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SOFTENING HYPERVISCOELASTICITY FOR MODELING RATE-DEPENDENT MATERIAL FAILURE

KONSTANTIN VOLOKH AND PAVEL TRAPPER

New models of viscoelastic solids at small and finite deformations are proposed that describe material failure by enforcing the energy limiter — the average bond energy. Basically, the bond energy defines the energy that is necessary to separate two attracting particles. In the case of a solid composed of many particles there exists a magnitude of the average bond energy that is necessary to separate particles in a small material volume. The average bond energy can be calculated if a statistical distribution of the bond density is known for a particular material. Alternatively, the average bond energy can be determined in macroscopic experiments if the energy limiter is introduced in a material constitutive model. Traditional viscoelastic models of materials do not have energy limiters and, consequently, they allow for unlimited energy accumulation under the strain increase. The latter is unphysical, of course, because no material can sustain large enough deformations without failure. The average bond energy is the energy limiter that controls material softening, which indicates failure. Thus, by limiting the stored energy we include a description of material failure in the constitutive model. Viscoelasticity including energy limiters can be called softening hyperviscoelasticity. We present two softening hyperviscoelasticity models for small and finite deformations. In all cases the elastic and viscoelastic responses are described by potentials with limiters, which control material softening. The models are studied in the case of simple shear and uniaxial tension. The results of the calculations show that softening hyperviscoelasticity can be used for analysis of rate-dependent failure of materials.

1. Introduction

Existing continuum mechanics approaches for modeling material failure can be divided in two groups: surface and bulk models. The surface models, pioneered by Barenblatt [1959], are called cohesive zone models (CZMs) in the modern literature. They present material surfaces — cohesive zones — where displacement discontinuities occur. The discontinuities are enhanced with constitutive laws relating normal and tangential displacement jumps with the corresponding tractions. There are plenty of proposals of constitutive equations for cohesive zones [Dugdale 1960; Rice and Wang 1989; Tvergaard and Hutchinson 1992; Xu and Needleman 1994; Camacho and Ortiz 1996]. All CZMs are constructed qualitatively as follows: tractions increase, reach a maximum, and then approach zero with increasing separation. Such a scenario is in harmony with our intuitive understanding of the rupture process. Since the work by Needleman [1987] CZMs are used increasingly in finite element simulations of many phenomena, such as crack tip plasticity and creep, crazing in polymers, adhesively bonded joints, interface cracks in bimaterials, delamination in composites and multilayers, and fast crack propagation in polymers. Cohesive zones can be inside finite elements or along their boundaries [Xu and Needleman 1994; Belytschko

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et al. 2001; De Borst 2001]. Crack nucleation, propagation, branching, kinking, and arrest are natural outcomes of the computations where the discontinuity surfaces are spread over the bulk material. This is in contrast to the traditional approach of fracture mechanics where stress analysis is separated from a description of the actual process of material failure. The CZM approach is natural for simulation of fracture at the *internal material interfaces* in polycrystals, composites, and multilayers. It is less natural for modeling bulk fracture because it leads to the simultaneous use of two material models for the same real material: one model describes the bulk while the other model describes a CZM imbedded in the bulk. Such a two-model approach is rather artificial physically. It seems preferable to incorporate a material failure law directly in the constitutive description of the bulk.

Remarkably, the first models of bulk failure — damage mechanics — proposed by Kachanov [1958] and Rabotnov [1963] for analysis of the gradual failure accumulation and propagation in *creep* and *fatigue* appeared almost simultaneously with the cohesive zone approach. The need to describe the failure *accumulation*, that is, evolution of the material microstructure, explains why damage mechanics is very similar to plasticity theories including the internal damage variable (inelastic strain), the critical threshold condition (yield surface), and the damage evolution equation (flow rule). The subsequent development of the formalism of damage mechanics [Kachanov 1986; Krajcinovic 1996; Skrzypek and Ganczarski 1999; Lemaitre and Desmorat 2005] left its physical origin well behind the mathematical and computational techniques and, eventually, led to the use of damage mechanics for the description of *any* bulk failure. Theoretically, the approach of damage mechanics is very flexible and allows reflecting physical processes triggering macroscopic damage at small length scales. Practically, the experimental calibration of damage theories is not trivial because it is difficult to measure the damage parameter directly. The experimental calibration should be implicit and include both the damage evolution equation and criticality condition.

A physically motivated alternative to damage mechanics in the cases of failure related with the bond rupture has been considered recently by Gao and Klein [1998] and Klein and Gao [1998] who showed how to mix the atomic/molecular and continuum descriptions in order to simulate material failure. They applied the Cauchy–Born rule linking micro and macro scales to empirical potentials, which include a possibility of the full atomic separation. The continuum-atomistic link led to the formulation of the macroscopic strain energy potentials allowing for the stress/strain softening and strain localization. The continuum-atomistic method is very effective at small length scales where purely atomistic analysis becomes computationally intensive. Unfortunately, a direct use of the continuum-atomistic method in macroscopic failure problems is not very feasible because its computer implementation includes a numerically involved procedure of the averaging of the interatomic potentials over a representative volume.

