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Using asymptotic homogenization, we derive the local problems and the corresponding homogenized coefficients of periodic thermomagnetoelectroelastic heterogeneous media. The theory is applied to obtain analytical expressions for all effective properties of an important class of periodic multilaminated composites. Universal relations involving homogenized thermal coefficients of two-phase laminated and fibrous piezoelectric/piezomagnetic periodic composites, with transversely isotropic constituents, are obtained. Theoretical evidence is shown for the existence of pyroelectric and pyromagnetic effects even if neither phase exhibits them. Numerical calculations and comparisons with others theories are included.

1. Introduction

A coupled effect is the capacity to convert system energy from one type to another (for instance, among magnetic, electric, mechanical, and thermal effects). Composites made of thermopiezoelectric and thermomagnetic components exhibit a magnetoelectric effect that is not present in their individual constituents. The knowledge of the global properties of such composites allows us to address the control of the response of smart structures whose phases are, in general, made of thermomagnetoelectroelastic (TMEE) materials.

Different methods have been employed to estimate the overall properties of TMEE periodic composites. For instance, Li and Dunn [1998b] developed a micromechanical methodology based on the mean field approach of Mori and Tanaka [1973] combined with the Eshelby tensor [Li and Dunn 1998a] to derive explicit formulae for the effective coefficients of two-phase laminated and fibrous composites. Aboudi [2001] employed a general micromechanical homogenization theory to investigate the global behavior of multiphase TMEE materials. His results show good agreement with those he derived in [Aboudi 1998], via the generalized method of cells, and those obtained in the work of Li and Dunn [1998b]. Using the theory of uniform fields in TMEE heterogeneous media by proper boundary conditions, universal relations between the effective properties of two-phase fibrous composites were derived in [Benveniste 1995]. In all these works, like in the present paper, the fully coupled constitutive laws for TMEE materials were used. Based on the asymptotic homogenization method, analytical expressions

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for magnetoelectroelastic unidirectional two-phase fibrous composites with circular cross section fibers and transversely isotropic constituents were derived in [Camacho-Montes et al. 2006].

In this paper, based on the asymptotic homogenization method, the general theory developed in [Bravo-Castillero et al. 2008], for magnetoelectroelastic heterogeneous media, is applied to determine analytical formulae for the effective properties of periodic multilaminated TMEE composites. The formulae obtained generalize those that appear in [Castillero et al. 1998] and in [Galka et al. 1996], where piezoelectric and thermopiezoelectric periodic composites, respectively, were studied. Here a general formula is presented in a unified way which is more adequate for computational implementation. This unified formula is specified for one example of multilayered composites with any finite number of transversely isotropic TMEE constituents to obtain all their effective coefficients. For the particular case of two-laminated media, such effective coefficients prove to satisfy the universal relations derived by Benveniste and Dvorak [1992] for orthotropic fibrous composites. Finally, by using a link between four local problems on the periodic cell for two-phase unidirectional fibrous TMEE composites with transversely isotropic constituents and arbitrary smoothness shape of the fibre cross section, universal relations involving pyromagnetic, pyroelectric, thermoelastic, elastic, piezoelectric, and piezomagnetic effective coefficients are derived [Benveniste 1995]. In particular, a proportion relating the pyromagnetic and pyroelectric components in the direction of the fibers is found. The constant of proportionality involves information relative to the piezoelectric and piezomagnetic constants of both phases.

2. Constitutive laws and equilibrium equations of TMEE materials

Let $\Omega \subset \mathbb{R}^3$ be open, bounded and sufficiently regular, with boundary $\partial\Omega$. The properties of a three-dimensional body, that is, Ω, made of a heterogeneous TMEE material are described by the elasticity four-order tensor c, the piezoelectric coupling third-order tensor e, the piezomagnetic coupling third-order tensor q, the electric permittivity second-order tensor κ, the magnetic permeability second-order tensor μ, the magnetoelectric second-order tensor α, the thermoelastic second-order tensor λ, the pyroelectric vector p, and the pyromagnetic vector m. Also, η stands for the heat conductivity second-order tensor, and $\beta = C_e/T_0$; C_e is the specific heat at constant strain per unit volume and T_0 is the reference (absolute) temperature. Thus, with Y the so-called unit periodic cell and the small parameter $\varepsilon > 0$, the material functions just introduced are εY-periodic in the local variable $y = x/\varepsilon$, and for each $x \in \Omega$ we write, for instance,

$$\begin{aligned}
&c_\varepsilon^{ijkl}(x) = c^{ijkl}(x/\varepsilon), \quad e_\varepsilon^{ikl}(x) = e^{ikl}(x/\varepsilon), \quad q_\varepsilon^{ikl}(x) = q^{ikl}(x/\varepsilon), \quad \kappa_\varepsilon^{ij}(x) = \kappa^{ij}(x/\varepsilon),\\
&\mu_\varepsilon^{ij}(x) = \mu^{ij}(x/\varepsilon), \quad \alpha_\varepsilon^{ij}(x) = \alpha^{ij}(x/\varepsilon), \quad \lambda_\varepsilon^{ij}(x) = \lambda^{ij}(x/\varepsilon), \quad p_\varepsilon^{i}(x) = p^{i}(x/\varepsilon),\\
&m_\varepsilon^{i}(x) = m^{i}(x/\varepsilon), \quad \eta_\varepsilon^{ij}(x) = \eta^{ij}(x/\varepsilon), \quad \beta_\varepsilon(x) = \beta(x/\varepsilon).
\end{aligned}$$

Latin indices run over the set $\{1, 2, 3\}$. The Einstein summation convention will be used throughout. The tensors of material functions satisfy the usual symmetry conditions:

$$\begin{aligned}
&c_\varepsilon^{ijkl} = c_\varepsilon^{jikl} = c_\varepsilon^{ijlk} = c_\varepsilon^{klij}, \quad e_\varepsilon^{ikl} = e_\varepsilon^{ilk}, \quad q_\varepsilon^{ikl} = q_\varepsilon^{ilk}, \quad \kappa_\varepsilon^{ij} = \kappa_\varepsilon^{ji},\\
&\mu_\varepsilon^{ij} = \mu_\varepsilon^{ji}, \qquad \alpha_\varepsilon^{ij} = \alpha_\varepsilon^{ji}, \quad \lambda_\varepsilon^{ij} = \lambda_\varepsilon^{ji}, \quad \eta_\varepsilon^{ij} = \eta_\varepsilon^{ji}.
\end{aligned}$$

We make the further assumption that *there exists a constant* $\delta > 0$ *such that, for any* $a \in \mathbb{R}^3$ *and any* *symmetric* 3×3 *matrix* X,

$$\left.\begin{aligned} c_\varepsilon^{ijkl}(x) X_{ij} X_{kl} &\geq \delta X_{ij} X_{ij} \\ \kappa_\varepsilon^{ij}(x) a_i a_j &\geq \delta a_i a_i \\ \mu_\varepsilon^{ij}(x) a_i a_j &\geq \delta a_i a_i \\ \lambda_\varepsilon^{ij}(x) a_i a_j &\geq \delta a_i a_i \\ \eta_\varepsilon^{ij}(x) a_i a_j &\geq \delta a_i a_i \end{aligned}\right\} \textit{for almost every } x \in \Omega.$$

For a fixed $\varepsilon > 0$, the TMEE behavior of this body is given by the elastic displacement field $u_\varepsilon = (u_\varepsilon^i)$, the electric potential φ_ε, the magnetic potential ψ_ε, and the entropy s_ε, which satisfy the equilibrium equations

$$\left.\begin{aligned} -\operatorname{div} \sigma_\varepsilon(u_\varepsilon, \varphi_\varepsilon, \psi_\varepsilon, s_\varepsilon) &= f \\ -\operatorname{div} D_\varepsilon(u_\varepsilon, \varphi_\varepsilon, \psi_\varepsilon, s_\varepsilon) &= 0 \\ -\operatorname{div} B_\varepsilon(u_\varepsilon, \varphi_\varepsilon, \psi_\varepsilon, s_\varepsilon) &= 0 \\ -\operatorname{div} T_\varepsilon(u_\varepsilon, \varphi_\varepsilon, \psi_\varepsilon, s_\varepsilon) &= 0 \end{aligned}\right\} \text{ in } \Omega, \tag{2-1}$$

where $\sigma_\varepsilon = (\sigma_\varepsilon^{ij})$ is the stress tensor, $D_\varepsilon = (D_\varepsilon^i)$ the electric displacement, $B_\varepsilon = (B_\varepsilon^i)$ the magnetic displacement, and $T_\varepsilon = (\eta_\varepsilon^{ij} \partial_j \theta_\varepsilon)$ the flux of the temperature θ_ε, Also $\partial_i = \partial / \partial x_i$, $(\operatorname{div} \sigma_\varepsilon)^i = \partial_j \sigma_\varepsilon^{ij}$, $\operatorname{div} D_\varepsilon = \partial_i D_\varepsilon^i$, $\operatorname{div} B_\varepsilon = \partial_i B_\varepsilon^i$, and $\operatorname{div} T_\varepsilon = \partial_i (\eta_\varepsilon^{ij} \partial_j \theta_\varepsilon)$, $x = (x_i) \in \Omega$. The constitutive equations are given by

