Detailed explanations of the analytical developments in the dynamic analysis of a fluttering beam, exposed to a high supersonic airflow along its axial direction, in presence of non-linear aerodynamic forces

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Abstract

This work aims to the explanations of the analytical developments in the dynamic analysis of a vibrating beam, simply supported or clamped at both ends, under the effects of a high supersonic airflow along its axial direction. A complete aerodynamic model of the “Piston Theory”, which takes into account also the non-linear components of the distributed aerodynamic transversal force, is utilized. The post-critical flutter behavior has been studied considering their influence on the vibration state solutions of a fluttering beam without aerodynamic damping. mode).

Three different schemes, two of which are semi-analytical, (based on the classical and well known Rayleigh-Ritz and Galerkin methods), and a numerical one (based on the finite element method (FEM)), have been herein exploited, like in previous author’s papers, where beam flutter models with linear aerodynamic analysis were utilized.

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More sophisticated models have been herein set-up, considering that a more accurate analysis is necessary with respect to the previous cases where the aerodynamic numerical model was limited within the framework of the quasi-steady linearized "Piston Theory", both for the coupling component between odd and even order vibrating modes, and for the aerodynamic damping component

Nomenclature

A_s	beam cross-section area
a_∞	sound speed
b_w	beam width
a_{ij} , c_{ij} b_{ijkl} , d_{ijkl}	coefficients determined by integrals in Ritz and FEM models
$c_{i_p i_a}$	generic coefficient of the non-dimensional flexural deflection series expansion in the generic i_e th element of FEM model
E	Young's modulus
F_a	resultant of the aerodynamic forces acting on both sides of the beam per unit length
$F_i^{(a)}$	generalized aerodynamic force acting on the i th degree of freedom
$f_i(\xi)$	generic trial describing function of the non-dimensional flexural deflection through the beam length
h	beam thickness

I	flexural moment of inertia
J_{sccc}, J_{sscc}	particular integrals utilized in the Galerkin method
k_{ij}	stiffness matrix elements
k_{ij}^{*}	linear structural and aerodynamic forces resultant matrix elements
L	beam length
M_∞	Mach number
m_{ij}	mass matrix elements
N	whole number of the degrees of freedom
N_E	whole number of the elements in FEM model
p_∞	unperturbed air pressure
P_a	force per unit axial length acting on a side of the beam profile
$Q_{i_p}^{(i_e)}$	generic degree of freedom in the i_e th element of FEM model
q	dynamic pressure
t	beam state evolution time
T_o	reference time
U_∞	airflow speed
u, w	beam points axial and flexural displacements, respectively
W	non-dimensional flexural displacement
W_i	generic coefficients of the non-dimensional flexural displacement series expansion

W_{i_e}	non-dimensional flexural displacement on a generic $(i_e + 1)$th section S_{i_e} of FEM model
x	beam axial coordinate

Greek symbols

α	non-dimensional beam axial parameter
β, γ_m	non-dimensional Mach numbers parameters
γ	ratio between the specific heat at constant pressure and volume, respectively
γ', γ''	non-dimensional aerodynamic damping coefficients
λ	non-dimensional mass distribution parameter
θ_{i_e}	rotation parameter in the generic $(i_e + 1)$th section S_{i_e} of FEM model
μ	mass per unit length
ξ	non-dimensional axial coordinate of the beam
ξ_{i_e}	non-dimensional axial coordinate of the $(i_e + 1)$th section S_{i_e} of the FEM model
ξ_n	normalized axial coordinate of a beam element in the FEM model
ρ_∞	unperturbed air density
σ	dimensional dynamic pressure parameter
σ_d	dimension-less dynamic pressure parameter
τ	non-dimensional time
φ_{ij}	integral of the product between the first derivatives of the generic describing functions $f_i(\xi)$ and $f_j(\xi)$

χ	normal speed down- or up-wash
ω	angular frequency
ω_d	dimension-less angular frequency parameter

Special symbols

∂	partial differentation
$\mathcal{H}_2(\mathcal{S})$	square summable functions space
$\mathcal{S}$	definition domain of the describing functions
$\mathbf{B},\mathbf{D}$	utilized matrices in the generalized form equation
$\mathbf{F}$	structural and linear aerodynamic forces matrix
$\mathbf{M}$	mass matrix
$\mathcal{T}$	kinetic energy
$\mathcal{U}$	potential energy
$\mathbf{W},\mathbf{Z}$	column vectors of the unknown variables and their time first derivatives
$\mathbf{W}^{(3)}$	column vector with elements the triple products between unknown variables
$\mathbf{W}^{(3\mathrm{d})}$	column vector with elements the triple products between two unknown variables and a time first derivative of a third variable

