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JAN ADAM KOŁODZIEJ AND MAGDALENA MIERZWICZAK

This paper describes the application of the method of fundamental solutions (MFS) and the collocation method to the simulation of the sloshing phenomenon on an ideal fluid in a two-dimensional rectangular vessel. The phenomenon is governed by the Laplace equation with respect to the velocity potential. The equation is solved with nonlinear boundary conditions. The velocity potential is approximated by a linear superposition of fundamental solutions with the appropriate coefficients at each time step.

1. Introduction

Free surface fluctuation, also called liquid sloshing, is the most prominent phenomenon of liquid motion in either stationary or moving tanks subjected to forced external perturbations. The problem of liquid sloshing inside of moving or stationary containers remains of great concern to the aerospace, civil, and nuclear engineers, physicists, and designers of road or ship tankers. This phenomenon can be observed on the oil vessel of a tanker sailing on the ocean; as the tanker oscillates, the oil may spill. Another example is the cooling of a water vessel in an atomic power reactor that is oscillated by an earthquake. It is important to study the motion of a liquid fuel inside a tank during a rocket launch. Excessive liquid sloshing may cause structural failure and manipulation loss, which can lead to loss of economic, human, and environmental resources.

Usually the sloshing problem is formulated as a two-dimensional initial boundary-value problem in terms of the velocity potential, assuming that the fluid is inviscid, incompressible, and that the flow is irrotational (such that the viscosity, which causes the rotational motion, may be negligible). In such a case, the governing equation is the Laplace equation, which is solved with the appropriate boundary conditions. The surface profile and the boundary conditions on the free surface are updated from the potential value and its derivatives. Due to the nonlinear boundary conditions on the free surface and because the free surface is not known a priori, this problem is difficult and is usually solved numerically. This phenomenon has been simulated by using boundary and finite element methods [Abe and Sakuraba 1999; Hamano et al. 2003; Cho and Lee 2004; Sriram et al. 2006].These methods, however, have some difficulties. The finite element method needs remeshing at each time step, and the computational time is very extensive. When we use the boundary element method with the boundary discretization alone, and therefore the mesh regeneration cost is cheaper, another difficulty arises due to the singularity of the fundamental solutions, and the calculation of the potential derivatives is somewhat troublesome.

Keywords: method of fundamental solutions (MFS), collocation points, source points, substantive (material) derivative, sloshing phenomenon.

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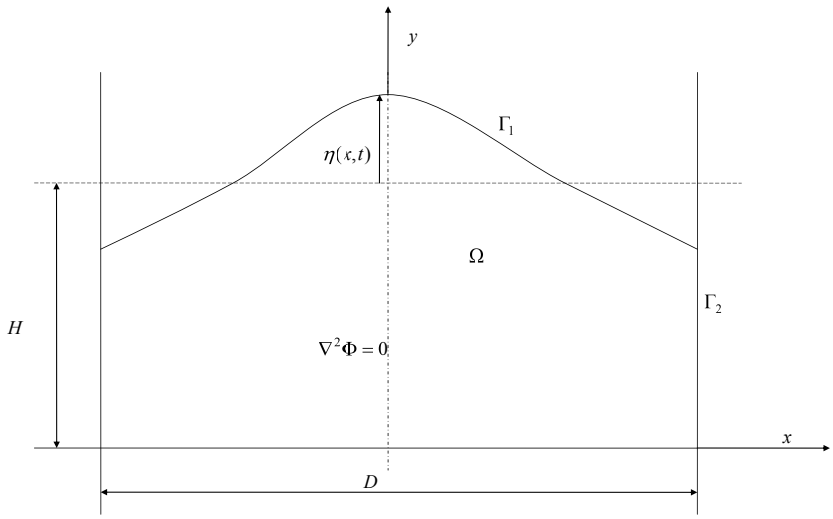

Figure 1. Rectangular vessel.

To overcome these difficulties, we propose using the meshless method of fundamental solutions (MFS). The velocity potential is approximated by the superposition of the fundamental solutions with unknown parameters that will be determined from the boundary condition. The calculation of the first and second derivatives is relatively easy. Therefore, we can treat the nonlinear boundary condition with Euler's extended algorithm [Kita et al. 2004].