In order to bypass the computational intensity of the continuum-atomistic method while preserving its sound physical basis the *softening hyperelasticity* approach was proposed by Volokh [2004; 2007a; 2007b]. The basic idea of the approach was to formulate an expression of the stored macroscopic energy, which would include the energy limiter — the average bond energy. Such a limiter automatically induces strain softening, that is, a material failure description, in the constitutive law. The softening hyperelasticity approach is computationally simple yet physically appealing. The approach proved itself in a number of problems varying from failure of brittle materials to rubbers and soft biological tissues [Volokh 2007a; 2007b; 2008a; 2008b; Trapper and Volokh 2008; Volokh and Trapper 2008; Volokh and Vorp 2008]. Besides Gei et al. [2004] used a variant of softening hyperelasticity for modeling plastic softening. One should be careful, however, with doing that because material failure during plastic

deformation is essentially due to microstructural changes rather than the bond rupture and the approach of energy limiters may not be applicable in this case on physical grounds.

It should be noted that softening hyperelasticity has been used for the prediction of the global material/structural instability in all mentioned works. To extend the approach to problems of dynamic failure propagation it is necessary to include rate-dependence in the constitutive description. In other words, the softening hyperelasticity should be extended to the softening hyperviscoelasticity, which is the main goal of the present work. Including viscosity in the constitutive framework is important physically. It is also important computationally because viscosity naturally regularizes the potentially ill conditioned problems related to tracking the propagation of dynamic failure.

The outline of the paper is as follows. Section 2 introduces the idea of the energy limiters providing a physical multiscale link for the phenomenological quantity of the average bond energy. Sections 3 and 4 present the softening hyperviscoelasticity theories for small and finite deformations respectively. A general discussion of the new theories is present in Section 5.

2. Energy limiters

To motivate the introduction of energy limiters and softening hyperelasticity we briefly describe the continuum-atomistic link. A more detailed exposition of the issue can be found in [Volokh and Trapper 2008; Trapper and Volokh 2008], for example.

Interaction of two particles (atoms, molecules, et cetera) can be described as

$$\psi(F) = \varphi(F) - \varphi_0, \qquad \varphi_0 = \min_L \varphi(F = 1). \tag{2-1}$$

Here ψ is the particle interaction potential; F is the one-dimensional deformation gradient mapping the distance between particles from the reference, L, to the current, l, state: $l = FL$. To be specific we choose the Lennard-Jones potential, for example, $\varphi(l) = 4\varepsilon((\sigma/l)^{12} - (\sigma/l)^6)$, where ε and σ are the bond energy and length constants accordingly. By direct computation we can find the energy limiter or the failure energy, Φ. Indeed, increasing deformation we cannot increase the energy unlimitedly:

$$\psi(F \to \infty) = -\varphi_0 = \Phi = \text{constant}\,. \tag{2-2}$$

Analogously to the case of the pair interaction it is possible to consider particle assemblies. Applying the assumption of applicability of continuum mechanics to the description of such assemblies, meaning using the Cauchy–Born rule, it is possible to derive a stored energy function analogously to (2-1),

$$\psi(\boldsymbol{C}) = \langle\varphi(\boldsymbol{C})\rangle - \langle\varphi\rangle_0, \qquad \langle\varphi\rangle_0 = \min_L \langle\varphi(\boldsymbol{C} = \boldsymbol{1})\rangle.$$

Here $\boldsymbol{C} = \boldsymbol{F}^T\boldsymbol{F}$ is the right Cauchy–Green deformation tensor and $\boldsymbol{F} = \partial\boldsymbol{x}/\partial\boldsymbol{X}$ is the deformation gradient of a generic material macroparticle of body Ω occupying position $\boldsymbol{X}$ at the reference state and position $\boldsymbol{x}(\boldsymbol{X})$ at the current state of deformation. The average means

$$\langle\varphi(\boldsymbol{C})\rangle = V_0^{-1}\int_{V_0^*} 4\varepsilon\left(\left(\frac{\sigma}{L\|\boldsymbol{C}\|}\right)^{12} - \left(\frac{\sigma}{L\|\boldsymbol{C}\|}\right)^{6}\right) D_V\, dV,$$

in the case of the Lennard-Jones potential, where the tensorial norm designates stretch in a bond direction, D_V is the volumetric bond density function, V_0^* is the integration volume defined by the range of influence of φ, and V_0 is the reference representative volume.