Subscripts

d	subscript referring to dimension-less parameters
i,j,k,l	subscripts referring to functions and unknown variables coefficients of the flexural displacement series expansion
i_e	subscript referring to the (i_e+1)th section S_{i_e} of the FEM model

i_p, j_p, k_p, l_p	subscripts characterizing the unknown variables and the coefficients of the flexural displacement series expansion in the i_eth element of FEM model
nd	subscript referring to non-dimensional variables
∞	subscript referring to the unperturbed airflow

Superscripts

(i_e)	superscript referring to the i_eth element of FEM model

acronyms

FEM	finite element method

1. Introduction

From many years steady and unsteady aerodynamic theory for aeroelastic panels flutter computations has received a lot of interest also in high supersonic speed regimen. It is useful to bring into perspective the main authors who developed studies for flutter analysis of panels exposed to a high supersonic flow.

Lighthill [1953] first proposed a "Piston Theory", which was proved to be an efficient and powerful tool for panel flutter analysis. It can be used to calculate the pressure on an airfoil in steady or unsteady motion with remarkable accuracy, even under non-isentropic conditions, whenever the flight Mach Number M_∞ has such an order of magnitude that $M_\infty^2 >> 1$. This Piston theory is quite attractive for flutter studies due to its simplicity in comparison with other supersonic theories.

This theory has been discussed by Ashley and Zartarian [1956], who made suggestions on following researches based on this new efficient aerodynamic tool, with particular regard to areas where computational labor can be reduced without losing the necessary accuracy.

Morgan et al. [1958] analyzed some of the theories for two-dimensional oscillatory wing structures, able to be applied for flutter computations in high Mach Number. The results obtained by the various aerodynamic theories have been compared for their flutter prediction in various Mach Number ranges. Also some possible refinements of the “Piston Theory” have been proposed for high Mach Numbers.

The heritage of the studies of the above mentioned authors enables us to know the complete expression of the aerodynamic transverse distributed force acting on a beam, which makes possible to determine its flutter dynamic response after appropriate approximations.

The object of this paper is to give detailed enough explanations of the utilized algorithms for this bean flutter dynamic analysis.

Three different schemes have been exploited for the flutter computational work, like in the case of the beam flutter analysis with linearized and idealized “Piston Theory” [Tizzi 1994; 2003]. First a numerical procedure [Tizzi 1994; 1996; 2003] which arises from the Rayleigh-Ritz method [Kantorovich and Krylov 1964; Mikhlin 1964; Reddy 1986], has been utilized, together with the finite element method (FEM) [Weaver and Johnston 1984; Reddy et al. 1988; Qin et al. 1993]. By knowing the structural and inertial forces potential functional and the aerodynamic generalized force, it has been possible to apply the Lagrange equations [Pars 1968] and derive the generalized governing equation in time, for which appropriate time-integration algorithms exist.

Then the Galerkin method [Kantorovich and Krylov 1964; Mikhlin 1964; Tizzi 1994; 2003] has been employed in the simply supported beam case, as in the Dowell's model [Dowell 1966; 1967], to validate the results of Ritz and FEM procedures.

The same procedures have been utilized in previous author's paper, but herein for the first time, these methods have been applied with non-linear aerodynamic forces presence.

2. Mathematical formulation

A vibrating beam exposed to a high supersonic flow along the x axis, previously analyzed with the linear aerodynamic model [Tizzi 1994; 2003], is considered and drawn in Fig.1.

The interest of the analysis herein developed is limited to the non-linear aerodynamic forces components of the "Piston Theory", considering that the terms of all structural and linear aerodynamic forces components have been sufficiently illustrated. The exact expression of the pressure acting on a beam element (unidimensional flow) is [Lightill 1953; Ashley and Zartarian 1956; Morgan et al. 1958]:

$$\frac{p}{p_\infty} = \left(1 + \frac{\gamma - 1}{2} \frac{\chi}{a_\infty} \right)^{\frac{2\gamma}{\gamma - 1}} \tag{1}$$

where: $\gamma = c_p / c_v$ is the ratio between the specific heat at constant pressure and volume, c_p and c_v, respectively, χ is the normal down- or up-wash, that is the component of the fluid velocity in the $z-$direction, normal to the beam profile, and $a_\infty = \sqrt{\gamma \frac{p_\infty}{\rho_\infty}}$ is the sound speed; furthermore p_∞ and ρ_∞ are the pressure and density of the unperturbed airflow. This pressure can be expressed in terms of series expansion function elements vs χ, as follows:

$$\frac{p}{p_\infty} = 1 + \gamma \frac{\chi}{a_\infty} + \frac{1}{4}\gamma(\gamma+1)\left(\frac{\chi}{a_\infty}\right)^2 + \frac{1}{12}\gamma(\gamma+1)\left(\frac{\chi}{a_\infty}\right)^3 + \dots \tag{2}$$

