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EXISTENCE OF ONE-COMPONENT RAYLEIGH WAVES, STONELEY WAVES, LOVE WAVES, SLIP WAVES AND ONE-COMPONENT WAVES IN A PLATE OR LAYERED PLATE

THOMAS C. T. TING

It is known that one-component surface (Rayleigh) waves exist in an anisotropic elastic half-space. Since the solution shows that the displacement normal to the free surface vanishes everywhere, a one-component surface wave is also a one-component slip wave in the half-space if the boundary of the half-space is a slippery surface. We show that no other one-component slip waves exist for the half-space. As to steady waves in a bimaterial that consists of two dissimilar anisotropic elastic materials, one-component slip waves can be constructed from two one-component surface waves. There are no other one-component slip waves for a bimaterial. By imposing the continuity of the displacement at the interface on the one-component slip wave, a one-component Stoneley wave is obtained. Although one-component waves for the half-space can also propagate in a homogeneous plate, we present new one-component waves in a plate for which the Stroh eigenvalue p is real. By superposition of the one-component waves in the layer and in the half-space, one-component Love waves can be constructed. Finally, we show that one-component waves can propagate in a layered plate.

1. Introduction

Surface (Rayleigh) waves in an anisotropic elastic half-space in general consist of two or three components. Even for an isotropic elastic material, surface waves consist of two components. It was first pointed out by Barnett et al. [1991] that one-component surface waves exist for certain special anisotropic elastic materials. Further study on one-component surface waves was done by Barnett and Chadwick [1991], Chadwick [1992], Norris [1992], Ting [1992], and Wang and Gundersen [1993]. The question of whether one-component steady waves exist in a bimaterial when the interface is in sliding contact or perfectly bonded remains open. Also open is the question of whether one-component waves exist in Love waves and in layered plates. The purpose of this paper is to address these questions.

The basic equations for steady waves in an anisotropic elastic medium based on the Stroh formalism [Stroh 1962; Barnett and Lothe 1973; Chadwick and Smith 1977; Ting 1996a] are outlined in Section 2. A modified version [Ting 2000] that is more suitable for steady waves is presented in Section 3. This version is employed to find one-component surface waves. The solution shows that the displacement normal to the free surface vanishes everywhere. Hence one-component Rayleigh waves are also one-component slip waves in the half-space when the boundary of the half-space is a slippery surface. We proved in Section 4 that there are no other one-component slip waves in the half-space. Steady state waves propagating in a bimaterial of dissimilar anisotropic materials are studied in Section 5. If the interface is in sliding

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contact, one can simply take two one-component surface waves for the two half-spaces that automatically satisfy the sliding contact conditions. Again we show that there are no other one-component slip waves for the bimaterial. As for the Stoneley waves for which the interface is perfectly bonded, one-component Stoneley waves can be obtained from the one-component slip waves in the bimaterial by choosing the material constants such that the displacement is continuous at the interface. This is presented in Section 6. Thus we obtain a one-component Stoneley wave that is also a one-component slip wave. The question of whether there are other one-component Stoneley waves is still open. The one-component waves obtained here are valid if the strain energy density is positive and nonzero. This means that the 5×5 matrix of the reduced elastic compliance $s'_{\alpha\beta}$ must be positive definite. In Section 7 we present conditions for positive definiteness of $s'_{\alpha\beta}$ for which one-component Rayleigh waves can propagate. Section 8 studies the existence of one-component waves in a homogeneous plate. Although one-component waves in the half-space can also propagate in the plate, we present new solutions in which the Stroh eigenvalue p is real. By superposition of one-component waves in a plate and in the half-space, one-component Love waves can be obtained. This is presented in Section 9. Finally, in Section 10, we show how to construct one-component waves in a layered plate. The interface between the layers can be perfectly bonded or in sliding contact.

2. Basic equations

In a fixed rectangular coordinate system $x_i (i = 1, 2, 3)$, the equation of motion is

$$\sigma_{ij,j} = \rho \ddot{u}_i, \tag{2.1}$$

where σ_{ij} is the stress, u_i is the displacement, ρ is mass density, the dot denotes differentiation with respect to time t and a comma denotes differentiation with respect to x_i. The stress-strain relation is

$$\sigma_{ij} = C_{ijks} u_{k,s}, \tag{2.2}$$

$$C_{ijks} = C_{jiks} = C_{ksij} = C_{ijsk}, \tag{2.3}$$

in which C_{ijks} is the elastic stiffness. The C_{ijks} is positive definite and possesses the full symmetry shown in Equation (2.3). The third equality in (2.3) is redundant because the first two imply the third [Ting 1996a].

For two-dimensional steady state motion in the x_1-direction with a constant wave speed $v > 0$, a general solution for the displacement $\mathbf{u}$ in (2.1) and (2.2) is

$$\mathbf{u} = \mathbf{a} e^{ikz}, \qquad z = x_1 + p x_2 - v t, \tag{2.4}$$

where $k > 0$ is the real wave number, t is the time, and p and $\mathbf{a}$ satisfy the equation

$$[\mathbf{Q} - X\mathbf{I} + p(\mathbf{R} + \mathbf{R}^T) + p^2 \mathbf{T}]\, \mathbf{a} = \mathbf{0}, \qquad X = \rho v^2. \tag{2.5}$$

In the above, the superscript T denotes the transpose, $\mathbf{I}$ is the identity matrix and

$$Q_{ik} = C_{i1k1}, \qquad R_{ik} = C_{i1k2}, \qquad T_{ik} = C_{i2k2}. \tag{2.6}$$

