Journal of
Mechanics of
Materials and Structures

ANALYSIS OF MULTIPLE AXISYMMETRIC ANNULAR CRACKS

Ebrahim Asadi, Shahriar Fariborz and Mojtaba Ayatollahi

Volume 4, № 1

January 2009

mathematical sciences publishers

JOURNAL OF MECHANICS OF MATERIALS AND STRUCTURES
Vol. 4, No. 1, 2009

ANALYSIS OF MULTIPLE AXISYMMETRIC ANNULAR CRACKS

EBRAHIM ASADI, SHAHRIAR FARIBORZ AND MOJTABA AYATOLLAHI

The solution of axisymmetric Volterra climb and glide dislocations in an infinite domain is obtained by means of the Hankel transforms. The distributed dislocation technique is used to construct integral equations for a system of coaxial annular cracks where the domain is under axisymmetric tensile load. These equations are solved numerically to obtain the dislocation density on the surfaces of the cracks. The dislocation densities are employed to determine stress intensity factors for annular and penny-shaped cracks.

1. Introduction

Large elastic bodies containing multiple interacting cracks situated far from the boundary may be considered as infinite regions weakened by cracks. An infinite domain containing a penny-shaped crack under axisymmetric tension is the simplest three-dimensional problem in fracture mechanics. The solution of this problem dates back to an article by Sneddon [1946], wherein the exact solution to the problem was derived. The solution of a penny-shaped crack under general loading in the form of Fourier series is rendered in the book by Kassir and Sih [1975]. Guidera and Lardner [1975] used the Somigliana formula to analyze a penny-shaped crack. The component of displacement discontinuity was presented as the solution of a system of three integral equations. A penny-shaped crack in a transversely isotropic infinite body subjected to arbitrary normal and shear tractions was solved by Fabrikant [1987]. Collins [1962] treated the problem of an infinite elastic solid containing two parallel penny-shaped cracks where the axis of symmetry of the problem passed through the centers of the cracks. In his study, the representation of displacement field devised by Green and Zerna [1954] was used to reduce the problem to the solution of a system of four Fredholm integral equations. The formulation, however, becomes extremely involved where the number of cracks increases. Isida et al. [1985] analyzed two elliptical parallel cracks by means of the body force method. Interaction among multiple penny-shaped cracks was studied by several investigators, see for example [Kachanov and Laures 1989] and for the most recently published article [Zhan and Wang 2006]. In the former article, the method developed by the first author for the analysis of several cracks was employed to study the interaction of arbitrarily located penny-shaped cracks in a three-dimensional body. In the latter study, the boundary collocation technique and average method for surface traction of cracks were used to solve the governing equations. The stress intensity factor for an annular crack situated in an infinite space under general loading was determined by Nied and Erdogan [1983] and by Selvadurai and Singh [1985] and Clements and Ang [1988] under axisymmetric normal loading. Eigenstrain solutions for axisymmetric crack problems in terms of Lipschitz–Hankel integrals was derived by Korsunsky [1995]. The stress fields are hypersingular at the eigenstrain ring yielding hypersingular integral equations for the ensuing crack problem.

Keywords: infinite domain, axisymmetric, annular crack, Volterra dislocation, dislocation density, Hankel transform.

In the present paper, utilizing the Popkovich–Neuber potentials, the solution of axisymmetric climb and glide edge dislocations is carried out by means of the Hankel transformation in an infinite isotropic domain. The stress components exhibit the well-known Cauchy-type singularity at dislocation location. The distributed dislocation method Hills et al. [1996] is employed to formulate integral equations for the dislocation density functions on a system of annular and/or penny-shaped coaxial cracks under the axisymmetric remote tensile load. These equations are of Cauchy singular type which are solved numerically. The modes I and II stress intensity factors at the crack edges are obtained and the interaction of two coaxial cracks is investigated. The interaction of annular cracks embedded in a half-space or strip under axisymmetric conditions may be analyzed by the procedure devised in this article.

2. Solution of the Volterra ring dislocation

In the linear theory of elasticity for isotropic materials neglecting the body force, the displacement vector $\underline{u}$ may be represented in terms of a harmonic vector $\underline{B}$ and a harmonic scalar B_0 that is, the well-known Popkovich–Neuber solution [Lur'e 1964] as

$$\underline{u} = \underline{B} - \frac{1}{4(1-\nu)} \operatorname{grad}(\underline{R} \cdot \underline{B} + B_0), \tag{1}$$

where ν is the Poisson's ratio of the material and $\underline{R}$ is the position vector. For axisymmetric problems it is convenient to utilize cylindrical coordinates and choose $\underline{B} = B_3 \underline{k}$, where $\underline{k}$ is the unit vector in axial direction. Therefore, the components of displacement vector by virtue of $\underline{R} = r\underline{e}_r + z\underline{e}_z$ yield

$$u_r = -\frac{1}{4(1-\nu)}\left(\frac{\partial B_0}{\partial r} + z\frac{\partial B_3}{\partial r}\right), \quad u_\theta = 0, \quad u_z = \frac{3-4\nu}{4(1-\nu)}B_3 - \frac{1}{4(1-\nu)}\left(\frac{\partial B_0}{\partial z} + z\frac{\partial B_3}{\partial z}\right). \tag{2}$$