The dimensional dynamic pressure parameter σ is also introduced:

$$\sigma = \frac{2q}{\beta} b_w \tag{3}$$

where: $\beta = \sqrt{M_\infty^2 - 1}$, $M_\infty = U_\infty / a_\infty$ is the Mach number, $q = \frac{1}{2}\rho_\infty U_\infty^2$ is the dynamic pressure, U_∞ is the airflow speed, and b_w is the beam width. Like in the previous author's paper [Tizzi 2003], the aerodynamic expressions of an infinite plate along the third not considered y axis, are applied, together with the structural constitutive relations of a beam; this hyphothesis can be accepted if the beam width b_w is very higher than L.

Since it is true that:

$$\sqrt{M_\infty^2 - 1} \cong M_\infty \tag{4}$$

if the Mach number is high enough, it is also true that:

$$\sigma = \frac{2q}{\beta} b_w \cong \gamma p_\infty M_\infty b_w \tag{5}$$

in view of the previously introduced expressions of q, M_∞ and a_∞.

Thus the force per unit axial length acting on each side of the beam profile can be evaluated from Eq. (2), and, by virtue of Eq. (5) and the expression of M_∞, it can be written as:

$$P_a = \Delta p b_w = \sigma \left[\frac{\chi}{U_\infty} + \frac{\gamma+1}{4} M_\infty \left(\frac{\chi}{U_\infty} \right)^2 + \frac{\gamma+1}{12} M_\infty^2 \left(\frac{\chi}{U_\infty} \right)^3 + \dots \right] \tag{6}$$

where $\Delta p = p - p_\infty$ is the pressure variation with respect to the unperturbed static conditions $p = p_\infty$.

The analysis has been developed only for symmetric cases, that is, both sides of the beam profile are exposed to the same airflow, and consequently only the odd powers give a contribution. In fact, taking into account that the normal speed component χ has opposite values on the upper and lower side, their effects sum-up for the odd powers and vanish for the even ones. Thence the resultant of the aerodynamic forces acting on both sides per unit length can be written as:

$$F_a = 2\sigma \left[\frac{\chi}{U_\infty} + \frac{\gamma+1}{12} M_\infty^2 \left(\frac{\chi}{U_\infty} \right)^3 + \dots \right] \tag{7}$$

The normal speed component due to the profile dynamics can be expressed as:

$$\chi = U_\infty \frac{\partial w}{\partial x} + \frac{\partial w}{\partial t} \tag{8}$$

where w is the beam points flexural displacement.

Substituting formula (8) of the normal fluid velocity into Eq. (7) gives the new expression of the resultant aerodynamic distributed force:

$$F_a = 2\sigma \left[\frac{\partial w}{\partial x} + \gamma_m \left(\frac{\partial w}{\partial x} \right)^3 + \dots \right] + 2\sigma \left\{ \frac{1}{U_\infty} \frac{\partial w}{\partial t} + \gamma_m \left[3 \left(\frac{\partial w}{\partial x} \right)^2 \frac{1}{U_\infty} \frac{\partial w}{\partial t} + \dots\dots \right] + \dots\dots \right\} \tag{9}$$

where: $$\gamma_m = \frac{\gamma + 1}{12} M_\infty^2 \tag{9a}$$

and the subsequent terms in the series expansion can be neglected. This is formed by two components: (1) the first F_{a1}, containing only spatial derivatives, is the coupling element between odd and even vibrating modes, which exists also without damping, (2) the second F_{a2}, containing also time derivatives, give rise to the aerodynamic dissipative force and damping. This second component is not considered for the undamped vibrating beam resolution.

The linear component of the distributed aerodynamic force in Eq. (9) is equal to the one obtained by Ashley [Bisplinghoff and Ashley 1975; Tizzi 2003], except for the presence of the ratio $\left(M_\infty^2 - 2\right) / \left(M_\infty^2 - 1\right)$ before the time derivative $\partial w / \partial t$, which is approximately equal to the unity for high Mach numbers.

