Journal of
Mechanics of
Materials and Structures

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Volume 3, № 6 June 2008

mathematical sciences publishers

JOURNAL OF MECHANICS OF MATERIALS AND STRUCTURES
Vol. 3, No. 6, 2008

NON-NEWTONIAN FLUID FLOW IN A POROUS MEDIUM

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This paper presents the properties of non-Newtonian fluid flow in a porous medium. A numerical study on Brinkman flow is considered. It is assumed that the flow is isothermal. The governing equations are included. The steady-state problem is considered. The problem is nonlinear, described by coupled equations and boundary conditions. To solve the problem, a method based on the method of fundamental solutions for solving nonlinear boundary problems is proposed. The numerical experiment is performed and results are discussed.

1. Introduction

Dynamic porous media analysis is a powerful tool used for solving many everyday engineering problems, such as earthquake engineering, soil-structure interaction, biomechanics, et cetera. Moreover, the non-Newtonian fluid flow in porous media is very important due to its practical engineering applications, such as oil recovery, food processing, and materials processing. NonNewtonian fluids in porous media exhibit a nonlinear behaviour that is different from that of Newtonian fluids. An analysis of flow behaviour of non-Newtonian fluids is presented by Skerget and Sames [1999]. The boundary domain integral method for the numerical simulation of unsteady incompressible Newtonian fluid flow is extended to analyse the effects of available non-Newtonian viscosity. The method was applied to the Rayleigh–Benard natural convection problem. The problem mentioned above was solved also in [Huang et al. 1999] using the finite element method. In [Bernal and Kindelan 2007] the problem of injecting a non-Newtonian fluid into a thin cavity was considered. Using the Hele–Shaw approximation the problem reduces to a moving boundary problem in which the pressure is described by a two-dimensional nonlinear, elliptic equation. Mesh-free methods are very well suited for the numerical solution of moving boundary problems since no remeshing is needed at each time step to correctly represent the boundary. Among these methods, Bernal and Kindelan [2007] have chosen the asymmetric RBF collocation method (Kansa's method), a mesh-free method. The numerical experiment is performed to test different boundary conditions. The other method to simulate the turbulent non-Newtonian flow was proposed in [Rudman and Blackburn 2006]. A spectral element Fourier method for direct numerical simulation of the turbulent flow of non-Newtonian fluids is described and the particular requirements for non-Newtonian rheology are discussed.

For non-Newtonian fluids the phenomena of natural convection in porous media has attracted more attention during recent years. The problem is discussed in the literature by many authors [Hadim 2006; Cheng 2006]. Some numerical methods have been proposed for solving the considered problem. The purpose of [Jecl and Skerget 2003] was to present the use of the boundary element method in the analysis of the natural convection in the porous cavity saturated by the non-Newtonian fluid. The results of

Keywords: non-Newtonian fluid, porous media, fundamental solutions method, Carreau model, Brinkman equation, method of fundamental solutions.

hydrodynamic and heat transfer evaluations were reported for the configuration in which the enclosure is heated from a side wall while the horizontal walls are insulated. The flow in the porous medium was modelled using the modified Brinkman extended Darcy model taking into account the nonDarcy viscous affects. Sarler et al. [2004] described the solution of a steady natural convection problem in porous media by the dual reciprocity boundary element method. The boundary element method for the coupled set of mass, momentum and energy equations in two dimensions was structured by the fundamental solution of the Laplace equation. Numerical examples were presented. The solution was assessed by comparison with reference results of the fine-mesh finite volume method.

The main purpose of this work is to consider an isothermal flow of non-Newtonian fluid in a porous medium. The problem is described by the equation of mass conservation and Brinkman equation. These equations give boundary value problem consisted of a system of nonlinear coupled equations and nonlinear coupled boundary conditions. The method of fundamental solutions (MFS) is implemented to solve the nonlinear problem. The algorithm for the nonlinear coupled equations with nonlinear boundary conditions is proposed and applied to the considered problem.

2. Problem description

The steady-state problem in a porous medium is considered. The porous medium is saturated with non-Newtonian fluid. The considered region is presented in Figure 1. The edges of the considered reservoir are insulated, except for two pieces of edge which are open. There is a difference between pressure on two open edges that causes the fluid to flow. The following assumptions are made:

(i) the only phase flowing is the fluid of constant composition;

(ii) The fluid is non-Newtonian;

(iii) flow is isothermal;

(iv) the permeability of the porous medium is constant and uniform;

(v) gravitational forces are neglected.