The constitutive relationships in axisymmetric problems of linear elasticity are

$$\begin{gathered}
\sigma_{rr} = \frac{2\mu}{1-2\nu}\left[(1-\nu)\frac{\partial u_r}{\partial r} + \nu\left(\frac{u_r}{r} + \frac{\partial u_z}{\partial z}\right)\right], \quad \sigma_{\theta\theta} = \frac{2\mu}{1-2\nu}\left[(1-\nu)\frac{u_r}{r} + \nu\left(\frac{\partial u_r}{\partial r} + \frac{\partial u_z}{\partial z}\right)\right], \\
\sigma_{zz} = \frac{2\mu}{1-2\nu}\left[(1-\nu)\frac{\partial u_z}{\partial z} + \nu\left(\frac{u_r}{r} + \frac{\partial u_r}{\partial r}\right)\right], \quad \sigma_{rz} = \mu\left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right), \quad \sigma_{r\theta} = \sigma_{\theta z} = 0,
\end{gathered} \tag{3}$$

where μ is the elastic shear modulus of the material. Substituting (2) into (3), we arrive at the stress components in terms of the potentials B_0 and B_3:

$$\begin{Bmatrix} \sigma_{rr} \\ \sigma_{\theta\theta} \\ \sigma_{zz} \\ \sigma_{rz} \end{Bmatrix} = \frac{\mu}{2(1-\nu)} \begin{bmatrix} 2\nu\dfrac{\partial B_3}{\partial z} - \dfrac{\partial^2}{\partial r^2}(B_0 + zB_3) \\ 2\nu\dfrac{\partial B_3}{\partial z} - \dfrac{1}{r}\dfrac{\partial}{\partial r}(B_0 + zB_3) \\ 2(1-\nu)\dfrac{\partial B_3}{\partial z} - \left(\dfrac{\partial^2 B_0}{\partial z^2} + z\dfrac{\partial^2 B_3}{\partial z^2}\right) \\ (1-2\nu)\dfrac{\partial B_3}{\partial r} - \dfrac{\partial}{\partial r}\left(\dfrac{\partial B_0}{\partial z} + z\dfrac{\partial B_3}{\partial z}\right) \end{bmatrix}. \tag{4}$$

The conditions representing a Volterra-type climb ring dislocation located at $r = a$, $z = 0$ in a three dimensional infinite domain with the line of dislocation in radial direction are

$$u_z(r, 0^+) - u_z(r, 0^-) = \delta_z H(a - r). \tag{5}$$

The conditions for a glide dislocation read as

$$u_r(r, 0^+) - u_r(r, 0^-) = \delta_r H(a - r), \tag{6}$$

where in (5) and (6), δ_z and δ_r designate the dislocation Burgers vectors and $H(\cdots)$ is the Heaviside step-function. Moreover, the continuity of traction vector on the line of dislocation requires that

$$\sigma_{zz}(r, 0^+) = \sigma_{zz}(r, 0^-) \quad \text{and} \quad \sigma_{rz}(r, 0^+) = \sigma_{rz}(r, 0^-). \tag{7}$$

For climb edge dislocation the problem is symmetric with respect to the plane $z = 0$, whereas it is antisymmetric for glide dislocation. Therefore, it is convenient to analyze the two problems separately. For the symmetric problem the half-space $z > 0$ is subjected to the boundary conditions

$$u_z(r, 0) = \frac{\delta_z}{2} H(a - r) \quad \text{and} \quad \sigma_{rz}(r, 0) = 0. \tag{8}$$

The boundary conditions for glide dislocation for the region $z > 0$ are

$$u_r(r, 0) = \frac{\delta_r}{2} H(a - r) \quad \text{and} \quad \sigma_{zz}(r, 0) = 0. \tag{9}$$

Consequently, the dislocation solutions for the climb and glide dislocations reduce to the solutions of two harmonic equations

$$\frac{\partial^2 B_i}{\partial r^2} + \frac{1}{r}\frac{\partial B_i}{\partial r} + \frac{\partial^2 B_i}{\partial z^2} = 0 \quad i = 0, 3, \quad z > 0, \tag{10}$$

subjected to boundary conditions (8) and (9), respectively. The solution to (10) is achieved by means of the Hankel transform. The Hankel transform of order ν of a sufficiently regular function $f(r)$ is defined [Sneddon 1972] as

$$F(\xi) = \int_0^\infty r f(r) J_\nu(\xi r)\, dr, \tag{11}$$

where $J_\nu(\cdots)$ is the Bessel function of first kind of order ν. The inversion of Hankel transform yields

$$f(r) = \int_0^\infty \xi F(\xi) J_\nu(\xi r)\, d\xi. \tag{12}$$

The zero order Hankel transform of (10), assuming that the potentials are $O(r^{-\alpha})$ as $r \to \infty$ for some $\alpha > 0.5$, leads to two second order ordinary differential equations

$$\frac{d^2 \bar{B}_i(\xi, z)}{dz^2} - \xi^2 \bar{B}_i(\xi, z) = 0 \quad i = 0, 3, \quad z > 0, \tag{13}$$

where $\bar{B}_0(\xi, z)$ and $\bar{B}_3(\xi, z)$ are zero order Hankel transforms of $B_0(r, z)$ and $B_3(r, z)$, respectively. The solution of (13) and (14), which is finite as $z \to \infty$, is readily known

$$\bar{B}_i(\xi, z) = Q_i(\xi) e^{-\xi z} \quad i = 0, 3. \tag{14}$$

We substitute boundary conditions (8) and (9) into (2) and (4), take the Hankel transform of the resultant equations and utilize (14) to obtain