It is necessary to recall the vibration governing equation of the fluttering beam [Tizzi 2003]:

$$EI\frac{\partial^4 w}{\partial x^4}+\mu\frac{\partial^2 w}{\partial t^2}-EA_s\,\frac{1}{2}\overline{\left(\frac{\partial w}{\partial x}\right)^2}\,\frac{\partial^2 w}{\partial x^2}+F_a=0 \tag{10}$$

where $A_s = b_w h$ is the cross-sectional area, h is the beam thickness, E is the Young's modulus, μ is the distributed mass per unit length, and $I = Eb_w h^3/12$ is the flexural moment of inertia.

Furthermore $\overline{\left(\frac{\partial w}{\partial x}\right)^2}$ has been previously defined as the mean square value of the flexural displacement first axial derivative over the whole beam length; the third term of Eq. (10), containing this mean square value, corresponds to the non-linear component of the transverse structural force, due to the beam axial stretching .

The axial inertia effects are being neglected, like in the previous analysis with linearized aerodynamic model, considering that the axial vibration frequencies are higher than the corresponding ones of the flutter vibration, and so the axial vibration frequency range is different from the flutter frequency range .

Substitution of the expression (9) of the aerodynamic distributed force into Eq. (10), with the assumed approximations, leads to the following governing equation:

$$EI\frac{\partial^4 w}{\partial x^4}+\mu\frac{\partial^2 w}{\partial t^2}-EA_s\,\frac{1}{2}\overline{\left(\frac{\partial w}{\partial x}\right)^2}\,\frac{\partial^2 w}{\partial x^2}+2\sigma\left[\frac{\partial w}{\partial x}+\gamma_m\left(\frac{\partial w}{\partial x}\right)^3\right]$$

$$+2\sigma\left\{\frac{1}{U_\infty}\frac{\partial w}{\partial t}+\gamma_m\left[3\left(\frac{\partial w}{\partial x}\right)^2\frac{1}{U_\infty}\frac{\partial w}{\partial t}\right]\right\}=0 \tag{11}$$

which can be reformulated in dimension-less form as:

$$\frac{\partial^4 W}{\partial \xi^4} + \lambda \frac{\partial^2 W}{\partial \tau^2} + \sigma_d \left[\frac{\partial W}{\partial \xi} + \gamma_m \left(\frac{\partial W}{\partial \xi} \right)^3 \right] - \frac{\alpha}{2} \overline{\left(\frac{\partial W}{\partial \xi} \right)^2} \frac{\partial^2 W}{\partial \xi^2} + \left[\gamma' + \gamma'' \left(\frac{\partial W}{\partial \xi} \right)^2 \right] \frac{\partial W}{\partial \tau} = 0 \qquad (12)$$

where the flexural displacement and the axial coordinate have been reformulated in non-dimensional form, and other dimension-less parameters have been introduced::

$$W(\xi, \tau) = \frac{w(x,t)}{L} \qquad \xi = \frac{x}{L} \qquad \lambda = \frac{\mu L^4}{EIT_o^2} \qquad \tau = \frac{t}{T_o} \qquad (12a)$$

$$\sigma_d = \frac{2\sigma L^3}{EI} \qquad \alpha = \frac{A_s L^2}{I} \qquad \gamma' = \frac{2\sigma}{U_\infty} \frac{L^4}{EIT_o} \qquad \gamma'' = 3\gamma' \gamma_m \qquad (12b)$$

and further T_o is a reference time.

The third and fifth terms in Eq. (12) refer to the non-dimensional equivalent form of the distributed aerodynamic force expression in Eq. (9):

$$(F_a)_{nd} = \sigma_d \left[\frac{\partial W}{\partial \xi} + \gamma_m \left(\frac{\partial W}{\partial \xi} \right)^3 \right] + \left[\gamma' + \gamma'' \left(\frac{\partial W}{\partial \xi} \right)^2 \right] \frac{\partial W}{\partial \tau} \qquad (13)$$

The Einstein's summation convention for repeated indices will be adopted in all the forthcoming expressions.