2. The governing equation's boundary and initial conditions

Let's consider an ideal fluid contained in a rectangular tank in two dimensions (Figure 1). For free vibrations of the fluid, the initial displacement of the free surface is assumed to be known. The object domain occupied by the fluid, the free surface of the fluid, and the wall of the vessel are indicated by $\Omega, \Gamma_1, \Gamma_2$, respectively. The x and y axes of the Cartesian coordinates are taken in the horizontal and vertical directions, respectively. The origin of coordinates is located in the center, at the bottom of the tank. The width of the tank is D, the height of the undisturbed fluid is H, $\eta(x,t)$ is the vertical displacement of the fluid surface (displacement from the undisturbed fluid level), and $\xi(x,t)$ is the horizontal displacement of a fluid element on the free surface. By simplifying the flow problem (the fluid is incompressible, the flow irrotational, and forces due to viscosity are neglected), fluid flow can be defined by the Laplace equation involving a velocity potential $\Phi(x, y)$,

$$\frac{\partial^2\Phi}{\partial x^2}+\frac{\partial^2\Phi}{\partial y^2}=0. \tag{2-1}$$

The boundary conditions are as follows:

(1) The bottom and lateral walls are rigid and flat, so that

$$\frac{\partial\Phi}{\partial y}=0 \quad \text{for } y=-H, \qquad \frac{\partial\Phi}{\partial x}=0 \quad \text{for } x=\pm\frac{D}{2}.$$

(2) The kinematic condition on the free surface states that a particle of fluid which is at some time on the free surface will always remain on the free surface. Since the equation of the surface is $y-\eta=0$, it follows that

$$\frac{D}{Dt}(y-\eta)=0.$$

This equation may be expanded to give

$$\frac{\partial \eta}{\partial t}+\frac{\partial \Phi}{\partial x}\frac{\partial \eta}{\partial x}-\frac{\partial \Phi}{\partial y}=0. \tag{2-2}$$

(3) The dynamic condition on the free surface is implemented through the Bernoulli equation for unsteady irrotational motion:

$$\frac{\partial \Phi}{\partial t}+\frac{P}{\rho}+\tfrac{1}{2}\nabla\Phi\circ\nabla\Phi+g\eta=0.$$

When the atmospheric pressure is taken as zero, the P will be zero, and

$$\frac{\partial \Phi}{\partial t}+\frac{1}{2}\left(\left(\frac{\partial \Phi}{\partial x}\right)^2+\left(\frac{\partial \Phi}{\partial y}\right)^2\right)+g\eta=0. \tag{2-3}$$

The initial conditions are the following (the initial displacement of the free surface is assumed to be known):

$$\xi\,(x,0)=\xi_0\,(x)\,,\ \ \eta\,(x,0)=\eta_0\,(x)\,. \tag{2-4}$$

3. Solving the sloshing phenomenon

The solution for the two-dimensional Laplace Equation (2-1) is given as follows:

$$\Phi\,(x,y,t)=\sum_{i=1}^{N}A_i\,(t)\ln\sqrt{(x-xs_i)^2+(y-ys_i)^2}, \tag{3-1}$$

where (xs_i, ys_i) are the source points placed outside of the considered domain, N is the total number of source points, $\{A_1\,(t)\,, A_2\,(t)\,, \ldots, A_n\,(t)\}$ denotes the vector of unknown functions of time, which is determined by the satisfaction of the appopriate boundary condition by means of the collocation method and initial conditions. It is easy to calculate the potential first and second-derivatives:

$$\frac{\partial \Phi}{\partial x}=\sum_{i=1}^{N}A_i\,(t)\,\frac{(x-xs_i)}{(x-xs_i)^2+(y-ys_i)^2},\qquad \frac{\partial \Phi}{\partial y}=\sum_{i=1}^{N}A_i\,(t)\,\frac{(y-ys_i)}{(x-xs_i)^2+(y-ys_i)^2},$$