$$Q_0(\xi) = (2\nu - 1)a\frac{J_1(\xi a)}{\xi^2}\delta_z, \quad Q_3(r, z) = a\frac{J_1(\xi a)}{\xi}\delta_z \tag{15}$$

for climb and

$$Q_0(\xi) = (\nu - 1)a\pi\frac{\eta(\xi, a)}{\xi^2}\delta_r \quad Q_3(r, z) = \frac{a\pi}{2}\frac{\eta(\xi, a)}{\xi}\delta_r \tag{16}$$

for glide dislocation. In (16) the function $\eta(\xi, a)$ is defined as

$$\eta(\xi, a) = J_0(\xi a)H_1(\xi a) - J_1(\xi a)H_0(\xi a), \tag{17}$$

where $H_\nu(\cdots)$ stands for the Struve function of order ν [Abramowitz and Stegun 1964]. The displacement and stress components in view of (14)–(16), (12), (2) and (4) yield

$$\begin{aligned}
u_r &= \frac{a}{4(1-\nu)}\int_0^\infty\left[\delta_z(2\nu-1+\xi z)J_1(\xi a)+\pi\delta_r\left(\nu-1+\frac{\xi z}{2}\right)\eta(\xi,a)\right]J_1(\xi r)e^{-\xi z}d\xi,\\
u_z &= \frac{a}{4(1-\nu)}\int_0^\infty\left[\delta_z(2(1-\nu)+\xi z)J_1(\xi a)+\frac{\pi\delta_r}{2}(1-2\nu+\xi z)\eta(\xi,a)\right]J_0(\xi r)e^{-\xi z}d\xi,\\
\sigma_{rr} &= \frac{\mu a}{2(1-\nu)}\left\{\delta_z\int_0^\infty J_1(\xi a)\left[\xi(\xi z-1)J_0(\xi r)+\frac{1}{r}(1-2\nu-\xi z)J_1(\xi r)\right]e^{-\xi z}d\xi\right.\\
&\qquad\left.+\frac{\pi\delta_r}{2}\int_0^\infty\eta(\xi,a)\left[\xi(\xi z-2)J_0(\xi r)+\frac{1}{r}(2(1-\nu)-\xi z)J_1(\xi r)\right]e^{-\xi z}d\xi\right\},\\
\sigma_{\theta\theta} &= \frac{\mu a}{2(1-\nu)}\left\{\delta_z\int_0^\infty J_1(\xi a)\left[-2\nu\xi J_0(\xi r)+\frac{1}{r}(2\nu-1+\xi z)J_1(\xi r)\right]e^{-\xi z}d\xi\right.\\
&\qquad\left.+\frac{\pi\delta_r}{2}\int_0^\infty\eta(\xi,a)\left[-2\xi\nu J_0(\xi r)+\frac{1}{r}(2(\nu-1)+\xi z)J_1(\xi r)\right]e^{-\xi z}d\xi\right\},\\
\sigma_{zz} &= \frac{-\mu a}{2(1-\nu)}\int_0^\infty\left[\delta_z\xi(1+\xi z)J_1(\xi a)+\frac{\pi\delta_r}{2}\xi^2 z\eta(\xi,a)\right]J_0(\xi r)e^{-\xi z}d\xi,\\
\sigma_{rz} &= \frac{\mu a}{2(1-\nu)}\int_0^\infty\left[-\delta_z\xi^2 zJ_1(\xi a)+\frac{\pi\delta_r}{2}\xi(1-\xi z)\eta(\xi,a)\right]J_1(\xi r)e^{-\xi z}d\xi, \quad z>0.
\end{aligned}\tag{18}$$

The stress and displacement fields for climb ring dislocation were obtained by Kroupa [1960] using the Galerkin solution of linear elasticity theory and solving the ensuing biharmonic equation. The solution in [Kroupa 1960] may be recovered by putting $\delta_r = 0$ in (18). In order to study the asymptotic behavior of stress components σ_{zz} and σ_{rz} at the dislocation location, we set $z = 0$ in the last two equations in (18), and arrive at

$$\sigma_{zz}(r, 0) = -\frac{\mu a\delta_z}{2(1-\nu)}\int_0^\infty \xi J_1(\xi a)J_0(\xi r)d\xi, \tag{19}$$

$$\sigma_{rz}(r,0)=\frac{\mu a\pi\delta_r}{4(1-\nu)}\int_0^\infty \xi J_1(\xi r)\eta(\xi,a)d\xi,\quad z>0.$$

