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Stan Chiriță and Michele Ciarletta

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SPATIAL EVOLUTION OF HARMONIC VIBRATIONS IN LINEAR ELASTICITY

STAN CHIRIȚĂ AND MICHELE CIARLETTA

In the present paper we consider a prismatic cylinder occupied by an anisotropic homogeneous compressible linear elastic material that is subject to zero body force and zero displacement on the lateral boundary. The elasticity tensor is strongly elliptic and the motion is induced by a harmonic time–dependent displacement specified pointwise over the base. We establish some spatial estimates for appropriate cross–sectional measures associated with the harmonic vibrations that describe how the corresponding amplitude evolves with respect to the axial distance at the excited base. The results are established for finite as well as for semi-infinite cylinders (where alternatives results of Phragmén-Lindelöf type are obtained) and the exciting frequencies can take appropriate low and high values. In fact, for the low frequency range the established spatial estimates are of exponential type, while for the high frequency range the spatial estimates are of a certain algebraic type.

1. Introduction

In the construction of buildings, bridges, aircraft, nuclear reactors and automobiles, the engineer must determine the depth to which local stresses, such as those produced by fasteners and at joints, or vibrations can penetrate girders, I-beams, braces and other similar structural elements. The determination of the extent of local or edge effects in structural systems allows the engineer to have a clear distinction between the global structure (where strength of materials approximations can be used) and the local excited portions which require a separate and more elaborate analysis based on some exact theories as that of linear elasticity. The standard procedure used in engineering practice to determine the extent of local stresses or edge effects is based on some form of the celebrated Saint Venant principle. A comprehensive surveys of contemporary research concerning Saint Venant principle can be found in [Horgan and Knowles 1983; Horgan 1989; 1996].

As regards elastic vibrations, it was observed in these papers that high frequency effects might be expected to propagate with little spatial attenuation (see also [Boley 1955; 1960]). It is outlined in [Horgan and Knowles 1983] that one would not expect to find unqualified decay estimates of the kind concerning Saint–Venant's principle in problems involving elastic wave propagation, even if the end loads are self-equilibrated at each instant. In this connection, Flavin and Knops [1987] have carried out an analysis of spatial decay for certain damped acoustic and elastodynamic problems in the low frequency range which substantiates the early work of Boley. These results are extended to linear anisotropic

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materials in [Flavin et al. 1990]. It should be noted that all of the investigations mentioned in the foregoing were concerned with elastic materials having a positive definite elasticity tensor.

In the present paper we address the question of spatial behavior of the harmonic vibrations in an anisotropic elastic cylinder under the condition of strong ellipticity for the elasticity tensor. In this respect, for vibrations in the low frequency range, our expected results describe exponential spatial estimates similar with those previously established by Flavin et al. [1987; 1990]. Moreover, for harmonic vibrations with appropriate high frequencies, the present results predict some algebraic spatial estimates, confirming the foregoing observations made by Boley in related context.

We consider a prismatic cylinder occupied by an anisotropic linear elastic material and subjected to zero body force and zero lateral boundary data and zero initial conditions. The motion is induced by a harmonic time–dependent displacement specified pointwise over the base and the other end is subjected to zero displacement (when a cylinder of finite extent is considered, to say). The elasticity tensor is assumed to be strongly elliptic and so a very large class of anisotropic elastic materials is considered, including those new materials with extreme and unusual physical properties like negative Poisson's ratio (that is, so called auxetic materials).

The primary purpose of the present paper is to examine how the amplitude of the harmonic vibration evolves with respect to the axial variable. To this end we associate with the amplitude of the harmonic vibration in concern, an appropriate cross–sectional integral function and further we prove that the strong ellipticity conditions assure that it is an acceptable measure. This is possible thanks to some appropriate auxiliary identities relating the amplitude of the harmonic vibrations. For these measures we are able to establish some differential inequalities whose integration allows us to obtain spatial estimates describing the spatial behavior of the amplitude in concern. In fact, when an identity of conservation energy type is used then certain exponential spatial estimates are obtained for all frequencies lower than a critical value. When a Rellich identity is involved then certain type of algebraic spatial estimates are established for appropriate high frequencies. All results are illustrated for transversely isotropic materials as well as for the rhombic systems.

2. Formulation of the problem

Consider a prismatic cylinder $B \subset \mathbb{R}^3$ whose bounded uniform cross–section $D \subset \mathbb{R}^2$ has piecewise continuously differentiable boundary ∂D. The origin of a rectangular Cartesian coordinate system is located in the cylinder's base and the positive x_3−axis is directed along that of the cylinder. It is convenient to introduce the further abbreviation

$$B_z = \{x \in B : z > x_3\} \tag{2.1}$$

and, moreover, we employ $D(x_3)$ to indicate that relevant quantities are to be evaluated over the cross–section whose distance from the origin is x_3.

The cylinder is occupied by an anisotropic compressible elastic material and is subject to a deformation in which the displacement field $u\,(x, t)$ is a smooth function satisfying the requirements of the classical dynamical theory of elasticity [Gurtin 1972]. The corresponding stress tensor $\mathbf{S}\,(x, t)$ has Cartesian components given by

$$S_{rs} = C_{rsmn}\varepsilon_{mn}, \tag{2.2}$$

where

$$\varepsilon_{mn} = \frac{1}{2}\left(u_{m,n} + u_{n,m}\right) \tag{2.3}$$

are the components of the strain tensor. Moreover, the constant elasticities C_{rsmn} possess the symmetries

$$C_{rsmn} = C_{mnrs} = C_{srmn}, \tag{2.4}$$

and satisfy the strong ellipticity condition

$$C_{rskl}m_r m_k n_s n_l > 0 \quad \text{for all nonzero vectors } (m_1, m_2, m_3), \ (n_1, n_2, n_3). \tag{2.5}$$

The cylinder is set in motion subject to a pointwise prescribed base harmonic time–dependent displacement, zero body–force and zero displacement on the lateral surface and the other end (when a finite cylinder is considered). Furthermore, the prescribed displacement is such that a classical solution exists on the interval $[0, \infty)$. Consequently, the problem to be considered is specified by

$$(C_{rskl}u_{k,l})_{,r} = \rho \ddot{u}_s, \quad (x, t) \in B \times [0, \infty), \tag{2.6}$$

$$u_r(x, 0) = u_r^0(x), \quad \dot{u}_r(x, 0) = \dot{u}_r^0(x), \quad x \in B, \tag{2.7}$$

$$u_r(x, t) = 0, \quad (x, t) \in \partial D \times [0, L] \times [0, \infty), \tag{2.8}$$

$$u_r(x, t) = f_r(x_1, x_2)\, e^{i\omega t}, \quad (x, t) \in D(0) \times [0, \infty), \tag{2.9}$$

$$u_r(x, t) = 0, \quad (x, t) \in D(L) \times [0, \infty), \tag{2.10}$$

in the case where L is finite (say). In the limiting case $L \to \infty$ a condition of the type (2.10) is unnecessary. In the above relations we have used a superposed dot for denoting differentiation with respect to time and a subscript comma indicates partial differentiation. Moreover, ρ is the constant positive mass density, ω is a positive constant (frequency of vibration), $u_r^0(x)$, $\dot{u}_r^0(x)$ and $f_r(x_1, x_2)$ are prescribed differentiable functions compatible with the initial and lateral boundary conditions and $i = \sqrt{-1}$ is the complex unit. We are interested in the study of the spatial behavior of the solution u_r of the above initial boundary value problem (2.6)–(2.10).

To this end we use the decomposition

$$u_r = U_r(x, t) + v_r(x)\, e^{i\omega t}, \tag{2.11}$$

where U_r (transient solution) satisfies the above initial boundary value problem with null boundary conditions and appropriate initial conditions, while v_r satisfies the boundary value problem

$$\left(C_{rskl}v_{k,l}\right)_{,r} + \rho\omega^2 v_s = 0, \quad x \in B, \tag{2.12}$$

$$v_r(x) = 0, \quad x \in \partial D \times [0, L], \tag{2.13}$$

$$v_r(x) = f_r(x_1, x_2), \quad x \in D(0), \tag{2.14}$$

$$v_r(x) = 0, \quad x \in D(L), \tag{2.15}$$

in the case where L is finite.

We note that the spatial behavior of the transient solution U_r can be described by the methods developed in [Chiriţă and Ciarletta 1999; Tibullo and Vaccaro 2008]. The exponential spatial decay of the amplitude v_r of the forced oscillation has been established in [Flavin and Knops 1987; Flavin et al. 1990; Knops 1991] for isotropic and anisotropic elastic materials with a positive definite elasticity tensor, provided the exciting frequency is less than a certain critical value. Considering an appropriate region filled with an isotropic elastic material an algebraical spatial decay of the amplitude of vibration has been established in [Chiriţă and Quintanilla 1996] under the assumption that the elasticity tensor is positive definite and without any restriction upon the frequency of vibration.

The main purpose of this paper consists of studying how the amplitude of harmonic vibration evolves with respect to the axial distance at the excited base, provided the strong ellipticity condition is assumed for the elasticity tensor. Under such hypotheses we will establish some appropriate algebraic and exponential spatial estimates describing the spatial decay of the amplitude of harmonic vibration. In fact, for all frequencies lower than a certain critical value we are able to establish exponential estimates describing how the amplitude evolves with the distance to the excited end. While, for all frequencies greater than an appropriate critical value, we can establish spatial estimates describing a specific algebraical behavior of the amplitude. Moreover, some alternatives of Phragmén–Lindelöf type are established for the semi-infinite cylinder.