$$\frac{\partial^2 \Phi}{\partial x^2}=\sum_{i=1}^{N}A_i\,(t)\,\frac{(y-ys_i)^2-(x-xs_i)^2}{\left((x-xs_i)^2+(y-ys_i)^2\right)^2},\qquad \frac{\partial^2 \Phi}{\partial y^2}=\sum_{i=1}^{N}A_i\,(t)\,\frac{(x-xs_i)^2-(y-ys_i)^2}{\left((x-xs_i)^2+(y-ys_i)^2\right)^2},$$

$$\frac{\partial^2 \Phi}{\partial x\partial y}=\sum_{i=1}^{N}A_i\,(t)\,\frac{-2\,(x-xs_i)\,(y-ys_i)}{\left((x-xs_i)^2+(y-ys_i)^2\right)^2},$$

Let's consider a collocation point in the free surface which moves from $\left(\xi^k, \eta^k\right)$ at time t to a new position $\left(\xi^{k+1}, \eta^{k+1}\right)$ at time $t + \triangle t$. If the time interval is small, the new position and the associated velocity potential can be obtained by Taylor series expansion:

$$\begin{aligned}
\Phi^{k+1} &= \Phi^k + \Delta t \frac{D\Phi}{Dt} + \tfrac{1}{2}(\Delta t)^2 \frac{D^2\Phi}{Dt^2}, \\
\xi^{k+1} &= \xi^k + \Delta t \frac{D\xi}{Dt} + \tfrac{1}{2}(\Delta t)^2 \frac{D^2\xi}{Dt^2}, \\
\eta^{k+1} &= \eta^k + \Delta t \frac{D\eta}{Dt} + \tfrac{1}{2}(\Delta t)^2 \frac{D^2\eta}{Dt^2}.
\end{aligned} \tag{3-2}$$

The substantive derivatives are estimated as follows:

$$\frac{D\Phi}{Dt} = \tfrac{1}{2}\nabla\Phi \circ \nabla\Phi - g\eta, \quad \frac{D\eta}{Dt} = \frac{\partial\Phi}{\partial y} = v_y, \quad \frac{D\xi}{Dt} = \frac{\partial\Phi}{\partial x} = v_x, \tag{3-3}$$

$$\begin{aligned}
\frac{D^2\eta}{Dt^2} &= \frac{D}{Dt}\left(\frac{D\eta}{Dt}\right) = \frac{\partial}{\partial y}\left(\frac{\partial\Phi}{\partial t}\right) + \frac{\partial\Phi}{\partial x}\frac{\partial^2\Phi}{\partial x\partial y} + \frac{\partial\Phi}{\partial y}\frac{\partial^2\Phi}{\partial y^2}, \\
\frac{D^2\xi}{Dt^2} &= \frac{D}{Dt}\left(\frac{D\xi}{Dt}\right) = \frac{\partial}{\partial x}\left(\frac{\partial\Phi}{\partial t}\right) + \frac{\partial\Phi}{\partial x}\frac{\partial^2\Phi}{\partial x^2} + \frac{\partial\Phi}{\partial y}\frac{\partial^2\Phi}{\partial x\partial y}, \\
\frac{D^2\Phi}{Dt^2} &= \frac{D}{Dt}\left(\frac{1}{2}\nabla\Phi \circ \nabla\Phi - g\eta\right) = v_x\frac{D^2\xi}{Dt^2} + v_y\frac{D^2\eta}{Dt^2} - g v_y,
\end{aligned} \tag{3-4}$$

The time derivative $\partial\Phi/\partial t = \Phi_t$ is calculated by solving the following boundary value problem using MFS, where the time derivative velocity potential is approximated by a linear combination of appropriate functions:

$$\begin{cases}
\nabla^2\Phi_t = 0 & (\text{in } \Omega), \\
\Phi_t = -\frac{1}{2}\left(v_x^2 + v_y^2\right) - gy & (\text{on } \Gamma_1), \\
\partial\Phi_t/\partial n = 0 & (\text{on } \Gamma_2),
\end{cases} \tag{3-5}$$

Computational accuracy is checked using the conservation of the fluid volume V in the container and of the total energy E, using the equations

$$V = \int_\Omega dV, \qquad E = \int_\Omega \left(\tfrac{1}{2}\nabla\Phi \cdot \nabla\Phi + gy\right) dV. \tag{3-6}$$

4. The algorithm

(1) Specify the initial profile of the free surface and the initial velocity potential on the free surface:

$$\Phi_0 = -g\eta_0.$$

(2) Initialize the time step: $k \leftarrow 0$.

