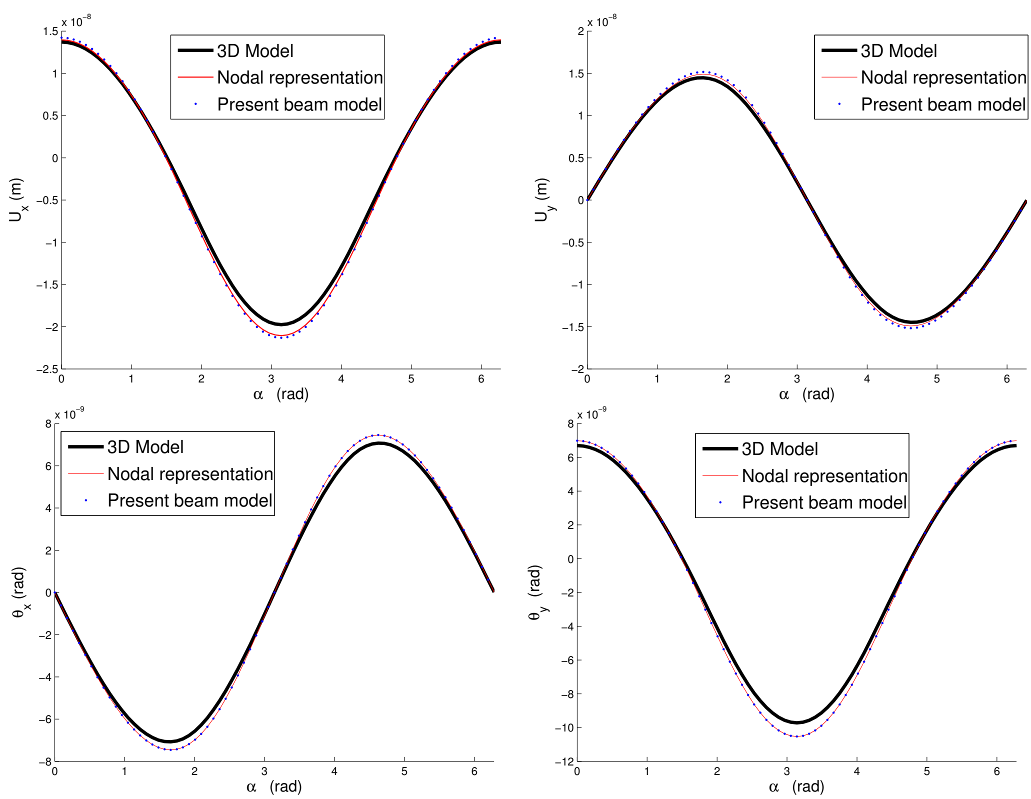


**Figure 5.** Model comparison to three-dimensional results and nodal modeling, $a/D = 0.50$.

## 5. Vibratory response of a cracked shaft

This section is devoted to exploring the vibratory response of a shaft with a cracked section under the effect of own weight. The system is composed of a beam of distributed mass $m$, circular section $S$, and diameter $D = 0.20$ m. The cracked section is located at mid-span. The structure is simply supported at node 1 and node 6 and submitted to the effects of its own weight. The shaft is rotating at the frequency $\Omega$ about the $(oz)$ axis. The system is divided into five elements of length $l = 4D$; see Figure 6. The elements 1–2, 2–3, 4–5 and 5–6 represent the structure's uncracked parts. Element 3–4 is a cracked beam finite element.

At $t = 0$, the crack is totally open; see Figure 6, right. We suppose that the stiffness matrix remains constant between two instants $t_n = nh$ and $t_{n+1} = (n+1)h$, where $h$ is the time step used for the numerical integration of the dynamical system. For the validity of this approximation, the time steps should be relatively small compared to the excitation period $(1/\Omega)$. In the low frequency domain, this difficulty is easily overcome. Thus, the stiffness matrix is updated at the end of each integration step. The HHT method is adopted for the numerical integration of the obtained system [Hilber et al. 1977] with $\alpha = \frac{1}{3}$, $\gamma = \frac{1}{2} + \alpha$, and $\beta = \frac{1}{4}(1+\alpha)^2$ which corresponds, in the linear analysis, to an unconditionally stable scheme with maximum precision.