Introducing the vector

$$\mathbf{b} = (\mathbf{R}^T + p\mathbf{T})\mathbf{a} = -p^{-1}(\mathbf{Q} - X\mathbf{I} + p\mathbf{R})\mathbf{a}, \tag{2.7}$$

where the second equality follows from (2.5), the stress computed from (2.2) and (2.4) can be written as

$$\sigma_{i1} = Xu_{i,1} - \phi_{i,2}, \qquad \sigma_{i2} = \phi_{i,1}, \tag{2.8}$$

where the vector

$$\boldsymbol{\phi} = \mathbf{b}e^{ikz} \tag{2.9}$$

is the stress function. The two equations in (2.7) can be written as

$$\mathbf{N}\boldsymbol{\xi} = p\boldsymbol{\xi}, \tag{2.10}$$

where [Ingebrigtsen and Tonning 1969]

$$\mathbf{N} = \begin{bmatrix} \mathbf{N}_1 & \mathbf{N}_2 \\ \mathbf{N}_3 + X\mathbf{I} & \mathbf{N}_1^T \end{bmatrix}, \qquad \boldsymbol{\xi} = \begin{bmatrix} \mathbf{a} \\ \mathbf{b} \end{bmatrix}, \tag{2.11}$$

and

$$\mathbf{N}_1 = -\mathbf{T}^{-1}\mathbf{R}^T, \qquad \mathbf{N}_2 = \mathbf{T}^{-1}, \qquad \mathbf{N}_3 = \mathbf{R}\mathbf{T}^{-1}\mathbf{R}^T - \mathbf{Q}. \tag{2.12}$$

The matrix $\mathbf{N}_2$ is symmetric and positive definite while $-\mathbf{N}_3$ is symmetric and positive semidefinite [Ting 1988; 1996a].

There are six eigenvalues p and six associated eigenvectors $\boldsymbol{\xi} = (\mathbf{a}, \mathbf{b})$. For a steady wave propagating in the half-space $x_2 \geq 0$, p must be complex with a positive imaginary part so that the displacement $\mathbf{u}$ computed from (2.4) vanishes at $x_2 = \infty$. If the six eigenvalues p are all complex, they consist of three pairs of complex conjugates. Let p_1, p_2, p_3 be the eigenvalues with a positive imaginary part. The general solution for the displacement $\mathbf{u}$ and the stress function $\boldsymbol{\phi}$ obtained from a superposition of (2.4) and (2.9) associated with p_1, p_2, p_3 is

$$\mathbf{u} = \sum_{m=1}^{3} q_m \mathbf{a}_m e^{ikz_m}, \qquad \boldsymbol{\phi} = \sum_{m=1}^{3} q_m \mathbf{b}_m e^{ikz_m}, \tag{2.13}$$

$$z_m = x_1 + p_m x_2 - vt, \tag{2.14}$$

where $(\mathbf{a}_m, \mathbf{b}_m)$ $(m = 1, 2, 3)$ are the eigenvectors associated with the eigenvalues p_m $(m = 1, 2, 3)$, and q_m are arbitrary constants. The q_m can be chosen such that the boundary condition at $x_2 = 0$ is satisfied.

If the boundary $x_2 = 0$ is a traction-free surface, the boundary condition is

$$\boldsymbol{\phi} = \mathbf{0} \qquad \text{at } x_2 = 0. \tag{2.15}$$

Equation (2.13) then gives

$$\sum_{m=1}^{3} q_m \mathbf{b}_m = \mathbf{0}. \tag{2.16}$$

In general, all three q_m (m=1,2,3) are needed to satisfy (2.16). We then have three-component surface waves. For certain anisotropic elastic materials of which isotropic materials are special cases, only two q_m are needed. We have two-component surface waves. Barnett et al. [1991] were the first ones to point out that there are anisotropic elastic materials for which one-component surface waves can propagate in the half-space. This is presented briefly in the next section.

3. One-component surface waves

The explicit expression of $\mathbf{N}_1$, $\mathbf{N}_2$, $\mathbf{N}_3$ in (2.12) was given in [Ting 1988] in terms of the elastic compliance $s_{\alpha\beta}$, and in [Barnett and Chadwick 1991] in terms of the elastic stiffness $C_{\alpha\beta}$. The $C_{\alpha\beta}$ is the contracted notation of C_{ijks}. $\mathbf{N}_1$, $\mathbf{N}_2$, $\mathbf{N}_3$ have a complicated expression in terms of $s_{\alpha\beta}$ or $C_{\alpha\beta}$. For the problems we will study here, an alternate equation to (2.10) is more convenient. It is shown in [Ting 2000] that (2.10) can be written in the form

$$\begin{bmatrix} Xs'_{61}-p & -1 & Xs'_{65} & s'_{66}-ps'_{61} & s'_{62} & s'_{64}-ps'_{65} \\ Xs'_{21} & -p & Xs'_{25} & s'_{26}-ps'_{21} & s'_{22} & s'_{24}-ps'_{25} \\ Xs'_{41} & 0 & Xs'_{45}-p & s'_{46}-ps'_{41} & s'_{42} & s'_{44}-ps'_{45} \\ Xs'_{11}-1 & 0 & Xs'_{15} & s'_{16}-ps'_{11} & s'_{12} & s'_{14}-ps'_{15} \\ 0 & X & 0 & -1 & -p & 0 \\ Xs'_{51} & 0 & Xs'_{55}-1 & s'_{56}-ps'_{51} & s'_{52} & s'_{54}-ps'_{55} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ b_1 \\ b_2 \\ b_3 \end{bmatrix} = \mathbf{0}, \tag{3.1}$$

where

$$s'_{\alpha\beta} = s_{\alpha\beta} - \frac{s_{\alpha 3}s_{3\beta}}{s_{33}} \tag{3.2}$$

is the *reduced elastic compliance*.

For a one-component free surface wave,

$$\sigma_{21} = \sigma_{22} = \sigma_{23} = 0, \tag{3.3}$$

at $x_2 = 0$. This means that, from (2.8) and (2.9), $\mathbf{b} = \mathbf{0}$ or

$$b_1 = b_2 = b_3 = 0. \tag{3.4}$$

Equation (3.1) reduces to

$$\begin{bmatrix} Xs'_{61}-p & -1 & Xs'_{65} \\ Xs'_{21} & -p & Xs'_{25} \\ Xs'_{41} & 0 & Xs'_{45}-p \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \mathbf{0}, \tag{3.5a}$$

and

$$\begin{bmatrix} Xs'_{11}-1 & 0 & Xs'_{15} \\ 0 & -X & 0 \\ Xs'_{51} & 0 & Xs'_{55}-1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \mathbf{0}. \tag{3.5b}$$