(3) Substitute $\Phi_k \leftarrow \Phi_0, \quad \eta_k \leftarrow \eta_0, \quad \xi_k \leftarrow \xi_0$.

(4) Solve the boundary-value problem for Φ give by

$$\begin{cases} \nabla^2\Phi = 0 & (\text{in } \Omega), \\ \Phi = \Phi_0 & (\text{on } \Gamma_1), \\ \partial\Phi/\partial n = 0 & (\text{on } \Gamma_2). \end{cases} \tag{4-1}$$

(5) Estimate the derivatives

$$\frac{\partial\Phi}{\partial x}, \quad \frac{\partial\Phi}{\partial y}, \quad \frac{\partial^2\Phi}{\partial x^2}, \quad \frac{\partial^2\Phi}{\partial y^2}, \quad \frac{\partial^2\Phi}{\partial x\partial y}.$$

(6) Solve the boundary-value problem for Φ_t given by

$$\begin{cases} \nabla^2\Phi_t = 0 & (\text{in } \Omega), \\ \Phi_t = -\frac{1}{2}\left(v_x^2 + v_y^2\right) - gy & (\text{on } \Gamma_1), \\ \partial\Phi_t/\partial n = 0 & (\text{on } \Gamma_2). \end{cases} \tag{4-2}$$

(7) Estimate the derivatives given in Equation (3-4):

$$\frac{D^2\eta}{Dt^2}, \quad \frac{D^2\xi}{Dt^2}, \quad \frac{D^2\Phi}{Dt^2}.$$

(8) Use Equation (3-2) to calculate Φ^{k+1}, ξ^{k+1}, η^{k+1}.

(9) Smooth the free surface profile using cubic spline interpolation.

(10) Correct the vertical positions of free surface fluid particles with respect to the constant volume of the fluid.

(11) Update the free surface profile and the boundary condition on it:

$$\Phi \leftarrow \Phi_{k+1}, \quad \eta \leftarrow \eta_{k+1}, \quad \xi \leftarrow \xi_0.$$

(12) Increment the time step $k \leftarrow k+1$.

(13) Go to step (4).

5. Numerical examples

We consider a two-dimensional rectangular vessel with an ideal fluid. The width of the tank is $D = 1$ m, and the height of the undisturbed fluid is $H = 1$ m. The initial form of the free surface is sinusoidal (Figure 1), and the initial amplitude is $A = 0.1$ m. The collocation and source points are placed uniformly on the whole boundary. The distance from the source points to the boundary walls or free surface is $s = 0.1$ m. Simulations are performed using different time-steps $\triangle t$, and different numbers of source and collocation points on the free surface and on the boundary tanks.