A series expansion for $W(\xi,\tau)$ in terms of function elements can be chosen:

$$W(\xi,\tau) = W_i(\tau) f_i(\xi) \qquad i = 1,2,\ldots,N \tag{14}$$

where each coefficient $W_i(\tau)$ is a Lagrangian degree of freedom and $f_i(\xi)$ are polynomial describing functions, belonging to the space of the square summable functions $\mathcal{H}_2(\mathcal{S})$, defined in the domain $\mathcal{S}$ (which in this case is the whole beam length). These satisfy only the geometric boundary conditions, as in the Ritz and FEM methods. The meaning of the coefficients $W_i(\tau)$ and the describing functions $f_i(\xi)$ are illustrated in Appendix A for the Ritz method, and Appendix B for FEM.

Eq. (12) can be transformed into its generalized equivalent form by the variational principle [Pars 1968], if the Lagrangian functional is introduced:

$$\mathcal{L} = \mathcal{T} - \mathcal{V} \tag{15}$$

where $\mathcal{T}$ is the kinetic energy and $\mathcal{V}$ is the strain energy. Thus the generic generalized ith governing equation can be written in the classical Lagrangian form:

$$\frac{d\left(\partial\mathcal{L}/\partial\dot{W}_i\right)}{d\tau} - \frac{\partial\mathcal{L}}{\partial W_i} + F_i^{(a)} = 0 \qquad i = 1,2,\ldots,N \tag{16}$$

where $F_i^{(a)}$ is the generalized aerodynamic force acting on the i th degree of freedom. It is possible to set a correspondence between each term of this generalized equation with each one of the governing Eq. (12).

The potential strain energy expression has been already determined [Tizzi 2003]:

$$\mathcal{U} = \frac{1}{2} k_{ij} W_i W_j + \frac{\alpha}{8} \varphi_{ij} \varphi_{kl} W_k W_l W_i W_j \qquad i, j, k, l = 1,2,\ldots, N \tag{17}$$

where:

$$\varphi_{ij} = \int_0^1 \frac{\partial f_i(\xi)}{\partial \xi} \frac{\partial f_j(\xi)}{\partial \xi} d\xi \tag{17a}$$

The stiffness matrix elements k_{ij} have been previously evaluated [Tizzi 1994]; obviously $k_{ij}\ddot{W}_j$ is the generalized linear structural force acting on the i th degree of freedom , which corresponds to the first term in Eq. (12). The elements of the second term at the second member of Eq. (17), corresponding to the non-linear contribution to the structural strain energy , are also known.

The expression of generalized aerodynamic force $F_i^{(a)}$ acting on the i th degree of freedom, corresponding to the third and fifth terms in Eq. (12), can be obtained by the use of the series expansion (14) in Eq. (13), and projecting the whole Eq. (13) onto the generic function element $f_i(\xi)$. Thus:

$$F_i^{(a)} = a_{ij} W_j + \gamma_m b_{ijkl} W_k W_l W_j + \gamma' c_{ij} \dot{W}_j + \gamma'' d_{ijkl} W_k W_l \dot{W}_j \tag{18}$$

$$j, k, l = 1,2 \ldots, N \qquad\qquad i = 1,2 \ldots, N$$

where the coefficients a_{ij} and c_{ij} are well known from the previous analysis [Tizzi 2003]:

$$a_{ij} = \sigma_d \int_0^1 f_i(\xi) \frac{\partial f_j(\xi)}{\partial \xi} d\xi \qquad\qquad c_{ij} = \int_0^1 f_i(\xi) f_j(\xi) d\xi \tag{18a}$$

and the new introduced ones are defined as:

$$b_{ijkl} = \sigma_d \int_0^1 f_i(\xi) \frac{\partial f_j(\xi)}{\partial \xi} \frac{\partial f_k(\xi)}{\partial \xi} \frac{\partial f_l(\xi)}{\partial \xi} d\xi \qquad d_{ijkl} = \sigma_d \int_0^1 f_i(\xi) f_j(\xi) \frac{\partial f_k(\xi)}{\partial \xi} \frac{\partial f_l(\xi)}{\partial \xi} d\xi \tag{18b}$$

The coefficients b_{ijkl} and d_{ijkl} have been determined both for the Ritz method and for FEM, in Appendix A and B, respectively.

Also the mass matrix elements m_{ij} in the kinetic energy expression:

$$\mathcal{T} = \frac{1}{2} m_{ij} \dot{W}_i \dot{W}_j \qquad \dot{W}_i = \partial W_i \,/\, /\partial \tau \qquad i, j = 1,2 \ldots, N \tag{19}$$

have been previously evaluated [Tizzi 1994]. It is true that:

$$m_{ij} = \lambda c_{ij} \tag{20}$$

where the non-dimensional coefficient λ has been defined in Eq. (12a). It is well known that $-m_{ij}\ddot{W}_j$ is the generalized inertial force acting on the ith degree of freedom, corresponding to the second term of the governing Eq. (12).