$$\begin{aligned} \sigma_\varepsilon^{ij}(u_\varepsilon, \varphi_\varepsilon, \psi_\varepsilon, s_\varepsilon) &= c_\varepsilon^{ijkl} s_{kj}(u_\varepsilon) + e_\varepsilon^{mij} \partial_m \varphi_\varepsilon + q_\varepsilon^{nij} \partial_n \psi_\varepsilon - \lambda_\varepsilon^{ij} s_\varepsilon, \\ D_\varepsilon^i(u_\varepsilon, \varphi_\varepsilon, \psi_\varepsilon, s_\varepsilon) &= e_\varepsilon^{ikl} s_{kl}(u_\varepsilon) - \kappa_\varepsilon^{im} \partial_m \varphi_\varepsilon - \alpha_\varepsilon^{in} \partial_n \psi_\varepsilon + p_\varepsilon^i s_\varepsilon, \\ B_\varepsilon^i(u_\varepsilon, \varphi_\varepsilon, \psi_\varepsilon, s_\varepsilon) &= q_\varepsilon^{ikl} s_{kl}(u_\varepsilon) - \alpha_\varepsilon^{im} \partial_m \varphi_\varepsilon - \mu_\varepsilon^{in} \partial_n \psi_\varepsilon + m_\varepsilon^i s_\varepsilon, \\ \theta_\varepsilon(u_\varepsilon, \varphi_\varepsilon, \psi_\varepsilon, s_\varepsilon) &= -\lambda_\varepsilon^{kl} s_{kl}(u_\varepsilon) + p_\varepsilon^m \partial_m \varphi_\varepsilon + m_\varepsilon^n \partial_n \psi_\varepsilon + \beta_\varepsilon s_\varepsilon, \end{aligned} \tag{2-2}$$

where $s_{kl}(u_\varepsilon)$, $\partial_k \varphi_\varepsilon$, and $\partial_k \psi_\varepsilon$ are, respectively, the linearized strain and the gradients of the electric and magnetic potentials, and $s_{kl}(u_\varepsilon) = \frac{1}{2}(\partial_k u_\varepsilon^l + \partial_l u_\varepsilon^k)$. In (2-1) we have a system of six partial differential equations for finding u_ε, φ_ε, ψ_ε, and s_ε. It has to be completed by suitable boundary conditions. For instance, we can assume homogeneous boundary conditions ($u_\varepsilon = 0$, $\varphi_\varepsilon = 0$, $\psi_\varepsilon = 0$, and $s_\varepsilon = 0$) on the external boundary $\partial\Omega$.

3. Homogenization

The method of two-scale asymptotic expansions will be applied in order to find the homogenized system. The solution of (2-1)–(2-2) can be sought in the form

$$\begin{aligned} u_\varepsilon(x) &= u^0(x, y) + \varepsilon u^1(x, y) + \dots, \\ \varphi_\varepsilon(x) &= \varphi^0(x, y) + \varepsilon \varphi^1(x, y) + \dots, \\ \psi_\varepsilon(x) &= \psi^0(x, y) + \varepsilon \psi^1(x, y) + \dots, \\ s_\varepsilon(x) &= s^0(x, y) + \varepsilon s^1(x, y) + \dots, \end{aligned} \tag{3-1}$$

where $u^0(x, y)$, $u^1(x, y)$, ..., $\varphi^0(x, y)$, $\varphi^1(x, y)$, ..., $\psi^0(x, y)$, $\psi^1(x, y)$, ..., $s^0(x, y)$, $s^1(x, y)$, ..., are Y-periodic functions with respect to the second variable $y = x/\varepsilon$. Similarly, as in the linear thermopiezolectric or magnetoelectroelastic problem (see, for instance, [Galka et al. 1996; Bravo-Castillero et al. 2008]), the functions $u^0(x, y)$, $\varphi^0(x, y)$, and $\psi^0(x, y)$ do not depend on y. However, in general, s^0 depends on both x and y. See [Galka et al. 1992] for a complete description in the thermopiezoelectric case. Due to the linearity of this problem and assuming both regularity of the inclusions' shapes and smoothness in the variation of coefficients, we have

$$\begin{aligned} u^1(x, y) &= s_{rt,x}(u^0(x))w^{rt}(y) + \partial_{m,x}\varphi^0(x)g^m(y) + \partial_{n,x}\psi^0(x)h^n(y) + \theta^h(x)\Gamma(y), \\ \varphi^1(x, y) &= s_{rt,x}(u^0(x))\zeta^{rt}(y) + \partial_{m,x}\varphi^0(x)\pi^m(y) + \partial_{n,x}\psi^0(x)\chi^n(y) + \theta^h(x)Q(y), \\ \psi^1(x, y) &= s_{rt,x}(u^0(x))\eta^{rt}(y) + \partial_{m,x}\varphi^0(x)\xi^m(y) + \partial_{n,x}\psi^0(x)\gamma^n(y) + \theta^h(x)R(y), \end{aligned} \tag{3-2}$$

where $\partial_{m,x}\phi = \partial\phi/\partial x_m$ and $s_{rt,x}(u) = \frac{1}{2}(\partial u_r/\partial x_t + \partial u_t/\partial x_r)$. The functions $u^0(x)$, $\varphi^0(x)$, and $\psi^0(x)$ are, respectively, the mechanical displacement field, the electric potential and the magnetic potential of the effective (homogenized) TMEE body while its temperature field θ^h is given by

$$\theta^h(x) = -\langle\lambda^{kl}\rangle(\partial_{l,x}u_k^0 + \partial_{k,y}u_k^1) + \langle p^k\rangle(\partial_{k,x}\varphi^0 + \partial_{k,y}\varphi^1) + \langle m^k\rangle(\partial_{k,x}\psi^0 + \partial_{k,y}\psi^1) + \langle\beta\rangle s^0(x, y),$$

where $\langle g\rangle = |Y|^{-1}\int_Y g(y)\,dy$ and the angle brackets denote averaging over the periodic cell Y. Note that in general s^0 depends not only on x but also on the microscopic variable y. The local functions w^{rt}, ζ^{rt}, η^{rt}; g^m, π^m, ξ^m; h^m, χ^m, γ^m; and Γ, Q, R are Y-periodic solutions of the following problems on the cell Y:

- Problem L_1^{rt}: Find the Y-periodic functions w^{rt}, ζ^{rt}, η^{rt} such that:

$$\left.\begin{aligned} -\partial_{j,y}\sigma^{ij}(w^{rt}, \zeta^{rt}, \eta^{rt}, 0) &= \partial_{j,y}c^{ijrt} \\ -\partial_{i,y}D^i(w^{rt}, \zeta^{rt}, \eta^{rt}, 0) &= \partial_{i,y}e^{irt} \\ -\partial_{i,y}B^i(w^{rt}, \zeta^{rt}, \eta^{rt}, 0) &= \partial_{i,y}q^{irt} \end{aligned}\right\} \text{ on } Y. \tag{3-3}$$

- Problem L_2^m: Find the Y-periodic functions g^m, π^m, ξ^m such that

$$\left.\begin{aligned} -\partial_{j,y}\sigma^{ij}(g^m, \pi^m, \xi^m, 0) &= \partial_{j,y}e^{mij} \\ -\partial_{i,y}D^i(g^m, \pi^m, \xi^m, 0) &= -\partial_{i,y}\kappa^{im} \\ -\partial_{i,y}B^i(g^m, \pi^m, \xi^m, 0) &= -\partial_{i,y}\alpha^{im} \end{aligned}\right\} \text{ on } Y. \tag{3-4}$$

- Problem L_3^m: Find the Y-periodic functions h^m, χ^m, γ^m such that

$$\left.\begin{aligned} -\partial_{j,y}\sigma^{ij}(h^m, \chi^m, \gamma^m, 0) &= \partial_{j,y}q^{mij} \\ -\partial_{i,y}D^i(h^m, \chi^m, \gamma^m, 0) &= -\partial_{i,y}\alpha^{im} \\ -\partial_{i,y}B^i(h^m, \chi^m, \gamma^m, 0) &= -\partial_{i,y}\mu^{im} \end{aligned}\right\} \text{ on } Y. \tag{3-5}$$

- Problem L_4: Find the Y-periodic functions Γ, Q, R such that

$$\left.\begin{aligned}-\partial_{j,y}\sigma^{ij}(\Gamma, Q, R, 0) &= -\partial_{j,y}\lambda^{ij}\\ -\partial_{i,y}D^{i}(\Gamma, Q, R, 0) &= -\partial_{i,y}p^{i}\\ -\partial_{i,y}B^{i}(\Gamma, Q, R, 0) &= -\partial_{i,y}m^{i}\end{aligned}\right\}\text{ on } Y. \tag{3-6}$$

- Problem L_5^k: Find the Y-periodic function T^k such that

$$-\partial_{i,y}(\eta^{ij}\partial_{j,y}T^k) = \partial_{i,y}\eta^{ik} \text{ on } Y. \tag{3-7}$$