Analogously to (2-2), we can find the energy limiter, Φ, increasing the deformation unlimitedly as

$$\Phi = \psi\big(\|\boldsymbol{C}\| \to \infty\big) = -\langle\varphi\rangle_0 = \text{constant}\,.$$

Thus, *the average bond energy sets a limit for the energy accumulation*. This conclusion generally does not depend on the choice of the particle potential and is valid for any interaction that includes a possible particle separation.

Contrary to the conclusion above traditional hyperelastic models of materials do not include the energy limiter. The stored energy of hyperelastic materials is defined as $\psi = W$. Here W is used for the strain energy of the *intact* material, which can be characterized as $\|\boldsymbol{C}\| \to \infty \Rightarrow \psi = W \to \infty$, where $\|\cdots\|$ is a tensorial norm.

In other words, the increasing strain increases the accumulated energy unlimitedly. Evidently, the consideration of only intact materials is restrictive and unphysical. The energy increase of a real material should be limited, as shown above:

$$\|\boldsymbol{C}\| \to \infty \;\Rightarrow\; \psi \to \Phi = \text{constant}, \tag{2-3}$$

where the average bond energy, $\Phi = \text{constant}$, can be called the *material failure energy*.

Equation (2-3) presents the fundamental idea of introducing a limiter of the stored energy in the elasticity theory. Such a limiter induces material softening, indicating material failure, automatically. *The choice of the limited stored energy expression should generally be material-specific*. Nonetheless, a somewhat universal formula [Volokh 2007b] can be introduced to enrich the already existing models of intact materials with the failure description

$$\psi(W) = \Phi - \Phi \exp\Big(\frac{-W}{\Phi}\Big), \tag{2-4}$$

where $\psi(W = 0) = 0$ and $\psi(W = \infty) = \Phi$.

Formula (2-4) obeys the condition $\|\boldsymbol{C}\| \to \infty \Rightarrow \psi(W(\boldsymbol{C})) \to \Phi$ and, in the case of the intact material behavior, $W \ll \Phi$, we have $\psi(W) \approx W$ preserving the features of the intact material.

Taking (2-4) into account, the constitutive equation can be written in the general form

$$\boldsymbol{\sigma} = 2J^{-1}\boldsymbol{F}\frac{\partial\psi}{\partial\boldsymbol{C}}\boldsymbol{F}^T = 2J^{-1}\boldsymbol{F}\frac{\partial W}{\partial\boldsymbol{C}}\boldsymbol{F}^T \exp\Big(\frac{-W}{\Phi}\Big), \tag{2-5}$$

where $\boldsymbol{\sigma}$ is the Cauchy stress tensor, $J = \det\boldsymbol{F}$, and the exponential multiplier enforces material softening. Constitutive equation (2-5) is especially effective for *incompressible soft materials undergoing finite deformations*. We strongly emphasize again, however, that *the best form of the energy function is not universal and should be material/problem-specific*.

In what follows we will extend the idea of the energy limiter to viscous deformations and examine the role of the rate-dependence in the description of material failure.

3. Softening hyperviscoelasticity at small deformations

We use a rheological model of the standard solid shown in Figure 1 as a prototype for the integral formulation of the constitutive law.

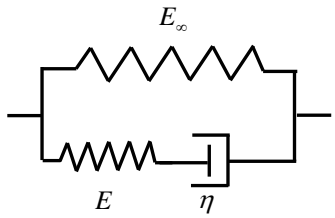

Figure 1. Rheological model of the standard solid.

3.1. ***Constitutive law.*** Following Simo and Hughes [1998] we define the hyperviscoelastic constitutive law in the form

$$\boldsymbol{\sigma}(t) = \frac{\partial \hat{\psi}(\varepsilon)}{\partial \varepsilon}\mathbf{1} + \int_{-\infty}^{t} m(t-\tau)\frac{\partial}{\partial \tau}\left(\operatorname{dev}\frac{\partial \overline{\psi}(\boldsymbol{e}(\tau))}{\partial \boldsymbol{e}}\right)d\tau, \tag{3-1}$$

where

$$\varepsilon = \operatorname{tr}\boldsymbol{\varepsilon}, \qquad \boldsymbol{e} = \operatorname{dev}\boldsymbol{\varepsilon} \equiv \boldsymbol{\varepsilon} - \tfrac{1}{3}\varepsilon\mathbf{1},$$

and $\boldsymbol{\varepsilon}$ is a linearized strain, that is, the symmetric part of the displacement gradient.

The relaxation function is defined in the form

$$m(t-\tau) = \beta_\infty + \beta\exp\left(-\frac{t-\tau}{\theta}\right), \qquad \beta_\infty = \frac{E_\infty}{E_\infty + E}, \quad \beta = \frac{E}{E_\infty + E}, \quad \theta = \frac{\eta}{E},$$

where β_∞ and β are dimensionless relative moduli and θ is the relaxation time (see Figure 1).