If the expressions of the strain and kinetic energy in Eqs. (17) and (19), respectively, along with the expression of the generalized aerodynamic force in Eq. (18), are substituted into Eq. (16), in view of the Lagrangian functional expression in Eq. (15), it is possible to achieve the equivalent generalized form of the governing Eq. (12), corresponding to the generic ith degree of freedom, as follows:

$$\left[k_{ij}^{*} + \frac{\alpha}{2}\varphi_{ij}\left(\varphi_{kl}W_k W_l\right)\right]W_j + \gamma_m b_{ijkl}W_k W_l W_j + m_{ij}\ddot{W}_j + \gamma' c_{ij}\dot{W}_j + \gamma'' d_{ijkl}W_k W_l \dot{W}_j = 0 \quad (21)$$

$$j,k,l = 1,2....,N \qquad i = 1,2....,N$$

The matrix elements k_{ij}^{*}, referring to both structural and aerodynamic linear forces, can be written as:

$$k_{ij}^{*} = k_{ij} + a_{ij} \quad (22)$$

Thus $k_{ij}^{*}W_j$ is the generalized linear structural–aerodynamic force acting on the same ith degree of freedom, which corresponds to the first term and the first part of the third term of the governing Eq. (12).

Further the term $(\varphi_{kl} W_k W_l)$ between brackets in Eq. (21), is the mean square value $\overline{\left(\frac{\partial W}{\partial \xi}\right)^2}$ of the first derivative $\frac{\partial W}{\partial \xi}$ over the whole beam length, which has been already introduced in Eq. (10). This is the reason for which it has been written separately in round brackets. As above mentioned, it takes into account the beam axial stretching, which gives rise to the non-linear component of the structural transverse force, as in the fourth term of Eq. (12).

The term containing the coefficient γ_m in Eq. (21) is equivalent to the second part of the third term of Eq. (12), and the terms containing γ' and γ'' are equivalent to the ones with the same coefficients in the governing Eq. (12).

The system of the generalized governing Eqs. (21), in view of Eq. (20), can also be written in matrix form:

$$\begin{aligned} \mathbf{Z} &= \dot{\mathbf{W}} \\ \dot{\mathbf{Z}} &= -\mathbf{M}^{-1}\mathbf{F}\mathbf{W} - \gamma_m \mathbf{M}^{-1}\mathbf{B}\mathbf{W}^{(3)} - \frac{\gamma'}{\lambda}\mathbf{Z} - \gamma'' \mathbf{M}^{-1}\mathbf{D}\mathbf{W}^{(3d)} \end{aligned} \qquad (23)$$

where: (1) $\mathbf{W}$ and $\mathbf{Z}$ are the column vectors of the coefficients W_j and their first derivatives $\dot{W}_j$ vs time τ, respectively, (2) $\mathbf{F}$ is the matrix whose elements are:

$$f_{ij} = k^*_{ij} + \frac{\alpha}{2}(\varphi_{kl} W_k W_l)\varphi_{ij} \qquad (24)$$

(3) $\mathbf{M}$ is the mass matrix, (4) $\mathbf{B}$ is a matrix with dimensions $N \times N^3$, whose elements are $b_{ij_{t3}} = b_{ijkl}$ (j_{t3} is the contraction of the three indices jkl and obviously $j_{t3} = 1,2\boldsymbol{....}N^3$), (5) $\mathbf{W}^{(3)}$ is the column vector with dimensions N^3, whose elements are the triple products $p_{jt3} = W_k W_l W_j$ between the coefficients of the series expansion in Eq. (14), (6) $\mathbf{D}$ is a matrix whose elements are $d_{ij_{t3}} = d_{ijkl}$ and with the same dimensions of $\mathbf{B}$, and at last (7) $\mathbf{W}^{(3d)}$ is a column vector with the same dimensions of $\mathbf{W}^{(3)}$, whose elements are the triple products $q_{jt3} = W_k W_l \dot{W}_j$.

Eqs. (21) and (23) are the same of the previous analysis with linearized aerodynamic forces, except for the presence of the terms with b_{ijkl} and d_{ijkl} in Eqs. (21), and the corresponding matrices $\mathbf{B}$ and $\mathbf{D}$ in Eq. (23), referring to the non-linear contribution of the “Piston Theory” to the aerodynamic forces. The system of Eqs. (23) can be integrated in time by appropriate algorithms.

