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ACOUSTIC EMISSION FROM A SURFACE-BREAKING CRACK IN A LAYER UNDER CYCLIC LOADING

JAN D. ACHENBACH

An isotropic, homogeneous, elastic layer is subjected to a time-harmonic tensile stress of constant amplitude parallel to the free surfaces. The cyclic stresses are assumed to generate a surface-breaking crack of length $l(t)$ which propagates normally to the top surface of the layer. The unloading of the crack faces generates acoustic emission, which is composed of Lamb waves. The elastodynamic reciprocity theorem for time-harmonic waves is used to determine the amplitudes of the radiated system of time-harmonic Lamb waves.

1. Introduction

One way of detecting a surface-breaking crack is by measuring the acoustic emission that is generated by nucleation and growth of the crack. To understand the acoustic emission requires the assistance of a suitable measurement model, for a proper interpretation of measured data. In this paper we provide such a model for nucleation, growth, and opening and closing of a surface-breaking crack in a layer which is subjected to cyclic loading.

The geometry is shown in Figure 1. The layer is homogeneous, isotropic, and linearly elastic. A two-dimensional geometry of plane strain is considered. Prior to crack nucleation the layer is in a state of stress defined by

$$\tau_{11} = \Delta\tau \sin(\omega\tau). \tag{1}$$

A standard way of calculating the acoustic emission from a crack in a stress-field defined by Equation (1) is to consider a crack whose faces are subjected to a compressive stress just equal and opposite to the stress given by Equation (1):

$$x_1 = 0^{\pm},\ -h \le z \le -h + l(t): \quad \tau_{11} = -\Delta\tau \sin(\omega t), \tag{2a}$$

$$\tau_{1z} = 0. \tag{2b}$$

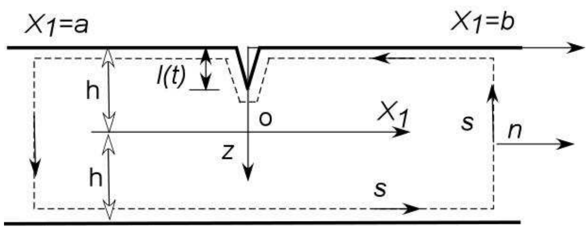

Figure 1. Surface-breaking crack in an elastic layer.

Keywords: layer, crack, cyclic loading, acoustic emission, reciprocity theorem.

If the two stress fields, Equations (1) and (2), are superimposed, the crack faces are rendered free of tractions. It then follows that the crack-face loading defined by Equations (2a), (2b) can be considered as the excitation that generates the acoustic emission due to the presence of traction-free crack faces. It should be noted that the crack length $l(t)$ is a function of time due to growth of the crack, but the effect of this time dependence is assumed to be of a quasistatic nature.

A loading of the form defined by Equations (2a), (2b) generates time-harmonic wave modes in a layer, known as Lamb waves. For a given frequency, ω, the dimensionless wave number kh, where h is the thickness of the layer, can be obtained from the Rayleigh–Lamb frequency equation. For a given frequency, this frequency equation has real-valued, imaginary and complex roots [Achenbach 1973], which define specific modes of wave motion. In this paper, we only consider the real-valued roots, since they define wave modes that are not attenuated, and that thus can be detected at some distance from their point of excitation.

There are two kinds of wave modes in a layer: modes whose displacements are either symmetric or antisymmetric with respect to the mid-plane of the layer. For the case at hand both symmetric and antisymmetric modes are generated. These modes propagate away from the plane of the crack in both the positive and negative x_1 direction.