Here $\partial_{m,y}\phi = \partial\phi/\partial y_m$. The homogenized problem can be written as

$$\left.\begin{aligned}-\operatorname{div}\bar{\sigma}(u^0, \varphi^0, \psi^0, \theta^h) &= f\\ -\operatorname{div}\overline{D}(u^0, \varphi^0, \psi^0, \theta^h) &= 0\\ -\operatorname{div}\overline{B}(u^0, \varphi^0, \psi^0, \theta^h) &= 0\\ -\operatorname{div}\overline{T}(u^0, \varphi^0, \psi^0, \theta^h) &= 0\end{aligned}\right\}\text{ on } \Omega, \tag{3-8}$$

with the homogeneous boundary conditions $u^0 = 0$, $\varphi^0 = 0$, $\psi^0 = 0$, and $\theta^h = 0$ on $\partial\Omega$. The effective constitutive laws are given by

$$\begin{aligned}\bar{\sigma}^{ij}(u^0, \varphi^0, \psi^0, \theta^h) &= \bar{c}^{ijkl}s_{kj}(u^0) + \bar{e}^{mij}\partial_m\varphi^0 + \bar{q}^{nij}\partial_n\psi^0 - \bar{\lambda}^{ij}\theta^h,\\ \overline{D}^{i}(u^0, \varphi^0, \psi^0, \theta^h) &= \bar{e}^{ikl}s_{kl}(u^0) - \bar{\kappa}^{im}\partial_m\varphi^0 - \bar{\alpha}^{in}\partial_n\psi^0 + \bar{p}^{i}\theta^h,\\ \overline{B}^{i}(u^0, \varphi^0, \psi^0, \theta^h) &= \bar{q}^{ikl}s_{kl}(u^0) - \bar{\alpha}^{im}\partial_m\varphi^0 - \bar{\mu}^{in}\partial_n\psi^0 + \overline{m}^{i}\theta^h,\\ \bar{\theta}(u^0, \varphi^0, \psi^0, \theta^h) &= -\bar{\lambda}^{kl}s_{kl}(u^0) + \bar{p}^{m}\partial_m\varphi^0 + \overline{m}^{n}\partial_n\psi^0 + \bar{\beta}\theta^h,\end{aligned} \tag{3-9}$$

where the bar indicates an effective property. The local problems must be completed with additional contact conditions on the interfaces between the constituents of the composite of interest. The homogenized effective coefficients have the definitions

$$\begin{aligned}\bar{c}^{ijrt} &= \langle c^{ijkl}[s_{kl,y}(w^{rt}) + \delta_{kl}^{rt}] + e^{kij}\partial_{k,y}\zeta^{rt} + q^{kij}\partial_{k,y}\eta^{rt}\rangle,\\ \bar{e}^{irt} &= \langle e^{ikl}[s_{kl,y}(w^{rt}) + \delta_{kl}^{rt}] - \kappa^{ik}\partial_{k,y}\zeta^{rt} - \alpha^{ik}\partial_{k,y}\eta^{rt}\rangle,\\ \bar{q}^{irt} &= \langle q^{ikl}[s_{kl,y}(w^{rt}) + \delta_{kl}^{rt}] - \alpha^{ik}\partial_{k,y}\zeta^{rt} - \mu^{ik}\partial_{k,y}\eta^{rt}\rangle,\\ \bar{e}^{mij} &= \langle c^{ijkl}s_{kl,y}(g^m) + e^{sij}[\partial_{s,y}\pi^m + \delta_s^m] + q^{sij}\partial_{s,y}\xi^m\rangle,\\ \bar{\kappa}^{im} &= \langle -e^{ikl}s_{kl,y}(g^m) + \kappa^{is}[\partial_{s,y}\pi^m + \delta_s^m] + \alpha^{is}\partial_{s,y}\xi^m\rangle,\\ \bar{\alpha}^{im} &= \langle -q^{ikl}s_{kl,y}(g^m) + \alpha^{is}[\partial_{s,y}\pi^m + \delta_s^m] + \mu^{is}\partial_{s,y}\xi^m\rangle,\\ \bar{\alpha}^{im} &= \langle -e^{ikl}s_{kl,y}(h^m) + \alpha^{ik}[\partial_{s,y}\gamma^m + \delta_k^m] + \kappa^{ik}\partial_{k,y}\chi^m\rangle,\\ \bar{q}^{mij} &= \langle c^{ijkl}s_{kl,y}(h^m) + q^{sij}[\partial_{s,y}\gamma^m + \delta_s^m] + e^{sij}\partial_{s,y}\chi^m\rangle,\\ \bar{\mu}^{im} &= \langle -q^{ikl}s_{kl,y}(h^m) + \alpha^{ik}\partial_{k,y}\chi^m + \mu^{ik}[\partial_{k,y}\gamma^m + \delta_k^m]\rangle,\end{aligned} \tag{3-10}$$

$$\bar{\lambda}^{ij} = \langle\lambda^{ij} - c^{ijkl}s_{kl,y}(\Gamma) - e^{kij}\partial_{k,y}Q - q^{ijk}\partial_{k,y}R\rangle, \quad \bar{p}^{i} = \langle p^i + e^{ikl}s_{kl,y}(\Gamma) - \kappa^{ik}\partial_{k,y}Q - \alpha^{ik}\partial_{k,y}R\rangle,$$

$$\overline{m}^{i} = \langle m^i + q^{ikl}s_{kl,y}(\Gamma) - \alpha^{ik}\partial_{k,y}Q - \mu^{ik}\partial_{k,y}R\rangle, \qquad \bar{\beta} = \langle\beta - \lambda^{kl}s_{kl,y}(\Gamma) + p^k\partial_{k,y}Q + m^k\partial_{k,y}R\rangle,$$

and

$$\bar{\eta}^{ij} = \langle \eta^{ij} + \eta^{il} \partial_{l,y} T^j \rangle, \tag{3-11}$$

where $\delta_{ij}^{kl} = \frac{1}{2}(\delta_j^i \delta_l^k + \delta_k^i \delta_l^j)$ and δ_l^j is the Kronecker delta.

4. Closed-form expressions for effective coefficients of multilayered TMEE composites

The local problems above, with equations (3-3)–(3-6), can be written in a unified way as follows: Find a Y-periodic $W_{b'}^{c't}$ such that

$$-\partial_m (C^{a'mb'n} \partial_n W_{b'}^{c't}) = \partial_m (C^{a'mc't}), \tag{4-1}$$

where

$$\begin{gathered} C^{imkn} \equiv c^{imkn}, \quad C^{im44} \equiv -\lambda^{im}, \quad C^{4mkn} \equiv e^{mkn}, \quad C^{4m4n} \equiv -\kappa^{mn}, \quad C^{4m44} \equiv p^m, \\ C^{4444} \equiv \beta, \quad C^{5mnk} \equiv q^{mnk}, \quad C^{5m4n} \equiv -\alpha^{mn}, \quad C^{5j44} \equiv m^j, \quad C^{5m5n} \equiv -\mu^{mn}, \\ W_k^{rt} \equiv w_k^{rt}, \quad W_4^{rt} \equiv \zeta^{rt}, \quad W_5^{rt} \equiv \eta^{rt}, \quad W_k^{4m} \equiv g_k^m, \quad W_4^{4m} \equiv \pi^m, \quad W_5^{4m} \equiv \xi^m, \\ W_k^{5m} \equiv f_k^m, \quad W_4^{5m} \equiv \xi^m, \quad W_5^{5m} \equiv \gamma^m, \quad W_k^{44} \equiv \Gamma_k, \quad W_4^{44} \equiv Q, \quad W_5^{44} \equiv R. \end{gathered}$$

The primed Latin indices run from 1 to 5.

The homogenized effective coefficients given by all the Equations (3-10) can be expressed by

$$\overline{C}^{a'mb't} = \langle C^{a'mb't} \rangle + \langle C^{a'mc'n} \partial_n W_{c'}^{b't} \rangle. \tag{4-2}$$

The unified formulation above is very convenient for some specific problems. For instance, let us consider the particular case of a laminated TMEE composite, made of cells which are periodically distributed along the axis y_1. Each cell may be made of any finite number of homogeneous TMEE layers. The axes of symmetry of each layer are parallel to each other and the y_1-axis is perpendicular to the layering. In this case, the material functions $C^{a'mb't}$ and the local functions $W^{b't}$ depend only on the fast variable y_1. Consequently, expressions (4-1) and (4-2) take the form

$$D_1 (C^{a'1b'1} D_1 W_{b'}^{c't}) = -D_1 (C^{a'1c't}), \tag{4-3}$$

$$\overline{C}^{a'mb't} = \langle C^{a'mb't} \rangle + \langle C^{a'mc'1} D_1 W_{c'}^{b't} \rangle, \tag{4-4}$$

where D_1 denotes the ordinary derivative in the generalized sense with respect to the y_1 coordinate. The angle brackets define the average per unit length of the relevant quantity over the periodic cell, that is, $\langle F \rangle = |Y|^{-1} \int_Y F(y_1)\, dy_1$, where $|Y|$ denotes the length of Y. For simplicity, a periodic unit cell Y will be considered. This is a one-dimensional homogenization problem which consists in finding the 1-periodic solution of (4-3), with an average of zero on Y, and satisfying the usual contact interface conditions; see, for instance, [Pobedrya 1984, Chapter 5].

Solving the system of ordinary differential equations defined by (4-3), taking into account perfect bonding conditions at the interfaces, and using (4-4), it is possible to obtain a general closed-form formula for all the TMEE effective coefficients:

$$\begin{aligned} \overline{C}^{a'mb't} = \langle C^{b'mb't} \rangle + \big\langle C^{b'mc'1} (C^{c'1d'1})^{-1} C^{d'1b't} \big\rangle \\ + \big\langle C^{a'mc'1} (C^{c'1d'1})^{-1} \big\rangle \big\langle (C^{d'1e'1})^{-1} \big\rangle^{-1} \big\langle (C^{e'1f'1})^{-1} C^{f'1b't} \big\rangle. \end{aligned} \tag{4-5}$$

Here $(C^{a'1b'1})^{-1}$ denotes the components of the inverse matrix of $(C^{a'1b'1})$. Equation (4-5) is a generalization of [Pobedrya 1984, Equation (1.11), p. 145] where the purely elastic case was investigated. From (4-5), for the particular case of a two-laminated TMEE composite, the following was derived:

$$\overline{C}^{a'mb't} = C_v^{a'mb't} - v_1(1-v_1)[\![C^{a'mc'1}]\!]B^{-1}_{c'd'}[\![C^{d'1b't}]\!], \tag{4-6}$$

where v_1 is the volume fraction of phase 1 ($BaTiO_3$); the material coefficients of such composite are piecewise constants, defined by

$$C^{a'mb't}(y_1) = \begin{cases} C_1^{a'mb't} & \text{for } y_1 \in (0, v_1), \\ C_2^{a'mb't} & \text{for } y_1 \in (v_1, 1), \end{cases}$$

and $C_v^{a'mb't} = v_1 C_1^{a'mb't} + (1-v_1)C_2^{a'mb't}$, $[\![C^{a'mc'1}]\!] = C_1^{a'mc'1} - C_2^{a'mc'1}$, a row vector for $a'm$ fixed, $[\![C^{d'1b't}]\!] = C_1^{d'1b't} - C_2^{d'1b't}$, a column vector for $b't$ fixed, and $[B_{c'd'}] = [v_1 C_2^{c'1d'1} - (1-v_1)C_1^{c'1d'1}]$, where $B^{-1}_{c'd'}$ is the inverse matrix of $B_{c'd'}$. Equation (4-6) is similar to [Galka et al. 1996, Equation (17), p. 138] for laminated thermopiezoelectric composites.