In the case of a simply supported beam it is easy to apply also the Galerkin method, as in the Dowell’s model [Dowell 1966; 1967], as shown in Appendix C. Tha advantages of the Galerkin procedure arise from the diagonal form of the mass matrix $\mathbf{M}$, due to the orthogonality between different describing function elements.

Appendix A

Only the parameters which refer to the non-linear aerodynamic forces are herein evaluated. The way to determine the others has been sufficiently illustrated in the previous author's papers [Tizzi 1994; 2003].

The generic describing function of the bending deflection $W(\xi,\tau)$ series expansion in Eq. (14) can be written as:

$$f_i(\xi) = \xi^i(1-\xi) \qquad \text{or} \qquad f_i(\xi) = \xi^{i+1}(1-\xi)^2 \tag{A.1}$$

for the simply supported or clamped at both ends beam, respectively.

This function can also be written in a more general concise form:

$$f_i(\xi) = s_{i_m}\xi^{i+i_m+i^*} \qquad i_m = 0,1..i_l \tag{A.2}$$

where $i_l = 1$ refers to the simply supported beam, whereas if $i_l = 2$ the beam clamped at both ends is considered. Furthermore it is assumed that:

$$i^* = i_l - 1 \qquad \text{and} \qquad s_0 = 1 \quad s_1 = -i_l \quad s_2 = 1 \tag{A.2a}$$

If the generic describing function in Eq. (A.2) is substituted into the expression of the coefficients b_{ijkl} in Eq. (18b), one obtains:

$$b_{ijkl} = \left(j + i^* + j_m\right)\left(k + i^* + k_m\right)\left(l + i^* + l_m\right) s_{i_m} s_{j_m} s_{k_m} s_{l_m} \frac{1}{i + i_m + j + j_m + k + k_m + l + l_m + 4i^* - 2}$$

$$i_m j_m, k_m, l_m = 0,1 \ldots i_l \qquad i, j, k, l = 1,2 \ldots N \qquad \text{(A.3)}$$

and the analogous expression of the other coefficients d_{ijkl} in Eq. (18b) can be written as:

$$d_{ijkl} = \left(k + i^* + k_m\right)\left(l + i^* + l_m\right) s_{i_m} s_{j_m} s_{k_m} s_{l_m} \frac{1}{i + i_m + j + j_m + k + k_m + l + l_m + 4i^* - 1}$$

$$i_m j_m, k_m, l_m = 0,1 \ldots i_l \qquad i, j, k, l = 1,2 \ldots N \qquad \text{(A.4)}$$

Appendix B

Also for FEM only the coefficients referring to the non-linear components of the aerodynamic forces are herein evaluated, because sufficient explanations to determine the other beam flutter parameters have been given in the previous mentioned author's papers.

The beam is divided into N_E elements and the generic i_eth element lies between the sections $S_{i_e - 1}$ and S_{i_e}, whose non-dimensional axial coordinates are $\xi_{i_e - 1} = \left(i_e - 1\right) / N_E$ and $\xi_{i_e} = i_e / N_E$, respectively. The non-dimensional normalized axial coordinate of a generic i_eth element can be introduced:

$$\xi_n = \left(\xi - \xi_{i_e - 1}\right) N_E \qquad \xi_{i_e - 1} \le \xi \le \xi_{i_e} \qquad 0 \le \xi_n \le 1 \tag{B.1}$$

The non-dimensional bending tranverse deflection $W(\xi)$ can be expressed vs the normalized axial coordinate ξ_n in the i_eth element in the classical and well known form [Tizzi 1994; 2003]:

$$W(\xi) = Q_{i_p}^{(i_e)} c_{i_p i_a} \xi_n^{i_a - 1} \qquad i_p, i_a = 1,2,3,4 \tag{B.2}$$

where the index i_p characterizes the nodal degree of freedom at the two delimiting sections $S_{i_e - 1}$ and S_{i_e}, that is:

$$Q_1^{(i_e)} = W_{i_e - 1} \qquad Q_2^{(i_e)} = \theta_{i_e - 1} \qquad Q_3^{(i_e)} = W_{i_e} \qquad Q_4^{(i_e)} = \theta_{i_e} \tag{B.3}$$

and W_{i_e}, θ_{i_e} are the flexural displacement and a rotation parameter (the true rotation divided by N_E), respectively, at the section S_{i_e}. The values of the coefficients $c_{i_p i_a}$ can be evaluated by imposing that the boundary conditions in Eq. (B.3) of the i_eth element are fulfilled. Thus the matrix $[\mathbf{C}]$, whose elements are the same coefficients $c_{i_p i_a}$, can be written in the well known form as:

$$[\mathbf{C}] = \begin{bmatrix} 1 & 0 & -3 & 2 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 3 & -2 \\ 0 & 0 & 1 & -1 \end{bmatrix} \tag{B.4}$$