These two integrals can be found in [Gradshteyn and Ryzhik 1980], and substituting their values gives

$$\begin{aligned}
\sigma_{zz}(r,0)&=\frac{\mu\delta_z a}{\pi(1-\nu)}\frac{E(r/a)}{r^2-a^2},\quad r<a,\\
\sigma_{rz}(r,0)&=\frac{\mu\pi a\delta_r}{4(1-\nu)}\Bigg\{\int_0^\infty \xi J_1(\xi r)\Big[J_0(\xi a)\Big(H_1(\xi a)-\frac{2}{\pi}-Y_1(\xi a)\Big)\\
&\qquad -J_1(\xi a)(H_0(\xi a)-\frac{2}{\pi\xi a}-Y_0(\xi a))\Big]d\xi\\
&\qquad -\frac{2}{\pi ar}+\frac{4}{\pi^2}E(r/a)\Big[\frac{a/r}{r^2-a^2}+\frac{1}{ar}\Big]\Bigg\},\quad r<a,\\
\sigma_{zz}(r,0)&=\frac{\mu\delta_z r}{\pi(1-\nu)}\Big[\frac{E(a/r)}{r^2-a^2}-\frac{K(a/r)}{r^2}\Big],\quad r>a,\\
\sigma_{rz}(r,0)&=\frac{\mu\pi a\delta_r}{4(1-\nu)}\Bigg\{\int_0^\infty \xi J_1(\xi r)\Big[J_0(\xi a)\Big(H_1(\xi a)-\frac{2}{\pi}-Y_1(\xi a)\Big)\\
&\qquad -J_1(\xi a)\Big(H_0(\xi a)-\frac{2}{\pi\xi a}-Y_0(\xi a)\Big)\Big]d\xi\\
&\qquad -\frac{2}{ra\pi}+\frac{4}{\pi^2}\Big[\frac{E(a/r)}{r^2-a^2}+\frac{E(a/r)-K(a/r)}{a^2}\Big]\Bigg\},\quad r>a.
\end{aligned}\tag{20}$$

In (20) $K(\cdots)$ and $E(\cdots)$ are the complete elliptic integrals of the first and second kind, and $Y_\nu(\cdots)$ is the Bessel function of the second kind of order ν. The integrals in (20) are regular; thus, the stress components are Cauchy singular as $r\to a$, which is a well-known feature of the stress fields caused by Volterra-type dislocations. It is noteworthy to mention that the assumption of axisymmetry of glide dislocation implies the vanishing of Burgers vector δ_r at the origin of coordinates leading to $\sigma_{rz}(0,0)=0$. Moreover, the asymptotic expansion of complete elliptic integral shows that $E(r)=\pi/2+O(r^2)$ as $r\to 0$. Therefore, the stress fields in (20) are bounded at the origin.

3. Axisymmetric crack formulation

Let climb and glide dislocations with densities $B_z(\rho)$ and $B_r(\rho)$ respectively be distributed on an annular crack situated at $z=z_0$ with inner radius ρ and outer radius $\rho+d\rho$. The axial and shear stress at a point with coordinates (r,z) due to the above distribution of dislocations on the crack surface are

$$\sigma_{zz}(r,z)=-\frac{\mu}{2(1-\nu)}\Bigg[\rho B_z d\rho\int_0^\infty \xi(1+\xi|z-z_0|)J_1(\xi\rho)J_0(\xi r)\,e^{-\xi|z-z_0|}d\xi\\
+\frac{\rho\pi B_r d\rho}{2}\int_0^\infty \xi^2\,\mathrm{sgn}(z-z_0)|z-z_0|\eta(\xi,\rho)J_0(\xi r)e^{-\xi|z-z_0|}d\xi\Bigg],$$

$$\sigma_{rz}(r,z)=\frac{\mu}{2(1-\nu)}\bigg[-\rho B_z d\rho\int_0^\infty \xi^2\,\mathrm{sgn}(z-z_0)\,|z-z_0|\,J_1(\xi\rho)J_1(\xi r)\,e^{-\xi|z-z_0|}d\xi \\ +\frac{\rho\pi B_r d\rho}{2}\int_0^\infty \xi(1-\xi|z-z_0|)J_1(\xi r)\eta(\xi,\rho)e^{-\xi|z-z_0|}d\xi\bigg].$$

Let the medium be weakened by N_1 annular and $N-N_1$ penny-shaped coaxial cracks situated at the axial coordinates z_j, $j=1,2,\dots,N$. The inner and outer radii of annular cracks are a_j and b_j, $j=1,2,\dots,N_1$ respectively and the radii of penny-shaped cracks are b_j, $j=N_1+1,\dots,N$. The cracks configurations may be expressed in parametric form as

$$r_j(s)=r_{cj}+L_js,\quad -1<s<1,\quad j=1,2,\dots,N, \tag{21}$$

where $r_{cj}=(b_j+a_j)/2$ and $L_j=(b_j-a_j)/2$. The traction components on the surface of i-th crack caused by dislocations distributed on all N cracks surfaces yield

$$\begin{aligned}\sigma_{zz}(r_i(s),z_i)&=\frac{\mu}{2(1-\nu)}\sum_{j=1}^{N}L_j\int_{-1}^{1}\Big[K_{zz}(r_i(s),r_j(t))B_{zj}(t)+K_{zr}(r_i(s),r_j(t))B_{rj}(t)\Big]dt,\\ \sigma_{rz}(r_i(s),z_i)&=\frac{\mu}{2(1-\nu)}\sum_{j=1}^{N}L_j\int_{-1}^{1}\Big[K_{rz}(r_i(s),r_j(t))B_{zj}(t)+K_{rr}(r_i(s),r_j(t))B_{rj}(t)\Big]dt,\end{aligned} \tag{22}$$