4.1. *Effective properties of a multilaminate with an orthotropic global behavior.* We now specialize formula (4-5) for the case of a multilaminated composite whose periodic unit cell can possess any finite number of homogeneous TMEE materials with transversely isotropic properties. Each phase is characterized by the following independent constants:

Five elastic constants:

$$C^{1111} = C^{2222} (\equiv c^{1111} = c^{2222}), \qquad C^{1133} = C^{2233} (\equiv c^{1133} = c^{2233}),$$
$$C^{1122} (\equiv c^{1122}), \qquad C^{3333} (\equiv c^{3333}), \qquad 2C^{1212} \equiv 2c^{1212} = (c^{1111} - c^{1122}).$$

Three piezoelectric constants:

$$C^{4311} = C^{4322} (\equiv e^{311} = e^{322}), \qquad C^{4333} (\equiv e^{333}), \qquad C^{4113} = C^{4223} (\equiv e^{113} = e^{223}).$$

Three piezomagnetic constants:

$$C^{5311} = C^{5322} (\equiv q^{311} = q^{322}), \qquad C^{5333} (\equiv q^{333}), \qquad C^{5113} = C^{5223} (\equiv q^{113} = q^{223}).$$

Two dielectric permittivity constants: $C^{4141} = C^{4242} (\equiv -\kappa^{11} = -\kappa^{22})$ and $C^{4343} (\equiv -\kappa^{33})$.

Two magnetoelectric constants: $C^{5141} = C^{5242} (\equiv -\alpha^{11} = -\alpha^{22})$ and $C^{5343} (\equiv -\alpha^{33})$.

Two magnetic permeability constants: $C^{5151} = C^{5252} (\equiv -\mu^{11} = -\mu^{22})$ and $C^{5353} (\equiv -\mu^{33})$.

Two thermal constants: $C^{1144} = C^{2244} (\equiv -\lambda^{11} = -\lambda^{22})$ and $C^{3344} (\equiv -\lambda^{33})$.

One pyroelectric constant: $C^{4344} (\equiv p^3)$.

One pyromagnetic constant: $C^{5344} (\equiv m^3)$.

The heat capacity: $C^{4444} (\equiv \beta)$.

Using (4-5), the effective coefficients for this composite material are:

Nine elastic effective constants:

$$\begin{aligned}
&\bar{c}^{1111} = 1/\langle 1/c^{1111}\rangle, \quad \bar{c}^{1122} = \langle c^{1122}/c^{1111}\rangle/\langle 1/c^{1111}\rangle, \quad \bar{c}^{1133} = \langle c^{1133}/c^{1111}\rangle/\langle 1/c^{1111}\rangle,\\
&\bar{c}^{2222} = \langle c^{1111}\rangle - \langle (c^{1122})^2/c^{1111}\rangle + \langle c^{1122}/c^{1111}\rangle^2/\langle 1/c^{1111}\rangle,\\
&\bar{c}^{2233} = \langle c^{1133}\rangle - \langle c^{1122}c^{1133}/c^{1111}\rangle + \langle c^{1122}/c^{1111}\rangle\langle c^{1133}/c^{1111}\rangle/\langle 1/c^{1111}\rangle,\\
&\bar{c}^{3333} = \langle c^{3333}\rangle - \langle (c^{1133})^2/c^{1111}\rangle + \langle c^{1133}/c^{1111}\rangle^2/\langle 1/c^{1111}\rangle,\\
&\bar{c}^{2323} = \langle c^{1313}\rangle, \qquad \bar{c}^{1313} = e_1^t\langle M_{13}^{-1}\rangle^{-1}e_1, \qquad \bar{c}^{1212} = 1/\langle 1/c^{1212}\rangle.
\end{aligned} \tag{4-7}$$

Five piezoelectric effective constants:

$$\begin{aligned}
&\bar{e}^{322} = \langle e^{311}\rangle + \langle e^{311}/c^{1111}\rangle\langle c^{1122}/c^{1111}\rangle/\langle 1/c^{1111}\rangle - \langle e^{311}c^{1122}/c^{1111}\rangle,\\
&\bar{e}^{333} = \langle e^{333}\rangle + \langle e^{311}/c^{1111}\rangle\langle c^{1133}/c^{1111}\rangle/\langle 1/c^{1111}\rangle - \langle e^{311}c^{1133}/c^{1111}\rangle,\\
&\bar{e}^{311} = \langle e^{311}/c^{1111}\rangle/\langle 1/c^{1111}\rangle, \qquad \bar{e}^{113} = e_2^t\langle M_{13}^{-1}\rangle^{-1}e_1, \qquad \bar{e}^{223} = \langle e^{113}\rangle.
\end{aligned} \tag{4-8}$$

Three dielectric permittivity effective constants:

$$\begin{aligned}
\bar{\kappa}^{11} &= -e_2^t\langle M_{13}^{-1}\rangle^{-1}e_2, \quad \bar{\kappa}^{22} = \langle \kappa^{11}\rangle,\\
\bar{\kappa}^{33} &= \langle \kappa^{33}\rangle + \langle e^{311}\rangle^2/\langle c^{1111}\rangle - \langle e^{311}/c^{1111}\rangle^2/\langle 1/c^{1111}\rangle.
\end{aligned} \tag{4-9}$$

Five piezomagnetic effective constants:

$$\begin{aligned}
&\bar{q}^{322} = \langle q^{311}\rangle + \langle q^{311}/c^{1111}\rangle\langle c^{1122}/c^{1111}\rangle/\langle 1/c^{1111}\rangle - \langle q^{311}c^{1122}/c^{1111}\rangle,\\
&\bar{q}^{333} = \langle q^{333}\rangle + \langle q^{311}/c^{1111}\rangle\langle c^{1133}/c^{1111}\rangle/\langle 1/c^{1111}\rangle - \langle q^{311}c^{1133}/c^{1111}\rangle,\\
&\bar{q}^{311} = \langle q^{311}/c^{1111}\rangle/\langle 1/c^{1111}\rangle, \qquad \bar{q}^{113} = e_3^t\langle M_{13}^{-1}\rangle^{-1}e_1, \qquad \bar{q}^{223} = \langle q^{113}\rangle.
\end{aligned} \tag{4-10}$$

Three magnetoelectric effective constants:

$$\begin{aligned}
\bar{\alpha}^{11} &= -e_3^t\langle M_{13}^{-1}\rangle^{-1}e_2, \qquad \bar{\alpha}^{22} = \langle \alpha^{11}\rangle,\\
\bar{\alpha}^{33} &= \langle \alpha^{33}\rangle + \langle q^{311}e^{311}/c^{1111}\rangle - \langle q^{311}/c^{1111}\rangle\langle e^{311}/c^{1111}\rangle/\langle 1/c^{1111}\rangle.
\end{aligned} \tag{4-11}$$

Three magnetic permeability effective constants:

$$\begin{aligned}
\bar{\mu}^{11} &= -e_3^t\langle M_{13}^{-1}\rangle^{-1}e_3, \qquad \bar{\mu}^{22} = \langle \mu^{11}\rangle,\\
\bar{\mu}^{33} &= \langle \mu^{33}\rangle + \langle (q^{311})^2/c^{1111}\rangle - \langle q^{311}/c^{1111}\rangle^2/\langle 1/c^{1111}\rangle.
\end{aligned} \tag{4-12}$$

Three thermoelastic effective constants:

$$\begin{aligned}
&\bar{\lambda}^{11} = \langle \lambda^{11}/c^{1111}\rangle/\langle 1/c^{1111}\rangle,\\
&\bar{\lambda}^{22} = \langle \lambda^{11}\rangle - \langle c^{1122}\lambda^{11}/c^{1111}\rangle + \langle c^{1122}/c^{1111}\rangle\langle \lambda^{11}/c^{1111}\rangle/\langle 1/c^{1111}\rangle,\\
&\bar{\lambda}^{33} = \langle \lambda^{33}\rangle - \langle c^{1133}\lambda^{11}/c^{1111}\rangle + \langle c^{1133}/c^{1111}\rangle\langle \lambda^{11}/c^{1111}\rangle/\langle 1/c^{1111}\rangle.
\end{aligned} \tag{4-13}$$

One pyroelectric effective constant:

$$\bar{p}^3 = \langle p^3\rangle + \langle e^{311}\lambda^{11}/c^{1111}\rangle - \langle e^{311}/c^{1111}\rangle\langle \lambda^{11}/c^{1111}\rangle/\langle 1/c^{1111}\rangle. \tag{4-14}$$

One pyromagnetic effective constant:

$$\bar{m}^3 = \langle m^3\rangle + \langle q^{311}\lambda^{11}/c^{1111}\rangle - \langle q^{311}/c^{1111}\rangle\langle\lambda^{11}/c^{1111}\rangle/\langle 1/c^{1111}\rangle. \tag{4-15}$$

Effective heat capacity:

$$\bar{\beta} = \langle\beta\rangle - \langle(\lambda^{11})^2/c^{1111}\rangle + \langle\lambda^{11}/c^{1111}\rangle^2/\langle 1/c^{1111}\rangle. \tag{4-16}$$

where e_i $(i = 1, 2, 3)$ are the vectors of the standard orthonormal basis for the Euclidean space R^3, and M_{13}^{-1} is the inverse matrix of

$$M_{13} = \begin{pmatrix} c^{1313} & e^{113} & q^{113} \\ e^{113} & -\kappa^{11} & -\alpha^{11} \\ q^{113} & -\alpha^{11} & -\mu^{11} \end{pmatrix}.$$