The second of the three scalar equations implicit in Equation (3.5b) demands that

$$a_2 = 0. \tag{3.6}$$

Since $\operatorname{Im} p > 0$ (otherwise $\mathbf{u}$ does not vanish at $x_2 = 8$) Equation $(3.5\text{a})_{1,3}$ tells us that a_1 and a_3 are nonzero and that the ratio of a_1 to a_3 cannot be real. It follows from $(3.5\text{a})_2$ and $(3.5\text{b})_{1,3}$ that

$$Xs'_{11} = 1 = Xs'_{55}, \qquad s'_{12} = s'_{15} = s'_{25} = 0. \tag{3.7}$$

Equation (3.5a) reduces to

$$\begin{bmatrix} Xs'_{16}-p & Xs'_{56} \\ Xs'_{14} & Xs'_{45}-p \end{bmatrix} \begin{bmatrix} a_1 \\ a_3 \end{bmatrix} = \mathbf{0}. \tag{3.8}$$

For a nontrivial solution for a_1 and a_3 we must have

$$(Xs'_{16} - p)(Xs'_{45} - p) - X^2 s'_{14} s'_{56} = 0, \tag{3.9}$$

or

$$p = \tfrac{1}{2} X \left(\varepsilon + i\sqrt{-\kappa}\right), \tag{3.10a}$$

where

$$\varepsilon = s'_{45} + s'_{16}, \qquad \kappa = (s'_{45} - s'_{16})^2 + 4s'_{14}s'_{56}. \tag{3.10b}$$

The p must be complex so that

$$\kappa = (s'_{45} - s'_{16})^2 + 4s'_{14}s'_{56} < 0. \tag{3.11}$$

This means that s'_{14} and s'_{56} cannot vanish and must have the opposite signs. Thus a one-component surface wave cannot propagate in monoclinic materials with the symmetry plane at $x_2 = 0$ or $x_3 = 0$ because s'_{14} and s'_{56} vanish for these materials [Barnett et al. 1991; Chadwick 1992].

The fact that $\mathbf{b} = \mathbf{0}$, that is, $\boldsymbol{\phi} = \mathbf{0}$, does not mean that all stresses vanish. Only σ_{i2} (i=1,2,3) vanish. σ_{11} and σ_{31} can be computed from $(2.8)_1$ while σ_{33} is from

$$0 = s_{31}\sigma_{11} + s_{33}\sigma_{33} + s_{35}\sigma_{31}, \tag{3.12}$$

which is the vanishing of the strain ε_{33}. Thus both the displacement and the stress are polarized on the plane $x_2 = 0$. It can be shown [Ting 1992] that the vectors (a_1, a_3) and (σ_{11}, σ_{31}) are orthogonal to each other.

The existence of Rayleigh waves in a general anisotropic elastic half-space has been studied by Barnett and Lothe [1978; 1987; 1989].

4. One-component slip waves in the half-space

If the boundary of the half-space is a slippery surface, we must have

$$u_2 = \sigma_{21} = \sigma_{23} = 0. \tag{4.1}$$

This means that

$$a_2 = b_1 = b_3 = 0. \tag{4.2}$$

Equation (4.2) is implied by (3.4) and (3.6). Hence the one-component surface waves presented in the previous section can also propagate in the half-space with a slippery surface.

Equations (3.4) and (3.6) have the restriction $b_2 = 0$ which is not required in (4.2). One may ask if there are other one-component slip waves in the half-space for which $b_2 \neq 0$. The answer is negative. The proof is very simple. Equation (3.1) consists of six scalar equations of which the fifth equation is

$$Xa_2 = b_1 + pb_2. \tag{4.3}$$

When (4.2) holds, (4.3) demands that $b_2 = 0$. Thus there are no one-component slip waves for the half-space for which $b_2 \neq 0$.

5. One-component slip waves in a bimaterial

Let the half-space $x_2 \geq 0$ be occupied by an anisotropic elastic material and the half-space $x_2 \leq 0$ be occupied by a different anisotropic elastic material. For a one-component wave, the solution for the material in $x_2 \geq 0$ is given in (2.4) and (2.9). The same solution applies for the material in $x_2 \leq 0$ and is written as

$$\hat{\mathbf{u}} = \hat{\mathbf{a}} e^{ik\hat{z}}, \qquad \hat{\boldsymbol{\phi}} = \hat{\mathbf{b}} e^{ik\hat{z}}, \tag{5.1}$$

where we have used the *hat* to distinguish the solution for the material in $x_2 \leq 0$ from the solution for the material in $x_2 \geq 0$. In Equation (5.1),

$$\hat{z} = x_1 + \hat{p} x_2 - v\,t, \qquad \operatorname{Im} \hat{p} < 0. \tag{5.2}$$

The imaginary part of $\hat{p}$ must be negative so that the displacement and the stress function vanish at $x_2 = -\infty$.

In this section we consider the case in which the interface $x_2 = 0$ is in sliding contact. This means that

$$b_1 = b_3 = 0, \qquad \hat{b}_1 = \hat{b}_3 = 0, \tag{5.3a}$$

$$b_2 = \hat{b}_2, \qquad a_2 = \hat{a}_2. \tag{5.3b}$$

Equation (4.3) gives, using (5.3a),

$$b_2 = p^{-1} X a_2, \qquad \hat{b}_2 = \hat{p}^{-1} \hat{X} \hat{a}_2, \tag{5.4}$$

or, by (5.3b),

$$(p^{-1} X - \hat{p}^{-1} \hat{X}) a_2 = 0. \tag{5.5}$$

But X and $\hat{X}$ are real, positive and nonzero while $\operatorname{Im} p > 0$ and $\operatorname{Im} \hat{p} < 0$. Hence $p^{-1} X - \hat{p}^{-1} \hat{X}$ cannot vanish. Thus Equation (5.5), and hence (5.4) and (5.3b) tells us that

$$a_2 = b_2 = \hat{a}_2 = \hat{b}_2 = 0. \tag{5.6}$$

The result is that a one-component slip wave in the bimaterial consists of two one-component Rayleigh waves in the two half-spaces. There are no other one-component slip waves.

In conclusion, a one-component slip wave exists in a bimaterial if the reduced elastic compliance of the material in $x_2 \geq 0$ satisfies the conditions given in (3.7) and (3.11). The eigenvalue p is given by (3.10). The reduced elastic compliance of the material in $x_2 \leq 0$ satisfies the same conditions given in (3.7) and (3.11). The eigenvalue $\hat{p}$ is given by

$$\hat{p} = \frac{\hat{X}}{2} \left(\hat{\varepsilon} - i\sqrt{-\hat{\kappa}} \right),$$

where the imaginary part is negative and

$$\hat{\varepsilon} = \hat{s}'_{45} + \hat{s}'_{16}, \qquad \hat{\kappa} = (\hat{s}'_{45} - \hat{s}'_{16})^2 + 4\hat{s}'_{14}\hat{s}'_{56} < 0.$$

Slip waves in a bimaterial that consists of two general anisotropic elastic half-spaces have been studied by Barnett et al. [1988].