Since the coefficients in the differential system (2.12) are real numbers, we can assume that v_r are real functions. Otherwise, we can proceed with the same method for the real part as well as for the imaginary part of v_r. So in what follows we shall consider the solution v_r to be real functions.

3. Some auxiliary identities

Before proceeding to derive *a priori* estimates for a solution to equations (2.12)–(2.15), we need some auxiliary identities concerning the equations (2.12), with the lateral boundary condition (2.13). Some of these are achieved via some Rellich–like identities (used for example in [Chiriţă et al. 2006; Chiriţă and Ciarletta 2008]).

Theorem 1. *Let v_r be a solution of the boundary value problem defined by relations* (2.12) *and* (2.13). *Then*

$$\int_{D(x_3)} \left(C_{rsmn} v_{r,s} v_{m,n} - \rho\omega^2 v_s v_s\right) da = \frac{d}{dx_3} \int_{D(x_3)} C_{3smn} v_{m,n} v_s \, da. \tag{3.1}$$

Proof. We form the identity

$$\int_{D(x_3)} v_s \left(\left(C_{rsmn} v_{m,n}\right)_{,r} + \rho\omega^2 v_s\right) da = 0. \tag{3.2}$$

Now integrate by parts in succession and use the boundary condition (2.13) to find (3.1). □

Theorem 2. *Let v_r be a solution of the boundary value problem defined by* (2.12) *and* (2.13). *Then*

$$\begin{aligned} \frac{1}{2} \int_{D(x_3)} &\left(C_{rsmn} v_{r,s} v_{m,n} - 3\rho\omega^2 v_s v_s\right) da \\ &= -\frac{d}{dx_3} \int_{D(x_3)} \left(x_\rho C_{3smn} v_{s,\rho} v_{m,n} + \tfrac{1}{2} x_3 \left(C_{r3m3} v_{r,3} v_{m,3} - C_{r\alpha m\beta} v_{r,\alpha} v_{m,\beta} + \rho\omega^2 v_s v_s\right)\right) da \\ &\qquad - \int_{\partial D(x_3)} \tfrac{1}{2} x_\rho n_\rho C_{r\alpha m\beta} n_\alpha n_\beta \frac{\partial v_r}{\partial n} \frac{\partial v_m}{\partial n} ds, \end{aligned} \tag{3.3}$$

where n_α are the components of the outward unit normal vector to ∂D and $\partial/\partial n$ represents the normal derivative.

Proof. We start with the identity

$$\int_{D(x_3)} x_p v_{s,p}\left(\left(C_{rskl}v_{k,l}\right)_{,r}+\rho\omega^2 v_s\right)da=0 \tag{3.4}$$

which can be written as

$$\int_{D(x_3)}\left(C_{rsmn}v_{r,s}v_{m,n}+x_p\left(\tfrac{1}{2}C_{rsmn}v_{r,s}v_{m,n}\right)_{,p}-x_p\left(\tfrac{1}{2}\rho\omega^2 v_s v_s\right)_{,p}\right)da$$
$$=\int_{D(x_3)}\left(x_pC_{rsmn}v_{m,n}v_{s,p}\right)_{,r}\,da; \tag{3.5}$$

moreover,

$$\frac{1}{2}\int_{D(x_3)}\left(C_{rsmn}v_{r,s}v_{m,n}-3\rho\omega^2 v_s v_s\right)da=-\int_{D(x_3)}\left(x_pv_{r,p}C_{rsmn}v_{m,n}\right)_{,s}\,da$$
$$+\int_{D(x_3)}\left(\tfrac{1}{2}x_pC_{rsmn}v_{r,s}v_{m,n}\right)_{,p}\,da-\int_{D(x_3)}\left(\tfrac{1}{2}x_p\rho\omega^2 v_r v_r\right)_{,p}\,da. \tag{3.6}$$

Using the divergence theorem and (2.13), we obtain from (3.6)

$$\frac{1}{2}\int_{D(x_3)}\left(C_{rsmn}v_{r,s}v_{m,n}-3\rho\omega^2 v_s v_s\right)da$$
$$=-\frac{d}{dx_3}\int_{D(x_3)}\left(C_{3smn}v_{m,n}x_pv_{s,p}-\tfrac{1}{2}x_3\left(C_{rsmn}v_{r,s}v_{m,n}-\rho\omega^2 v_s v_s\right)\right)da$$
$$+\int_{\partial D(x_3)}\left(\tfrac{1}{2}x_\rho n_\rho C_{rsmn}v_{r,s}v_{m,n}-x_pv_{r,p}C_{r\rho mn}v_{m,n}n_\rho\right)ds. \tag{3.7}$$

At this point we note that the boundary condition (2.13) implies

$$v_{r,3}=0,\quad x\in\partial D\times[0,L]. \tag{3.8}$$

Moreover, we write $v_{r,\alpha}$ on the curve $\partial D(x_3)$ as $v_{r,\alpha}=n_\alpha\left(\partial v_r/\partial n\right)+\tau_\alpha\left(\partial v_r/\partial\tau\right)$, where τ_α are the components of the tangential unit vector, $\partial/\partial n$ is the normal derivative and $\partial/\partial\tau$ is the tangential derivative. In view of the boundary condition (2.13) we have $(\partial v_r/\partial\tau)=0$ on $\partial D(x_3)$ and hence we deduce that $v_{r,\alpha}=n_\alpha\left(\partial v_r/\partial n\right)$ on $\partial D(x_3)$. Thus, we obtain

$$\int_{\partial D(x_3)}\left(x_\rho n_\rho C_{r\alpha m\beta}v_{r,\alpha}v_{m,\beta}-2x_\alpha v_{r,\alpha}C_{r\rho m\beta}v_{m,\beta}n_\rho\right)ds=-\int_{\partial D(x_3)}x_\rho n_\rho C_{r\alpha m\beta}n_\alpha n_\beta\frac{\partial v_r}{\partial n}\frac{\partial v_m}{\partial n}ds. \tag{3.9}$$

Substituting (3.9) into (3.7) we obtain (3.3). □

By combining these two theorems we obtain the following result.

Theorem 3. *Let v_r be a solution of the boundary value problem defined by* (2.12) *and* (2.13). *Then*

$$\frac{d}{dx_3}\int_{D(x_3)}\left(2C_{3rmn}v_{m,n}v_r+2x_\rho C_{3smn}v_{s,\rho}v_{m,n}+x_3(C_{r3m3}v_{r,3}v_{m,3}-C_{r\alpha m\beta}v_{r,\alpha}v_{m,\beta}+\rho\omega^2 v_s v_s)\right)da$$
$$=\int_{D(x_3)}\left(C_{rsmn}v_{r,s}v_{m,n}+\rho\omega^2 v_s v_s\right)da-\int_{\partial D(x_3)}x_\rho n_\rho C_{r\alpha m\beta}n_\alpha n_\beta\frac{\partial v_r}{\partial n}\frac{\partial v_m}{\partial n}ds. \tag{3.10}$$

Remarks. (1) By using the integration by parts and the lateral boundary condition (2.13) we can establish the identities

$$\int_{D(x_3)} v_{r,3} v_{m,\beta}\, da - \int_{D(x_3)} v_{r,\beta} v_{m,3}\, da = \frac{d}{dx_3} \int_{D(x_3)} v_r v_{m,\beta}\, da, \tag{3.11}$$

$$\int_{D(x_3)} v_{r,3} v_{m,\beta}\, da - \int_{D(x_3)} v_{r,\beta} v_{m,3}\, da = -\frac{d}{dx_3} \int_{D(x_3)} v_{r,\beta} v_m\, da. \tag{3.12}$$

(2) Theorem 1 can be viewed in connection with a (dynamic) virtual work expression, while Theorems 2 and 3 are closer to the mathematical Rellich identity often used in the study of structural stability. See [Chiriţă et al. 2006; Chiriţă and Ciarletta 2008], for example.

4. Some spatial estimates for appropriate low frequencies

Throughout this section we will study the spatial evolution of the amplitude v_r by starting with the identity established in Theorem 1. To this end we combine the identity (3.1) with (3.11) and (3.12) in the same manner like that used in [Chiriţă and Ciarletta 2006]. Our objective consists of finding measures of the amplitude that are able to furnish information on the spatial evolution of the amplitude v_r for the entire class of anisotropic strongly elliptic elastic materials. Since such task can be too complex for general anisotropic elastic materials we will proceed to pursue our method for some particularly important classes of anisotropic materials, namely those of transversely isotropic and rhombic systems. We recall that for these systems we have established explicit necessary and sufficient conditions in [Chiriţă et al. 2007] characterizing the strong ellipticity condition.

4.1. ***Transversely isotropic materials.*** Many natural and man–made materials are classified as *transversely isotropic* (or *hexagonal*). Such materials are characterized by the fact that one can find a line that allows a rotation of the material about it without changing its properties. The plane, which is perpendicular to this line (the axis of rotational symmetry) is called a *plane of elastic symmetry* or *plane of isotropy*. A modern example for such a material are laminates made of randomly oriented chopped fibers that are in general placed in a certain plane. The effective material properties for a bundled structure have no profound direction in that plane, which then becomes a plane of elastic symmetry. Hence, each plane that contains the axis of rotation is a plane of symmetry, and therefore, transversely isotropic material admits an infinite number of elastic symmetries.