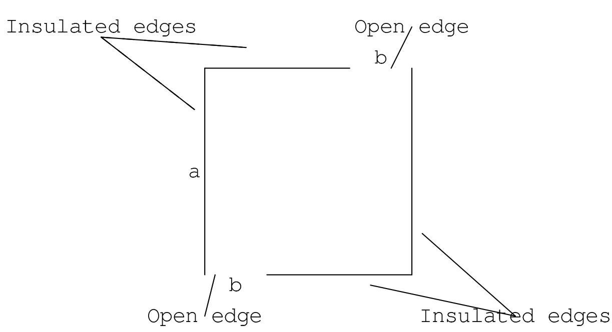

Figure 1. Geometry of the porous medium.

3. The non-Newtonian fluid

To introduce the equations governing the non-Newtonian fluid flow, some general auxiliary parameters are described. The shear-thinning non-Newtonian fluids are ones whose rheology is described by a generalised Newtonian model. Such fluids have an isotropic viscosity that is a function of flow properties. Extra stress tensor S is commonly known as a tensor which is related to the deformation rate by the constitutive equation

$$S_{ij} = \alpha\delta_{ij} + \beta G_{ij} + \gamma G_{ij} G_{ij}, \tag{1}$$

where α, β and γ are functions of three scalar invariants of G_{ij}

$$I_1 = G_{ii}, \qquad I_2 = G_{ij} G_{ji}, \qquad I_3 = G_{ij} G_{jk} G_{ki}, \tag{2}$$

so

$$\alpha = \alpha(I_1, I_2, I_3), \qquad \beta = \beta\big(I_1, I_2, I_3\big), \qquad \gamma = \gamma\big(I_1, I_2, I_3\big), \tag{3}$$

and the deformation rate tensor (the rate of strain tensor) is defined as

$$\mathbf{G} = \nabla\mathbf{v} + \big(\nabla\mathbf{v}\big)^T, \tag{4}$$

where $\mathbf{v}$ is the velocity field.

Equation (1) is the most general formula for the extra stress of the viscous shear flow. Such fluids are usually called Reiner–Rivlin fluids. For incompressible materials, I_1 equals zero and α, β and γ are considered as functions of I_2 and I_3. It is recognised that it is far too general to solve a specific flow problem. In order to numerically solve different types of flow problems, there have been many constitutive models in the fluid flow literature proposed by investigators. There is, however, a fairly large category of fluids for which the velocity is not independent of the rate of shear and these fluids are referred to as non-Newtonian. If the viscosity is considered as a function of the invariant I_2 many more practical flow problems can be solved. Then $\mathbf{S} = \eta(I_2)\mathbf{G}$, which represents a generalised Newtonian fluid. The shear rate is defined by $\dot{\gamma} = \sqrt{\frac{1}{2} I_2}$.

Several models are known for non-Newtonian fluids, such as power law fluids or Carreau fluids. In this paper the Carreau fluid is considered. The viscosity for the model is described by the formula

$$\mu = \mu_\infty + \big(\mu_0 - \mu_\infty\big)\big(1 + (A\dot{\gamma})^2\big)^{\frac{B-1}{2}},$$

where μ_0 and μ_∞ are asymptotic viscosities (measured in Pa) at large and small strain rates, respectively. A, B are fluid-specific constants (measured in s^{-1}) determined by plotting the observed viscosity as a function of strain rate on a log-log plot, for example. The Carreau model is particularly well-suited for certain dilute, aqueous, polymer solutions and melts.

4. The equations of non-Newtonian fluid flow in porous media

The motion of the fluid in porous media is described by the Brinkman equation for a viscous incompressible isothermal fluid (the momentum equation), the continuity equation and the thermal diffusion equation. The Brinkman equation is in the form

$$\rho_0 \frac{\partial \mathbf{v}}{\partial t} = -\nabla p + \nabla \cdot \mathbf{S} + \frac{\eta}{k}\mathbf{v} + \mathbf{F}, \tag{5}$$

where ρ_0 is the mass density of the fluid, $\mathbf{v}$ is the velocity field, p is the pressure, μ is the viscosity, k is the permeability of the porous structure. In the case of incompressible fluid the equation of the continuity reads