6. One-component Stoneley waves

We study in this section the case in which the interface $x_2 = 0$ is perfectly bonded. This means that

$$\hat{a}_i = a_i, \quad \hat{b}_i = b_i, \qquad i = 1, 2, 3. \tag{6.1}$$

The one-component slip waves in a bimaterial presented in the previous section satisfy the interface conditions (6.1) except

$$\hat{a}_1 = a_1, \qquad \hat{a}_3 = a_3. \tag{6.2}$$

If we can choose the elastic constants of the bimaterial such that (6.2) holds, we have a one-component Stoneley wave, albeit a rather strange one, because it is also a slip wave. It suffices to consider the condition

$$\frac{\hat{a}_3}{\hat{a}_1} = \frac{a_3}{a_1}. \tag{6.3}$$

From $(3.8)_1$ we have

$$\frac{a_3}{a_1} = \frac{p - Xs'_{16}}{Xs'_{56}} = \frac{ps'_{11} - s'_{16}}{s'_{56}}, \tag{6.4}$$

where the second equality follows from (3.7). Substitution of p from (3.10) yields

$$\frac{a_3}{a_1} = \frac{(s'_{45} - s'_{16}) + i\sqrt{-(s'_{45} - s'_{16})^2 - 4s'_{14}s'_{56}}}{2s'_{56}}. \tag{6.5}$$

For the material in $x_2 \leq 0$, $\operatorname{Im}\hat{p} < 0$ so that

$$\frac{\hat{a}_3}{\hat{a}_1} = \frac{(\hat{s}'_{45} - \hat{s}'_{16}) - i\sqrt{-(\hat{s}'_{45} - \hat{s}'_{16})^2 - 4\hat{s}'_{14}\hat{s}'_{56}}}{2\hat{s}'_{56}}. \tag{6.6}$$

Equation (6.3) holds if

$$\hat{s}'_{56}s'_{56} < 0 \qquad \text{and} \qquad \frac{\hat{s}'_{45} - \hat{s}'_{16}}{\hat{s}'_{56}} = \frac{s'_{45} - s'_{16}}{s'_{56}}, \qquad \frac{\hat{s}'_{14}}{\hat{s}'_{56}} = \frac{s'_{14}}{s'_{56}}. \tag{6.7}$$

In particular, the equations in (6.7) hold when

$$\hat{s}'_{56} = -s'_{56}, \qquad \hat{s}'_{45} = -s'_{45}, \qquad \hat{s}'_{16} = -s'_{16}, \qquad \hat{s}'_{14} = -s'_{14}. \tag{6.8}$$

The wave speed v obtained from $(2.5)_2$ and (3.7) is

$$v^2 = (\rho s'_{11})^{-1}. \tag{6.9}$$

This must be the same for the half-space $x_2 \leq 0$ so that we must have

$$\hat{\rho}\hat{s}'_{11} = \rho s'_{11}. \tag{6.10}$$

Equation (6.10) is the additional condition that is required for a one-component Stoneley wave to propagate in the bimaterial. It is also a slip wave.

To see if there is one-component Stoneley wave that is not a slip wave, we have to solve (3.1) for p, $\mathbf{a}$ and $\mathbf{b}$ in terms of X. The same solutions apply to the material in $x_2 \leq 0$, with $\operatorname{Im}\hat{p} < 0$. Imposing the conditions that $\hat{\mathbf{a}} = \mathbf{a}$, $\hat{\mathbf{b}} = \mathbf{b}$ and $\hat{X}/\hat{\rho} = X/\rho$, one obtains restrictions imposed on the reduced elastic compliances $s'_{\alpha\beta}$ and $\hat{s}'_{\alpha\beta}$. This is easier said than done because the algebra would be too complicated. Hence the question is open if there exist one-component Stoneley waves that are not slip waves.

The existence of Stoneley waves in a general anisotropic elastic bimaterial was studied by Barnett et al. [1985] and Chadwick and Currie [1974].

It should be noted that the application of (4.3) for the materials in $x_2 \geq 0$ and $x_2 \leq 0$ leads to, using (6.1),

$$Xa_2 = b_1 + pb_2, \qquad \hat{X}a_2 = b_1 + \hat{p}b_2. \tag{6.11}$$

The following are obvious:

(i) If $a_2 = 0$, we must have $b_1 = b_2 = 0$.

(ii) If $b_1 = 0$, following the discussion in (5.5) and (5.6) we must have $a_2 = b_2 = 0$.

(iii) If $b_2 = 0$, we must have either $X = \hat{X}$ (which means $\rho = \hat{\rho}$) or $a_2 = b_1 = 0$.

Thus, the vanishing of any one of a_2, b_1, b_2 implies the vanishing of the remaining two with the exception of when $b_2 = 0$ and $X = \hat{X}$.

7. Positive definiteness of strain energy density

The above solutions for one-component waves are valid if, under the restrictions imposed on the elastic constants, the strain energy density is positive and nonzero. The strain energy density is positive and nonzero if the 6×6 symmetric matrix of the elastic stiffness $C_{\alpha\beta}$ or the 6×6 symmetric matrix of the elastic compliance $s_{\alpha\beta}$ is positive definite. For the one-component Rayleigh waves, Barnett et al. 1991 provide a numerical example of a 6×6 symmetric matrix $s_{\alpha\beta}$ that is positive definite subject to the conditions (3.7) and (3.11). We will present here a set of general conditions for which the strain energy density is positive definite under the restrictions (3.7) and (3.11).