As we can observe the corresponding homogenized material behaves as a TMEE material with orthorhombic symmetry (2 mm). From the equations involving M_{13}^{-1} in (4-7), (4-8), (4-9), (4-10), (4-11), and (4-12), we can find the expression

$$\overline{M}_{13} = \langle M_{13}^{-1}\rangle^{-1}, \qquad \overline{M}_{13} = \begin{pmatrix} \bar{c}^{1313} & \bar{e}^{113} & \bar{q}^{113} \\ \bar{e}^{113} & -\bar{\kappa}^{11} & -\bar{\alpha}^{11} \\ \bar{q}^{113} & -\bar{\alpha}^{11} & -\bar{\mu}^{11} \end{pmatrix}.$$

Consequently,

$$\frac{\bar{c}^{1313}}{\Delta_{11}} = -\frac{\bar{e}^{131}}{\Delta_{12}} = -\frac{\bar{\kappa}^{11}}{\Delta_{22}} = \frac{\bar{q}^{131}}{\Delta_{13}} = \frac{\bar{\alpha}^{11}}{\Delta_{23}} = -\frac{\bar{\mu}^{11}}{\Delta_{33}} = \frac{1}{\Delta}, \tag{4-17}$$

where Δ is the determinant of the $\langle M_{13}^{-1}\rangle$ matrix, and Δ_{ij} is the minor obtained by excluding the i-th row and j-th column. From (4-17), one can observe that if one of the six effective coefficients is known then it is possible to calculate the other ones.

4.2. *Two-laminated TMEE composites: Benveniste–Dvorak type relations.* In this section we illustrate how Equations (4-7)–(4-16), for the case of two-laminated composites, can be used to derive universal relations of the type obtained in [Benveniste and Dvorak 1992]. In fact, from these formulae we can obtain the following expressions for the effective coefficients:

$$\bar{c}^{1111}-c_v^{1111} = -K[\![c^{1111}]\!]^2, \qquad \bar{c}^{1122}-c_v^{1122} = K[\![c^{1111}]\!][\![c^{1122}]\!], \qquad \bar{c}^{2222}-c_v^{2222} = K[\![c^{1122}]\!]^2, \tag{4-18}$$

$$\bar{c}^{1133}-c_v^{1133} = K[\![c^{1111}]\!][\![c^{1133}]\!], \quad \bar{c}^{2233}-c_v^{1133} = K[\![c^{1122}]\!][\![c^{1133}]\!], \quad \bar{c}^{3333}-c_v^{3333} = K[\![c^{1133}]\!]^2, \tag{4-19}$$

$$\bar{e}^{311}-e_v^{311} = K[\![c^{1111}]\!][\![e^{311}]\!], \qquad \bar{e}^{322}-e_v^{311} = K[\![c^{1122}]\!][\![e^{311}]\!], \qquad \bar{e}^{333}-e_v^{333} = K[\![c^{1133}]\!][\![e^{311}]\!], \tag{4-20}$$

$$\bar{q}^{311}-q_v^{311} = K[\![c^{1111}]\!][\![q^{311}]\!], \qquad \bar{q}^{322}-q_v^{311} = K[\![c^{1122}]\!][\![q^{311}]\!], \qquad \bar{q}^{333}-q_v^{333} = K[\![c^{1133}]\!][\![q^{311}]\!], \tag{4-21}$$

$$\bar{\kappa}^{33}-\kappa_v^{33} = -K[\![e^{311}]\!]^2, \qquad \bar{\alpha}^{33}-\alpha_v^{33} = -K[\![e^{311}]\!][\![q^{311}]\!], \qquad \bar{\mu}^{33}-\mu_v^{33} = -K[\![q^{311}]\!]^2, \tag{4-22}$$

$$\bar{\lambda}^{11}-\lambda_v^{11} = K[\![c^{1111}]\!][\![\lambda^{11}]\!], \qquad \bar{\lambda}^{22}-\lambda_v^{11} = K[\![c^{1122}]\!][\![\lambda^{11}]\!], \qquad \bar{\lambda}^{33}-\lambda_v^{33} = K[\![c^{1133}]\!][\![\lambda^{11}]\!], \tag{4-23}$$

$$\bar{p}^{3}-p_v^{3} = -K[\![e^{311}]\!][\![\lambda^{11}]\!], \qquad \bar{m}^{3}-m_v^{3} = -K[\![q^{311}]\!][\![\lambda^{11}]\!], \qquad \bar{\beta}-\beta_v = K[\![\lambda^{11}]\!]^2, \tag{4-24}$$

where $K = \upsilon_1(1-\upsilon_1)/(\upsilon_1 c_2^{1111} + (1-\upsilon_1)c_1^{1111})$. Eliminating K from $(4\text{-}18)_2$, $(4\text{-}18)_3$, and $(4\text{-}19)_2$, and then again from $(4\text{-}19)_{1,2,3}$, from $(4\text{-}20)_{1,2,3}$, from $(4\text{-}21)_{1,2,3}$, and from $(4\text{-}23)_{1,2,3}$, we derive the universal relations

$$\begin{aligned}\frac{[\![c^{1111}+c^{1122}]\!]}{[\![c^{1133}]\!]} &= \frac{\bar{c}^{2222}+\bar{c}^{1122}-(c_\nu^{2222}+c_\nu^{1122})}{\bar{c}^{2233}-c_\nu^{1133}} = \frac{\bar{c}^{1133}+\bar{c}^{2233}-2c_\nu^{1133}}{\bar{c}^{3333}-c_\nu^{3333}} \\ &= \frac{\bar{e}^{311}+\bar{e}^{322}-2e_\nu^{311}}{\bar{e}^{333}-e_\nu^{333}} = \frac{\bar{q}^{311}+\bar{q}^{322}-2q_\nu^{311}}{\bar{q}^{333}-q_\nu^{333}} = \frac{\bar{\lambda}^{11}+\bar{\lambda}^{22}-2\lambda_\nu^{11}}{\bar{\lambda}^{33}-\lambda_\nu^{33}}.\end{aligned} \tag{4-25}$$

Analogously, from expressions (4-18)$_{2,3}$, (4-19)$_{2,3}$, (4-20)$_{1,2}$, (4-21)$_{2,3}$, (4-22)$_{1,3}$, (4-23)$_{1,2,3}$, one can obtain the common constant $[\![e^{311}]\!]/[\![c^{1133}]\!]$, and hence the following relations connecting the effective properties:

$$\frac{[\![e^{311}]\!]}{[\![c^{1133}]\!]} = \frac{\bar{e}^{311}-e_\nu^{311}}{\bar{c}^{1133}-c_\nu^{1133}} = \frac{\bar{e}^{322}-e_\nu^{311}}{\bar{c}^{2233}-c_\nu^{1133}} = \frac{\bar{e}^{333}-e_\nu^{333}}{\bar{c}^{3333}-c_\nu^{3333}} = \frac{\kappa_\nu^{33}-\bar{\kappa}^{33}}{\bar{e}^{333}-e_\nu^{333}} = \frac{\alpha_\nu^{33}-\bar{\alpha}^{33}}{\bar{q}^{333}-q_\nu^{333}} = \frac{p_\nu^{3}-\bar{p}^{3}}{\bar{\lambda}^{33}-\lambda_\nu^{33}}. \tag{4-26}$$

The following relations are obtained by manipulating (4-18)$_2$, (4-19)$_2$, (4-20)$_{1,3}$, (4-21)$_2$, (4-22)$_{1,2}$, (4-23)$_2$, (4-24)$_1$:

$$\frac{[\![q^{311}]\!]}{[\![c^{1133}]\!]} = \frac{\bar{q}^{311}-q_\nu^{311}}{\bar{c}^{1133}-c_\nu^{1133}} = \frac{\bar{q}^{333}-q_\nu^{333}}{\bar{c}^{3333}-c_\nu^{3333}} = \frac{\mu_\nu^{33}-\bar{\mu}^{33}}{\bar{q}^{333}-q_\nu^{333}} = \frac{\alpha_\nu^{33}-\bar{\alpha}^{33}}{\bar{e}^{333}-e_\nu^{333}} = \frac{m_\nu^{3}-\bar{m}^{3}}{\bar{\lambda}^{33}-\lambda_\nu^{33}}. \tag{4-27}$$

The universal relations (28) and (29) of [Benveniste and Dvorak 1992] are contained respectively in (4-25) and (4-26). Similarly (12) and (13) of [Benveniste 1995] are included in (4-27). On the other hand these equations illustrate the interrelation between effective thermal terms with other individuals and global elastic, piezoelectric, and piezomagnetic properties. From (4-26) and (4-27) the following relation of proportionality among effective pyromagnetic and pyroelectric properties can be produced:

$$\frac{[\![e^{311}]\!]}{[\![q^{311}]\!]} = \frac{\bar{p}^{3}-p_\nu^{3}}{\bar{m}^{3}-m_\nu^{3}}. \tag{4-28}$$

This relation can also be obtained from the explicit expressions for the effective moduli $\bar{p}^3$ and $\bar{m}^3$ given in [Li and Dunn 1998b, p. 409] for fibrous (circular cylinder) composites.

4.3. ***Two-phase TMEE fibrous composites: Benveniste type exact connections.*** Next we determine exact relations (à la [Benveniste 1995]) between the elastic, piezoelectric, piezomagnetic, and thermal effective moduli of two-phase periodic fibrous composite systems characterized by a cylindrical geometry and consisting also of transversely isotropic TMEE constituents. The axis of the cylinder coincides with the axis x_3. Here such exact connections will be derived in a different way, without solving any local problem, based on certain links among the solutions of the local problems L_1^{33}, L_2^3, L_3^3, and L_4. In order to show that, these local problems will be presented in a compact form as the problem $L^{(q)}$ with $q = 1, 2, 3, 4$. The two-phase periodic cell is denoted by Y, while Σ represents the contact interface between the matrix Y_1 and the inclusion Y_2 (see Figure 1).