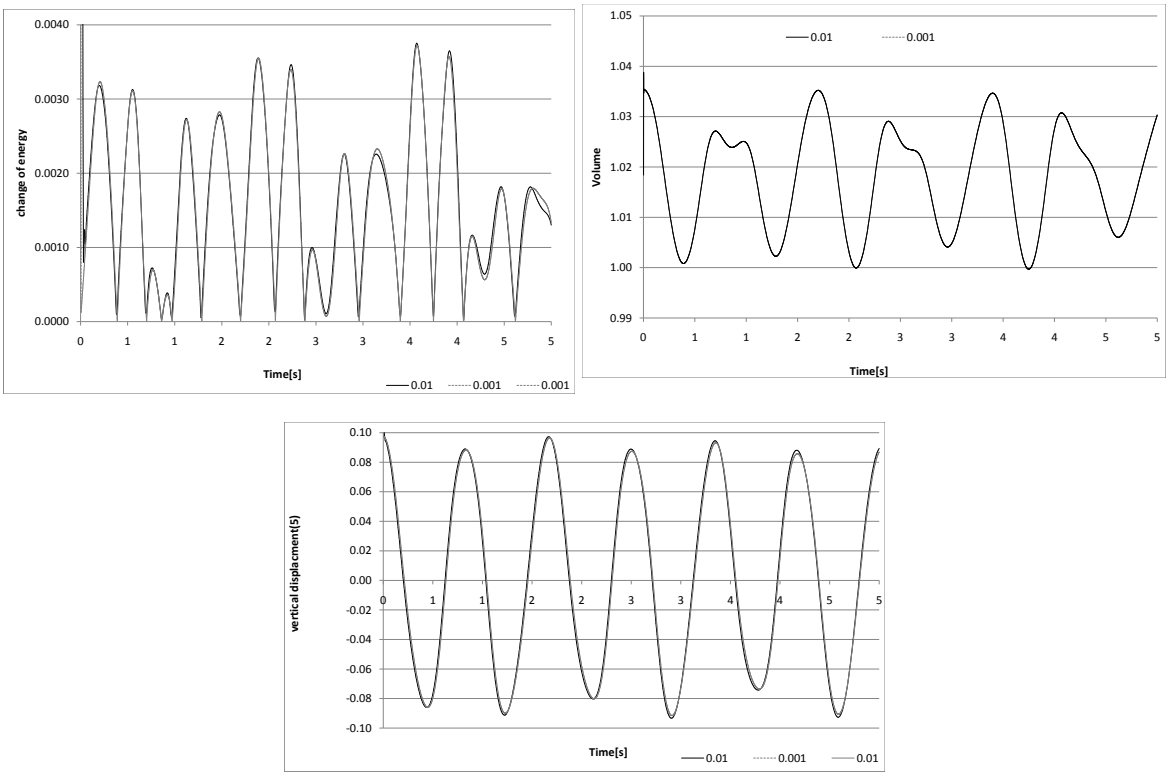

Figure 2. Top: Fluctuation of energy (left) and fluid volume (right) as a function of time. Bottom: Vertical displacement of point number 5 in the free surface. A total of 44 collocation and source points were used (see text).

6. Results

For a set of simulations with 11 collocation and source points on each of the tank's three walls plus the free surface (for a total of 44 collocation points), we compare simulation results for time step $\triangle t = 0.01$ and $\triangle t = 0.001$. Figure 2 shows the history of volume, the change in energy, and the vertical displacement for a free surface point, for different time steps. We see that the results are similar, and for equal time steps we obtained good results.

For the next case, when $\triangle t = 0.005$ and 11 (source, collocation) points are placed on the free surface and 7 (source, collocation) points are placed on the bottom, right, and left walls of the tank (for a total of 31 collocation and source points), the results are presented in Figure 3. Whether or not we observe fluctuations in the total fluid volume or energy, the algorithm is stable for many time steps.

When we take a time step of $\triangle t = 0.01$ and 15 (source, collocation) points in free surface, 11 (source, collocation) points on the bottom, right and left wall of the tank (for a total of 48 collocation and source points), the algorithm loses its stability after a number of time steps (see Figure 4).

Taking the last time step equal to $\triangle t = 0.001$ with the same numbers of collocation and source points, the algorithm is still unstable after some time steps, at which point the total energy rapidly increases (Figure 5).