2. Time-harmonic waves in an elastic layer

In a two-dimensional geometry, relative to the x_1z coordinate system of Figure 1, and for the case of plane strain, the modes of symmetric motion, propagating away from the plane of the crack in a homogeneous, isotropic, linearly elastic layer may be represented by (see [Achenbach 2003, p. 150])

$$u_1^n = \pm i A_n^S V_S^n(z) e^{\pm i k_n x_1}, \tag{3}$$

$$u_z^n = A_n^S W_S^n(z) e^{\pm i k_n x_1}. \tag{4}$$

In these expressions, $\exp(-i\omega t)$ has been omitted. The index n defines the n-th mode, and k_n is the wave number of that mode. The plus or minus sign applies to propagation in the positive or negative x_1-direction, respectively. The mode shapes for the *symmetric* modes are

$$V_S^n(z) = s_1 \cos(pz) + s_2 \cos(qz), \tag{5}$$

$$W_S^n(z) = s_3 \sin(pz) + s_4 \sin(qz), \tag{6}$$

with

$$s_1 = 2\cos(qh), \qquad s_2 = -\left[(k_n^2 - q^2)/k_n^2\right]\cos(ph), \tag{7}$$

$$s_3 = -2(p/k_n)\cos(qh), \qquad s_4 = -\left[(k_n^2 - q^2)/qk_n\right]\cos(ph), \tag{8}$$

where

$$p^2 = \frac{\omega^2}{c_L^2} - k_n^2 \quad \text{and} \quad q^2 = \frac{\omega^2}{c_T^2} - k_n^2. \tag{9}$$

Equation (9) shows the dependence of p and q on n. For simplicity of notation, this dependence will not be explicitly indicated until the next section, when we replace p and q by

$$p_n^2 = \frac{\omega^2}{c_L^2} - k_n^2 \quad \text{and} \quad q_n^2 = \frac{\omega^2}{c_T^2} - k_n^2.$$

The relevant corresponding stresses are (see [Achenbach 1973, p. 150])

$$\tau_{1z}^n = \pm i A_n^S T_{1z}^{Sn}(z) e^{\pm i k_n x_1}, \tag{10}$$
$$\tau_{zz}^n = A_n^S T_{zz}^{Sn}(z) e^{\pm i k_n x_1}, \tag{11}$$
$$\tau_{11}^n = A_n^S T_{11}^{Sn}(z) e^{\pm i k_n x_1}. \tag{12}$$

In these expressions

$$T_{1z}^{Sn}(z) = \mu[s_5 \sin(pz) + s_6 \sin(qz)], \tag{13}$$
$$T_{zz}^{Sn}(z) = \mu[s_7 \cos(pz) + s_8 \cos(qz)], \tag{14}$$
$$T_{11}^{Sn}(z) = \mu[s_9 \cos(pz) + s_{10} \cos(qz)], \tag{15}$$

where

$$s_5 = 4p\cos(qh), \qquad s_6 = [(k_n^2 - q^2)^2/qk_n^2]\cos(ph), \tag{16}$$
$$s_7 = [2(k_n^2 - q^2)/k_n]\cos(qh), \qquad s_8 = -[2(k_n^2 - q^2)/k_n]\cos(ph), \tag{17}$$
$$s_9 = [2(2p^2 - k_n^2 - q^2)/k_n]\cos(qh), \qquad s_{10} = [2(k_n^2 - q^2)/k_n]\cos(ph). \tag{18}$$

The condition that the faces of the layer are free of surface tractions yields the well-known Rayleigh–Lamb frequency equation for symmetric modes

$$\frac{\tan(qh)}{\tan(ph)} = -\frac{4pqk^2}{(q^2 - k^2)^2}, \tag{19}$$

which relates k_n and ω.

Similarly, for the *antisymmetric* modes we have

$$u_1^n = \pm i A_n^A V_A^n(z) e^{\pm i k_n x_1}, \tag{20}$$
$$u_z^n = A_n^A W_A^n(z) e^{\pm i k_n x_1}, \tag{21}$$
$$\tau_{1z}^n = \pm i A_n^A T_{1z}^{An}(z) e^{\pm i k_n x_1}, \tag{22}$$
$$\tau_{zz}^n = A_n^A T_{zz}^{An}(z) e^{\pm i k_n x_1}, \tag{23}$$
$$\tau_{11}^n = A_n^A T_{11}^{An}(z) e^{\pm i k_n x_1}. \tag{24}$$