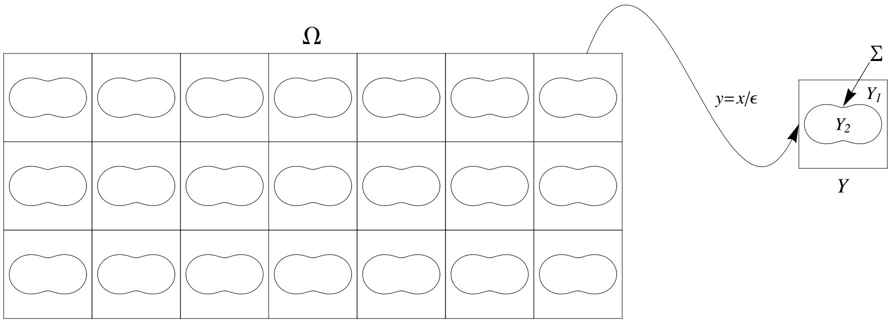

Figure 1. A part of a periodic domain and the unit cell.

The cell problem $L^{(q)}$ consists in finding the Y-periodic functions $w^{(q)}$, $\zeta^{(q)}$, and $\eta^{(q)}$, with an average of zero on Y, satisfying the following equations and continuity conditions on Y:

$$\left.\begin{aligned}\partial_{j,y}\sigma^{ij}(w_\alpha^{(q)},\zeta_\alpha^{(q)},\eta_\alpha^{((q)},0)&=0\\ \partial_{i,y}D^i(w_\alpha^{(q)},\zeta_\alpha^{(q)},\eta_\alpha^{(q)},0)&=0\\ \partial_{i,y}B^i(w_\alpha^{(q)},\zeta_\alpha^{(q)},\eta_\alpha^{(q)},0)&=0\end{aligned}\right\}\text{ in }Y_\alpha,\tag{4-29}$$

$$\left.\begin{aligned}[\![w^{(q)}]\!]&=0\\ [\![\zeta^{(q)}]\!]&=0\\ [\![\eta^{(q)}]\!]&=0\\ [\![\sigma^{1\delta}(w_\alpha^{(q)},\zeta_\alpha^{(q)},\eta_\alpha^{(q)},0)n_\delta]\!]&=\kappa^{(q)}n_1\\ [\![\sigma^{2\delta}(w_\alpha^{(q)},\zeta_\alpha^{(q)},\eta_\alpha^{(q)},0)n_\delta]\!]&=\kappa^{(q)}n_2\\ [\![\sigma^{3\delta}(w_\alpha^{(q)},\zeta_\alpha^{(q)},\eta_\alpha^{(q)},0)n_\delta]\!]&=0\\ [\![D^{\delta}(w_\alpha^{(q)},\zeta_\alpha^{(q)},\eta_\alpha^{(q)},0)n_\delta]\!]&=0\\ [\![B^{\delta}(w_\alpha^{(q)},\zeta_\alpha^{(q)},\eta_\alpha^{(q)},0)n_\delta]\!]&=0\end{aligned}\right\}\text{ on }\Sigma,\tag{4-30}$$

where $w^{(1)}\equiv w^{33}$, $\zeta^{(1)}\equiv\zeta^{33}$, $\eta^{(1)}\equiv\eta^{33}$; $w^{(2)}\equiv g^3$, $\zeta^{(2)}\equiv\pi^3$, $\eta^{(2)}\equiv\chi^3$; $w^{(3)}\equiv h^3$, $\zeta^{(3)}\equiv\xi^3$, $\eta^{(3)}\equiv\gamma^3$; and $w^{(4)}\equiv\Gamma$, $\zeta^{(4)}\equiv Q$, $\eta^{(4)}\equiv R$ are, respectively, the solutions of the local problems L_1^{33}, L_2^3, L_3^3, and L_4. The jump $\kappa^{(p)}$ on the interface Σ is defined for each local problem by the constants $\kappa^{(1)}=-[\![c^{3311}]\!]$, $\kappa^{(2)}=-[\![e^{311}]\!]$, $\kappa^{(3)}=-[\![q^{311}]\!]$, and $\kappa^{(4)}=[\![\lambda^{11}]\!]$. The structure of the problems (4-29)–(4-30) is very similar to the corresponding ones for elastic [Guinovart-Díaz et al. 2001; Rodríguez-Ramos et al. 2001]) and piezoelectric [Bravo-Castillero et al. 2001; Sabina et al. 2001] unidirectional fibrous composites. The unique nonzero solutions of these problems are the elastic plane-strain local functions $w_1^{(q)}(y_1,y_2)$ and $w_2^{(q)}(y_1,y_2)$, for $q=1,2,3,4$, which are connected by

$$w_\alpha^{(2)}=\frac{[\![e^{311}]\!]}{[\![c^{1133}]\!]}w_\alpha^{(1)},\qquad w_\alpha^{(3)}=\frac{[\![q^{311}]\!]}{[\![c^{1133}]\!]}w_\alpha^{(1)},\qquad w_\alpha^{(4)}=-\frac{[\![\lambda^{11}]\!]}{[\![c^{1133}]\!]}w_\alpha^{(1)},\tag{4-31}$$

Taking into account the above considerations and notations, and using (3-10), the following expressions for effective properties can be obtained:

- From the problem L_1^{33} ($\equiv L^{(1)}$):

$$
\begin{aligned}
\bar{c}^{1133} &= \langle c^{1133}\rangle + \langle c^{1111}\partial_{1,1}w_1^{(1)} + c^{1122}\partial_{2,2}w_2^{(1)}\rangle,\\
\bar{c}^{3333} &= \langle c^{3333}\rangle + \langle c^{1133}(\partial_{1,1}w_1^{(1)} + \partial_{2,2}w_2^{(1)})\rangle,\\
\bar{e}^{333} &= \langle e^{333}\rangle + \langle e^{311}(\partial_{1,1}w_1^{(1)} + \partial_{2,2}w_2^{(1)})\rangle,\\
\bar{q}^{333} &= \langle q^{333}\rangle + \langle q^{311}(\partial_{1,1}w_1^{(1)} + \partial_{2,2}w_2^{(1)})\rangle.
\end{aligned}
\tag{4-32}
$$

- From the problem L_2^{3} ($\equiv L^{(2)}$):

$$
\begin{aligned}
\bar{\kappa}^{33} &= \langle \kappa^{33}\rangle - \langle e^{311}(\partial_{1,1}w_1^{(2)} + \partial_{2,2}w_2^{(2)})\rangle,\\
\bar{\alpha}^{33} &= \langle \alpha^{33}\rangle - \langle q^{311}(\partial_{1,1}w_1^{(2)} + \partial_{2,2}w_2^{(2)})\rangle,\\
\bar{e}^{311} &= \langle e^{311}\rangle + \langle c^{1111}\partial_{1,1}w_1^{(2)} + c^{1122}\partial_{2,2}w_2^{(2)}\rangle.
\end{aligned}
\tag{4-33}
$$

- From the problem L_3^{3} ($\equiv L^{(3)}$):

$$
\begin{aligned}
\bar{\mu}^{33} &= \langle \mu^{33}\rangle - \langle q^{311}(\partial_{1,1}w_1^{(3)} + \partial_{2,2}w_2^{(3)})\rangle,\\
\bar{q}^{311} &= \langle q^{311}\rangle + \langle c^{1111}\partial_{1,1}w_1^{(3)} + c^{1122}\partial_{2,2}w_2^{(3)}\rangle.
\end{aligned}
\tag{4-34}
$$

- From the problem L_4 ($\equiv L^{(4)}$):

$$
\begin{aligned}
\bar{\lambda}^{11} &= \langle \lambda^{11}\rangle - \langle c^{1111}\partial_{1,1}w_1^{(4)} + c^{1122}\partial_{2,2}w_2^{(4)}\rangle,\\
\bar{\lambda}^{33} &= \langle \lambda^{33}\rangle - \langle c^{1133}(\partial_{1,1}w_1^{(4)} + \partial_{2,2}w_2^{(4)})\rangle,\\
\bar{p}^{3} &= \langle p^{3}\rangle + \langle e^{311}(\partial_{1,1}w_1^{(4)} + \partial_{2,2}w_2^{(4)})\rangle,\\
\bar{m}^{3} &= \langle m^{3}\rangle + \langle q^{311}(\partial_{1,1}w_1^{(4)} + \partial_{2,2}w_2^{(4)})\rangle,\\
\bar{\beta} &= \langle \beta\rangle - \langle \lambda^{11}(\partial_{1,1}w_1^{(4)} + \partial_{2,2}w_2^{(4)})\rangle.
\end{aligned}
\tag{4-35}
$$

Now, combining (4-32) and (4-35) and making use of $(4\text{-}31)_3$, the following universal relations can be obtained:

$$
\frac{[\![c^{1133}]\!]}{[\![\lambda^{11}]\!]} = \frac{\bar{c}^{1133} - c_\nu^{1133}}{\bar{\lambda}^{11} - \lambda_\nu^{11}} = \frac{\bar{c}^{3333} - c_\nu^{3333}}{\bar{\lambda}^{33} - \lambda_\nu^{33}} = \frac{e_\nu^{333} - \bar{e}^{333}}{\bar{p}^{3} - p_\nu^{3}} = \frac{q_\nu^{333} - \bar{q}^{333}}{\bar{m}^{3} - m_\nu^{3}}.
\tag{4-36}
$$