The elastic potential is decomposed into the volumetric and distortional parts accordingly as

$$\psi(\boldsymbol{\varepsilon}) = \hat{\psi}(\varepsilon) + \overline{\psi}(\boldsymbol{e}), \tag{3-2}$$

and the hyperelastic constitutive law is derived as

$$\boldsymbol{\sigma} = \frac{\partial \psi}{\partial \boldsymbol{\varepsilon}} = \frac{\partial \hat{\psi}}{\partial \varepsilon}\mathbf{1} + \operatorname{dev}\frac{\partial \overline{\psi}}{\partial \boldsymbol{e}}. \tag{3-3}$$

We define the separate potentials with softening that have not been considered in the literature yet,

$$\hat{\psi}(\varepsilon) = \Phi_1 - \Phi_1\left(1 + \sqrt{\frac{K}{\Phi_1}}\varepsilon\right)\exp\left(-\sqrt{\frac{K}{\Phi_1}}\varepsilon\right), \qquad \overline{\psi}(\boldsymbol{e}) = \Phi_2 - \Phi_2\exp\left(-\frac{\mu}{\Phi_2}\boldsymbol{e}:\boldsymbol{e}\right), \tag{3-4}$$

where K and μ are the bulk and shear moduli of the isotropic Hookean solid and Φ_1 and Φ_2 are the failure energies for volumetric and distortional deformations. By introducing different failure constants we increase the flexibility of the phenomenological description of material failure.

Substituting (3-4) in (3-3) we have

$$\boldsymbol{\sigma} = \tilde{K}\varepsilon\mathbf{1} + 2\tilde{\mu}\boldsymbol{e}, \qquad \tilde{K} = K\exp\left(-\sqrt{\frac{K}{\Phi_1}}\varepsilon\right), \qquad \tilde{\mu} = \mu\exp\left(-\frac{\mu}{\Phi_2}\boldsymbol{e}:\boldsymbol{e}\right). \tag{3-5}$$

Linearization of these equations leads to the classical linear elasticity with $\tilde{K} = K$ and $\tilde{\mu} = \mu$.

The motivation for the specific forms of the softening hyperelastic potentials (3-4) comes from the consideration of two simple deformations.

Firstly, in the case of hydrostatic tension (3-5)$_1$ takes the form

$$\sigma = \sigma_{11} = \sigma_{22} = \sigma_{33} = K\varepsilon \exp\left(-\sqrt{\frac{K}{\Phi_1}}\varepsilon\right),$$

and its graph is shown in Figure 2, left.

Evidently, the hydrostatic compression does not lead to material failure while the hydrostatic tension does. The maximum point on the tension branch of the curve corresponds to the onset of static instability when the material failure starts propagating.

Secondly, in the case of pure shear, e_{12}, (3-5)$_1$ takes the form

$$\sigma_{12} = 2\mu e_{12} \exp\left(-\frac{\mu}{\Phi_2} e_{12}^2\right),$$

and its graph is shown in Figure 2, right.

Evidently, the skew-symmetry of the failure response is desirable and expected.

3.2. ***Simple shear.*** In this subsection we examine rate-dependent response of the model described above in the case of simple shear

$$\sigma_{12}(t) = 2\mu \int_0^t \left[\beta_\infty + \beta \exp\left(-\frac{t-\tau}{\theta}\right)\right] \frac{\partial}{\partial \tau}\left[e_{12} \exp\left(-\frac{\mu}{\Phi_2} e_{12}^2\right)\right] d\tau, \tag{3-6}$$

where there is no stressing until $t = 0$.

Further simplifications are due to the assumption of the constant stretch/strain rate as $\dot{\gamma} = \text{constant}$. The latter assumption leads to the simple formulae for time

$$t = \frac{e_{12}}{\dot{\gamma}}, \qquad \tau = \frac{\xi_{12}}{\dot{\gamma}},$$