Necessary and sufficient conditions for strong ellipticity to hold for a transversely isotropic linearly elastic solid are established in [Chiriţă et al. 2007; Chiriţă 2006]. In this connection we recall the standard notation

$$\begin{gathered} c_{ij} = C_{iijj}, \quad i, j \in \{1,2,3\} \quad \text{(not summed)}, \quad c_{22} = c_{11}, \quad c_{23} = c_{13}, \\ c_{44} = c_{55} = C_{2323} = C_{1313}, \quad c_{66} = C_{1212} = \tfrac{1}{2}(c_{11} - c_{12}), \end{gathered} \tag{4.1}$$

corresponding to the direction of transverse isotropy coinciding with the x_3 coordinate axis. Apart from terms obtained by use of the symmetries (2.4), these are the only nonzero components C_{ijkl}. Then the necessary and sufficient conditions for strong ellipticity to hold are (*loc. cit.*)

$$c_{11} > 0, \quad c_{33} > 0, \quad c_{55} > 0, \quad c_{11} > c_{12}, \quad |c_{13} + c_{55}| < c_{55} + \sqrt{c_{11} c_{33}}. \tag{4.2}$$

Combining relations (3.1), (3.11) and (3.12) and using (4.1), we obtain

$$\frac{d}{dx_3}\int_{D(x_3)}\big(v_\alpha((c_{55}-\varkappa)v_{3,\alpha}+c_{55}v_{\alpha,3})+v_3(c_{33}v_{3,3}+(c_{13}+\varkappa)(v_{1,1}+v_{2,2}))\big)\,da$$
$$=\int_{D(x_3)}\big(c_{66}(v_{1,2}-v_{2,1})^2-\rho\omega^2 v_\alpha v_\alpha\big)\,da+\int_{D(x_3)}\big(c_{11}(v_{1,1}+v_{2,2})^2+c_{33}v_{3,3}^2+2(c_{13}+\varkappa)(v_{1,1}+v_{2,2})v_{3,3}\big)\,da$$
$$+\int_{D(x_3)}\big((c_{55}(v_{3,1}^2+v_{1,3}^2)+2(c_{55}-\varkappa)v_{1,3}v_{3,1})+(c_{55}(v_{3,2}^2+v_{2,3}^2)+2(c_{55}-\varkappa)v_{2,3}v_{3,2})-\rho\omega^2 v_3^2\big)\,da, \tag{4.3}$$

where $\varkappa \in (0, 2c_{55})$ is a positive parameter at our disposal.

Now we choose the parameter $\varkappa$ in such a way that

$$\max\left(-c_{13}-\sqrt{c_{11}c_{33}},0\right) < \varkappa < \min\left(2c_{55},-c_{13}+\sqrt{c_{11}c_{33}}\right), \tag{4.4}$$

so

$$|c_{13}+\varkappa| < \sqrt{c_{11}c_{33}}, \quad |c_{55}-\varkappa| < c_{55}. \tag{4.5}$$

We deduce that

$$\begin{aligned} c_{55}\left(v_{3,1}^2+v_{1,3}^2\right)+2\left(c_{55}-\varkappa\right)v_{1,3}v_{3,1} &\geq \nu_1\left(v_{3,1}^2+v_{1,3}^2\right), \\ c_{55}\left(v_{3,2}^2+v_{2,3}^2\right)+2\left(c_{55}-\varkappa\right)v_{2,3}v_{3,2} &\geq \nu_1\left(v_{3,2}^2+v_{2,3}^2\right), \end{aligned} \tag{4.6}$$

where

$$\nu_1 = \min\left(\varkappa,\ 2c_{55}-\varkappa\right). \tag{4.7}$$

Moreover, we have

$$c_{11}(v_{1,1}+v_{2,2})^2+c_{33}v_{3,3}^2+2(c_{13}+\varkappa)(v_{1,1}+v_{2,2})v_{3,3} \geq \nu_2\left((v_{1,1}+v_{2,2})^2+v_{3,3}^2\right), \tag{4.8}$$

where

$$\nu_2 = \tfrac{1}{2}\big(c_{11}+c_{33}-\sqrt{(c_{11}-c_{33})^2+4(c_{13}+\varkappa)^2}\,\big). \tag{4.9}$$

On the other hand, in view of the boundary condition (2.13), we obtain

$$\int_{D(x_3)} v_{\alpha,\beta}v_{\alpha,\beta}\,da \geq \lambda\int_{D(x_3)} v_\alpha v_\alpha\,da, \qquad \int_{D(x_3)} v_{3,\beta}v_{3,\beta}\,da \geq \lambda\int_{D(x_3)} v_3^2\,da, \tag{4.10}$$

where $\lambda > 0$ is the first eigenvalue in the two-dimensional clamped membrane eigenvalue problem for the cross section $D(x_3)$.

At this instant we introduce the critical frequency

$$\omega_1 = \sqrt{\frac{\lambda}{\rho}\min\left(\nu_1, \min\left(c_{66}, \nu_2\right)\right)} \tag{4.11}$$

and then assume that the frequency of vibration ω is lower than ω_1, that is

$$0 < \omega < \omega_1. \tag{4.12}$$

Throughout in the remainder of this subsection we will assume that relations (4.4) and (4.12) hold true. Then we introduce the function

$$I_\varkappa(x_3) = -\int_{D(x_3)}\big[v_\alpha\big((c_{55}-\varkappa)v_{3,\alpha}+c_{55}v_{\alpha,3}\big)+v_3\big(c_{33}v_{3,3}+(c_{13}+\varkappa)v_{\alpha,\alpha}\big)\big]\,da \tag{4.13}$$

for all $x_3 \in [0, L]$ and note that relations (4.3)–(4.12) imply

$$-\frac{dI_\varkappa}{dx_3}(x_3) \geq \min(c_{66}, \nu_2)\left(1-\frac{\omega^2}{\omega_1^2}\right)\int_{D(x_3)} v_{\alpha,\beta}v_{\alpha,\beta}\,da$$
$$+\nu_2\int_{D(x_3)} v_{3,3}^2\,da+\nu_1\int_{D(x_3)} v_{\alpha,3}v_{\alpha,3}\,da+\nu_1\left(1-\frac{\omega^2}{\omega_1^2}\right)\int_{D(x_3)} v_{3,\beta}v_{3,\beta}\,da \geq 0 \quad (4.14)$$

and hence $I_\varkappa(x_3)$ is a nonincreasing function with respect to x_3 on $[0, L]$.

We have now all preliminary material in order to state and proof the following result.

Theorem 4. *Let v_r be the amplitude of a harmonic vibration whose frequency is lower than the critical frequency ω_1 given by* (4.11). *Then, for every $\varkappa$ satisfying* (4.4)*, the cross section integral $I_\varkappa(x_3)$ as defined by* (4.13) *is an acceptable measure of the amplitude v_r (that is, $I_\varkappa(x_3) \geq 0$ and $I_\varkappa(x_3) = 0$ implies that $v_r = 0$) and it satisfies the spatial decay estimate*

$$0 \leq I_\varkappa(x_3) \leq I_\varkappa(0)\, e^{-\sigma_1 x_3} \quad \textit{for all} \;\; x_3 \in [0, L], \tag{4.15}$$

where σ_1 is given by

$$\frac{1}{\sigma_1} = \frac{1}{\sqrt{\lambda}}\max\left\{\frac{c_{55}+\sqrt{c_{11}c_{33}}}{\min(c_{66}, \nu_2)\left(1-\omega^2/\omega_1^2\right)}, \frac{c_{33}}{2\nu_2}, \frac{c_{55}}{2\nu_1}, \frac{c_{33}+c_{55}+\sqrt{c_{11}c_{33}}}{2\nu_1\left(1-\omega^2/\omega_1^2\right)}\right\}. \tag{4.16}$$

Proof. On the basis of the end boundary condition (2.15) and relation (4.13) we deduce that $I_\varkappa(L) = 0$, so that we have

$$I_\varkappa(x_3) \geq 0 \quad \text{for all} \;\; x_3 \in [0, L]. \tag{4.17}$$

Thus, $I_\varkappa(x_3)$ represents an acceptable measure for the amplitude v_r of the harmonic vibration.