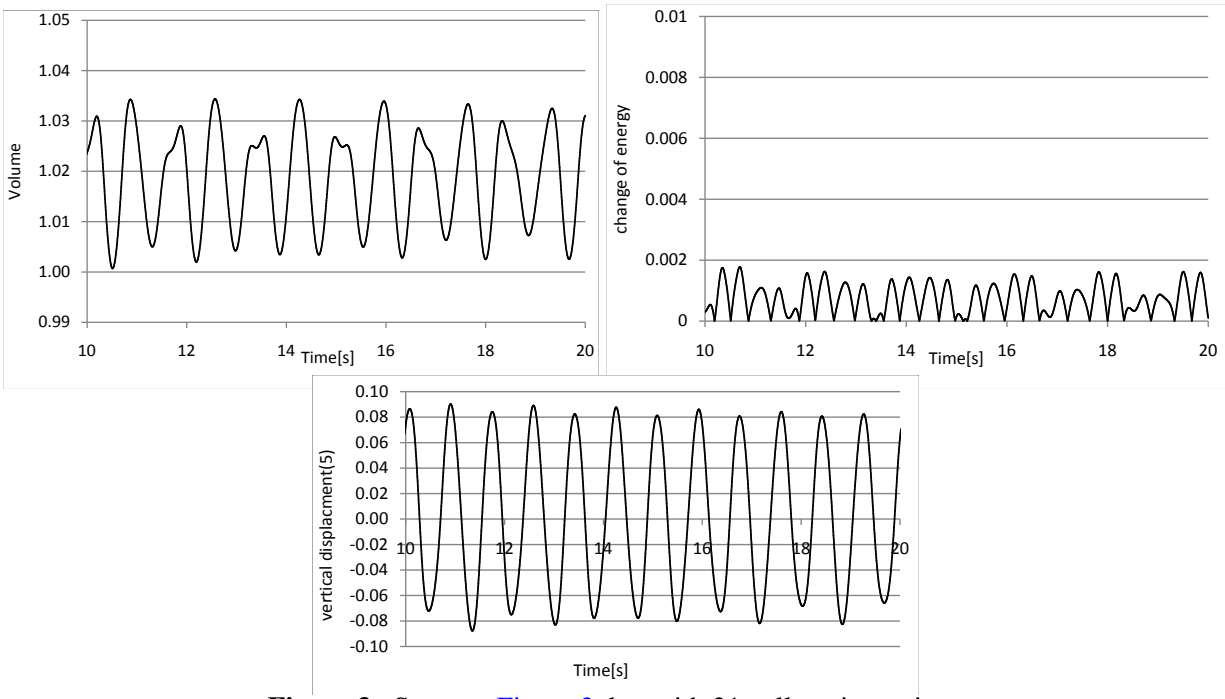

Figure 3. Same as Figure 2, but with 31 collocation points.