The index i, characterizing the generic coefficient W_i and the generic describing function $f_i(\xi)$ of the series expansion in Eq. (14), is connected with the above introduced indices i_e and i_p via the introduced relations in the same previous author's paper [Tizzi 2003]. The generic polynomial describing function $f_i(\xi)$ of the bending displacement in the i_eth element of FEM model is the same introduced in Eq. (B.2):

$$f_i(\xi) = c_{i_p i_a} \xi_n^{i_a - 1} \qquad i = i(i_e, i_p) \tag{B.5}$$

$$i_p, i_a = 1,2,3,4 \qquad i_e = 1,2 \ldots N_E \qquad i = 1,2 \ldots N$$

where $N = 2N_E$ for the simply supported beam and $N = 2(N_E - 1)$ for the beam clamped at both ends. The generic Lagrangian degree of freedom is $Q_{i_p}^{(i_e)}$, whose meaning has been explained in Eq. (B.3). Thus, by the use of the describing function in Eq. (B.5), the expression of the coefficients b_{ijkl} and d_{ijkl}, introduced in Eq. (18b), can be evaluated in the i_eth beam element and written as:

$$b_{ijkl} = N_E^2 j_a k_a l_a c_{i_p i_a} c_{j_p j_a} c_{k_p k_a} c_{l_p l_a} \frac{1}{i_a + j_a + k_a + l_a - 6} \tag{B.6}$$

$$i_a = 1,2,3,4 \qquad\qquad j_a, k_a, l_a = 2,3,4$$

$$d_{ijkl} = N_E k_a l_a c_{i_p i_a} c_{j_p j_a} c_{k_p k_a} c_{l_p l_a} \frac{1}{i_a + j_a + k_a + l_a - 5} \quad . \tag{B.7}$$

$$i_a, j_a = 1,2,3,4 \qquad\qquad k_a, l_a = 2,3,4$$

$$i_p, j_p, k_p, l_p = 1,2,3,4 \quad i = i(i_e, i_p) \quad j = j(i_e, j_p) \quad k = k(i_e, k_p) \quad l = l(i_e, l_p) \tag{B.7a}$$

taking into account that $d\xi = d\xi_n / N_E$ from Eq. (B.1). By summing-up the effects of all beam elements it is possible to know the resulting value of these coefficients, which can be substituted into the generalized Eq. (21) to find the requested solution.

Appendix C

Also with Galerkin method the interest of computational work is limited to the non-linear aerodynamic forces components, because the other parameters are already known.

The chosen describing function of the transverse deflection series expansion is the trigonometric type:

$$W(\xi, \tau) = W_i(\tau)\mathbf{sin}(i\pi\xi) \tag{C.1}$$

This series expansion can be substituted into Eq. (12) , and then by pre-multiplying by the generic describing function $\mathbf{sin}(i\pi\xi)$ and integrating throughout the beam length, it is possible to achieve the dynamic governing equation:

$$\left[\text{linear terms} + \text{non-linear structural-forces}\right] + \gamma_m J_{sccc} W_k W_l W_j + \gamma'' J_{sscc} W_k W_l \dot{W}_j = 0 \qquad \text{(C.2)}$$

$$j, k, l = 1,2....N$$

where:

$$J_{sccc} = (j\pi)(k\pi)(l\pi)\int_0^1 \mathbf{sin}(i\pi\xi)\mathbf{cos}(j\pi\xi)\mathbf{cos}(k\pi\xi)\mathbf{cos}(l\pi\xi)d\xi \qquad \text{(C.2a)}$$

$$J_{sscc} = (k\pi)(l\pi)\int_0^1 \mathbf{sin}(i\pi\xi)\mathbf{sin}(j\pi\xi)\mathbf{cos}(k\pi\xi)\mathbf{cos}(l\pi\xi)d\xi \qquad \text{(C.2b)}$$

The two introduced integrals can be easily determined.

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