The displacement mode shapes are

$$V_A^n(z) = a_1 \sin(pz) + a_2 \sin(qz), \tag{25}$$
$$W_A^n(z) = a_3 \cos(pz) + a_4 \cos(qz), \tag{26}$$

where

$$a_1 = 2\sin(qh), \qquad a_2 = -[(k_n^2 - q^2)/k_n^2]\sin(ph), \tag{27}$$

$$a_3 = 2(p/k_n)\sin(qh), \qquad a_4 = [(k_n^2 - q^2)/qk_n]\sin(ph); \tag{28}$$

also

$$T_{1z}^{An}(z) = \mu[a_5\cos(pz) + a_6\cos(qz)], \tag{29}$$

$$T_{zz}^{An}(z) = \mu[a_7\sin(pz) + a_8\sin(qz)], \tag{30}$$

$$T_{11}^{An}(z) = \mu[a_9\sin(pz) + a_{10}\sin(qz)]. \tag{31}$$

In these expressions

$$a_5 = -4p\sin(qh), \qquad a_6 = -\big[(k_n^2 - q^2)^2/qk_n^2\big]\sin(ph), \tag{32}$$

$$a_7 = \big[2(k_n^2 - q^2)/k_n\big]\sin(qh), \qquad a_8 = -\big[2(k_n^2 - q^2)/k_n\big]\sin(ph), \tag{33}$$

$$a_9 = \big[2(2p^2 - k_n^2 - q^2)/k_n\big]\sin(qh), \qquad a_{10} = \big[2(k_n^2 - q^2)/k_n\big]\sin(ph). \tag{34}$$

For the antisymmetric modes the Rayleigh–Lamb frequency equation is

$$\frac{\tan(qh)}{\tan(ph)} = -\frac{(q^2 - k^2)^2}{4pqk^2}. \tag{35}$$

It was shown in [Achenbach 2003, p. 152] that the following orthogonality relation holds for the symmetric modes

$$I_{mn}^S = 0 \quad \text{for} \quad m \neq n, \tag{36}$$

where

$$I_{mn}^S = \int_{-h}^{h} \big[T_{11}^{Sm}(z)V_S^n(z) - T_{1z}^{Sn}(z)W_S^m(z)\big]\,dz. \tag{37}$$

Similarly, for the antisymmetric modes we have

$$I_{mn}^A = 0 \quad \text{for} \quad m \neq n, \tag{38}$$

where

$$I_{mn}^A = \int_{-h}^{h} \big[T_{11}^{Am}(z)V_A^n(z) - T_{1z}^{An}(z)W_A^m(z)\big]\,dz. \tag{39}$$

3. Acoustic emission from a surface-breaking crack

In this section, we will consider the steady part of the response generated by Equation (2). We will consider a crack-face loading of the form

$$\tau_{11} = \Delta\tau e^{-i\omega t}, \tag{40}$$

$$\tau_{1z} = 0, \tag{41}$$

with the understanding that the imaginary part of the solution applies. The loading defined by Equations (40)–(41) will generate a system of symmetric and antisymmetric modes. By virtue of the general forms discussed in the previous section this system of wave modes may be represented by

$$u_1^A(x_1,z) = \pm i\left\{\sum_{m=0}^{\infty} \mathrm{A}_m^S V_S^m(z)e^{\pm ik_m^S x_1} + \sum_{m=0}^{\infty} \mathrm{A}_m^A V_A^m(z)e^{\pm ik_m^A x_1}\right\}, \tag{42}$$

$$u_z^A(x_1,z) = \left\{\sum_{m=0}^{\infty} \mathrm{A}_m^S W_S^m(z)e^{\pm ik_m^S x_1} + \sum_{m=0}^{\infty} \mathrm{A}_m^A W_A^m(z)e^{\pm ik_m^A x_1}\right\}, \tag{43}$$

where the plus and minus signs are for propagation in the positive and negative x_1-directions, that is, for $x_1 > 0$ and $x_1 < 0$, respectively. The relevant corresponding stresses are