Now, by using the Schwarz and arithmetic-geometric mean inequalities, from (4.5) and (4.13) we obtain the estimate

$$|I_\varkappa(x_3)| \leq \frac{1}{\sqrt{\lambda}}\left(c_{55}+\sqrt{c_{11}c_{33}}\right)\int_{D(x_3)} v_{\alpha,\beta}v_{\alpha,\beta}\,da+\frac{1}{2\sqrt{\lambda}}c_{33}\int_{D(x_3)} v_{3,3}^2\,da$$
$$+\frac{1}{2\sqrt{\lambda}}c_{55}\int_{D(x_3)} v_{\alpha,3}v_{\alpha,3}\,da+\frac{1}{2\sqrt{\lambda}}\left(c_{33}+c_{55}+\sqrt{c_{11}c_{33}}\right)\int_{D(x_3)} v_{3,\beta}v_{3,\beta}\,da. \quad (4.18)$$

By combining (4.14) and (4.18) we obtain the first order differential inequality

$$\frac{dI_\varkappa}{dx_3}(x_3)+\sigma_1 I_\varkappa(x_3) \leq 0 \quad \text{for all} \;\; x_3 \in [0, L], \tag{4.19}$$

which, when integrated, furnishes the exponential spatial decay estimate (4.15). □

4.2. ***Rhombic materials.*** Suppose the cylinder is filled with a rhombic elastic material with the group $\mathscr{C}_3$ generated by $\mathbf{R}_{e_3}^{\pi}$, $\mathbf{R}_{e_2}^{\pi}$ (here $\mathbf{R}_e^{\theta}$ is the orthogonal tensor corresponding to a right–handed rotation through the angle $\theta \in (0, 2\pi)$, about an axis in the direction of the unit vector $\mathbf{e}$). According to Gurtin

[1972], this class of materials is characterized by

$$\begin{gathered} C_{1123}=C_{1131}=C_{1112}=C_{2223}=C_{2231}=C_{2212}=0, \\ C_{3323}=C_{3331}=C_{3312}=C_{2331}=C_{2312}=C_{3112}=0, \\ c_{11}=C_{1111}, \quad c_{22}=C_{2222}, \quad c_{33}=C_{3333}, \quad c_{12}=C_{1122}, \quad c_{23}=C_{2233}, \\ c_{31}=C_{3311}, \quad c_{44}=C_{2323}, \quad c_{55}=C_{1313}, \quad c_{66}=C_{1212}. \end{gathered} \tag{4.20}$$

The strong ellipticity condition (2.5) becomes

$$\begin{aligned} c_{11}n_1^2m_1^2+c_{22}n_2^2m_2^2+c_{33}n_3^2m_3^2+c_{66}\left(n_1m_2+n_2m_1\right)^2+c_{44}\left(n_3m_2+n_2m_3\right)^2 \\ +c_{55}\left(n_1m_3+n_3m_1\right)^2+2c_{12}n_1m_1n_2m_2+2c_{23}n_2m_2n_3m_3+2c_{31}n_3m_3n_1m_1>0, \end{aligned} \tag{4.21}$$

for all nonzero vectors (m_1, m_2, m_3) and (n_1, n_2, n_3). It is equivalent to the conditions [Chiriţă et al. 2007]

$$c_{11}>0, \quad c_{22}>0, \quad c_{33}>0, \quad c_{44}>0, \quad c_{55}>0, \quad c_{66}>0, \tag{4.22}$$

$$\begin{gathered} -2c_{66}+\varkappa_3^i\sqrt{c_{11}c_{22}}<c_{12}<\varkappa_3^s\sqrt{c_{11}c_{22}}, \qquad -2c_{44}+\varkappa_1^i\sqrt{c_{22}c_{33}}<c_{23}<\varkappa_1^s\sqrt{c_{22}c_{33}}, \\ -2c_{55}+\varkappa_2^i\sqrt{c_{11}c_{33}}<c_{13}<\varkappa_2^s\sqrt{c_{11}c_{33}}, \end{gathered} \tag{4.23}$$

where $\left(\varkappa_1^i, \varkappa_1^s\right)$, $\left(\varkappa_2^i, \varkappa_2^s\right)$ and $\left(\varkappa_3^i, \varkappa_3^s\right)$ are solutions with respect to x, y and z of the equation $x^2+y^2+z^2-2xyz-1=0$, satisfying $|x|<1,\ |y|<1, |z|<1$ and

$$x\in\left\{\frac{c_{23}}{\sqrt{c_{22}c_{33}}}, \frac{c_{23}+2c_{44}}{\sqrt{c_{22}c_{33}}}\right\}, \quad y\in\left\{\frac{c_{13}}{\sqrt{c_{11}c_{33}}}, \frac{c_{13}+2c_{55}}{\sqrt{c_{11}c_{33}}}\right\}, \quad z\in\left\{\frac{c_{12}}{\sqrt{c_{11}c_{22}}}, \frac{c_{12}+2c_{66}}{\sqrt{c_{11}c_{22}}}\right\}. \tag{4.24}$$

This statement is equivalent with the relation (4.22) and all points $P(x, y, z)$, with coordinates satisfying (4.24) lie inside the region limited by the surface $S(x, y, z)\equiv x^2+y^2+z^2-2xyz-1=0$, with $|x|<1$, $|y|<1$, $|z|<1$.

In the case of a rhombic material the relation (4.3) is replaced by

$$\begin{aligned} \frac{d}{dx_3}\int_{D(x_3)}\big[v_1\left(c_{55}v_{1,3}+(c_{55}-\varkappa_2)v_{3,1}\right)+v_2\left(c_{44}v_{2,3}+(c_{44}-\varkappa_1)v_{3,2}\right) \\ +v_3\left((c_{13}+\varkappa_2)v_{1,1}+(c_{23}+\varkappa_1)v_{2,2}+c_{33}v_{3,3}\right)\big]da \\ =\int_{D(x_3)}\left[c_{11}v_{1,1}^2+c_{22}v_{2,2}^2+c_{33}v_{3,3}^2+2(c_{12}+\varkappa_3)v_{1,1}v_{2,2}+2(c_{13}+\varkappa_2)v_{1,1}v_{3,3}+2(c_{23}+\varkappa_1)v_{2,2}v_{3,3}\right]da \\ +\int_{D(x_3)}\left[c_{66}(v_{1,2}^2+v_{2,1}^2)+2(c_{66}-\varkappa_3)v_{1,2}v_{2,1}\right]da+\int_{D(x_3)}\left[c_{55}(v_{3,1}^2+v_{1,3}^2)+2(c_{55}-\varkappa_2)v_{1,3}v_{3,1}\right]da \\ +\int_{D(x_3)}\left[c_{44}(v_{3,2}^2+v_{2,3}^2)+2(c_{44}-\varkappa_1)v_{2,3}v_{3,2}\right]da-\int_{D(x_3)}\rho\omega^2v_sv_s\,da, \end{aligned} \tag{4.25}$$

where $\varkappa_1\in[0, 2c_{44}]$, $\varkappa_2\in[0, 2c_{55}]$ and $\varkappa_3\in[0, 2c_{66}]$ are positive parameters at our disposal. In view of the assumptions (4.22) and (4.23) we can choose $\varkappa_1\in[0, 2c_{44}]$, $\varkappa_2\in[0, 2c_{55}]$, $\varkappa_3\in[0, 2c_{66}]$ so that $P(x, y, z)$, with coordinates

$$x=\frac{c_{23}+\varkappa_1}{\sqrt{c_{22}c_{33}}}, \qquad y=\frac{c_{13}+\varkappa_2}{\sqrt{c_{11}c_{33}}}, \qquad z=\frac{c_{12}+\varkappa_3}{\sqrt{c_{11}c_{22}}},$$

lies inside the region limited by the surface $S(x, y, z)$. With these choices we have

$$c_{44}\left(v_{3,2}^2+v_{2,3}^2\right)+2\left(c_{44}-\varkappa_1\right)v_{2,3}v_{3,2}\geq\xi_1\left(v_{3,2}^2+v_{2,3}^2\right), \tag{4.26}$$

$$c_{55}\left(v_{3,1}^2+v_{1,3}^2\right)+2\left(c_{55}-\varkappa_2\right)v_{1,3}v_{3,1}\geq\xi_2\left(v_{3,1}^2+v_{1,3}^2\right), \tag{4.27}$$

$$c_{66}\left(v_{1,2}^2+v_{2,1}^2\right)+2\left(c_{66}-\varkappa_3\right)v_{1,2}v_{2,1}\geq\xi_3\left(v_{1,2}^2+v_{2,1}^2\right), \tag{4.28}$$

$$\begin{aligned}c_{11}v_{1,1}^2+c_{22}v_{2,2}^2+c_{33}v_{3,3}^2+2\left(c_{12}+\varkappa_3\right)v_{1,1}v_{2,2}&\\+2\left(c_{13}+\varkappa_2\right)v_{1,1}v_{3,3}+2\left(c_{23}+\varkappa_1\right)v_{2,2}v_{3,3}&\geq\xi_4\left(v_{1,1}^2+v_{2,2}^2+v_{3,3}^2\right),\end{aligned} \tag{4.29}$$

where

$$\xi_1=\min\left(2c_{44}-\varkappa_1,\varkappa_1\right),\quad \xi_2=\min\left(2c_{55}-\varkappa_2,\varkappa_2\right),\quad \xi_3=\min\left(2c_{66}-\varkappa_3,\varkappa_3\right) \tag{4.30}$$

and ξ_4 is the lowest positive eigenvalue of the 3×3 matrix

$$\begin{pmatrix} c_{11} & c_{12}+\varkappa_3 & c_{13}+\varkappa_2\\ c_{12}+\varkappa_3 & c_{22} & c_{23}+\varkappa_1\\ c_{13}+\varkappa_2 & c_{23}+\varkappa_1 & c_{33}\end{pmatrix}. \tag{4.31}$$