where the kernels in the above equations are

$$\begin{aligned}K_{zz}(r_i(s),r_j(t))&=\int_0^\infty -\xi r_j(t)(1+\xi|z_i-z_j|\,)J_1(\xi r_j(t))J_0(\xi r_i(s))\,e^{-\xi|z_i-z_j|}d\xi,\\ K_{zr}(r_i(s),r_j(t))&=\frac{\pi}{2}\int_0^\infty -\xi^2 r_j(t)\,\mathrm{sgn}(z_i-z_j)|z_i-z_j|\eta(\xi,r_j(t))J_0(\xi r_i(s))e^{-\xi|z_i-z_j|}d\xi,\\ K_{rz}(r_i(s),r_j(t))&=\int_0^\infty -\xi^2 r_j(t)\,\mathrm{sgn}(z_i-z_j)|z_i-z_j|\,J_1(\xi r_j(t))J_1(\xi r_i(s))\,e^{-\xi|z_i-z_j|}d\xi,\\ K_{rr}(r_i(s),r_j(t))&=\frac{\pi}{2}\int_0^\infty \xi r_j(t)(1-\xi|z_i-z_j|)\eta(\xi,r_j(t))J_1(\xi r_i(s))e^{-\xi|z_i-z_j|}d\xi.\end{aligned} \tag{23}$$

Since stress components (20) are Cauchy singular at the dislocation location, the system of integral equations(22) for the density functions are Cauchy singular for $i=j$ as $s\to t$. Employing the definition of the dislocation density function, the crack opening displacement for an annular crack becomes

$$u_{kj}^{+}(s)-u_{kj}^{-}(s)=L_j\int_{-1}^{s}B_{kj}(t)\,dt,\quad k=z,r. \tag{24}$$

The displacement field is single-valued away from the crack surfaces. Thus, the dislocation density for the j-th annular crack is subjected to the closure requirement

$$\int_{-1}^{1}B_{kj}(t)\,dt=0,\quad j=1,2,\dots,N_1,\quad k=z,r. \tag{25}$$

To obtain the dislocation density, the integral equations (22) and (25) are to be solved simultaneously. The stress fields exhibit a square-root singularity at the crack tips. Therefore, the dislocation densities for annular cracks are taken as

$$B_{kj}(t) = \frac{g_{kj}(t)}{\sqrt{1-t^2}}, \quad -1 \le t \le 1, \quad j = 1, 2, \ldots, N_1, \quad k = z, r. \tag{26}$$

A penny-shaped crack is considered as an annular edge crack. Taking the embedded crack tip at $t = 1$, the dislocation density functions for penny-shaped cracks may be written as

$$B_{kj}(t) = g_{kj}(t)\sqrt{\frac{1+t}{1-t}}, \quad -1 \le t \le 1, \quad j = N_1+1, \ldots, N, \quad k = z, r. \tag{27}$$

The functions $g_{kj}(t)$ in (26)–(27) are continuous in $-1 \le t \le 1$. The numerical solution of integral equations (22) and (25) is carried out by the procedure developed in [Faal et al. 2006]. Substituting (26) and (27) into (22) and (25) and discretizing the domain $-1 \le t \le 1$ by $n+1$ segments, the integral equations are reduced to the following system of $N \times n$ linear algebraic equations:

$$\begin{bmatrix} H_{11} & H_{12} & H_{13} & \ldots & H_{1N} \\ H_{21} & H_{22} & H_{23} & \ldots & H_{2N} \\ H_{31} & H_{32} & H_{33} & \ldots & H_{3N} \\ \vdots & \vdots & \vdots & \ddots & \ldots \\ H_{N1} & H_{N2} & H_{N3} & \ldots & H_{NN} \end{bmatrix} \begin{bmatrix} \boldsymbol{g}_1(t_k) \\ \boldsymbol{g}_2(t_k) \\ \boldsymbol{g}_3(t_k) \\ \vdots \\ \boldsymbol{g}_N(t_k) \end{bmatrix} = \begin{bmatrix} \boldsymbol{q}_1(s_r) \\ \boldsymbol{q}_2(s_r) \\ \boldsymbol{q}_3(s_r) \\ \vdots \\ \boldsymbol{q}_N(s_r) \end{bmatrix}, \tag{28}$$

where the collocation points are

$$s_r = \cos\left(\frac{\pi r}{n}\right), \quad r = 1, \ldots, n-1 \quad \text{and} \quad t_k = \cos\left(\pi\frac{2k-1}{2n}\right), \quad k = 1, \ldots, n. \tag{29}$$

The components of the matrix and vectors in (28) are

$$H_{ij} = \begin{bmatrix} A_{j1}k_{ij}(s_1,t_1) & A_{j2}k_{ij}(s_1,t_2) & \cdots & A_{jn}k_{ij}(s_1,t_n) \\ A_{j1}k_{ij}(s_2,t_1) & A_{j2}k_{ij}(s_2,t_2) & \cdots & A_{jn}k_{ij}(s_2,t_n) \\ \vdots & \vdots & & \vdots \\ A_{j1}k_{ij}(s_{n-1},t_1) & A_{j2}k_{ij}(s_{n-1},t_2) & \cdots & A_{jn}k_{ij}(s_{n-1},t_n) \\ A_{j1}B_{ij}(t_1) & A_{j2}B_{ij}(t_2) & \ldots & A_{jn}B_{ij}(t_n) \end{bmatrix},$$

$$\boldsymbol{g}_j(t_k) = \begin{bmatrix} g_{zj}(t_1) & g_{rj}(t_1) & \cdots & g_{zj}(t_n) & g_{rj}(t_n) \end{bmatrix}^T, \quad j = 1, \ldots, N, \tag{30}$$