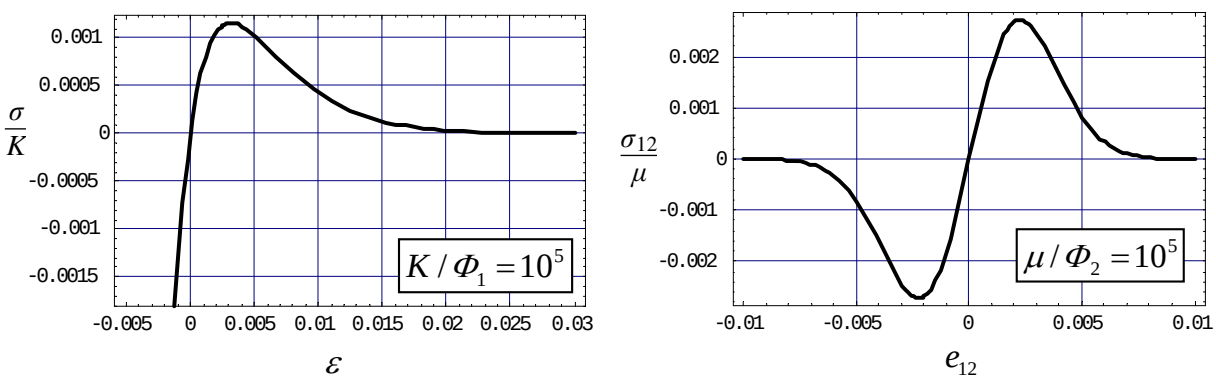

Figure 2. Left: Hydrostatic tension σ/K. Right: Simple shear σ_{12}/μ.

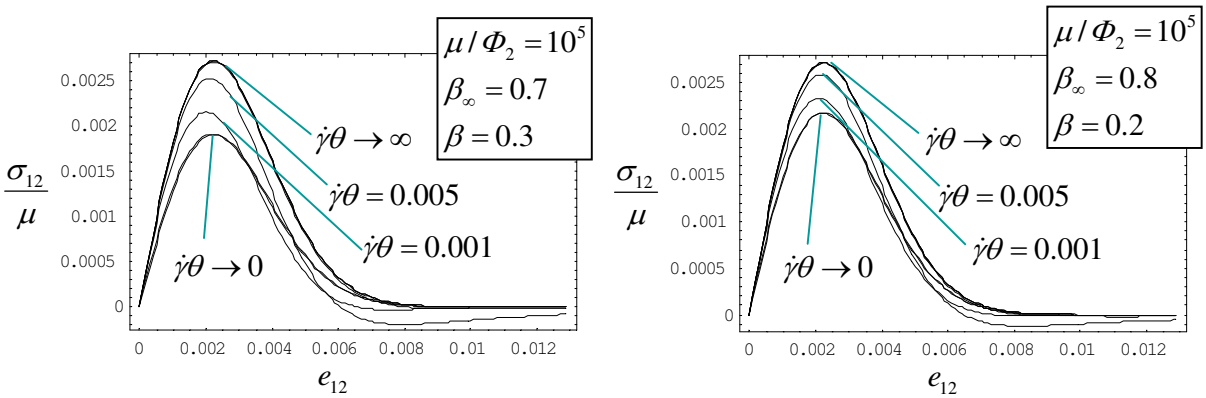

Figure 3. Simple shear for various strain rates.

where $\xi_{12} = e_{12}(\tau)$ and, consequently, (3-6) takes the form

$$\sigma_{12}(t) = 2\mu \int_0^{e_{12}} \left[\beta_\infty + \beta \exp\left(-\frac{e_{12} - \xi_{12}}{\theta\dot\gamma}\right)\right] \frac{\partial}{\partial \xi_{12}} \left[\xi_{12} \exp\left(-\frac{\mu}{\Phi_2}\xi_{12}^2\right)\right] d\xi_{12}. \tag{3-7}$$

Stress-strain curves defined by (3-7) are present in Figure 3 for different strain rates.

Evidently, material stiffness and strength — the curve maximum — increase with the increasing deformation rate for a given relaxation time. This conclusion is expected intuitively. Moreover, the stable (prior to failure) branches of the response curves are limited by the curve corresponding to $\dot\gamma\theta \to 0$ from the bottom and $\dot\gamma\theta \to \infty$ from the top. Physically the limit cases correspond to the very slow quasistatic response and the fastest instantaneous response of the material accordingly. Interestingly, the range of the strength variation depends on the relative contribution of the elastic springs in the rheological model shown in Figure 1. The greater the contribution of the spring corresponding to the dashpot, E, the larger the strength range.

4. Softening hyperviscoelasticity at finite deformations

The integral hyperviscoelasticity formulation for small deformations considered in the previous section can be equivalently reformulated in the differential form [Simo and Hughes 1998]. Unfortunately, in the case of finite deformation the integral and differential formulations are not necessarily equivalent in general. There are plenty of integral formulations of nonlinear viscoelasticity. We should mention, however, that the foundations of the theory have been set by Green and Rivlin [1957; 1959] and Green et al. [1959]. Further developments are reviewed in [Lockett 1972; Carreau et al. 1997; Hoo Fatt and Ouyang 2007], for example. There are also numerous differential formulations of nonlinear viscoelasticity based on the introduction of internal variables and their evolution equations. The most popular scheme includes the multiplicative decomposition of the deformation gradient into elastic and inelastic parts [Lubliner 1985; Lion 1996; Govindjee and Reese 1997; 1998; Bergström and Boyce 1998; Huber and Tsakmakis 2000; Amin et al. 2006; Hoo Fatt and Ouyang 2008]. Despite its popularity the scheme including the multiplicative decomposition of the deformation gradient is not entirely perfect: the intermediate elastically-relaxed configuration cannot be determined uniquely. Indeed, it is always possible to superimpose a local rotation on such a configuration without violating the multiplicative decomposition.