So we have to introduce the function

$$\begin{aligned}J_\varkappa(x_3)=-\int_{D(x_3)}\Big[v_1\big((c_{55}-\varkappa_2)v_{3,1}+c_{55}v_{1,3}\big)+v_2\big((c_{44}-\varkappa_1)v_{3,2}+c_{44}v_{2,3}\big)&\\+v_3\big((c_{13}+\varkappa_2)v_{1,1}+(c_{23}+\varkappa_1)v_{2,2}+c_{33}v_{3,3}\big)\Big]\,da&\end{aligned} \tag{4.32}$$

and note that identity (4.25) and relations (4.10) and (4.26)–(4.29) imply

$$\begin{aligned}-\frac{dJ_\varkappa}{dx_3}(x_3)\geq\int_{D(x_3)}\left(\xi_4\left(v_{1,1}^2+v_{2,2}^2\right)+\xi_3\left(v_{1,2}^2+v_{2,1}^2\right)-\frac{\rho\omega^2}{\lambda}v_{\alpha,\beta}v_{\alpha,\beta}\right)da&\\+\int_{D(x_3)}\left(\xi_1v_{3,2}^2+\xi_2v_{3,1}^2-\frac{\rho\omega^2}{\lambda}v_{3,\alpha}v_{3,\alpha}\right)da+\int_{D(x_3)}\left(\xi_1v_{2,3}^2+\xi_2v_{1,3}^2+\xi_4v_{3,3}^2\right)da.&\end{aligned} \tag{4.33}$$

At this point we introduce the critical frequency

$$\omega_2=\sqrt{\frac{\lambda}{\rho}\min\left(\min\left(\xi_1,\xi_2\right),\min\left(\xi_3,\xi_4\right)\right)} \tag{4.34}$$

and assume that the vibration frequency ω is lower than ω_2:

$$0<\omega<\omega_2. \tag{4.35}$$

Thus, we have

$$\begin{aligned}-\frac{dJ_\varkappa}{dx_3}(x_3)\geq\min\left(\xi_3,\xi_4\right)\left(1-\frac{\omega^2}{\omega_2^2}\right)\int_{D(x_3)}v_{\alpha,\beta}v_{\alpha,\beta}\,da+\min\left(\xi_1,\xi_2\right)\left(1-\frac{\omega^2}{\omega_2^2}\right)\int_{D(x_3)}v_{3,\alpha}v_{3,\alpha}\,da&\\+\int_{D(x_3)}\left(\xi_1v_{2,3}^2+\xi_2v_{1,3}^2+\xi_4v_{3,3}^2\right)da\geq 0.&\end{aligned} \tag{4.36}$$

Consequently, $J_\varkappa(x_3)$ is a nonincreasing function with respect to x_3 on $[0, L]$.

Theorem 5. *Let v_r be the amplitude of a harmonic vibration whose frequency is lower than the critical frequency ω_2 of* (4.34). *Then the cross section integral $J_\varkappa(x_3)$ as defined by* (4.32) *is an acceptable measure of the amplitude v_r (that is, $J_\varkappa(x_3) \geq 0$ and $J_\varkappa(x_3) = 0$ implies that $v_r = 0$) and it satisfies the spatial decay estimate*

$$0 \leq J_\varkappa(x_3) \leq J_\varkappa(0)\, e^{-\sigma_2 x_3} \quad \textit{for all } x_3 \in [0, L], \tag{4.37}$$

where σ_2 is given by

$$\frac{1}{\sigma_2} = \frac{1}{2\sqrt{\lambda}} \max \left\{ \frac{\max\left(2c_{55} + \sqrt{c_{11}c_{33}},\, 2c_{44} + \sqrt{c_{22}c_{33}}\right)}{\min(\xi_3, \xi_4)\left(1 - \omega^2/\omega_2^2\right)}, \right.$$
$$\left. \frac{c_{44}}{\xi_1}, \frac{c_{55}}{\xi_2}, \frac{c_{33}}{\xi_4}, \frac{\max(c_{44}, c_{55}) + \sqrt{c_{33}}\left(\sqrt{c_{11}} + \sqrt{c_{22}} + \sqrt{c_{33}}\right)}{\min(\xi_1, \xi_2)\left(1 - \omega^2/\omega_2^2\right)} \right\}. \tag{4.38}$$

Proof. On the basis of the end boundary condition (2.15) and relation (4.32) we deduce that $J_\varkappa(L) = 0$, so that we have

$$J_\varkappa(x_3) \geq 0 \quad \text{for all } x_3 \in [0, L]. \tag{4.39}$$

Thus, $J_\varkappa(x_3)$ represents an acceptable measure for the amplitude v_r of the harmonic vibration.

We further note that

$$|c_{44} - \varkappa_1| < c_{44}, \quad |c_{55} - \varkappa_2| < c_{55}, \quad |c_{13} + \varkappa_2| < \sqrt{c_{11}c_{33}}, \quad |c_{23} + \varkappa_1| < \sqrt{c_{22}c_{33}}. \tag{4.40}$$

On this basis and by using the Schwarz and arithmetic-geometric mean inequalities and (4.10), we obtain from (4.32) the estimate

$$\begin{aligned} |J_\varkappa(x_3)| \leq{}& \frac{1}{2\sqrt{\lambda}} \max\left(2c_{55} + \sqrt{c_{11}c_{33}},\, 2c_{44} + \sqrt{c_{22}c_{33}}\right) \int_{D(x_3)} v_{\alpha,\beta} v_{\alpha,\beta}\, da \\ &+ \frac{1}{2\sqrt{\lambda}} \left[\max(c_{44}, c_{55}) + \sqrt{c_{33}}\left(\sqrt{c_{11}} + \sqrt{c_{22}} + \sqrt{c_{33}}\right)\right] \int_{D(x_3)} v_{3,\alpha} v_{3,\alpha}\, da \\ &+ \frac{1}{2\sqrt{\lambda}} \int_{D(x_3)} \left(c_{33} v_{3,3}^2 + c_{44} v_{2,3}^2 + c_{55} v_{1,3}^2\right) da. \end{aligned} \tag{4.41}$$

By combining (4.36), (4.38) and (4.41) we obtain the first order differential inequality

$$\frac{dJ_\varkappa}{dx_3}(x_3) + \sigma_2 J_\varkappa(x_3) \leq 0 \quad \text{for all } x_3 \in [0, L], \tag{4.42}$$

whose integration furnishes the spatial decay expressed by (4.37). □

The analysis of this section can be extended to the case of a semi-infinite cylinder, that is the case when $L \to \infty$. We shall exemplify this for the case of measure $J_\varkappa(x_3)$. In view of (4.25) and (4.32), by an integration $[x_3, L]$, we obtain

$$J_\varkappa(x_3) - J_\varkappa(L) = E(x_3, L), \tag{4.43}$$

where

$$E(x_3, L) = \int_{B(x_3,L)} \Big(c_{11} v_{1,1}^2 + c_{22} v_{2,2}^2 + c_{33} v_{3,3}^2 + 2(c_{12}+\varkappa_3) v_{1,1} v_{2,2} + 2(c_{13}+\varkappa_2) v_{1,1} v_{3,3} \\ + 2(c_{23}+\varkappa_1) v_{2,2} v_{3,3} + c_{66}(v_{1,2}^2 + v_{2,1}^2) + 2(c_{66}-\varkappa_3) v_{1,2} v_{2,1} + c_{55}(v_{3,1}^2 + v_{1,3}^2) \\ + 2(c_{55}-\varkappa_2) v_{1,3} v_{3,1} + c_{44}(v_{3,2}^2 + v_{2,3}^2) + 2(c_{44}-\varkappa_1) v_{2,3} v_{3,2} - \rho\omega^2 v_s v_s \Big) dv \geq 0, \quad (4.44)$$

with $B(x_3, L) = B_{x_3} \backslash B_L$ and B_{x_3} is defined by relation (2.1). We conclude that $J_\varkappa(\infty) = \lim_{L\to\infty} J_\varkappa(L)$ exists and is finite if and only if there is finite the energetic measure $E(x_3) = \lim_{L\to\infty} E(x_3, L)$ associated with the amplitude v_r in the cylinder B_{x_3}. Since $J_\varkappa(x_3)$ is a nonincreasing function with respect to x_3, there are the only two possibilities: (a) $J_\varkappa(x_3) \geq 0$ for all $x_3 \in [0, \infty)$ or (b) there exists $x_3^* \in [0, \infty)$ so that $J_\varkappa\left(x_3^*\right) < 0$.

In the case (a) we can apply the same procedure as in the above to obtain the spatial decay estimate (4.37). So in what follows we shall consider the case (b), that is we will suppose that $J_\varkappa\left(x_3^*\right) < 0$. Then we have

$$J_\varkappa(x_3) < 0 \quad \text{for all } \; x_3 \in [x_3^*, \infty), \quad (4.45)$$

so that (4.36), (4.38), (4.41) and (4.45) now give

$$\frac{dJ_\varkappa}{dx_3}(x_3) - \sigma_2 J_\varkappa(x_3) \leq 0 \quad \text{for all } \; x_3 \in [0, L], \quad (4.46)$$

which implies

$$-J_\varkappa(x_3) \geq -J_\varkappa\left(x_3^*\right) e^{\sigma_2\left(x_3 - x_3^*\right)} \quad \text{for all } \; x_3 \in [x_3^*, \infty) \quad (4.47)$$

and hence $J_\varkappa(\infty) = -\infty$ and the energetic measure $E(x_3)$ is infinite.

We may summarize this analysis in the following alternative of Phragmén–Lindelöf type result.