With this modeling, the unilateral contact conditions between the crack lips are taken into account exactly. In fact, when the crack is totally closed, we obtain $\mathbf{K_e}(\varphi_e) = \mathbf{0}$ and the cracked element stiffness matrix is the one of an uncracked element. Furthermore, the time step used here is $\sim 10^{-3}$ s, which is at least 20 to 100 times smaller than the ones used for a penalization-implicit approach when the cracked section is modeled by a nodal element (element of length zero). In this later approch, the time step depends on the value given to the penalization constant and it is always less than $\sim 10^{-5}$ s to ensure calculus convergence.

The vibratory response shows superharmonic resonance phenomena when the rotating frequency passes through submultiples of the critical speed $w_1$; that is, for $\xi \approx w_1/n$, the shaft orbit and the phase diagram are composed of $n$ interlaced loops (see Figure 7, where the viscous damping $d$ is 5%). The vibration amplitude of the $n$-fold harmonic reaches higher levels at this rotating frequency (Figure 8).

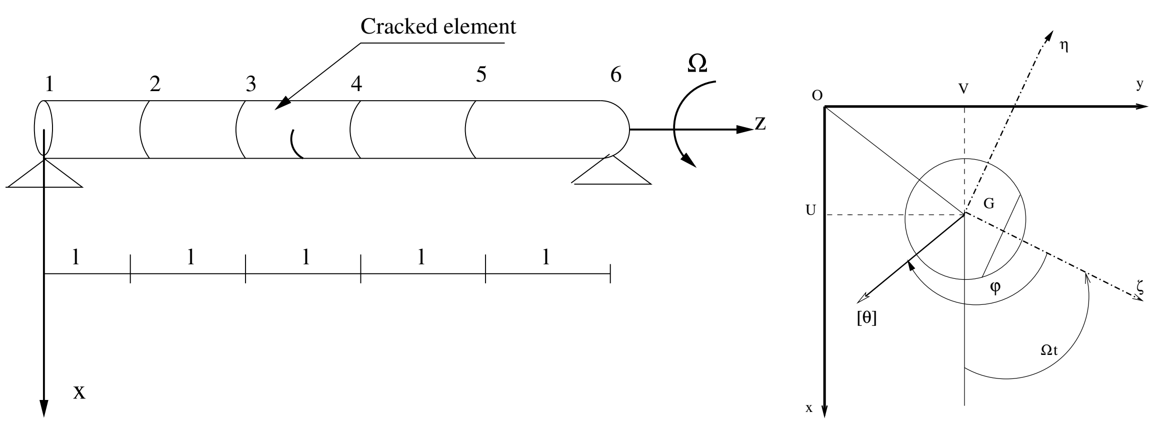


**Figure 6.** Finite element modeling of the cracked shaft (left) and the cracked section (right).

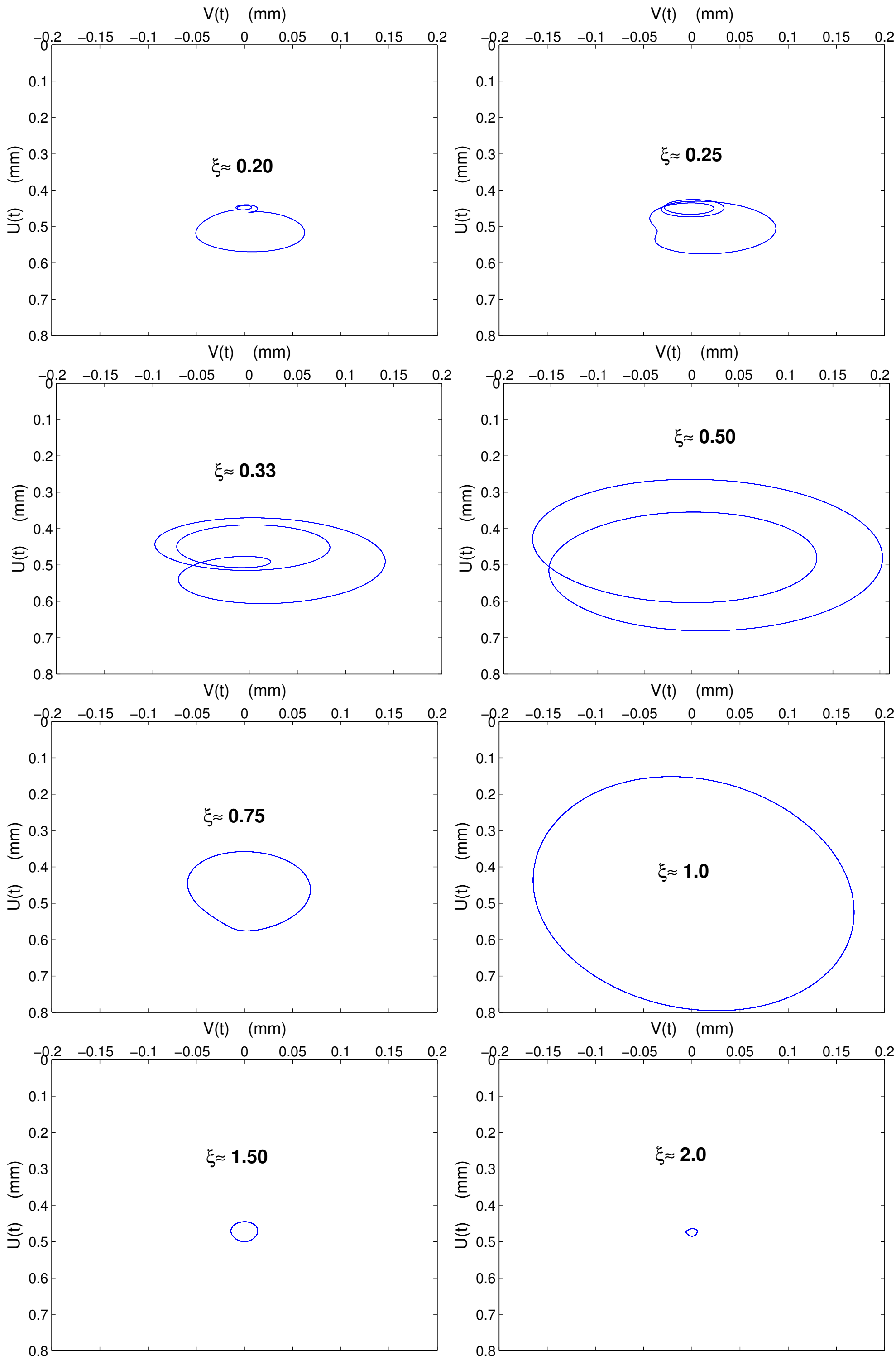


**Figure 7.** Examples of node 2 orbits, $a/D = 0.50$, $d = 0.05$.

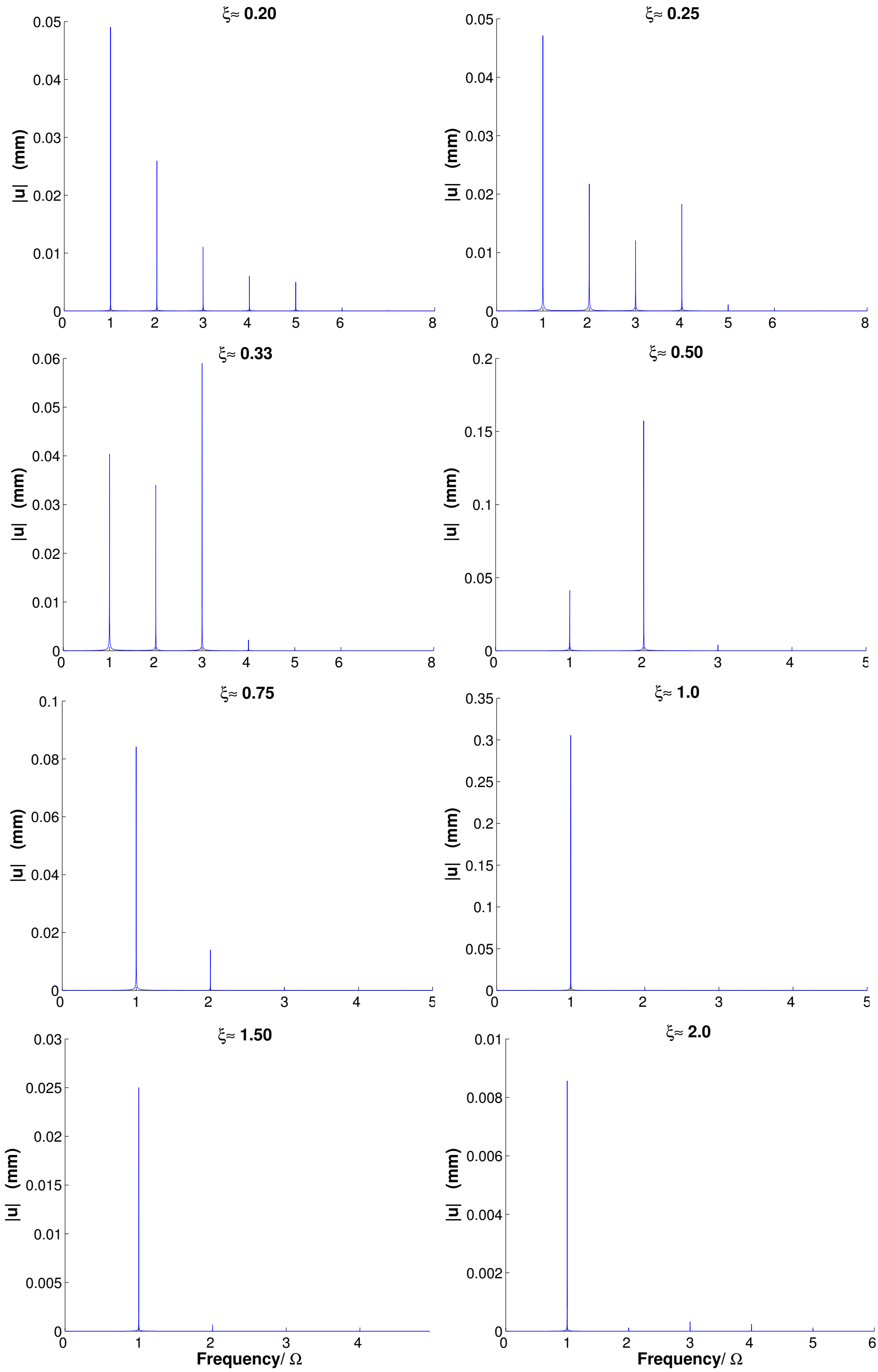


**Figure 8.** Examples of node 3 $u(t)$ amplitude spectra, $a/D = 0.50$, $d = 0.05$.

## 6. Stability analysis

The stability of a cracked shaft is analyzed using the Floquet method [Floquet 1879; Nayfeh and Mook 1979; Nayfeh and Balachandran 1994]. This method was used by Gasch [1976], Meng and Gasch [2000], and El Arem and Nguyen [2006] for the stability analysis of a two parameter cracked rotating shaft. The first step of this section consists of approximating the $\mathbf{K_e}(\varphi_e)$ terms by a classical function of the crack depth $a$ and angle $\varphi_e$. We have noticed that the function $k_{xx}(\varphi_e)$ for different straight tip crack depths shows that it could be approximated by

$$k_{xx}(\varphi_e) \approx k_{max} \sin^4\Big(\frac{\varphi_e}{2} - \frac{\pi}{4}\Big) \exp\Big(\sin\Big(\frac{\varphi_e}{2} - \frac{\pi}{4}\Big)\Big)^4,$$

where $k_{max}$ is given by $k_{max} \approx 3.43(a/D)^{2.73}$. These approximations remain satisfactory for cracks of depth going up to 30% of the diameter of the shaft; see Figure 9.

By using the Floquet theory, we have investigated, numerically, the stability of a cracked rotating shaft with a straight tip crack at mid-span; see Figure 6. The results, shown in Figure 10, show three principal areas of instability. The first corresponds to a $(0 < \xi < 0.5)$ superharmonic resonance phenomenon. The second is located around the exact resonance $(\xi = 1)$ and the third area (around $\xi = 2$) corresponds to a subharmonic resonance. It is important to note that even for weak viscous damping $(d \approx 1\%)$ the stability of the cracked shaft is only slightly affected. The zones of instabilities appear for $\Omega$ near the first critical speed $(\xi \approx 1)$ and twice the critical speed $(\xi \approx 2)$ and correspond to deep cracks $(\frac{a}{D} > 25\%)$.