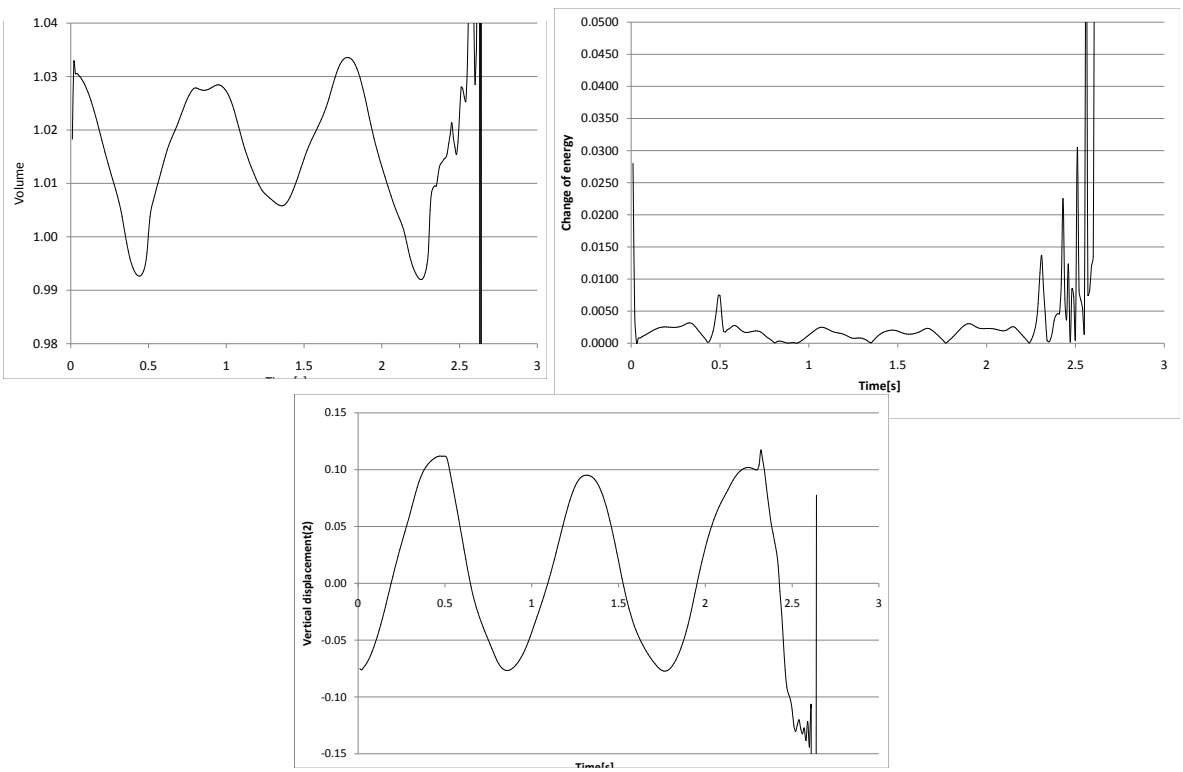

Figure 4. Same as Figure 2, but with 48 collocation points.

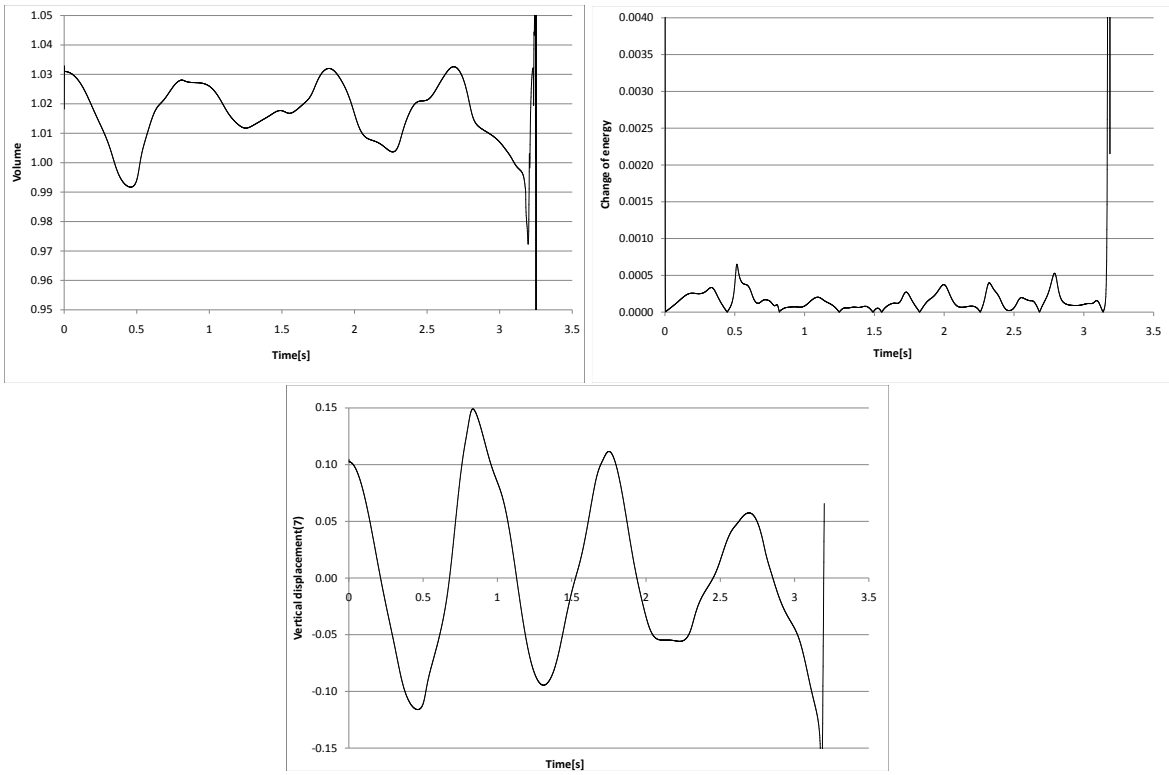

Figure 5. Same as Figure 4, but with $\triangle t = 0.001$.