The matrices $C_{\alpha\beta}$ and $s_{\alpha\beta}$ are the inverse of each other. For two-dimensional deformations of an anisotropic elastic body, the elements of the third column, and hence the third row, of $C_{\alpha\beta}$ do not appear in the analysis. Let $C^o_{\alpha\beta}$ be the 5×5 matrix obtained from $C_{\alpha\beta}$ by deleting the third row and the third column. It can be shown that $C^o_{\alpha\beta}$ and the 5×5 matrix of the reduced elastic compliance $s'_{\alpha\beta}$ are the inverse of each other [Ting 1996a; 1996b]. Both $C^o_{\alpha\beta}$ and $s'_{\alpha\beta}$ must be positive definite. When a positive definite $s'_{\alpha\beta}$ is obtained, taking the inverse of $s'_{\alpha\beta}$ gives a positive definite $C^o_{\alpha\beta}$. By inserting $C_{3\beta}$ $(\beta = 1, 2, \ldots, 6)$ in the third row and $C_{\alpha 3}$ $(\alpha = 1, 2, \ldots, 6)$ in the third column we obtain $C_{\alpha\beta}$. One can choose C_{33} large enough (see below) to obtain a positive definite $C_{\alpha\beta}$. The inverse of $C_{\alpha\beta}$ provides a positive definite $s_{\alpha\beta}$. Hence it suffices to consider the positive definiteness of the 5×5 matrix $s'_{\alpha\beta}$.

A matrix is positive definite if and only if the leading principal minors of the matrix are positive and nonzero [Hohn 1965]. For the 5×5 matrix of the reduced elastic compliance $s'_{\alpha\beta}$ that provides one-component Rayleigh waves in the half-space, we have, from (3.7),

$$s'_{\alpha\beta} = \begin{bmatrix} s'_{11} & 0 & s'_{14} & 0 & s'_{16} \\ & s'_{22} & s'_{24} & 0 & s'_{26} \\ & & s'_{44} & s'_{45} & s'_{46} \\ & & & s'_{11} & s'_{56} \\ & & & & s'_{66} \end{bmatrix}. \tag{7.1}$$

Only the upper triangle of the matrix is shown because the matrix is symmetric. To study the conditions for the matrix to be positive definite, we will move the fourth column to the second column and the fourth row to the second row. Thus the new matrix is

$$s^*_{\alpha\beta} = \begin{bmatrix} s'_{11} & 0 & 0 & s'_{14} & s'_{16} \\ & s'_{11} & 0 & s'_{54} & s'_{56} \\ & & s'_{22} & s'_{24} & s'_{26} \\ & & & s'_{44} & s'_{46} \\ & & & & s'_{66} \end{bmatrix}. \tag{7.2}$$

If the matrix $s'_{\alpha\beta}$ in (7.1) is positive definite, so is the matrix $s^*_{\alpha\beta}$ in (7.2) and vice versa. We can prescribe the off-diagonal elements of $s^*_{\alpha\beta}$ arbitrarily subject to condition (3.11). We can then choose, as shown below, the diagonal elements s'_{11}, s'_{22}, s'_{44} and s'_{66}, in that order, such that all leading principal minors of $s^*_{\alpha\beta}$ are positive and nonzero.

Let Δ_1, Δ_2, ..., be the first, second, ..., principal minors of $s^*_{\alpha\beta}$. We have

$$\Delta_1 = s'_{11}, \qquad \Delta_2 = (s'_{11})^2, \qquad \Delta_3 = s'_{22}(s'_{11})^2, \qquad \Delta_4 = \begin{vmatrix} s'_{11} & 0 & 0 & s'_{14} \\ & s'_{11} & 0 & s'_{54} \\ & & s'_{22} & s'_{24} \\ & & & s'_{44} \end{vmatrix}, \qquad \Delta_5 = \left|s^*_{\sigma\beta}\right|. \tag{7.3}$$

The matrix $s^*_{\alpha\beta}$ is positive definite if Δ_α $(\alpha = 1, 2, \ldots, 5)$ are all positive and nonzero. The first three equations in (7.3) tell us that we must have

$$s'_{11} > 0, \qquad s'_{22} > 0. \tag{7.4}$$

We show below that we can always choose s'_{44} and s'_{55} such that $\Delta_4 > 0$ and $\Delta_5 > 0$.

Let $\mathbf{U}$ be a symmetric square matrix that is divided as

$$\mathbf{U} = \begin{bmatrix} \mathbf{E} & \mathbf{G} \\ \mathbf{G}^T & \mathbf{F} \end{bmatrix}, \tag{7.5}$$

where $\mathbf{E}$ and $\mathbf{F}$ are symmetric square matrices. If $\mathbf{U}$ is positive definite, so are $\mathbf{E}$ and $\mathbf{F}$, and the inverse of $\mathbf{E}$ exists. From

$$\mathbf{U} = \begin{bmatrix} \mathbf{E} & \mathbf{G} \\ \mathbf{G}^T & \mathbf{F} \end{bmatrix} = \begin{bmatrix} \mathbf{E} & 0 \\ \mathbf{G}^T & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{I} & \mathbf{E}^{-1}\mathbf{G} \\ 0 & \mathbf{F} - \mathbf{G}^T\mathbf{E}^{-1}\mathbf{G} \end{bmatrix}, \tag{7.6}$$

we have

$$|\mathbf{U}| = |\mathbf{E}| \cdot \left|\mathbf{F} - \mathbf{G}^T\mathbf{E}^{-1}\mathbf{G}\right|. \tag{7.7}$$

The determinant of a larger matrix is replaced by a product of the determinants of smaller matrices. We pay the price because we must compute the inverse of E [Ting 1996b].

In computing Δ_4 in Equation (7.3), we choose

$$\mathbf{E} = \begin{bmatrix} s'_{11} & 0 & 0 \\ & s'_{11} & 0 \\ & & s'_{22} \end{bmatrix}, \qquad \mathbf{G} = \begin{bmatrix} s'_{14} \\ s'_{54} \\ s'_{24} \end{bmatrix}, \qquad \mathbf{F} = s'_{44}. \tag{7.8}$$

We then have

$$\Delta_4 = \Delta_3[s'_{44} - (s'^2_{14} + s'^2_{54})(s'_{11})^{-1} - s'^2_{24}(s'_{22})^{-1}]. \tag{7.9}$$

Hence $\Delta_4 > 0$ if

$$s'_{44} > (s'^2_{14} + s'^2_{54})(s'_{11})^{-1} + s'^2_{24}(s'_{22})^{-1}. \tag{7.10}$$