## 7. Conclusions

A method for the construction of a cracked beam finite element is presented. The crack breathing phenomenon is finely described since the flexibility due to the presence of the cracks is deduced from three-dimensional finite element calculations taking into account the conditions of unilateral contact between

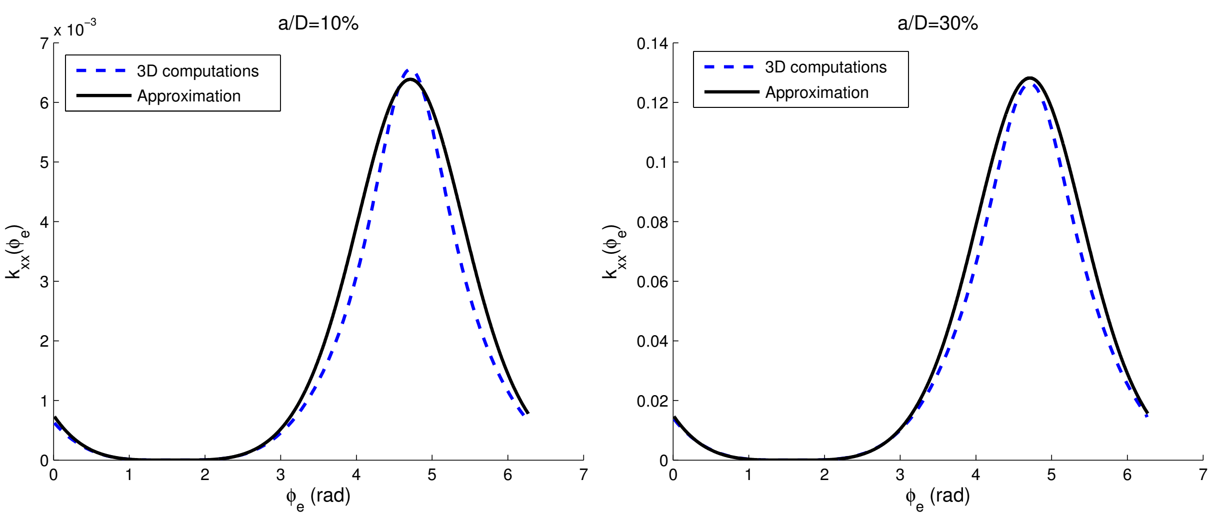


**Figure 9.** $k_{xx}(\varphi_e)$ approximation for different straight crack depths.

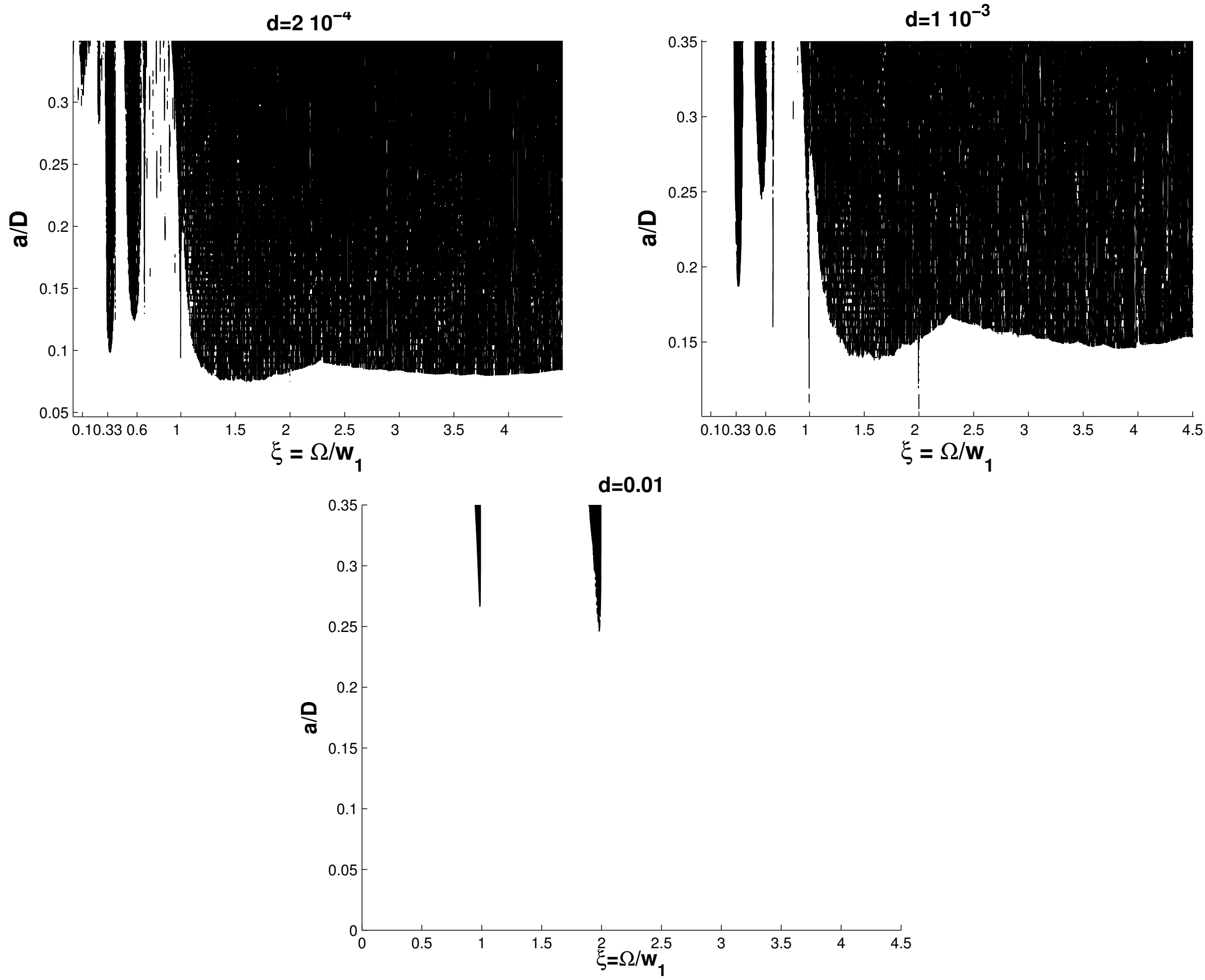


**Figure 10.** Stable and unstable (hatched) regions.

the crack lips as originally developed by Andrieux and Varé [2002]. The precise descriptions of the loss of stiffness and of the progressive closure or opening of the cracks are of fundamental importance.

The approach is quite simple and comprehensive and can be applied to any geometry of cracks. It is important to note the considerable gain in computational efforts when compared with the use of the technique of penalization.

Indeed this technique, when used when the cracked section is modeled by a nodal element [El Arem 2006], leads to the appearance of very high numerical frequencies (without physical significance). The time steps considered for the temporal numerical integration of the dynamic system are then very small and the computation costs are, consequently, very important. When considering the conditions of contact between the crack lips, this model has the following advantages compared to the nodal representation of the cracked section and the technique of penalization:

- It takes into account in an exact way the phenomenon of cracks breathing mechanism.
- It reduces the calculation costs by 20 to 100 times.

To reduce the difficulty of the problem in the study of cracked shafts, researchers often assumed that the amplitude of the vibrations due to the presence of cracks is weak compared to those of the vibrations due to permanent loads [El Arem and Nguyen 2006; Gasch 1976; Mayes and Davies 1976; 1980; Davies and Mayes 1984; Henry and Okah-Avae 1976; Gasch 1993]. The present model is more general and allows us to overcome the limitations of such an assumption.

The stability analysis of the cracked shaft of Figure 6 was carried out using the Floquet theory. Figure 10 shows stable and unstable zones in the $(\xi, a/D)$ plane for different viscous damping coefficients. We have noticed that the instability zones disappear for $d \geq 3\%$. For real turbines shafts, where the viscous damping is about 3% to 4% [Lalanne and Ferraris 1990], it can be deduced that the effects of a cracked section at mid-span on the stability of the structure is negligible when $a/D$ is less than 35% .

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