$$\boldsymbol{q}_j(s_r) = \begin{bmatrix} \sigma_{zz}(r_j(s_1), z_j) & \sigma_{rz}(r_j(s_1), z_j) & \cdots & \sigma_{zz}(r_j(s_{n-1}), z_j)\sigma_{rz}(r_j(s_{n-1}), z_j) \end{bmatrix}^T, \quad j = 1, 2, \ldots, N_1,$$

$$\boldsymbol{q}_j(s_r) = \begin{bmatrix} \sigma_{zz}(r_j(s_1), z_j) & \sigma_{rz}(r_j(s_1), z_j) & \cdots & \sigma_{zz}(r_j(s_{n-1}), z_j) & \sigma_{rz}(r_j(s_{n-1}), z_j) & \sigma_{zz}(r_j(-1), z_j) & \sigma_{rz}(r_j(-1), z_j) \end{bmatrix}^T, \quad j = N_1+1, \ldots, N,$$

where superscript T stands for transposition and

$$A_{jk} = \frac{\pi}{n} \begin{cases} 1, & j = 1, 2, \dots, N_1, \\ 1 + t_k, & j = N_1 + 1, \dots, N, \quad k = 1, 2, \dots, n, \end{cases}$$
$$B_{ij} = \frac{\pi}{n} \begin{cases} \delta_{ij} L_i, & j = 1, 2, \dots, N_1, \\ k_{ij}(-1, t_k), & j = N_1 + 1, \dots, N, \quad k = 1, 2, \dots, n, \end{cases} \tag{31}$$
$$k_{ij}(s_r, t_k) = L_j \begin{bmatrix} K_{zz}(r_i(s), r_j(t)) & K_{zr}(r_i(s), r_j(t)) \\ K_{rz}(r_i(s), r_j(t)) & K_{rr}(r_i(s), r_j(t)) \end{bmatrix}.$$

In (31), δ_{ij} in the B_{ij} is the Kronecker delta and the components of matrix $k_{ij}(s_r, t_k)$ are defined in (23). The modes I and II stress intensity factors for an annular crack with inner and outer radii, a and b, respectively, are defined as

$$\begin{Bmatrix} k_I^a \\ k_{II}^a \end{Bmatrix} = \lim_{r \to a^-} \sqrt{2(a-r)} \begin{Bmatrix} \sigma_{zz}(r,0) \\ \sigma_{rz}(r,0) \end{Bmatrix} \quad \text{and} \quad \begin{Bmatrix} k_I^b \\ k_{II}^b \end{Bmatrix} = \lim_{r \to b^+} \sqrt{2(r-b)} \begin{Bmatrix} \sigma_{zz}(r,0) \\ \sigma_{rz}(r,0) \end{Bmatrix}. \tag{32}$$

Substituting the axial and shear stress components into (32) yields

$$\begin{Bmatrix} k_I^{aj} \\ k_{II}^{aj} \end{Bmatrix} = \frac{\sqrt{L_j}}{2(1-\nu)} \begin{Bmatrix} g_{zj}(-1) \\ g_{rj}(-1) \end{Bmatrix} \quad \text{and} \quad \begin{Bmatrix} k_I^{bj} \\ k_{II}^{bj} \end{Bmatrix} = -\frac{\sqrt{L_j}}{2(1-\nu)} \begin{Bmatrix} g_{zj}(1) \\ g_{rj}(1) \end{Bmatrix}, \quad j = 1, 2, \dots, N_1. \tag{33}$$

Analogously, for the penny-shaped crack, stress intensity factors becomes

$$\begin{Bmatrix} k_I^j \\ k_{II}^j \end{Bmatrix} = -\frac{\sqrt{L_j}}{\sqrt{2}(1-\nu)} \begin{Bmatrix} g_{zj}(1) \\ g_{rj}(1) \end{Bmatrix}, \quad j = N_1 + 1, \dots, N. \tag{34}$$

The solution of the system (28) should be substituted into (33) and (34) to determine stress intensity factors.

4. Numerical results

In what follows, the Poisson's ratio of the medium $\nu = 0.25$ and remote constant tensile traction σ_0 is applied in the axial direction. In the first example, we consider an annular crack with inner and outer radii a and b, respectively. The nondimensional stress intensity factors $K/\overline{K}$, where $\overline{K} = \sigma_0 \sqrt{(b-a)/2}$ for different crack aspect ratios a/b together with the results obtained in [10, 12], are given in Table 1.

The nondimensional stress intensity factors K_I/K_I^0 and K_{II}/K_I^0 of two parallel penny-shaped cracks with radius a, where $K_I^0 = \sigma_0 \sqrt{a}/\pi$ is the stress intensity factor of a penny-shaped crack with radius a situated in an infinite domain, are presented in Table 2 for different values of distance d between the cracks and compared against those of [Isida et al. 1985; Kachanov and Laures 1989; Zhan and Wang 2006]. The interaction of parallel cracks results in the mode II stress intensity factor which decays by increasing the distance between cracks. As it may be observed, except for K_I/K_I^0 where $d/2a = 0.05$, the results of the above two examples are in excellent agreement with the cited references confirming the validity of the methodology.