Theorem 6. *In the context of a semi-infinite cylinder made of a rhombic elastic material, for all harmonic vibrations with frequency lower than the critical value ω_2, the amplitude v_r either has a finite energetic measure $E(x_3)$ and then we have*

$$E(x_3) \leq E(0)\, e^{-\sigma_2 x_3} \quad \textit{for all } \; x_3 \in [0, \infty), \quad (4.48)$$

or it has an infinite energetic measure and then $-J_\varkappa(x_3)$ goes to infinity faster than the exponential $e^{\sigma_2\left(x_3 - x_3^*\right)}$.

5. Spatial estimates for appropriate high frequencies

Throughout this section we will study the spatial evolution of the amplitude v_r by starting with the identity established in Theorem 3. To this end we note that the strong ellipticity condition (2.5) implies that

$$C_{k3l3}\zeta_k\zeta_l > 0 \quad \text{for all nonzero vectors } \zeta_r, \quad (5.1)$$

and

$$C_{r\alpha s\beta} m_r m_s n_\alpha n_\beta > 0 \quad \text{for all nonzero vectors } \; (m_1, m_2, m_3), \;\; (n_1, n_2). \quad (5.2)$$

We further assume that ∂D is star shaped with respect to the origin so that $x_\rho n_\rho \geq h_0 > 0$, with h_0 constant.

On this basis we observe that

$$0 \le \int_{\partial D(x_3)} x_\rho n_\rho C_{r\alpha m\beta} n_\alpha n_\beta \frac{\partial v_r}{\partial n}\frac{\partial v_m}{\partial n} ds \le dC \int_{\partial D(x_3)} \frac{\partial v_r}{\partial n}\frac{\partial v_r}{\partial n} ds, \tag{5.3}$$

where $C = \sqrt{C_{r\alpha m\beta}C_{r\alpha m\beta}}$ and

$$d = \sup_{(x_1,x_2)\in\partial D} \sqrt{x_\alpha x_\alpha}. \tag{5.4}$$

Further, we introduce

$$m_0 = \max_{x_3\in[0,L]} \frac{\displaystyle\int_{\partial D} \frac{\partial v_r}{\partial n}\frac{\partial v_r}{\partial n} ds}{\displaystyle\int_D v_r v_r \, da}, \qquad \omega^* = \frac{1}{\rho} dC m_0, \tag{5.5}$$

and assume that

$$\omega \ge \omega^*. \tag{5.6}$$

Whereas one cannot, in general, obtain m_0 explicitly, we have the crude bound

$$m_0 \le m_1, \quad \text{with} \quad m_1 = \sup_{v_i\in H_0^1(D)} \frac{\displaystyle\int_{\partial D} \frac{\partial v_r}{\partial n}\frac{\partial v_r}{\partial n} ds}{\displaystyle\int_D v_r v_r \, da}. \tag{5.7}$$

So, when m_1 is finite, we can take $\omega^* = \dfrac{1}{\rho} dC m_1$ and obtain an explicit critical value for the frequency of vibration.

Then the identity (3.10), relations (5.3) and (5.6), and the definition of m_0 in (5.5) give

$$\frac{d}{dx_3}\int_{D(x_3)} \left[2C_{3rmn}v_{m,n}v_r + 2x_\rho C_{3smn}v_{s,\rho}v_{m,n} + x_3\left(C_{r3m3}v_{r,3}v_{m,3} - C_{r\alpha m\beta}v_{r,\alpha}v_{m,\beta} + \rho\omega^2 v_s v_s\right)\right] da \ge \int_{D(x_3)} C_{rsmn}v_{r,s}v_{m,n}\, da. \tag{5.8}$$

Our objective now is to find measures of the amplitude that are able to furnish information on the spatial evolution of the amplitude v_r for the entire class of anisotropic strongly elliptic elastic materials. We now pursue our method for transversely isotropic and rhombic systems.

5.1. ***Transversely isotropic materials.*** We first consider the class of transversely isotropic materials as defined in Section 4.1. Relations (5.8), (3.11) and (3.12) give

$$\begin{aligned}
\frac{d}{dx_3}\int_{D(x_3)} \Big(& v_\alpha\big((2c_{55}-\varkappa)v_{3,\alpha}+2c_{55}v_{\alpha,3}\big) + v_3\big(2c_{33}v_{3,3}+(2c_{13}+\varkappa)v_{\alpha,\alpha}\big) \\
& + 2c_{55}x_\rho v_{\alpha,\rho}(v_{3,\alpha}+v_{\alpha,3}) + 2x_\rho v_{3,\rho}(c_{13}v_{\alpha,\alpha}+c_{33}v_{3,3}) \\
& + x_3\big[c_{55}(v_{1,3}^2+v_{2,3}^2)+c_{33}v_{3,3}^2-c_{66}(v_{1,2}-v_{2,1})^2-c_{11}(v_{1,1}+v_{2,2})^2 \\
& -2c_{13}v_{\alpha,\alpha}v_{3,3}-c_{55}v_{3,\alpha}v_{3,\alpha}-2c_{55}v_{\alpha,3}v_{3,\alpha}+\rho\omega^2 v_s v_s\big]\Big)\, da \\
\ge \int_{D(x_3)} & c_{66}(v_{1,2}-v_{2,1})^2\, da + \int_{D(x_3)} \big[c_{11}(v_{1,1}+v_{2,2})^2+c_{33}v_{3,3}^2+2(c_{13}+\varkappa)(v_{1,1}+v_{2,2})v_{3,3}\big]\, da \\
& + \int_{D(x_3)} \big[c_{55}(v_{3,1}^2+v_{1,3}^2)+2(c_{55}-\varkappa)v_{1,3}v_{3,1}+c_{55}(v_{3,2}^2+v_{2,3}^2)+2(c_{55}-\varkappa)v_{2,3}v_{3,2}\big]\, da,
\end{aligned} \tag{5.9}$$

where $\varkappa \in (0, 2c_{55})$ is a positive parameter chosen in such way to satisfy relation (4.4). Therefore, we can introduce the function

$$\begin{aligned}\mathcal{I}_\varkappa(x_3) = -\int_{D(x_3)} \Big(& v_\alpha\big((2c_{55}-\varkappa)v_{3,\alpha} + 2c_{55}v_{\alpha,3}\big) + v_3\big(2c_{33}v_{3,3} + (2c_{13}+\varkappa)v_{\alpha,\alpha}\big) \\ & + 2c_{55}x_\rho v_{\alpha,\rho}(v_{3,\alpha}+v_{\alpha,3}) + 2x_\rho v_{3,\rho}(c_{13}v_{\alpha,\alpha} + c_{33}v_{3,3}) \\ & + x_3\big[c_{55}(v_{1,3}^2 + v_{2,3}^2) + c_{33}v_{3,3}^2 - c_{66}(v_{1,2}-v_{2,1})^2 - c_{11}(v_{1,1}+v_{2,2})^2 \\ & - 2c_{13}v_{\alpha,\alpha}v_{3,3} - c_{55}v_{3,\alpha}v_{3,\alpha} - 2c_{55}v_{\alpha,3}v_{3,\alpha} + \rho\omega^2 v_s v_s\big]\Big)\, da \end{aligned} \tag{5.10}$$

and note that relations (4.4), (4.6), (4.8) and (5.9) imply

$$\begin{aligned}-\frac{d\mathcal{I}_\varkappa}{dx_3}(x_3) &\geq \min(c_{66}, \nu_2)\int_{D(x_3)} v_{\alpha,\beta}v_{\alpha,\beta}\, da + \nu_2\int_{D(x_3)} v_{3,3}^2\, da + \nu_1\int_{D(x_3)} v_{\alpha,3}v_{\alpha,3}\, da + \nu_1\int_{D(x_3)} v_{3,\beta}v_{3,\beta}\, da \\ &\geq 0.\end{aligned} \tag{5.11}$$

Thus, $\mathcal{I}_\varkappa(x_3)$ is a nonincreasing function with respect to x_3 on $[0, L]$.

Theorem 7. *Let v_r be the amplitude of a harmonic vibration whose frequency is greater than the critical frequency*

$$\omega_1^* = \frac{dm_0}{\rho}\sqrt{2c_{11}^2 + 2c_{12}^2 + (c_{11}-c_{12})^2 + 2c_{55}^2}.$$

Then the cross section integral $\mathcal{I}_\varkappa(x_3)$ as defined by (5.10) *is an acceptable measure of the amplitude v_r (that is, $\mathcal{I}_\varkappa(x_3) \geq 0$ and $\mathcal{I}_\varkappa(x_3) = 0$ implies that $v_r = 0$) and it satisfies the spatial decay estimate*

$$0 \leq \mathcal{I}_\varkappa(x_3) \leq \mathcal{I}_\varkappa(0)\left(1 + \frac{\beta}{\alpha}x_3\right)^{-1/\beta} \quad \textit{for all } x_3 \in [0, L], \tag{5.12}$$

where α and β are positive constants computable in terms of the elastic coefficients, λ, d, ω and ρ.

Proof. On the basis of the end boundary condition (2.15) and relation (5.10) we deduce that $\mathcal{I}_\varkappa(L) = 0$, so that we have

$$\mathcal{I}_\varkappa(x_3) \geq 0 \quad \text{for all } x_3 \in [0, L]. \tag{5.13}$$

Thus, $\mathcal{I}_\varkappa(x_3)$ represents an acceptable measure for the amplitude v_r of the harmonic vibration.