The nonuniqueness of the multiplicative decomposition is often eliminated by a specific and explicit choice of the deformation or by the use of certain computational schemes, which regularize the problem implicitly. Unfortunately, the artificial regularizations cannot improve the general formulation. Since the elastically-relaxed configuration is not unique one may question the necessity to define it. Instead of looking for a specific elastically-relaxed *configuration* it is possible to look only for a family of such configurations enjoying the same *metric*. In the latter case there is no need in the use of the nonunique multiplicative decomposition of the deformation gradient and it is enough to track the evolution of the metric tensor of possible elastically-relaxed configurations. Such a line of thought was pioneered by Eckart [1948] and further developed in [Leonov 1976; Rubin 1994; Rubin and Bodner 2002]. It is worth mentioning that the refusal to look for a unique elastically-relaxed configuration is justified by the fact that such a configuration is incompatible and, consequently, unobservable physically.

4.1. ***Constitutive law.*** We will use the rheological model shown in Figure 1 as a prototype for the nonlinear model too. The springs should be thought of as nonlinear in this case. We extend (3-1) to finite deformations directly following Simo and Hughes [1998]:

$$\boldsymbol{\tau}(t) = J\frac{\partial\hat{\psi}(\varepsilon)}{\partial\varepsilon}\mathbf{1} + \int_{-\infty}^{t} m(t-\tau)\,\frac{\partial}{\partial\tau}\left(\operatorname{dev}\left[2\overline{\boldsymbol{F}}(\tau)\frac{\partial\overline{\psi}\big(\overline{\boldsymbol{C}}(\tau)\big)}{\partial\overline{\boldsymbol{C}}}\overline{\boldsymbol{F}}^{T}(\tau)\right]\right)d\tau,$$

where $\boldsymbol{\tau} = J\boldsymbol{\sigma}$ is the so-called Kirchhoff stress tensor and

$$\varepsilon = J = \det\boldsymbol{F}, \qquad \overline{\boldsymbol{F}} = J^{-1/3}\boldsymbol{F}\,(\det\overline{\boldsymbol{F}} = 1), \qquad \overline{\boldsymbol{C}} = \overline{\boldsymbol{F}}^{T}\overline{\boldsymbol{F}}. \tag{4-1}$$

The stored energy is also decomposed analogously to (3-2) as $\psi(\boldsymbol{C}) = \hat{\psi}(\varepsilon) + \overline{\psi}(\overline{\boldsymbol{C}})$, and the hyperelastic constitutive law is derived as

$$\boldsymbol{\tau} = J\frac{\partial\hat{\psi}}{\partial\varepsilon}\mathbf{1} + \operatorname{dev}\left[2\overline{\boldsymbol{F}}\frac{\partial\overline{\psi}}{\partial\overline{\boldsymbol{C}}}\overline{\boldsymbol{F}}^{T}\right].$$

We mention that the idea to extend the volumetric-distortional decomposition of small strains to the case of large strains based on the volume-preserving deformation gradient, $(4\text{-}1)_2$, is due to Flory [1961].

Though the formulation above is the most general the majority of soft materials undergoing finite deformations are incompressible. The latter means that the analytical formulation can be simplified as

$$\varepsilon = J = \det\boldsymbol{F} = 1, \tag{4-2}$$

$$\overline{\boldsymbol{F}} = \boldsymbol{F}, \qquad \overline{\boldsymbol{C}} = \boldsymbol{C}, \qquad \psi(\boldsymbol{C}) = \overline{\psi}(\overline{\boldsymbol{C}}), \qquad \boldsymbol{\sigma} = \boldsymbol{\tau} = -p\mathbf{1} + \operatorname{dev}\left[2\boldsymbol{F}\frac{\partial\psi}{\partial\boldsymbol{C}}\boldsymbol{F}^{T}\right], \tag{4-3}$$

and

$$\boldsymbol{\sigma}(t) = \boldsymbol{\tau}(t) = -p\mathbf{1} + \int_{-\infty}^{t} m(t-\tau)\,\frac{\partial}{\partial\tau}\left(\operatorname{dev}\left[2\boldsymbol{F}(\tau)\frac{\partial\psi\big(\boldsymbol{C}(\tau)\big)}{\partial\boldsymbol{C}}\boldsymbol{F}^{T}(\tau)\right]\right)d\tau, \tag{4-4}$$

where the indefinite Lagrange multiplier, p, is used to enforce the incompressibility condition (4-2).