For Δ_5 we choose

$$\mathbf{E} = \begin{bmatrix} s'_{11} & 0 & 0 & s'_{14} \\ & s'_{11} & 0 & s'_{54} \\ & & s'_{22} & s'_{24} \\ & & & s'_{44} \end{bmatrix}, \qquad \mathbf{G} = \begin{bmatrix} s'_{16} \\ s'_{56} \\ s'_{26} \\ s'_{46} \end{bmatrix}, \qquad \mathbf{F} = s'_{66}. \tag{7.11}$$

Hence $\Delta_5 > 0$ if we choose s'_{66} such that

$$s'_{66} > \mathbf{G}^T\mathbf{E}^{-1}\mathbf{G}, \tag{7.12}$$

where

$$\mathbf{E}^{-1} = \frac{1}{\Delta_4}\begin{bmatrix} s'_{11}(s'_{22}s'_{44} - s'^2_{24}) - s'_{22}s'^2_{45} & -s'_{14}s'_{22}s'_{45} & -s'_{11}s'_{14}s'_{24} & s'_{11}s'_{22}s'_{14} \\ & s'_{11}(s'_{22}s'_{44} - s'^2_{24}) - s'_{22}s'^2_{14} & -s'_{11}s'_{24}s'_{45} & -s'_{11}s'_{22}s'_{45} \\ & & s'_{11}(s'_{11}s'_{44} - s'^2_{14} - s'^2_{45}) & s'_{11}s'^2_{24} \\ & & & s'^2_{11}s'_{22} \end{bmatrix}$$

Thus one-component Rayleigh waves presented earlier are valid if the conditions (7.4), (7.10) and (7.12) are satisfied.

The above illustration shows that, after the off-diagonal elements of the matrix are specified, one can always choose the diagonal elements of the matrix in succession, so that the Δ_i are positive and nonzero. The same procedure can be employed to find a positive definite matrix $s'_{\alpha\beta}$ for materials for which other one-component steady waves can propagate.

8. One-component waves in a plate

Consider a plate of uniform thickness h whose mid-plane is parallel to the plane $x_2 = 0$. Let the surfaces of the plate be traction-free or a slippery surface. The one-component Rayleigh waves presented in (3.5)–(3.11) apply here. We have

$$\mathbf{u} = \mathbf{a}e^{ikz}, \qquad z = x_1 + px_2 - vt. \tag{8.1}$$

The imaginary part of p need not be positive. Hence (3.1) and (6.5) are replaced by

$$p^{\pm} = \tfrac{1}{2}X\left(\varepsilon \pm i\sqrt{-\kappa}\right), \tag{8.2}$$

$$\left(\frac{a_3}{a_1}\right)^{\pm} = \frac{(s'_{45} - s'_{16}) \pm i\sqrt{-(s'_{45} - s'_{16})^2 - 4s'_{14}s'_{56}}}{2s'_{56}}. \tag{8.3}$$

However, these two are not the only solutions because the Stroh eigenvalue p need not be complex for a plate of finite thickness. Below, we will present new solutions associated with a real p.

When p is real, (3.11) is replaced by

$$\kappa = (s'_{45} - s'_{16})^2 + 4s'_{14}s'_{56} \geq 0. \tag{8.4}$$

From Equation (3.5b) we have $a_2 = 0$ so that (3.6) remains valid. Since p is real, (3.5a) suggests that a_1 and a_3 are also real. For a nontrivial solution of a_1, a_3 from (3.5b) we must have

$$(Xs'_{11} - 1)(Xs'_{55} - 1) - (Xs'_{15})^2 = 0. \tag{8.5}$$

It gives two solutions

$$X^{\pm} = \frac{(s'_{55} + s'_{11}) \pm \sqrt{(s'_{55} - s'_{11})^2 + (2s'_{15})^2}}{2(s'_{11}s'_{55} - s'_{15}s'_{15})} > 0. \tag{8.6}$$

They are both real and positive. Equation (3.5b) also gives

$$\frac{a_3}{a_1} = \frac{1 - Xs'_{11}}{Xs'_{15}}. \tag{8.7}$$

Use of Equation (8.6) leads to

$$\frac{a_3}{a_1} = \frac{(s'_{55} - s'_{11}) \mp \sqrt{(s'_{55} - s'_{11})^2 + (2s'_{15})^2}}{2s'_{15}} = \tan\theta^{\pm}, \tag{8.8}$$

say, for $X = X^{\pm}$. The displacement (u_1, u_3) is polarized along a straight line on a plane parallel to $x_2 = 0$. The line makes an angle θ^+ (or θ^-) with the x_1-axis when $X = X^+$ (or X^-). It is known that the polarization vectors a associated with two different X are orthogonal to each other. Indeed, one can show that

$$(\tan\theta^+)(\tan\theta^-) = -1, \tag{8.9}$$

so that θ^+ and θ^- differ by ninety degrees.

With $a_2 = 0$, Equation $(3.5a)_2$ gives

$$\frac{a_3}{a_1} = \frac{-s'_{12}}{s'_{25}}. \tag{8.10}$$

Equation $(3.5a)_{1,3}$ reduces to (3.8). It has a nontrivial solution for a_1, a_3 if (3.9) holds,

$$p^{\pm} = \frac{X}{2}\left\{(s'_{45} + s'_{16}) \pm \sqrt{(s'_{45} - s'_{16})^2 + 4s'_{14}s'_{56}}\right\}. \tag{8.11}$$

The two roots are real in view of (8.4). The a_1, a_3 obtained from $(3.8)_1$ is

$$\frac{a_3}{a_1} = \frac{p - Xs'_{16}}{Xs'_{56}}, \tag{8.12}$$

or, using (8.11),

$$\frac{a_3}{a_1} = \frac{(s'_{45} - s'_{16}) \pm \sqrt{(s'_{45} - s'_{16})^2 + 4s'_{14}s'_{56}}}{2s'_{56}}, \tag{8.13}$$

for $p = p^{\pm}$. We therefore have four sets of solutions associated with (X^+, p^+), (X^+, p^-), (X^-, p^+) and (X^-, p^-). Equations (8.8), (8.10) and (8.13) should be compatible. Hence the conditions for the one-component waves are

$$\frac{(s'_{55}-s'_{11})\mp\sqrt{(s'_{55}-s'_{11})^2+(2s'_{15})^2}}{2s'_{15}}=\frac{-s'_{12}}{s'_{25}}, \tag{8.14}$$

for $X = X^{\pm}$ and

$$\frac{(s'_{45}-s'_{16})\pm\sqrt{(s'_{45}-s'_{16})^2+4s'_{14}s'_{56}}}{2s'_{56}}=\frac{-s'_{12}}{s'_{25}}, \tag{8.15}$$

for $p = p^{\pm}$. These are in addition to (8.4).