On the other hand, by using the Schwarz and arithmetic-geometric mean inequalities, from (4.10), (5.4) and (5.10) we obtain the estimates

$$\begin{aligned}&\left|\int_{D(x_3)} \big[v_\alpha\big((2c_{55}-\varkappa)v_{3,\alpha} + 2c_{55}v_{\alpha,3}\big) + v_3\big(2c_{33}v_{3,3} + (2c_{13}+\varkappa)v_{\alpha,\alpha}\big)\big]\, da\right| \\ &\quad \leq \frac{2}{\sqrt{\lambda}}(2c_{55} + \sqrt{c_{11}c_{33}})\int_{D(x_3)} v_{\alpha,\beta}v_{\alpha,\beta}\, da + \frac{c_{55}}{\sqrt{\lambda}}\int_{D(x_3)} v_{\alpha,3}v_{\alpha,3}\, da \\ &\qquad + \frac{1}{\sqrt{\lambda}}(2c_{55} + c_{33} + \sqrt{c_{11}c_{33}})\int_{D(x_3)} v_{3,\alpha}v_{3,\alpha}\, da + \frac{c_{33}}{\sqrt{\lambda}}\int_{D(x_3)} v_{3,3}^2\, da,\end{aligned} \tag{5.14}$$

$$\left|\int_{D(x_3)} \left[2c_{55}x_\rho v_{\alpha,\rho}(v_{3,\alpha}+v_{\alpha,3})+2x_\rho v_{3,\rho}(c_{13}v_{\alpha,\alpha}+c_{33}v_{3,3})\right] da\right|$$
$$\le d(c_{55}+2\,|c_{13}|)\int_{D(x_3)} v_{\alpha,\beta}v_{\alpha,\beta}\,da+2dc_{55}\int_{D(x_3)} v_{\alpha,3}v_{\alpha,3}\,da$$
$$+d(2c_{55}+2c_{33}+|c_{13}|)\int_{D(x_3)} v_{3,\alpha}v_{3,\alpha}\,da+2dc_{33}\int_{D(x_3)} v_{3,3}^2\,da, \quad (5.15)$$

$$\left|\int_{D(x_3)} \left[c_{55}(v_{1,3}^2+v_{2,3}^2)+c_{33}v_{3,3}^2-c_{66}(v_{1,2}-v_{2,1})^2-c_{11}(v_{1,1}+v_{2,2})^2\right.\right.$$
$$\left.\left.-2c_{13}v_{\alpha,\alpha}v_{3,3}-c_{55}v_{3,\alpha}v_{3,\alpha}-2c_{55}v_{\alpha,3}v_{3,\alpha}+\rho\omega^2 v_s v_s\right] da\right|$$
$$\le\left(\max(c_{11},c_{66})+2\,|c_{13}|+\frac{\rho\omega^2}{\lambda}\right)\int_{D(x_3)} v_{\alpha,\beta}v_{\alpha,\beta}\,da+2c_{55}\int_{D(x_3)} v_{\alpha,3}v_{\alpha,3}\,da$$
$$+\left(2c_{55}+\frac{\rho\omega^2}{\lambda}\right)\int_{D(x_3)} v_{3,\alpha}v_{3,\alpha}\,da+(c_{33}+|c_{13}|)\int_{D(x_3)} v_{3,3}^2\,da. \quad (5.16)$$

Therefore, if we use the estimates (5.14)–(5.16) in (5.10) and then use (5.11), we obtain the differential inequality

$$|\mathcal{I}_\varkappa(x_3)|\le-(\alpha+\beta x_3)\frac{d\mathcal{I}_\varkappa}{dx_3}(x_3)\quad\text{for all}\ \ x_3\in[0,L], \quad (5.17)$$

where

$$\alpha=\max\left\{\frac{1}{\min(c_{66},\nu_2)}\left[\left(\frac{4}{\sqrt{\lambda}}+d\right)c_{55}+\frac{2}{\sqrt{\lambda}}\sqrt{c_{11}c_{33}}+2d\,|c_{13}|\right],\right.$$
$$\frac{1}{\nu_1}\left[2\left(\frac{1}{\sqrt{\lambda}}+d\right)c_{55}+\left(\frac{1}{\sqrt{\lambda}}+2d\right)c_{33}+\frac{1}{\sqrt{\lambda}}\sqrt{c_{11}c_{33}}+d\,|c_{13}|\right],$$
$$\left.\frac{1}{\nu_1}\left(\frac{1}{\sqrt{\lambda}}+2d\right)c_{55},\ \frac{1}{\nu_2}\left(\frac{1}{\sqrt{\lambda}}+2d\right)c_{33}\right\}, \quad (5.18)$$

$$\beta=\max\left\{\frac{1}{\min\,(c_{66},\nu_2)}\left(\max(c_{11},c_{66})+2\,|c_{13}|+\frac{\rho\omega^2}{\lambda}\right),\ \frac{1}{\nu_1}\left(2c_{55}+\frac{\rho\omega^2}{\lambda}\right),\ \frac{2c_{55}}{\nu_1},\ \frac{c_{33}+|c_{13}|}{\nu_2}\right\}. \quad (5.19)$$

To integrate the differential inequality (5.17) we write it in the form

$$\frac{d\mathcal{I}_\varkappa}{dx_3}(x_3)+\frac{d}{dx_3}\left(\int_0^{x_3}\frac{1}{\alpha+\beta t}dt\right)\mathcal{I}_\varkappa(x_3)\le 0\quad\text{for all}\ \ x_3\in[0,L], \quad (5.20)$$

which, multiplied by $\exp\left(\int_0^{x_3} dt/(\alpha+\beta t)\right)$ and then integrated with respect to x_3, gives the estimate (5.12). □

5.2. ***Rhombic materials.*** For a rhombic material, we proceed similarly. Relation (5.8) combined with (3.11) and (3.12) gives

$$
\begin{aligned}
-\frac{d}{dx_3}&\mathcal{I}_\varkappa(x_3)\\
&\geq \int_{D(x_3)} \bigl(c_{11}v_{1,1}^2+c_{22}v_{2,2}^2+c_{33}v_{3,3}^2+2(c_{12}+\varkappa_3)v_{1,1}v_{2,2}+2(c_{13}+\varkappa_2)v_{1,1}v_{3,3}+2(c_{23}+\varkappa_1)v_{2,2}v_{3,3}\bigr)\,da\\
&\quad+\int_{D(x_3)}\bigl(c_{66}v_{1,2}^2+c_{66}v_{2,1}^2+2(c_{66}-\varkappa_3)v_{1,2}v_{2,1}\bigr)\,da+\int_{D(x_3)}\bigl(c_{55}v_{3,1}^2+c_{55}v_{1,3}^2+2(c_{55}-\varkappa_2)v_{1,3}v_{3,1}\bigr)\,da\\
&\qquad+\int_{D(x_3)}\bigl(c_{44}v_{3,2}^2+c_{44}v_{2,3}^2+2(c_{44}-\varkappa_1)v_{2,3}v_{3,2}\bigr)\,da, \qquad (5.21)
\end{aligned}
$$

where $\varkappa_1$, $\varkappa_2$, $\varkappa_3$ satisfy the conditions requested in Section 4.2 and we have introduced the function

$$
\begin{aligned}
\mathcal{I}_\varkappa(x_3) = -\int_{D(x_3)}\Bigl(&v_1\bigl((2c_{55}-\varkappa_2)v_{3,1}+2c_{55}v_{1,3}\bigr)+v_2\bigl((2c_{44}-\varkappa_1)v_{3,2}+2c_{44}v_{2,3}\bigr)\\
&+v_3\bigl((2c_{13}+\varkappa_2)v_{1,1}+(2c_{23}+\varkappa_1)v_{2,2}+2c_{33}v_{3,3}\bigr)+2c_{55}x_\rho v_{1,\rho}(v_{3,1}+v_{1,3})\\
&+2c_{44}x_\rho v_{2,\rho}(v_{3,2}+v_{2,3})+2x_\rho v_{3,\rho}(c_{13}v_{1,1}+c_{23}v_{2,2}+c_{33}v_{3,3})\\
&+x_3\bigl[c_{55}v_{1,3}^2+c_{44}v_{2,3}^2+c_{33}v_{3,3}^2-c_{66}(v_{1,2}+v_{2,1})^2\\
&\quad-(c_{11}v_{1,1}^2+c_{22}v_{2,2}^2+2c_{12}v_{1,1}v_{2,2}+c_{55}v_{3,1}^2+c_{44}v_{3,2}^2)+\rho\omega^2 v_s v_s\bigr]\Bigr)\,da. \qquad (5.22)
\end{aligned}
$$

Now (4.26)–(4.29) and (5.21) give

$$
\begin{aligned}
-\frac{d\mathcal{I}_\varkappa}{dx_3}(x_3) &\geq \int_{D(x_3)}\bigl(\xi_4(v_{1,1}^2+v_{2,2}^2)+\xi_3(v_{1,2}^2+v_{2,1}^2)+\xi_1 v_{3,2}^2+\xi_2 v_{3,1}^2\bigr)\,da+\int_{D(x_3)}\bigl(\xi_1 v_{2,3}^2+\xi_2 v_{1,3}^2+\xi_4 v_{3,3}^2\bigr)\,da\\
&\geq 0; \qquad (5.23)
\end{aligned}
$$

hence $\mathcal{I}_\varkappa(x_3)$ is a nonincreasing function with respect to x_3 on $[0, L]$. Moreover, by means of the end condition (2.15) and relation (5.22) we obtain $\mathcal{I}_\varkappa(L) = 0$ and hence $\mathcal{I}_\varkappa(x_3) \geq 0$ for all $x_3 \in [0, L]$, that is $\mathcal{I}_\varkappa(x_3)$ is a measure of the amplitude of the harmonic vibration.