We further use a stored energy with softening that was calibrated for analysis of the material of the abdominal aortic aneurysm [Volokh and Vorp 2008]

$$\psi(I_1) = \Phi - \Phi\exp\left[-\frac{\alpha_1}{\Phi}(I_1-3) - \frac{\alpha_2}{\Phi}(I_1-3)^2\right], \tag{4-5}$$

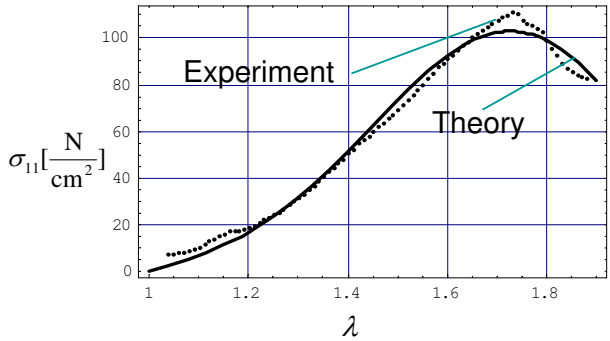

Figure 4. Theory versus experiment for the uniaxial tension test [Volokh and Vorp 2008].

where $I_1 = \operatorname{tr} \boldsymbol{C}$, α_1 and α_2 are the elasticity constants of the material, and Φ is the failure energy, which is another material constant controlling its softening.

The uniaxial tension test results are shown in Figure 4, where model (4-5) was fitted with the following constants: $\alpha_1 = 10.3\,\mathrm{N/cm^2}$, $\alpha_2 = 18.0\,\mathrm{N/cm^2}$, and $\Phi = 40.2\,\mathrm{N/cm^2}$.

Substituting (4-5) in (4-4) we have

$$\boldsymbol{\sigma}(t) = -p\mathbf{1} + \int_{-\infty}^{t} m(t-\tau)\,\frac{\partial}{\partial \tau}\Big(\operatorname{dev}\big[2\psi_1(\tau)\boldsymbol{B}(\tau)\big]\Big)\,d\tau,$$

where $\boldsymbol{B} = \boldsymbol{F}\boldsymbol{F}^T$ and

$$\psi_1 \equiv \frac{\partial \psi}{\partial I_1} = \big[\alpha_1 + 2\alpha_2(I_1 - 3)\big]\exp\Big[-\frac{\alpha_1}{\Phi}(I_1 - 3) - \frac{\alpha_2}{\Phi}(I_1 - 3)^2\Big].$$

4.2. ***Uniaxial tension.*** In the case of the uniaxial tension we have the following simplifications within the Cartesian coordinate framework $\{\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3\}$:

$$\boldsymbol{F} = \lambda \boldsymbol{k}_1 \otimes \boldsymbol{k}_1 + \lambda^{-1/2}(\boldsymbol{k}_2 \otimes \boldsymbol{k}_2 + \boldsymbol{k}_3 \otimes \boldsymbol{k}_3), \qquad \boldsymbol{B} = \lambda^2 \boldsymbol{k}_1 \otimes \boldsymbol{k}_1 + \lambda^{-1}(\boldsymbol{k}_2 \otimes \boldsymbol{k}_2 + \boldsymbol{k}_3 \otimes \boldsymbol{k}_3),$$

where λ is the axial stretch.

The nontrivial stress components accordingly take the forms

$$\sigma_{11}(t) = -p + \frac{4}{3}\int_{-\infty}^{t} m(t-\tau)\,\frac{\partial}{\partial \tau}\Big(\psi_1(\tau)\big[\lambda^2(\tau) - \lambda^{-1}(\tau)\big]\Big)\,d\tau, \tag{4-6}$$

$$\sigma_{22}(t) = \sigma_{33}(t) = -p - \frac{2}{3}\int_{-\infty}^{t} m(t-\tau)\,\frac{\partial}{\partial \tau}\Big(2\psi_1(\tau)\big[\lambda^2(\tau) - \lambda^{-1}(\tau)\big]\Big)\,d\tau. \tag{4-7}$$

Since $\sigma_{22}(t) = \sigma_{33}(t) = 0$ we can find the Lagrange multiplier from (4-7) and substitute it in (4-6) getting the final formula

$$\sigma_{11}(t) = 2\int_{0}^{t} m(t-\tau)\,\frac{\partial}{\partial \tau}\Big(\psi_1(\tau)\big[\lambda^2(\tau) - \lambda^{-1}(\tau)\big]\Big)\,d\tau, \tag{4-8}$$

where $\xi = \lambda(\tau)$ and the lower integration boundary has been shifted assuming no stressing before time $t = 0$.