The one-component waves in plates with real p are the exceptional bulk waves studied by Alshits and Lothe [1979]. They employed a different derivation. The derivation presented above provides explicit expression of the solution for X and p, and the restrictions (8.14), (8.15) and (8.4) on $s'_{\alpha\beta}$ so that one-component waves with real p can propagate in a plate.

The existence of one-component waves in plates with real p is not guaranteed unless the 5×5 matrix $s'_{\alpha\beta}$ is positive definite, subject to (8.14), (8.15) and (8.4). Since the diagonal elements s'_{11} and s'_{55} appear in (8.14), we will rearrange the 5×5 matrix $s'_{\alpha\beta}$ and consider

$$s^*_{\alpha\beta}=\begin{bmatrix} s'_{11} & s'_{15} & s'_{12} & s'_{14} & s'_{16} \\ & s'_{55} & s'_{52} & s'_{54} & s'_{56} \\ & & s'_{22} & s'_{24} & s'_{26} \\ & & & s'_{44} & s'_{46} \\ & & & & s'_{66} \end{bmatrix}. \tag{8.16}$$

The first two principal minors are

$$\Delta_1 = s'_{11}, \qquad \Delta_2 = s'_{11}s'_{55}-(s'_{15})^2. \tag{8.17}$$

We can choose s'_{11}, s'_{15} and s'_{55} such that

$$s'_{11} > 0, \qquad s'_{55} > (s'_{15})^2/s'_{11}. \tag{8.18}$$

We next choose the remaining off-diagonal elements such that (8.4), (8.14) and (8.15) are satisfied. Following the procedure illustrated in Section 7, we can choose s'_{22}, s'_{44} and s'_{66} in that order such that Δ_3, Δ_4 and Δ_5 are positive and nonzero.

As a special case, let

$$s'_{11} = s'_{55}, \qquad s'_{14}s'_{56} = 0. \tag{8.19}$$

Equations (8.6) and (8.8) reduce to

$$X^{\pm} = (s'_{11}\mp s'_{15})^{-1}, \qquad a_3 = \mp a_1, \tag{8.20}$$

and (8.11) simplifies to

$$p^{+} = Xs'_{45}, \qquad p^{-} = Xs'_{16}. \tag{8.21}$$

For $X = (s'_{11}-s'_{15})^{-1}$ and $p = Xs'_{45}$, we have $a_3 = -a_1$ and (3.5a) is satisfied if

$$s'_{14} = s'_{12}-s'_{25} = s'_{16}-s'_{45}-s'_{56} = 0. \tag{8.22}$$

For $X = (s'_{11} - s'_{15})^{-1}$ and $p = Xs'_{16}$, we have $a_3 = -a_1$ and (3.5a) is satisfied if

$$s'_{56} = s'_{12} - s'_{25} = s'_{16} - s'_{45} + s'_{14} = 0. \tag{8.23}$$

For $X = (s'_{11} + s'_{15})^{-1}$ and $p = Xs'_{45}$, we have $a_3 = a_1$ and (3.5a) is satisfied if

$$s'_{14} = s'_{12} + s'_{25} = s'_{16} - s'_{45} + s'_{56} = 0. \tag{8.24}$$

For $X = (s'_{11} + s'_{15})^{-1}$ and $p = Xs'_{16}$, we have $a_3 = a_1$ and (3.5a) is satisfied if

$$s'_{56} = s'_{12} + s'_{25} = s'_{16} - s'_{45} - s'_{14} = 0. \tag{8.25}$$

If the material is isotropic, there are two solutions, $X = 1/s'_{11}$ and $X = 1/s'_{55}$, while $p = 0$. We have

$$(a_1, a_3) = (1, 0), \qquad s'_{12} = 0, \qquad \text{for } X = 1/s'_{11} = 2\mu, \tag{8.26}$$

where μ is the shear modulus and

$$(a_1, a_3) = (0, 1), \qquad \text{for } X = 1/s'_{55} = \mu. \tag{8.27}$$

The solution in (8.26) is a longitudinal wave that leaves $\sigma_{22} = 0$. s'_{12} must vanish. Otherwise σ_{22} does not vanish. The solution in (8.27) is a horizontally polarized shear wave.

Unlike the one-component waves associated with a complex p, the materials for which one-component waves associated with a real p can propagate can be any anisotropic elastic material, including the isotropic materials.

Steady waves in a plate of general anisotropy have been extensively studied in the literature. The readers may consult the references listed in the more recent studies by Shuvalov [2000; 2004] and Ting [2008].

9. One-component Love waves

Let an anisotropic elastic layer of thickness $\hat{h}$ occupy the region $0 \geq x_2 \geq -\hat{h}$. It is attached to the half-space $x_2 \geq 0$ of different anisotropic elastic material. The surface $x_2 = -\hat{h}$ of the layer can be traction-free or a slippery surface.

If the interface $x_2 = 0$ between the layer and the half-space is in sliding contact, the solution for one-component Rayleigh waves obtained in Section 4 applies for the half-space while the solutions for a homogeneous plate given in Section 8 apply to the layer here. As shown in Section 8, there are two possible solutions for the layer when p is complex and four possible solutions when p is real.

If the interface between the layer and the half-space is perfectly bonded, we have to impose the continuity of the displacement at the interface between the layer and the half-space at $x_2 = 0$ and demand that the wave speed v be the same in the layer and the half-space. The wave speed is the same if (6.10) holds, where the hat refers to materials in the layer. The displacement is continuous if

$$\frac{a_3}{a_1} = \frac{\hat{a}_3}{\hat{a}_1}. \tag{9.1}$$

(a_3/a_1) for the half-space is given in (6.5), which is complex. $(\hat{a}_3/\hat{a}_1)$ for the layer can be either one of the two solutions associated with a complex $\hat{p}$ presented in (8.3). The imaginary part of $\hat{p}$ can be positive or negative. We cannot employ the solutions associated with a real $\hat{p}$ for the layer because the displacement would not be complex.