Note that Equations (4-36) involve all thermal global coefficients with the exception of $\bar{\beta}$. However, if the interface Σ is smooth enough so that Green's formula can be applied, then, from $(4\text{-}31)_3$, $(4\text{-}32)_3$, $(4\text{-}32)_4$, and $(4\text{-}35)_{3,4,5}$ one can obtain

$$
\begin{gathered}
\bar{e}^{333} - e_\nu^{333} = -[\![e^{311}]\!]\Pi, \qquad \bar{q}^{333} - q_\nu^{333} = -[\![q^{311}]\!]\Pi, \qquad \bar{\beta} - \beta_\nu = -\frac{[\![\lambda^{11}]\!]^2}{[\![c^{1133}]\!]}\Pi,\\
\bar{p}^{3} - p_\nu^{3} = \frac{[\![\lambda^{11}]\!][\![e^{311}]\!]}{[\![c^{1133}]\!]}\Pi, \qquad \bar{m}^{3} - m_\nu^{3} = \frac{[\![\lambda^{11}]\!][\![q^{311}]\!]}{[\![c^{1133}]\!]}\Pi,
\end{gathered}
\tag{4-37}
$$

Parameters	Units	$BaTiO_3$	$CoFe_2O_4$	epoxy
c^{1111}	GPa	166.	286.	5.53
c^{1122}	GPa	77.	173.	2.97
c^{1133}	GPa	78.	170.5	2.97
c^{3333}	GPa	162.	269.5	5.53
c^{1313}	GPa	43.	45.3	0
e^{113}	C/m^2	11.6	0	0
e^{311}	C/m^2	−4.4	0	0
e^{333}	C/m^2	18.6	0	0
κ^{11}	$10^{-10}\, C^2/Nm^2$	112.	0.8	1
κ^{33}	$10^{-10}\, C^2/Nm^2$	126.	0.93	1
q^{113}	N/Am	0	550.	0
q^{311}	N/Am	0	580.3	0
q^{333}	N/Am	0	699.7	0
μ^{11}	$10^{-6}\, Ns^2/C^2$	5.	−590.	1
μ^{33}	$10^{-6}\, Ns^2/C^2$	10.	157.	1

Table 1. Material properties used in the calculations. Taken from [Lee et al. 2005].

where $\Pi = \int_\Sigma (w_1^{(1)} dy_2 - w_2^{(1)} dy_1)$. Eliminating Π from these equations, it is possible to obtain relations involving β:

$$\frac{[\![c^{1133}]\!]}{[\![\lambda^{11}]\!]} = \frac{[\![\lambda^{11}]\!]}{[\![e^{311}]\!]} \frac{e_\nu^{333} - \bar{e}^{333}}{\bar{\beta} - \beta_\nu} = \frac{[\![\lambda^{11}]\!]}{[\![q^{311}]\!]} \frac{q_\nu^{333} - \bar{q}^{333}}{\bar{\beta} - \beta_\nu}. \tag{4-38}$$

In equations (4-36) and (4-38) nine effective properties are involved. The knowledge of one fixes the values of the others eight. In a similar way, other relations can be derived. For instance, by manipulating $(4\text{-}37)_{3,4}$ we can derive (4-28). On the other hand, combining $(4\text{-}31)_1$, $(4\text{-}32)_3$, and $(4\text{-}33)_2$, or again $(4\text{-}31)_2$, $(4\text{-}32)_4$, and (4-34) one can find the relationships

$$\frac{[\![c^{1133}]\!]}{[\![q^{311}]\!]} = \frac{\bar{e}^{333} - e_\nu^{333}}{\alpha_\nu^{33} - \bar{\alpha}^{33}}, \qquad \frac{[\![c^{1133}]\!]}{[\![e^{311}]\!]} = \frac{\bar{q}^{333} - q_\nu^{333}}{\alpha_\nu^{33} - \bar{\alpha}^{33}}. \tag{4-39}$$

These equations coincide with (13) and (15) of [Benveniste 1995]. Finally, it is interesting to observe that working with expressions (4-31)–(4-35) one can get relations (4-26) and (4-27). All these relations are valid independently of the geometrical cross section of the fibers.

5. Numerical examples

The closed-form formulae for the effective properties of TMEE multilaminated composites, summarized in Section 4.1, were analytically checked in Section 4.2 by means of the derivation (from such formulae) of the universal relations of [Benveniste and Dvorak 1992] and [Benveniste 1995]. For the case of a binary laminated composite, with transversely isotropic piezoelectric constituents, Equations (4-7)–(4-9) yield [Benveniste and Dvorak 1992, (47), p. 1309].

Since TMEE multilaminated composites could be considered as a limit case of unidirectional fibrous composites, formulas of the type described can be useful for checking numerical codes.

To illustrate the performance of the formulae for three-phase magnetoelectroelastic composites, we present the results for a three-phase laminate made of a piezoelectric phase ($BaTiO_3$), a piezomagnetic phase ($CoFe_2O_4$), and an isotropic linear elastic phase (epoxy). The material properties are given in Table 1. The volume fraction υ_3 of the epoxy phase is fixed at 0.4.

In Figures 2 and 3, all effective properties (elastic, piezoelectric, piezomagnetic, electric permittivity, magnetic permeability, and magnetoelectric) of these composites are plotted against the piezomagnetic volume fraction. In Figure 2 we observe that the curves for $\bar{c}^{1111}$, $\bar{c}^{1133}$, $\bar{c}^{1313}$, $\bar{c}^{2323}$, and $\bar{c}^{3333}$ show the same trend as those appearing in [Lee et al. 2005, Figure 17], where a three-phase fibrous magneto-electroelastic composite was investigated via a finite element model. The same figure also shows that the coefficient $\bar{c}^{1212}$ agrees better with the corresponding one from [Lee et al. 2005, Figure 18] than the one derived from the Mori–Tanaka method of [Li and Dunn 1998b]. The rest of the elastic effective properties $\bar{c}^{2222}$, $\bar{c}^{2233}$, and $\bar{c}^{1122}$ also have a linear behavior but cannot be compared because the global behavior of the three-laminate (orthorhombic 2 mm) is different that of the three-phase fibrous composite (tetragonal 4 mm) of [Lee et al. 2005].

A similar situation can be observed in Figure 3, which shows the effective piezoelectric ($\bar{e}^{333}$, $\bar{e}^{113}$ and $\bar{e}^{311}$), piezomagnetic ($\bar{q}^{333}$, $\bar{q}^{311}$ and $\bar{q}^{113}$), dielectric ($\bar{\kappa}^{33}$, $\bar{\kappa}^{11}$), and magnetic ($\bar{\mu}^{33}$, $\bar{\mu}^{11}$) constants to be practically the same as those in [Lee et al. 2005, Figures 19–22, pp. 810–811]. Finally, the piezomagnetic

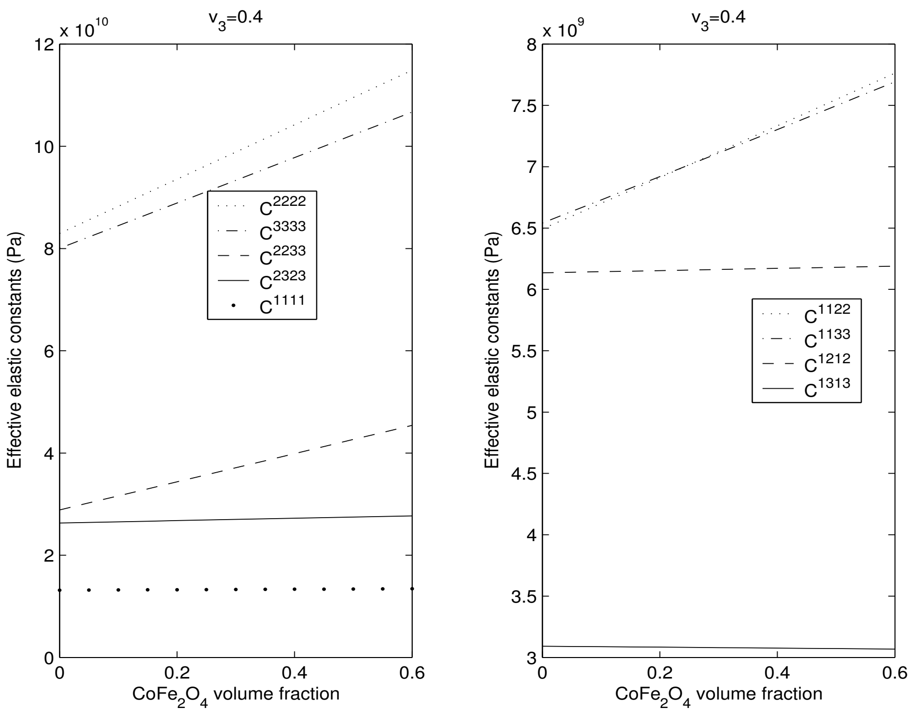

Figure 2. Effective elastic properties of a three-phase magnetoelectroelastic laminated composite versus volume fraction of piezomagnetic phase, for $\upsilon_3 = 0.4$.