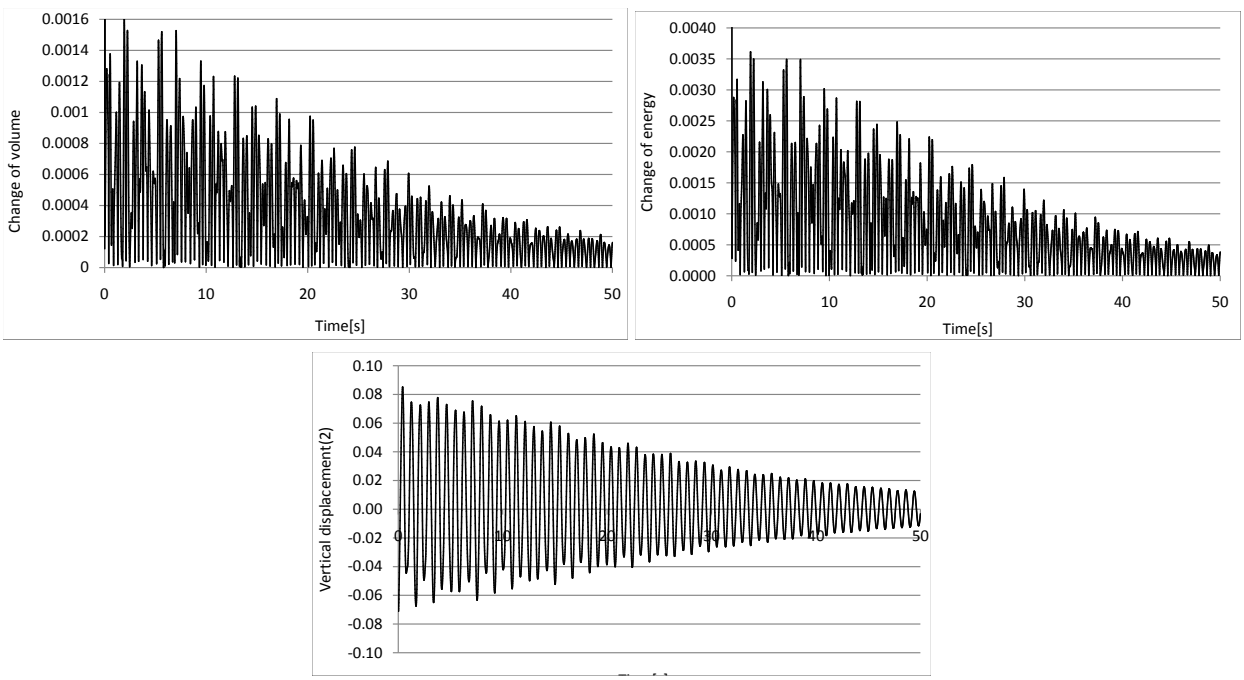

Figure 5. Same as Figure 4, but with 60 collocation points.

The algorithm becomes stable when we increase the number of collocation and source points (15) on the bottom, right and left wall of the tank at $\triangle t = 0.01$ (see Figure 5).

7. Conclusions

We applied a meshless numerical method to simulate the sloshing phenomenon. The method is "meshless" or "element-free", thus simplifying the geometric representation of the solution domain and eliminating the need for constructing and booking element connectivity. This comes in handy particularly when a moving boundary is involved, which may require frequent remeshing. Using the fundamental solution of the Laplace equation and locating the source points outside the computational domain, the problem is solved by collocation of a few boundary points. We used a mixed Eulerian and Lagrangian method (ALE). The numerical simulation was validated by checking the accuracy, including the errors in total volume and energy, and the convergence of the simulation was studied by changing the number of collocation and source points while varying the time interval. The results presented show that for some input parameters the algorithm is stable (Figures 2 and 3), but for others, after some number of time steps (300), the algorithm becomes unstable (Figures 4 and 5). In the last example, we observed damping of the change of energy and damping of the free surface motion for many time steps (Figure 5). We suppose that the reason for this simulation behavior is the lack of an energy correction. In future algorithms, conservation of energy will also be investigated.

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