$$\nabla \cdot \mathbf{v} = 0. \tag{6}$$

The considered problem is two-dimensional, so the system of equations (5), (6) has the form

$$\begin{aligned}
\eta \nabla^2 v_1 &= -\frac{\partial p}{\partial x_1} + 2\left(\frac{\partial \eta}{\partial x_1}\frac{\partial v_1}{\partial x_1} + \frac{\partial \eta}{\partial x_2}\left(\frac{\partial v_1}{\partial x_2} + \frac{\partial v_2}{\partial x_1}\right)\right) - \frac{\eta}{k} v_1, \\
\eta \nabla^2 v_2 &= -\frac{\partial p}{\partial x_2} + 2\left(\frac{\partial \eta}{\partial x_2}\frac{\partial v_2}{\partial x_2} + \frac{\partial \eta}{\partial x_1}\left(\frac{\partial v_1}{\partial x_2} + \frac{\partial v_2}{\partial x_1}\right)\right) - \frac{\eta}{k} v_2, \\
\nabla^2 p &= 2\left(\frac{\partial^2 \eta}{\partial x_1^2}\frac{\partial v_1}{\partial x_1} + \frac{\partial^2 \eta}{\partial x_2^2}\frac{\partial v_2}{\partial x_2} + \frac{\partial^2 \eta}{\partial x_1 \partial x_2}\left(\frac{\partial v_1}{\partial x_2} + \frac{\partial v_2}{\partial x_1}\right)\right) \\
&\quad + \frac{\partial \eta}{\partial x_1}\left(\nabla^2 v_1 + \frac{\partial^2 v_1}{\partial x_2^2} + \frac{v_1}{k}\right) + \frac{\partial \eta}{\partial x_2}\left(\nabla^2 v_2 + \frac{\partial^2 v_2}{\partial x_1^2} + \frac{v_2}{k}\right).
\end{aligned} \tag{7}$$

The boundary conditions are detailed below. For the boundaries of the region the no-slip condition is applied. This means that

$$v_1 = 0, \qquad v_2 = 0, \tag{8}$$

for

$$\big\{(x, y) \mid \big((b < x < a) \cap (y = 0)\big) \cup \big((x = a) \cap (0 < y < a)\big) \cup \big((0 < x < a - b) \cap (y = a)\big) \cup \big((x = 0) \cap (0 < y < a)\big)\big\}.$$

The velocities on the open edges should meet the conditions

$$\frac{\partial v_1}{\partial n} = 0, \qquad \frac{\partial v_2}{\partial n} = 0, \tag{9}$$

for $\big\{(x, y) \mid \big((0 < x < b) \cap (y = 0)\big) \cup \big((a - b < x < a) \cap (y = a)\big)\big\}$. The boundary condition for the pressure field at insulated edges is

$$\nabla p - \nabla \cdot \eta\big(\nabla \mathbf{v} + (\nabla \mathbf{v})^T\big) = 0, \tag{10}$$

for

$$\big\{(x, y) \mid \big((b < x < a) \cap (y = 0)\big) \cup \big((x = a) \cap (0 < y < a)\big) \cup \big((0 < x < a - b) \cap (y = a)\big) \cup \big((x = 0) \cap (0 < y < a)\big)\big\},$$

where $\mathbf{v} = (v_1, v_2)$. The flow is imposed by pressure difference on both open edges. Therefore, the boundary conditions are

$$p = p_1 \quad \text{for} \quad \big\{(x, y) \mid \big((0 < x < b) \cap (y = 0)\big)\big\}, \tag{11}$$

$$p = p_2 \quad \text{for} \quad \big\{(x, y) \mid \big((a - b < x < a) \cap (y = a)\big)\big\}. \tag{12}$$

Moreover, the condition $p_1 > p_2$ has to be introduced. The problem consisting of equations (7) and boundary conditions (8)–(12) was solved in this paper using the method of fundamental solutions supported by Picard iterations.