Love waves for which the layer and the half-space are general anisotropic elastic materials have been studied by Shuvalov and Every [2002]. Ting [2009] considered the case in which the interface can be either perfectly bonded or in a sliding contact.

10. One-component waves in a layered plate

Consider first a plate that consists of two layers. One layer has thickness h and occupies the region $0 \leq x_2 \leq h$. The solution is given by

$$\mathbf{u} = \mathbf{a}\, e^{ikz}, \tag{10.1}$$

where

$$z = x_1 + p x_2 - v t. \tag{10.2}$$

The other layer has the thickness $\hat{h}$ and occupies the region $-\hat{h} \leq x_2 \leq 0$. The solution is given by

$$\hat{\mathbf{u}} = \hat{\mathbf{a}}\, e^{ik\hat{z}}, \tag{10.3}$$

where

$$\hat{z} = x_1 + \hat{p} x_2 - v t. \tag{10.4}$$

The surfaces at $x_2 = h$ and $x_2 = -\hat{h}$ of the plate can be traction-free or slippery surfaces.

If the interface at $x_2 = 0$ is in sliding contact, any one of the solutions for a homogeneous plate given in Section 8 applies to each layer. There are two possible solutions when p is complex and four possible solutions when p is real for the layer. Thus there are six possible solutions for each layer in the plate.

If the interface between the layer and the half-space is perfectly bonded, we have to impose the continuity of the displacement at the interface $x_2 = 0$ and demand that the wave speed v be the same in the two layers. The wave speed is the same if

$$X/\rho = \hat{X}/\hat{\rho}. \tag{10.5}$$

The displacement is continuous if Equation (9.1) holds. For (9.1) to hold, it is necessary that the p in both layers be either complex or real.

For a plate that consists of n layers, the problem is simple if all interfaces between the layers are in sliding contact. In this case, each layer can have any one of the six one-component waves (two associated with a complex p and four associated with a real p) presented in Section 8. The solution in each layer is independent of the solutions in other layers.

If some of the interfaces are perfectly bonded, we have to impose the continuity of the displacement across the perfectly bonded interfaces. Let h_1, h_2,..., h_n be the thicknesses of the layers that occupy the regions

$$\begin{gathered} h_1 \geq x_2 \geq 0, \\ h_1 + h_2 \geq x_2 \geq h_1, \\ \dots \\ h_1 + h_2 + \dots + h_n \geq x_2 \geq h_1 + h_2 + \dots + h_{n-1}, \end{gathered} \tag{10.6}$$

respectively. Let the interface between the m-th and $(m+1)$-th layers be perfectly bonded. The solution for the one-component wave shown in (2.4) can have a more general expression. For the m-th layer, we let

$$\mathbf{u}^{(m)} = \mathbf{a}^{(m)} e^{ikz^{(m)}}, \qquad m = 1, 2, \ldots, n, \tag{10.7}$$

where

$$z^{(m)} = x_1 + p^{(m)}(x_2 - \alpha_m) + \beta_m - vt. \tag{10.8}$$

The α_m and β_m are constants to be determined. The solution for the $(m+1)$-th layer is obtained by replacing m by $(m+1)$ in (10.7) and (10.8). At the interface between the m-th and $(m+1)$-th layers,

$$x_2 = \sum_{i=1}^{m} h_i, \tag{10.9}$$

so that (10.8) gives

$$\begin{aligned} z^{(m)} &= x_1 + p^{(m)} \sum_{i=1}^{m} h_i - p^{(m)} \alpha_m + \beta_m - vt, \\ z^{(m+1)} &= x_1 + p^{(m+1)} \sum_{i=1}^{m} h_i - p^{(m+1)} \alpha_{m+1} + \beta_{m+1} - vt. \end{aligned} \tag{10.10}$$

To enforce the continuity of the displacement at the interface, $z^{(m)}$ and $z^{(m+1)}$ in (10.10) must be identical. They are identical if we set

$$\alpha_m = \sum_{i=1}^{m-1} h_i, \qquad \beta_m = \sum_{i=1}^{m-1} p^{(i)} h_i. \tag{10.11}$$

Equation (10.10) then gives

$$z^{(m)} = x_1 + \beta_{m+1} - vt = z^{(m+1)}. \tag{10.12}$$

The continuity of the displacement at the interface is

$$\frac{a_3^{(m)}}{a_1^{(m)}} = \frac{a_3^{(m+1)}}{a_1^{(m+1)}}. \tag{10.13}$$

The Stroh eigenvalues $p^{(m)}$ and $p^{(m+1)}$ must be either complex or real. If they are complex, there are two possible solutions as shown in Section 8 for each of the two layers. If they are real, there are four possible solutions for each of the two layers. The wave speed v must be the same for the two layers. This means that

$$\frac{X^{(m)}}{\rho^{(m)}} = \frac{X^{(m+1)}}{\rho^{(m+1)}}. \tag{10.14}$$

Equations (10.13) and (10.14) must be satisfied for the two layers whose interface is perfectly bonded.

Thus, it is not difficult to see that one-component waves can propagate in a special layered plate in which the interfaces can be in sliding contact or perfectly bonded. The dispersion equations for a general anisotropic elastic layered plate can be found in [Alshits et al. 2003] and [Ting 2008].

Remarks

We have shown that one-component steady waves can propagate in the half-space in which the boundary surface can be traction-free or a slippery surface. We have also shown that one-component steady waves can propagate in a bimaterial for which the interface can be in sliding contact or perfectly bonded. The basic solution common to all one-component steady waves is the one-component surface wave. It has the characteristic that the stress σ_{i2} ($i = 1, 2, 3$) and the displacement u_2 vanish everywhere. Thus any plane that is parallel to $x_2 = 0$ is a free surface and also a slippery surface. Because of this, the half-space need not be infinite in extent. We could have a plate of finite thickness h whose surfaces are either traction-free or slippery surfaces. One-component steady waves can propagate in the plate and the wave is *not* dispersive. Likewise, the bimaterial need not be infinite in extent. If one of the two half-spaces is finite we have one-component Love waves. If both half-spaces are finite, we have a plate that consists of two layers. Again one-component steady waves can propagate in the two-layered plate and the wave is not dispersive. For the plate of finite thickness, the Stroh eigenvalue need not be complex. We present new one-component waves that can propagate in a plate and a layered plate that consists of any number of layers. These waves are not dispersive.

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