The applicability of the procedure is demonstrated by solving two examples with more complicated geometries. Two concentric cracks, a penny-shaped crack surrounded by an annular crack are considered. The dimensions of the annular crack, $b/2$ and b, remain fixed, whereas the radius of the penny-shaped

a/b	Present		[Clements and Ang 1988]		[Nied and Erdogan 1983]	
	$K^a/\overline{K}$	$K^b/\overline{K}$	$K^a/\overline{K}$	$K^b/\overline{K}$	$K^a/\overline{K}$	$K^b/\overline{K}$
0.01	5.7720	0.9013	5.784	0.901	5.922	0.900
0.1	1.9698	0.9091	1.972	0.909	1.972	0.909
0.2	1.5005	0.9180	1.502	0.918	1.502	0.918
0.3	1.3091	0.9270	1.310	0.927	1.310	0.927
0.4	1.2035	0.9363	1.204	0.936	1.204	0.936
0.5	1.1363	0.9461	1.137	0.946	1.137	0.946
0.6	1.0907	0.9559	1.091	0.956	1.089	0.957
0.7	1.0576	0.9662	1.058	0.966	1.057	0.967
0.8	1.0329	0.9768	1.033	0.977	1.032	0.978
0.9	1.0141	0.9880	1.015	0.988	1.014	0.988
0.99	1.0008	0.9983	1.001	0.998	1.001	0.99

Table 1. Stress intensity factors of an annular crack.

$d/2a$	Present		[ZW 2006]		[I+ 1985]		[KL 1989]
	K_I/K_I^0	K_{II}/K_I^0	K_I/K_I^0	K_{II}/K_I^0	K_I/K_I^0	K_{II}/K_I^0	K_I/K_I^0
0.05	0.6966	0.1923	———	———	———	———	0.7386
0.15	0.7351	0.1623	———	———	———	———	———
0.25	0.7671	0.1381	0.7678	0.1382	0.7759	0.1390	0.7678
0.35	0.7950	0.1173	0.7955	0.1173	———	———	0.7898
0.5	0.8313	0.0903	0.8316	0.0903	0.8356	0.0910	0.8249
0.75	0.8810	0.0551	0.8813	0.0551	0.8828	0.0549	0.8781
1	0.9185	0.0322	0.9187	0.0322	0.9189	0.0325	0.9176
1.5	0.9617	0.0114	0.9616	0.0114	0.9613	0.0115	0.9614
2	0.9841	0.0040	0.9802	———	0.9802	0.0041	———
5	1.0000	0.0000	0.9983	———	0.9990	———	———

Table 2. Interaction of two parallel identical penny-shaped cracks under normal loading. [ZW 2006] = [Zhan and Wang 2006]; [I+ 1985] = [Isida et al. 1985]; [KL 1989] = [Kachanov and Laures 1989].

crack, a, changes. Figure 1 shows the nondimensional mode I stress intensity factors K/K_0 of the two cracks, where $K_0 = \sigma_0\sqrt{b/\pi}$. The variation of K/K_0 at the outer edge of the annular crack is negligible, which may be attributed to the large distance between this edge and the penny-shaped crack.

In the last example two interacting identical annular cracks with $a/b = 0.5$ are considered. The dimensionless stress intensity factors at the inner and outer edges are given in Table 3. For $d/b \geq 5$ the interaction vanishes and the problem reduces to an annular crack in infinite medium. It is, however, interesting to note that the mode II stress intensity factor at the inner edge of cracks does not decrease monotonically with increasing distance between cracks. For cracks with the present dimensions, $K^a_{II}/\overline{K}$ has a local maximum at $d/b \simeq 1.5$.

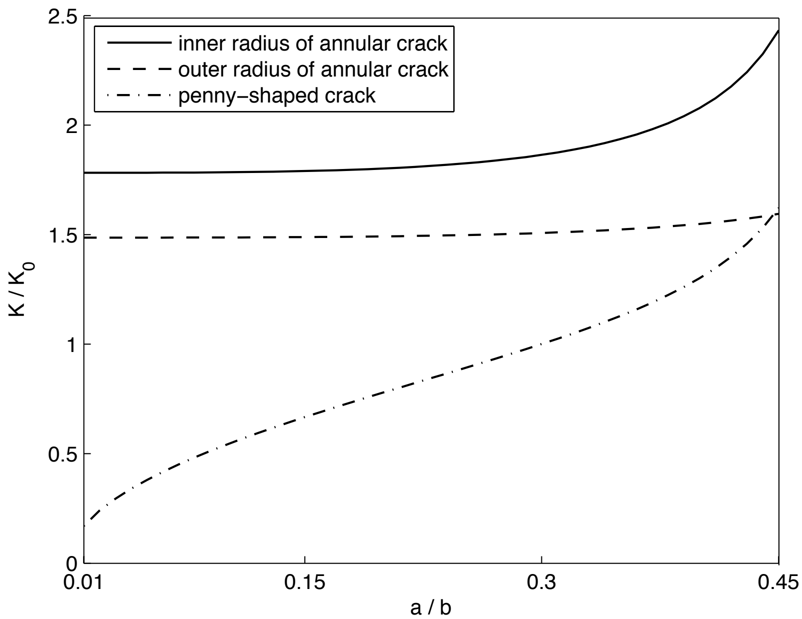

Figure 1. Stress intensity factors of two interacting concentric cracks.

d/b	$K_I^a/\overline{K}$	$K_I^b/\overline{K}$	$K_{II}^a/\overline{K}$	$K_{II}^b/\overline{K}$
0.05	0.7937	0.6637	0.2049	0.1771
0.15	0.8372	0.7053	0.1566	0.1423
0.25	0.8729	0.7386	0.1179	0.1176
0.35	0.9049	0.7676	0.0842	0.0961
0.5	0.9481	0.8059	0.0427	0.0704
0.75	1.0057	0.8530	0.0025	0.0446
1	1.0414	0.8810	0.0129	0.0315
1.5	1.0790	0.9079	0.0162	0.0191
2	1.0977	0.9230	0.0087	0.0088
5	1.1362	0.9461	0.0000	0.0000

Table 3. Interaction of two parallel identical annular cracks under normal loading for $a/b = 0.5$.