5. Method of fundamental solutions for nonlinear problems

The nonlinear problem is written in a general form

$$A_n \mathbf{u}(\mathbf{x}) = f_n(\mathbf{x}), \tag{13}$$

for $\mathbf{x} \in \Omega$, where N_e is the number of equations, $n = 1, \dots, N_e$, A_n is a nonlinear partial differential operator, f_n are known functions and Ω is a region in which the equations are determined. The coordinates of the points are given by $\mathbf{x} = \big(x_1, \dots, x_N\big)$. The solution requires to calculate $\mathbf{u}(\mathbf{x}) = \big(u_1(\mathbf{x}), \dots u_{N_e}(\mathbf{x})\big)$. For the considered problem the boundary conditions are given by

$$B_l \mathbf{u}(\mathbf{x}) = g_l(\mathbf{x}), \tag{14}$$

for $\mathbf{x} \in \Gamma$ and $l = 1, \dots, N_{bc}$, where Γ is the boundary of the region Ω and N_{bc} is a number of all boundary conditions defined for the considered problem.

In the case that the nonlinear operator can be written as a sum of linear and nonlinear operators, the method of Picard iterations is applied. The nonlinear operator A_n is rewritten as

$$A_n = L_n + N_n, \tag{15}$$

where L_n is the linear partial differential operator of A_n and N_n is a nonlinear partial differential operator. The system of differential equations (13) is written as a system of linear differential equations. The nonlinearity of equation is added to the inhomogeneous part of the equation. Therefore, the considered system of equations has the form

$$L_n \mathbf{u}(\mathbf{x}) = f_n(\mathbf{x}) - N_n \mathbf{u}(\mathbf{x}), \tag{16}$$

for $\mathbf{x} \in \Omega$, where $n = 1, \dots, N_e$. Of course, the boundary conditions (14) are still valid. The proposed transformation of the system of coupled nonlinear equations gives the system of quasilinear equations in implicit form. In order to solve such a system of equations the Picard iterations are implemented. The iterative fashion of the considered system of equations is given as

$$L_n \mathbf{u}^{(k)}(\mathbf{x}) = f_n(\mathbf{x}) - N_n \mathbf{u}^{(k-1)}(\mathbf{x}), \tag{17}$$

for $\mathbf{x} \in \Omega$, where $n = 1, \dots, N_e$ and $k = 1, 2, \dots$. Each of the equations determined in k-th iteration step is solved with the method of fundamental solutions with boundary conditions

$$B_l \mathbf{u}^{(k)}(\mathbf{x}) = g_l(\mathbf{x}), \tag{18}$$

for $\mathbf{x} \in \Gamma$ and $l = 1, \dots, N_{bc}$. The inhomogeneous part of each equation is approximated by radial basis functions and polynomials.

The iterative process begins with initial approximations of the solution, which is obtained by solving the auxiliary boundary value problem

$$L_n \mathbf{u}^{(0)}(\mathbf{x}) = f_n(\mathbf{x}), \tag{19}$$

for $n = 1, \dots, N_e$. The set of equations (19) can be viewed as a system of uncoupled linear equations. Each equation is solved by method of fundamental solutions with proper boundary condition

$$B_l \mathbf{u}^{(0)}(\mathbf{x}) = g_l(\mathbf{x}), \tag{20}$$

for $\mathbf{x} \in \Gamma$ and $l = 1, \dots, N_{bc}$. If the functions $f_n(\mathbf{x})$ do not equal zero, they are approximated by radial basis functions and polynomials. The iterative process has to be stopped if the obtained results reach demanded accuracy. There are some criteria. In this paper the convergence is defined by thresholding the error of obtained solution

$$E_a = \max_{1 \le i \le N_C} \max_{1 \le n \le N_e} \left| u_n^{(k)}\left(x_i^C\right) - u_n^{(k-1)}\left(x_i^C\right) \right|, \tag{21}$$

where $\left\{\mathbf{x}_i^C\right\}_{i=1}^{N_C}$ is a set of trial points with arbitrary chosen number of trial points N_C. Than the condition to stop the iteration problem is

$$E_a < \epsilon, \tag{22}$$

where ϵ denotes a threshold which a small number such as 10^{-5}.