$$\tau_{11}^A(x_1,z) = \left\{\sum_{m=0}^{\infty} \mathrm{A}_m^S T_{11}^{Sm}(z)e^{\pm ik_m^S x_1} + \sum_{m=0}^{\infty} \mathrm{A}_m^A T_{11}^{Am}(z)e^{\pm ik_m^A x_1}\right\}, \tag{44}$$

$$\tau_{1z}^A(x_1,z) = \pm i\left\{\sum_{m=0}^{\infty} \mathrm{A}_m^S T_{1z}^{Sm}(z)e^{\pm ik_m^S x_1} + \sum_{m=0}^{\infty} \mathrm{A}_m^A T_{1z}^{Am}(z)e^{\pm ik_m^A x_1}\right\}. \tag{45}$$

In these expressions the terms p and q are redefined as

$$p_m^2 = \frac{\omega^2}{c_L^2} - k_m^2 \quad \text{and} \quad q_m^2 = \frac{\omega^2}{c_T^2} - k_m^2. \tag{46}$$

4. Application of the reciprocity theorem

Following a method developed in [Achenbach 2003], the Lamb waves generated by acoustic emission from a surface breaking crack will be obtained by an application of the elastodynamic reciprocity theorem for time-harmonic wave motion. For a body of region V and surface S this reciprocity theorem (see [Achenbach 2003, p. 92]) may be written as

$$\int_V (f_j^A u_j^B - f_j^B u_j^A)dV = \int_S (\tau_{ij}^B u_j^A - \tau_{ij}^A u_j^B)n_i\, dS, \tag{47}$$

where n_i are the components of the outward normal to S, and f_j, u_j, and τ_{ij} are the components of body forces, displacements, and stresses. In Equation (47), the superscripts denote two elastodynamic solutions for the same body, State A and State B, where in the present application State A is the desired solution of the acoustic emission problem and State B is an auxiliary solution, or in our terminology, a virtual wave.

The coefficients A_m^S and A_m^A will be determined by application of the reciprocity theorem. For this application the contour S, shown in Figure 1 has a loop Σ around the crack. State A is the solution of the acoustic emission problem defined by Equations (42)–(45). For state B, the virtual wave, we select the n-th symmetric mode propagating in the positive x_1-direction

$$u_1^B(x_1, z) = i B_n^S V_S^n(z) e^{ik_n^S x_1}, \tag{48}$$

$$u_z^B(x_1, z) = B_n^S W_S^n(z) e^{ik_n^S x_1}, \tag{49}$$

$$\tau_{11}^B(x_1, z) = B_n^S T_{11}^{Sn}(z) e^{ik_n^S x_1}, \tag{50}$$

$$\tau_{1z}^B(x_1, z) = i B_n^S T_{1z}^{Sn}(z) e^{ik_n^S x_1}. \tag{51}$$

Since there are no body forces in this application of the reciprocity theorem, Equation (47) reduces to

$$\int_S (\tau_{ij}^B u_j^A - \tau_{ij}^A u_j^B) n_i \, dS = 0. \tag{52}$$

For the contour shown in Figure 1, there are possible contributions from the line $x_1 = a$, $-h \le z \le h$, which we call J_1, from $x_1 = b$, $-h \le z \le h$, called J_2, and from σ, the contour around the crack, called J_3. There are no contributions from the free surfaces at $z = \pm h$.

First let us consider the contribution from $x_1 = a$. Along $x_1 = a$ we have

$$J_1 = -\int_{-h}^{h} F_{AB}(x_1, z)\big|_{x_1=a} \, dz, \tag{53}$$

where

$$F_{AB} = \tau_{11}^B u_1^A + \tau_{1z}^B u_z^A - \tau_{11}^A u_1^B - \tau_{1z}^A u_z^B. \tag{54}$$