Further simplifications are due to the assumption of the constant stretch/strain rate

$$\dot{\xi} = \dot{\lambda} = \dot{\gamma} = \text{constant}\,.$$

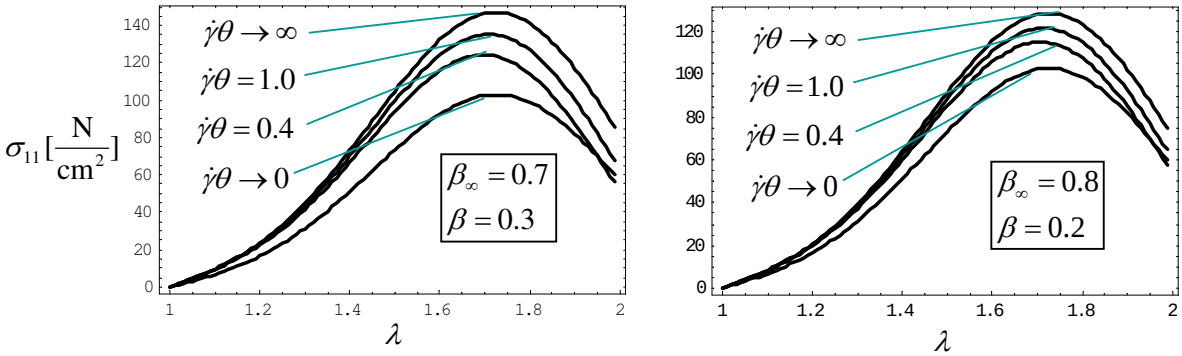

Figure 5. Uniaxial tension for various strain rates.

The latter assumption leads to the simple formulae

$$t = \frac{\lambda - 1}{\dot{\gamma}}, \qquad \tau = \frac{\xi - 1}{\dot{\gamma}},$$

for the times; consequently, (4-8) takes the form

$$\sigma_{11}(t) = 2\int_1^\lambda \left[\beta_\infty + \beta \exp\left(-\frac{\lambda - \xi}{\dot{\gamma}\theta}\right)\right] \frac{\partial}{\partial \xi}\left[\psi_1(\xi)(\xi^2 - \xi^{-1})\right] d\xi. \tag{4-9}$$

Computations based on (4-9) are shown in Figure 5 for various stretch rates. We also scaled the material constants $\alpha_1 \to \alpha_1/\beta_\infty$, $\alpha_2 \to \alpha_2/\beta_\infty$, and $\Phi \to \Phi/\beta_\infty$ in order to preserve the form of the quasistatic curve, $\dot{\gamma}\theta \to 0$, shown in Figure 5.

As in the case of small deformations, material stiffness and strength increase with the increasing deformation rate for a given relaxation time. The stable (prior to failure) branches of the response curves are limited by the curve corresponding to $\dot{\gamma}\theta \to 0$ from the bottom and $\dot{\gamma}\theta \to \infty$ from the top. The range of the strength variation depends on the relative contribution of the elastic springs in the rheological model shown in Figure 1. The greater the contribution of the spring corresponding to the dashpot the larger the strength range. There is a complete analogy between the cases of small and finite deformations as expected.

5. Discussion

A new approach for modeling rate-dependent failure of materials has been proposed. Its basic idea is the introduction of energy limiters in the constitutive description of materials. Such limiters control softening providing a failure account. The energy limiters were introduced in the models of isotropic Hookean solids, which are suitable for a description of quasibrittle failure in ceramics, concrete, glass, or even metals at high-velocity dynamic processes where the plastic deformation can be ignored. Besides, the energy limiters were considered for soft materials at finite deformations. It should be clearly realized that the proposed approach is suitable for materials whose failure is due to the bond rupture and cannot be used for materials that fail due to large plastic deformations. If failure is accompanied by a gradual accumulation of inelastic deformations then the approach of damage mechanics is probably more relevant and should be used.

Though we used essentially different material models in the considered examples of small and finite deformations the qualitative results are very similar. Particularly, we observed that the increasing rate of deformations leads to the increase of both stiffness and strength of the material. Such a conclusion corresponds well to the experimental observations. Moreover, while the existing viscoelasticity theories can describe the phenomenon of material stiffening under increasing deformation rate it is for the first time that the presented approach clearly predicts the increase of the material strength (and not only stiffening) with the increase of the deformation rate.

Concerning the limitations of the presented computations it should be emphasized that the considered examples include only proportional loading and there are no returning deformation waves. Such waves may lead to the material healing within the hyperelastic framework, which should be suppressed. To avoid the material healing in the finite element computations it is possible, for example, to reduce the values of the material parameters by orders of magnitude in the elements, which reached the critical failure energy or arrived at zero stresses. This circle of questions is beyond the scope of the present note where only a principal possibility of inducing a failure description in nonlinear viscoelasticity has been considered. Nonetheless, the finite element implementation of the proposed method is important and will be considered in a separate work.

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