6. Numerical experiment

For considered problem of non-Newtonian fluid flow in porous media the following notation is introduced

$$\mathbf{x} = (x_1, x_2), \qquad \mathbf{u}(\mathbf{x}) = \big(u_1(\mathbf{x}), u_2(\mathbf{x}), u_3(\mathbf{x})\big) = \big(v_1(\mathbf{x}), v_2(\mathbf{x}), p(\mathbf{x})\big). \tag{23}$$

Then, the set (13) is rewritten as

$$A_1\mathbf{u} = f_1(\mathbf{x}), \qquad A_2\mathbf{u} = f_2(\mathbf{x}), \qquad A_3\mathbf{u} = f_3(\mathbf{x}), \tag{24}$$

with proper boundary conditions. The boundary conditions (14) are the conditions coupling the flow velocities and pressure in porous media. Moreover, one of these conditions consists of a nonlinear operator. Therefore, at every iteration step it is modified using the solution of previous iteration step. In the considered method the set (17) is rewritten in the form of iterative equations

$$\begin{aligned} \nabla^2 u_1^{(k)}(\mathbf{x}) &= f_1(\mathbf{x}) - N_1\mathbf{u}^{(k-1)}(\mathbf{x}), \\ \nabla^2 u_2^{(k)}(\mathbf{x}) &= f_2(\mathbf{x}) - N_2\mathbf{u}^{(k-1)}(\mathbf{x}), \\ \nabla^2 u_3^{(k)}(\mathbf{x}) &= f_3(\mathbf{x}) - N_3\mathbf{u}^{(k-1)}(\mathbf{x}), \end{aligned} \tag{25}$$

for $k = 1, 2, \dots$.

Equation (25) are Poisson equations. The inhomogeneous part in each equation is a sum of the functions of independent variables ($f_1(\mathbf{x})$, $f_2(\mathbf{x})$, $f_3(\mathbf{x})$) and the part determined by nonlinear operators ($N_1\mathbf{u}(\mathbf{x})$, $N_2\mathbf{u}(\mathbf{x})$, $N_3\mathbf{u}(\mathbf{x})$). The system of equations (25) is solved with modified boundary conditions at each iteration step. At the k-th step of procedure the boundary conditions are computed with equations given above. On the boundary the no-slip condition is

$$\begin{gathered} u_1^{(k)} = 0, \qquad u_2^{(k)} = 0, \\ \nabla u_3^{(k)} - \nabla \cdot \eta\Big(\nabla(u_1^{(k)}, u_2^{(k)}) + \big(\nabla(u_1^{(k)}, u_2^{(k)})\big)^T\Big) = 0, \end{gathered} \tag{26}$$

for

$$\{(x, y) \mid ((b < x < a) \cap (y = 0)) \cup ((x = a) \cap (0 < y < a)) \cup ((0 < x < a - b) \cap (y = a)) \cup ((x = 0) \cap (0 < y < a))\}.$$

In the case of an open edge on a boundary the normal flow is defined with

$$\frac{\partial u_1^{(k)}}{\partial y} = 0, \qquad \frac{\partial u_2^{(k)}}{\partial y} = 0, \tag{27}$$

for $\{(x, y) \mid ((0 < x < b) \cap (y = 0)) \cup ((a - b < x < a) \cap (y = a))\}$. Pressure is given as

$$u_3^{(k)} = p_1, \tag{28}$$

for $\{(x, y) \mid ((a - b < x < a) \cap (y = a))\}$. At the beginning of the iterative procedure the initial values of unknown variables are set. In the considered case initial values are chosen as

$$u_1^{(0)} = 0, \qquad u_2^{(0)} = 0, \qquad u_3^{(0)} = 0, \tag{29}$$

$$\frac{\partial u_1^{(0)}}{\partial x} = 0, \qquad \frac{\partial u_1^{(0)}}{\partial y} = 0, \qquad \frac{\partial u_2^{(0)}}{\partial x} = 0, \qquad \frac{\partial u_2^{(0)}}{\partial y} = 0. \tag{30}$$

Then the system of Poisson equations is obtained

$$\nabla^2 u_1^{(1)} = f_1(\mathbf{x}), \qquad \nabla^2 u_2^{(1)} = f_2(\mathbf{x}), \qquad \nabla^2 u_3^{(1)} = f_3(\mathbf{x}) - N_3 \mathbf{u}^{(1)}(\mathbf{x}), \tag{31}$$

with the boundary conditions (26)–(28) applied with $k = 1$.

The solution is obtained in five iterations. Figures 2, 3 and 4 show, respectively, the vertical component of a velocity field, the horizontal component of the velocity field and the pressure field in a porous medium. It can be observed on the graphs that the boundary conditions for velocity and pressure are met.