By using the Schwarz and arithmetic-geometric mean inequalities and with the aid of (4.10), (4.40) and (5.4), we obtain

$$
\begin{aligned}
&\biggl|\int_{D(x_3)}\Bigl(v_1\bigl((2c_{55}-\varkappa_2)v_{3,1}+2c_{55}v_{1,3}\bigr)+v_2\bigl((2c_{44}-\varkappa_1)v_{3,2}+2c_{44}v_{2,3}\bigr)\\
&\qquad\qquad+v_3\bigl((2c_{13}+\varkappa_2)v_{1,1}+(2c_{23}+\varkappa_1)v_{2,2}+2c_{33}v_{3,3}\bigr)\Bigr)\,da\biggr|\\
&\leq \frac{1}{\sqrt{\lambda}}\int_{D(x_3)}\Bigl[\bigl(3c_{55}+\sqrt{c_{11}c_{33}}\bigr)v_{1,1}^2+\bigl(3c_{44}+\sqrt{c_{22}c_{33}}\bigr)v_{2,2}^2+2c_{55}v_{1,2}^2+2c_{44}v_{2,1}^2\\
&\qquad+\bigl(c_{33}+c_{44}+2c_{55}+\sqrt{c_{11}c_{33}}+\sqrt{c_{22}c_{33}}\bigr)v_{3,1}^2\\
&\qquad+\bigl(c_{33}+2c_{44}+c_{55}+\sqrt{c_{11}c_{33}}+\sqrt{c_{22}c_{33}}\bigr)v_{3,2}^2+c_{55}v_{1,3}^2+c_{44}v_{2,3}^2+c_{33}v_{3,3}^2\Bigr]\,da, \qquad (5.24)
\end{aligned}
$$

$$
\begin{aligned}
&\biggl|\int_{D(x_3)}\bigl(2c_{55}x_\rho v_{1,\rho}(v_{3,1}+v_{1,3})+2c_{44}x_\rho v_{2,\rho}(v_{3,2}+v_{2,3})+2x_\rho v_{3,\rho}(c_{13}v_{1,1}+c_{23}v_{2,2}+c_{33}v_{3,3})\bigr)\,da\biggr|\\
&\leq d\int_{D(x_3)}\Bigl[(c_{55}+|c_{13}|)v_{1,1}^2+(c_{44}+|c_{23}|)v_{2,2}^2+c_{55}v_{1,2}^2+c_{44}v_{2,1}^2+(c_{33}+2c_{55}+|c_{13}|+|c_{23}|)v_{3,1}^2\\
&\qquad+(c_{33}+2c_{44}+|c_{13}|+|c_{23}|)v_{3,2}^2+2c_{55}v_{1,3}^2+2c_{44}v_{2,3}^2+c_{33}v_{3,3}^2\Bigr]\,da, \qquad (5.25)
\end{aligned}
$$

$$
\begin{aligned}
&\left|\int_{D(x_3)}\big[c_{55}v_{1,3}^2+c_{44}v_{2,3}^2+c_{33}v_{3,3}^2-c_{66}(v_{1,2}+v_{2,1})^2\right.\\
&\qquad\qquad\left.-(c_{11}v_{1,1}^2+c_{22}v_{2,2}^2+2c_{12}v_{1,1}v_{2,2}+c_{55}v_{3,1}^2+c_{44}v_{3,2}^2)+\rho\omega^2v_sv_s\big]\,da\right|\\
&\quad\le\int_{D(x_3)}\bigg[\Big(c_{11}+|c_{12}|+\frac{\rho\omega^2}{\lambda}\Big)v_{1,1}^2+\Big(c_{22}+|c_{12}|+\frac{\rho\omega^2}{\lambda}\Big)v_{2,2}^2+\Big(2c_{66}+\frac{\rho\omega^2}{\lambda}\Big)(v_{1,2}^2+v_{2,1}^2)\\
&\qquad\qquad+\Big(c_{55}+\frac{\rho\omega^2}{\lambda}\Big)v_{3,1}^2+\Big(c_{44}+\frac{\rho\omega^2}{\lambda}\Big)v_{3,2}^2+c_{55}v_{1,3}^2+c_{44}v_{2,3}^2+c_{33}v_{3,3}^2\bigg]\,da. \qquad (5.26)
\end{aligned}
$$

To conclude, we obtain from (5.22)–(5.26) a first order differential inequality of type (5.17), where now we have

$$
\begin{aligned}
\alpha=\max\bigg\{&\frac{1}{\xi_4}\Big(\frac{1}{\sqrt{\lambda}}\left(3c_{55}+\sqrt{c_{11}c_{33}}\right)+d\,(c_{55}+|c_{13}|)\Big),\ \frac{1}{\xi_3}c_{55}\Big(\frac{2}{\sqrt{\lambda}}+d\Big),\ \frac{1}{\xi_3}c_{44}\Big(\frac{2}{\sqrt{\lambda}}+d\Big),\\
&\frac{1}{\xi_4}\Big(\frac{1}{\sqrt{\lambda}}\left(3c_{44}+\sqrt{c_{22}c_{33}}\right)+d\,(c_{44}+|c_{23}|)\Big),\ \frac{1}{\xi_2}c_{55}\Big(\frac{1}{\sqrt{\lambda}}+2d\Big),\ \frac{1}{\xi_1}c_{44}\Big(\frac{1}{\sqrt{\lambda}}+2d\Big),\\
&\frac{1}{\xi_2}\Big(\frac{1}{\sqrt{\lambda}}\left(c_{44}+2c_{55}+c_{33}+\sqrt{c_{11}c_{33}}+\sqrt{c_{22}c_{33}}\right)+d\,(c_{33}+2c_{55}+|c_{13}|+|c_{23}|)\Big),\\
&\frac{1}{\xi_1}\Big(\frac{1}{\sqrt{\lambda}}\left(2c_{44}+c_{55}+c_{33}+\sqrt{c_{11}c_{33}}+\sqrt{c_{22}c_{33}}\right)+d\,(c_{33}+2c_{44}+|c_{13}|+|c_{23}|)\Big),\\
&\frac{1}{\xi_4}c_{33}\Big(\frac{1}{\sqrt{\lambda}}+d\Big)\bigg\}, \qquad (5.27)
\end{aligned}
$$

$$
\begin{aligned}
\beta=\max\bigg\{&\frac{1}{\xi_4}\Big(c_{11}+|c_{12}|+\frac{\rho\omega^2}{\lambda}\Big),\frac{1}{\xi_4}\Big(c_{22}+|c_{12}|+\frac{\rho\omega^2}{\lambda}\Big),\ \frac{1}{\xi_3}\Big(2c_{66}+\frac{\rho\omega^2}{\lambda}\Big),\\
&\frac{1}{\xi_2}\Big(c_{55}+\frac{\rho\omega^2}{\lambda}\Big),\ \frac{1}{\xi_1}\Big(c_{44}+\frac{\rho\omega^2}{\lambda}\Big),\ \frac{1}{\xi_4}c_{33}\bigg\}. \qquad (5.28)
\end{aligned}
$$

Therefore, the spatial evolution of the amplitude is described by the estimate (5.12), where α and β are given now by relations (5.27) and (5.28) and ω_1^* is replaced by

$$
\omega_2^*=\frac{dm_0}{\rho}\sqrt{c_{11}^2+c_{22}^2+c_{44}^2+c_{55}^2+2c_{12}^2+4c_{66}^2}. \qquad (5.29)
$$

The analysis of this section can be extended to a semi-infinite cylinder using the procedure developed at the end of the above section.

6. Concluding remarks

We have addressed some exponential and algebraic spatial estimates for describing how the amplitude of a harmonic vibration evolves in an anisotropic elastic cylinder. The discussion is based on the assumption regarding the strong ellipticity of the elasticity tensor. This hypothesis allows us to obtain results valid for a very large class of anisotropic elastic materials, including auxetic materials (which, having a negative Poisson's ratio or negative stiffness, expand laterally when stretched in contrast to ordinary materials; see [Park and Lakes 2007], for example).

Exponential spatial decay estimates are predicted for harmonic vibrations whose frequency is lower than a certain critical value, as defined by relations (4.11) and (4.34), for example. However, as we can see from relations (4.15), (4.16), (4.37) and (4.38), these estimates fall to give information regarding the spatial evolution for harmonic vibrations with frequency close to the critical value.

On the other hand, the algebraic spatial estimate (5.12) proves how the spatial behavior evolves in the case of harmonic vibrations with frequency greater than the critical value ω^* as defined in (5.5).

The extent to which our present results cover the entire range of frequencies remains open question.

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STAN CHIRIȚĂ: schirita@uaic.ro
Faculty of Mathematics, Al.I. Cuza University of Iași, Blvd. Carol I, nr. 11, 700506 - Iași, Romania

MICHELE CIARLETTA: ciarlett@diima.unisa.it
Dipartimento di Ingegneria dell'Informazione e Matematica Applicata (DIIMA), Università di Salerno, Via Ponte Don Melillo, 84084 Fisciano (SA), Italy