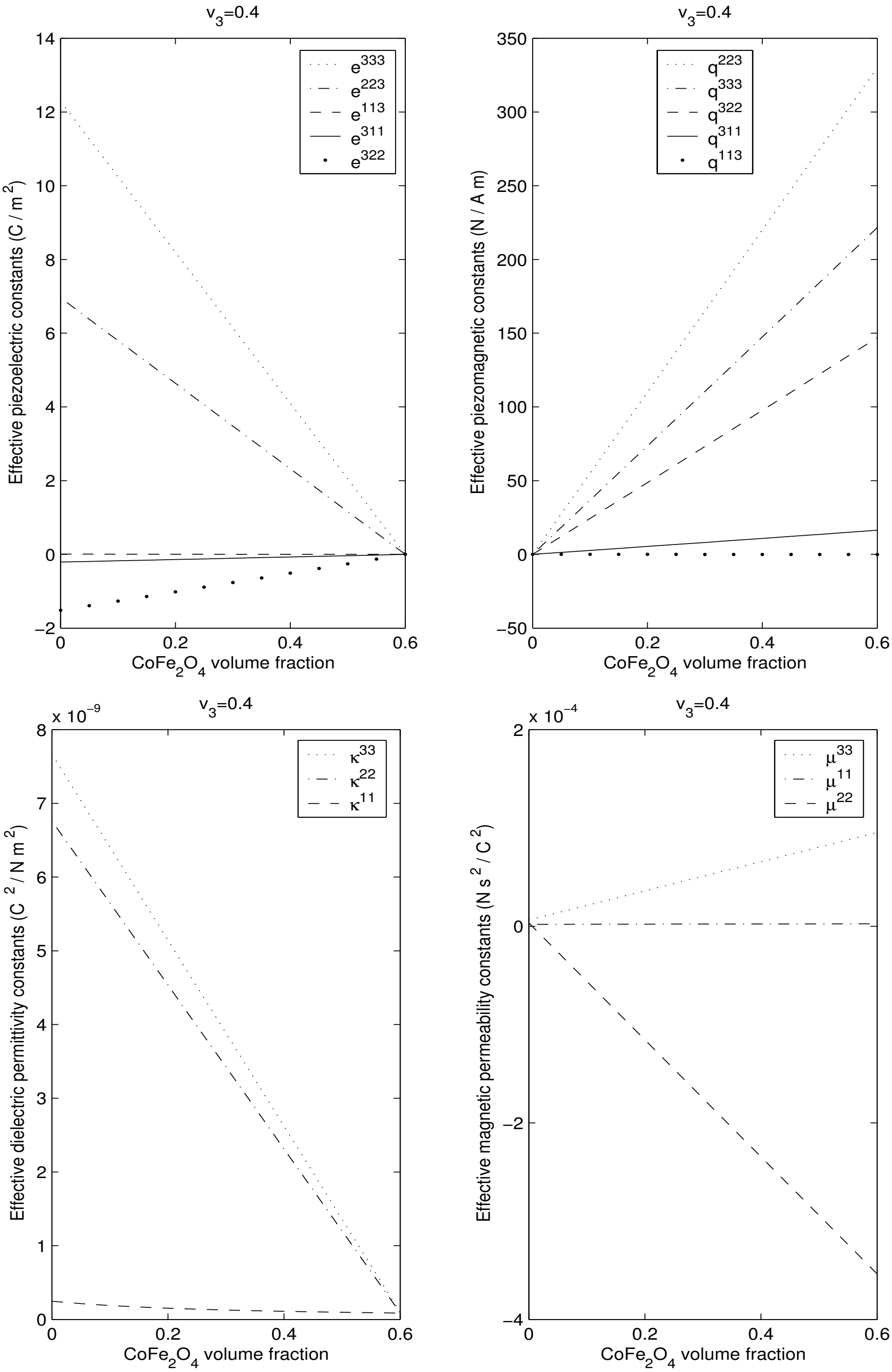

Figure 3. Effective piezoelectric (top left), and piezomagnetic (top right), dielectric (bottom left) and magnetic permeability (bottom right) properties of a three-phase magnetoelectroelastic laminated composite versus volume fraction of piezomagnetic phase, for $v_3 = 0.4$.

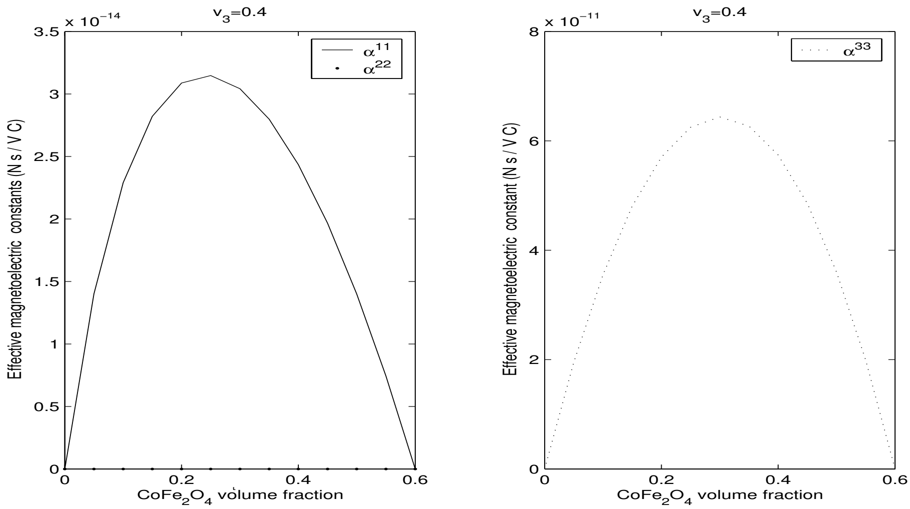

Figure 4. Effective magnetoelectric properties of a three-phase magnetoelectroelastic laminated composite versus volume fraction of piezomagnetic phase, for $\upsilon_3 = 0.4$.

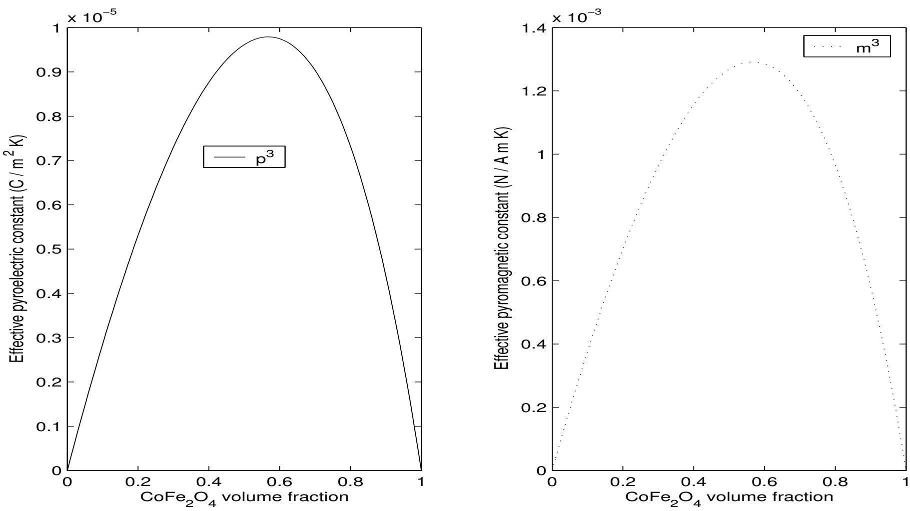

Figure 5. Effective pyroelectric and pyromagnetic properties of a two-phase $BaTiO_3$-$CoFe_2O_4$ TMEE laminated composite versus volume fraction of piezomagnetic phase.

effective constants ($\bar{\alpha}^{11}$ and $\bar{\alpha}^{33}$) illustrated in Figure 4 have the same tendency (magnetoelectric effect) as those in [Lee et al. 2005, Figures 23 and 24, p. 812].

Figure 5 illustrates the behavior of the pyroelectric and pyromagnetic effective constants of a two-phase ($BaTiO_3$-$CoFe_2O_4$) TMEE laminated composite against the piezomagnetic volume fraction. The data for the thermal expansion constants of the constituents were taken from [Ootao and Tanigawa 2005, p. 476]; they are $\theta^{11} = \theta^{22} = 15.7 \times 10^{-6}\,\mathrm{K}^{-1}$, $\theta^{33} = 6.4 \times 10^{-6}\,\mathrm{K}^{-1}$ ($BaTiO_3$), and $\theta^{11} = \theta^{22} = \theta^{33} = 10 \times 10^{-6}\,\mathrm{K}^{-1}$ ($CoFe_2O_4$) where $\lambda^{ij} = c^{ijkl}\theta^{kl}$. In this figure, the existence of pyroelectric and pyromagnetic effects is apparently, though neither phase by itself exhibits them.

6. Concluding remarks

In this paper, based on the asymptotic homogenization method, a description of the derivation of the local problems and the formulae to obtain all homogenized effective coefficients of a thermomagneto-electroelastic (TMEE) periodic heterogeneous media are given. The general homogenization model is applied to obtain closed-form formulae for effective (elastic, piezoelectric, piezomagnetic, dielectric, magnetic, magnetoelectric, thermoelastic, pyroelectric, pyromagnetic, and heat capacity) coefficients of periodic multilaminated composites with any finite number of transversely isotropic TMEE constituents. Such formulae are specified for the case of a two-phase laminated composite with an orthotropic global behavior which satisfies the universal relations of [Benveniste and Dvorak 1992]. These relations illustrate the interrelation among magnetoelectroelastic and thermal effective properties. In particular, (4-28) shows the proportionality connecting the pyroelectric and pyromagnetic effective coefficients with the proportionality constant given by the ratio of the piezoelectric and piezomagnetic individual properties. Another application of the general homogenization model is devoted to obtaining universal relations (4-36) and (4-38)–(4-39) for two-phase periodic unidirectional fibrous composites with TMEE transversely isotropic individual phases. The derivation of such universal relations does not require the solution of any local problem, and is based on certain links, given by (4-31), among the solutions of four local problems which are expressed in a compact form by (4-29)–(4-30). Several universal relations reported in [Benveniste and Dvorak 1992; Benveniste 1995] are recovered here following a different method. Some numerical calculations for three-phase laminated magnetoelectroelastic show a good concordance with similar results obtained for three-phase fibrous composites in [Ootao and Tanigawa 2005]. The magnetoelectric effect expressed by (4-28) is illustrated in Figure 5. The analytical formulae and universal relations of this work can be useful for checking numerical code.

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