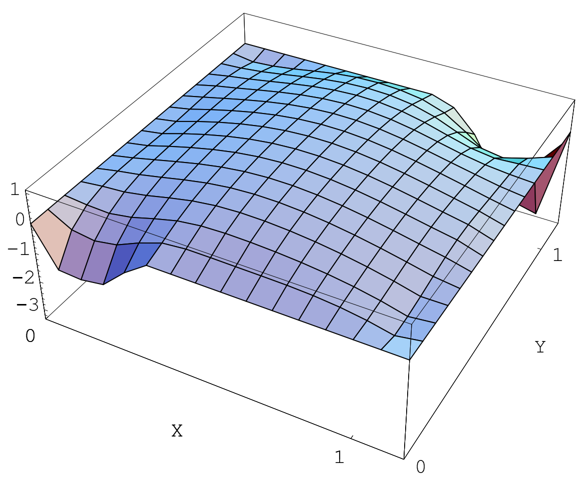

Figure 2. Component v_1 of velocity of non-Newtonian fluid in the porous medium.

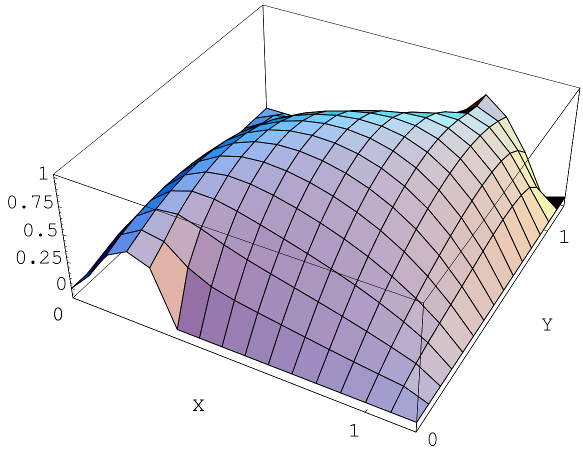

Figure 3. Component v_2 of velocity of non-Newtonian fluid in the porous medium.

The velocity of a non-Newtonian fluid flow in a porous medium, presented in Figure 2 and Figure 3, has the maximum value at the centre point of the considered region. On the boundary with a no-slip condition, velocity equals zero. This fact is observed in Figures 2 and 3. This shows that the implemented method meets the imposed boundary conditions. Near the open edges the component of velocity has positive values, indicating the direction of the flow. Of course, the fluid flows from the edge of higher pressure to the edge of lower pressure. At some distance from the open edges the component changes sign and becomes negative. This results in turbulence. Figure 4, consisting of the pressure field in considered region, confirms fulfilling boundary conditions determined in the problem. On the open edges, the values of pressure have been imposed and kept during implementation of method of fundamental solutions.

The results of numerical experiment show that the numerical method implemented for considered problem is sufficient and correct for nonlinear problems. The method is supported by Picard iterations and

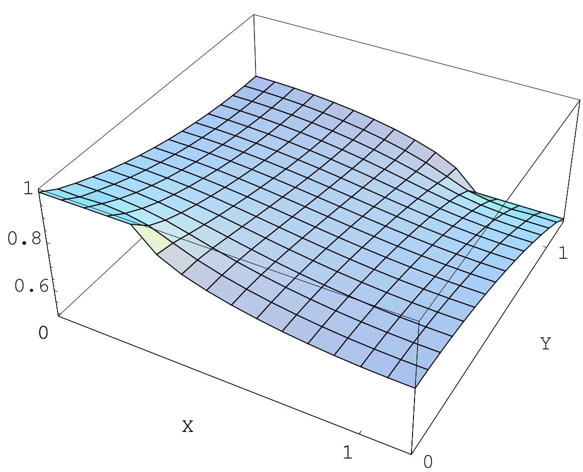

Figure 4. Pressure field in the porous medium.

method of fundamental solutions. For the considered problem the iteration process has been convergent. The satisfactory precision of obtained results has been achieved in five iterations. The presented results are compatible with expected ones.

7. Conclusions

In this paper the flow of non-Newtonian fluid in a porous medium has been considered. The governing equations were written and applied for two dimensional problem. Than the numerical algorithm has been proposed to solve the considered problem. The implementation of the method of fundamental solutions for the system of nonlinear coupled equations with nonlinear coupled boundary conditions has been presented. The numerical experiment, performed for the considered problem gives proper results, compatible with expected ones.

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Received 7 Feb 2008. Accepted 25 Mar 2008.

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