Substitution of the summations given by Equations (42)–(45) for State A, and the n-th virtual mode given by Equations (48)–(51) for State B, in Equation (53) yields after some manipulation

$$J_1 = \sum_{m=0}^{\infty} i A_m^S B_n^S [I_{mn}^S + I_{nm}^S] e^{i(k_m - k_n)a} + \sum_{m=0}^{\infty} i A_m^A B_n^S [I_{mn}^{AS} + I_{nm}^{AS}] e^{i(k_m - k_n)a}, \tag{55}$$

where I_{mn}^S and I_{nm}^S are defined by Equation (37), and I_{mn}^{AS} is given by

$$I_{mn}^{AS} = \int_{-h}^{h} \left[T_{11}^{Am}(z) V_S^n(z) - T_{1z}^{An}(z) W_S^n(z) \right] dz, \tag{56}$$

with an analogous definition for I_{nm}^{AS}. Inspection of Equations (31), (5), (29), (6) shows that the terms in Equation (56) are

$$\mu \left[a_9 \sin(p_m z) + a_{10} \sin(q_m z) \right] \left[s_1 \cos(p_n z) + s_2 \cos(q_n z) \right] \tag{57}$$

and

$$\mu \left[a_5 \cos(p_n z) + a_6 \cos(q_n z) \right] \left[s_3 \sin(p_m z) + s_4 \sin(q_m z) \right]. \tag{58}$$

These terms are of the general forms

$$\sin(\alpha z) \cos(\beta z) = \tfrac{1}{2} \left[\sin(\alpha + \beta) z + \sin(\alpha - \beta) z \right].$$

Such terms when integrated over z from $z = -h$ to $z = h$ yield zero. Hence the second summation in Equation (55) vanishes. In addition, by virtue of the results given by Equation (36), the first summation in Equation (55) only produces a nonzero result when $n = m$. Consequently we find

$$J_1 = 2i A_m^S B_m^S I_{mm}^S. \tag{59}$$

A computation of the contribution from $x_1 = b$ leads to

$$J_2 = 0. \tag{60}$$

This is in agreement with the observation made in [Achenbach 2003, p. 101] that contributions are obtained only when States A and B are counterpropagating. Across $x_1 = b$ the two states both propagate in the positive x_1 direction.

Next we consider the contribution from the integration along Σ. Here it should be understood that the virtual wave defined as State B is for a layer without a surface-breaking crack. State B applies to a cracked layer if the faces of the crack are subjected to tractions equal to the ones that would exist in the uncracked layer for State B. This implies that for State B the displacements and the stresses are continuous across the crack faces.

In terms of the coordinate s, which increases in the counterclockwise direction, the contributions from the two line elements that make up Σ require careful consideration. Along $x_1 = 0^+$, $-h \le z \le -h + l$, where l is the length of the crack, the integration is similar to the one along $x_1 = a$, $-h \le z < h$. We obtain

$$J_3^+ = -\int_{-h}^{-h+l} F_{AB}(x_1, z)\big|_{x_1=0^+}\, dz. \tag{61}$$

However, along $x_1 = 0^-$, $-h \le z \le -h + l$, some more care must be exercised. We find

$$J_3^- = \int_{-h}^{-h+l} F_{AB}(x_1, z)\big|_{x_1=0^-}\, dz, \tag{62}$$

and thus

$$J_3 = \int_{-h}^{-h+l} \big\{F_{AB}(x_1, z)_{x_1=0^-} - F_{AB}(x_1, z)\big|_{x_1=0^+}\big\}\, dz. \tag{63}$$

From the definition of $F_{AB}(x_1, z)$ it follows that the integral of Equation (63) is composed of differences of equivalent quantities on the two crack faces. It can also be noted that the crack opens up symmetrically and thus $u_1^{A+} = -u_1^{A-}$ and $u_z^{A+} = u_z^{A-}$. Also $u_1^{B+} = u_1^{B-}$ and $u_z^{B+} = u_z^{B-}$. In addition $\tau_{11}^{B+} = \tau_{11}^{B-}$, $\tau_{1z}^{B+} = \tau_{1z}^{B-}$, $\tau_{11}^{A+} = \tau_{11}^{A-}$ and $\tau_{1z}^{A-} = \tau_{1z}^{A+} = 0$. As a consequence, only the first terms in F_{AB} on both sides remain, and Equation (63) reduces to

$$J_3 = -\int_{-h}^{-h+l} \tau_{11}^B(0, z)\, \Delta u_1(0, z)\, dz, \tag{64}$$

where

$$\Delta u_1(0, z) = u_1^A(0^+, z) - u_1^A(0^-, z) = 2u_1^A(0^+, z). \tag{65}$$

The quantity Δu_1 is called the crack opening displacement (COD).