References

[Abramowitz and Stegun 1964] M. Abramowitz and I. A. Stegun (editors), *Handbook of mathematical functions with formulas, graphs, and mathematical tables*, National Bureau of Standards Applied Mathematics Series **55**, U.S. Government Printing Office, Washington, DC, 1964.

[Clements and Ang 1988] D. L. Clements and W. T. Ang, "Stress intensity factors for the circular annulus crack", *Int. J. Eng. Sci.* **26**:4 (1988), 325–329.

[Collins 1962] W. D. Collins, "Some axially symmetric stress distributions in elastic solids containing penny-shaped cracks, I: Cracks in an infinite solid and a thick plate", *Proc. R. Soc. Lond. A* **266**:1326 (1962), 359–386.

[Faal et al. 2006] R. T. Faal, S. J. Fariborz, and H. R. Daghyani, "Antiplane deformation of orthotropic strips with multiple defects", *J. Mech. Mater. Struct.* **1**:7 (2006), 1097–1114.

[Fabrikant 1987] V. I. Fabrikant, "Penny-shaped crack revisited: closed-form solutions", *Philos. Mag. A* **56**:2 (1987), 191–207.

[Gradshteyn and Ryzhik 1980] I. S. Gradshteyn and I. M. Ryzhik, *Table of integrals, series, and products*, Corr. and enl. ed., edited by A. Jeffrey, Academic Press, New York, 1980.

[Green and Zerna 1954] A. E. Green and W. Zerna, *Theoretical elasticity*, Clarendon, Oxford, 1954.

[Guidera and Lardner 1975] J. T. Guidera and R. W. Lardner, "Penny-shaped cracks", *J. Elasticity* **5**:1 (1975), 59–73.

[Hills et al. 1996] D. A. Hills, P. A. Kelly, D. N. Dai, and A. M. Korsunsky, *Solution of crack problems: the distributed dislocation technique*, Solid Mechanics and its Applications **44**, Kluwer, Dordrecht, 1996.

[Isida et al. 1985] M. Isida, K. Hirota, H. Noguchi, and T. Yoshida, "Two parallel elliptical cracks in an infinite solid subjected to tension", *Int. J. Fract.* **27**:1 (1985), 31–48.

[Kachanov and Laures 1989] M. Kachanov and J.-P. Laures, "Three-dimensional problems of strongly interacting arbitrarily located penny-shaped cracks", *Int. J. Fract.* **41**:4 (1989), 289–313.

[Kassir and Sih 1975] M. K. Kassir and G. C. Sih, *Three-dimensional crack problems: a new selection of crack solutions in three-dimensional elasticity*, Mechanics of Fracture **2**, Noordhoff, Leyden, 1975.

[Korsunsky 1995] A. M. Korsunsky, "Fundamental eigenstrain solutions for axisymmetric crack problems", *J. Mech. Phys. Solids* **43**:8 (1995), 1221–1241.

[Kroupa 1960] F. Kroupa, "Circular edge dislocation loop", *Czech. J. Phys.* **10**:4 (1960), 284–293.

[Lur'e 1964] A. I. Lur'e, *Three-dimensional problem of the theory of elasticity*, Interscience, New York, 1964.

[Nied and Erdogan 1983] H. F. Nied and F. Erdogan, "The elasticity problem for a thick-walled cylinder containing a circumferential crack", *Int. J. Fract.* **22**:4 (1983), 277–301.

[Selvadurai and Singh 1985] A. P. S. Selvadurai and B. M. Singh, "The annular crack problem for an isotropic elastic solid", *Q. J. Mech. Appl. Math.* **38**:2 (1985), 233–243.

[Sneddon 1946] I. N. Sneddon, "The distribution of stress in the neighbourhood of a crack in an elastic solid", *Proc. R. Soc. Lond. A* **187**:1009 (1946), 229–260.

[Sneddon 1972] I. N. Sneddon, *The use of integral transforms*, McGraw-Hill, New York, 1972.

[Zhan and Wang 2006] S. Zhan and T. Wang, "Interactions of penny-shaped cracks in three-dimensional solids", *Acta Mech. Sinica* **22**:4 (2006), 341–353.

Received 5 Jul 2008. Revised 1 Nov 2008. Accepted 13 Nov 2008.

EBRAHIM ASADI: easadi8@yahoo.com
Department of Mechanical Engineering, Amirkabir University of Technology (Tehran Polytechnic), Hafez Avenue, 424, Tehran 15875-4413, Iran

SHAHRIAR FARIBORZ: sjfariborz@yahoo.com
Department of Mechanical Engineering, Amirkabir University of Technology (Tehran Polytechnic), Hafez Avenue, 424, Tehran 15875-4413, Iran

MOJTABA AYATOLLAHI: MAyatollahi@yahoo.com
Department of Mechanical Engineering, Zanjan University, Zanjan 45195-313, Iran