The expression for J_3 can be further simplified by substitution of $\tau_{11}^B(0, z)$ from Equation (50). The result is

$$J_3 = -B_m^S \int_{-h}^{-h+l} T_{11}^{Sm}(z)\, \Delta u_1(0, z)\, dz. \tag{66}$$

From the condition that the total contour integral must vanish, the quantity A_m^S can then by solved from $J_1 + J_2 + J_3 = 0$ and Equations (59) and (66) as

$$A_m^S = \frac{-i}{2I_{mm}^S} \int_{-h}^{-h+l} T_{11}^{Sm}(z) \Delta u_1(0, z) dz. \tag{67}$$

In Equation (65) $u_1^A(0^+, z)$ is the normal displacement on the crack faces. It should be noted that the crack-opening displacement $\Delta u_1(x_1, z)$ is, as yet, unknown.

By selecting the n-th antisymmetric mode as the virtual wave we obtain in a totally analogous manner

$$A_m^A = \frac{-i}{2I_{mm}^A} \int_{-h}^{-h+l} T_{11}^{AM}(z) \Delta u_1(0_1 z) dz. \tag{68}$$

5. Crack-opening displacement

The crack-opening displacement, Equation (65), due to loading of the crack faces by a uniform pressure, can be calculated numerically by using, for example, the boundary element method. A quasistatic approximation to the COD can, however, be computed by using a simple idea from fracture mechanics based on energy considerations.

The propagation of a crack requires a certain amount of energy, which can be calculated provided that the stress-intensity factor is known. For a surface-breaking crack of length l in a layer of thickness $2h$, uniformly loaded by uniform pressure $\Delta\tau$ on its crack faces, the stress-intensity factor is of the general form

$$K_I = \Delta\tau(\pi l)^{1/2} F(l/h). \tag{69}$$

The energy required to form this crack can be looked up in a book on fracture mechanics, for example [Kanninen and Popelar 1985], as

$$G = \int_0^l \frac{1-\upsilon^2}{E} K_I^2 \, dl. \tag{70}$$

The energy that was provided to open the crack is

$$U = \tfrac{1}{2}\Delta\tau \int_{-h}^{-h+l} \Delta u_1(0, z) dz = \frac{1}{2} \Delta\tau \overline{\Delta u_1} \, l, \tag{71}$$

where

$$\overline{\Delta u_1} = \frac{1}{l} \int_{-h}^{-h+l} \Delta u_1(0, z) dz. \tag{72}$$

The integral in Equation (71) is the crack opening volume (COV). The equality of U and G yields an expression for $\overline{\Delta u_1}$ as

$$\overline{\Delta u_1} = \frac{G}{\frac{1}{2}\Delta\tau l}. \tag{73}$$

By the use of Equation (73) the expressions for A_m^S and A_m^A can now be written as

$$A_m^S = -i\frac{\overline{\Delta u}}{2I_{mm}^S}\int_{-h}^{-h+l} T_{11}^{Sm}(z)\,dz, \tag{74}$$

$$A_m^A = -i\frac{\overline{\Delta u}}{2I_{mm}^A}\int_{-h}^{-h+l} T_{11}^{Am}(z)\,dz. \tag{75}$$

Substitution in Equations (42)–(45) yields expressions whose imaginary parts are the radiated fields of acoustic emission.

6. Conclusions

The steady state solutions as given by Equations (42)–(45) are indeed obtained in a very simple manner. They represent propagating waves which appear as superpositions on the quasistatic displacement generated by the externally applied stress field.

The analogous problem for a surface-breaking crack in a half-space under cyclic loading is discussed in [Achenbach 2008].

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