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In this paper, the problem of wave propagation in periodic structured composites is studied, and a dispersive asymptotic method for the description of these dynamic processes is proposed. Assuming a single-frequency dependence of the solution for the one dimensional wave equation in a periodic composite material, higher-order terms in the asymptotic expansion for the displacement functions are studied. Nonuniformity is eliminated by finding a suitable regular asymptotic expansion for the perturbation frequency. Only two spatial scales are considered, and the equivalence of this method and the introduction of multiple slow temporal scales is shown, in good agreement with previous approaches. For a selection of boundary problems, analytic solutions are given and graphically illustrated. The problem of failures is also discussed, and some illustrative calculations are presented.

1. Introduction

Due to their importance in industry and their wide range of applications, many attempts have been made to describe the global behavior of composite materials. In elastodynamics, for example, if a traveling signal has scale comparable to the size of the material's heterogeneities, successive wave reflections and refractions take place at the interfaces. Significant wave dispersion then results, leading to distortions of the pulse shape and wave front.

The introduction of multiple scales and the methods of asymptotic homogenization [Bensoussan et al. 1978; Pobedria 1984; Bakhvalov and Panasenko 1989] has been helpful in treating a particularly important problem, the prediction of global or effective properties for composites which small-scale heterogeneities. Asymptotic analysis, as a powerful mathematical tool in dealing with problems involving small parameters, plays a fundamental role in bridging the small and large scales relevant to models of composite materials [Sánchez-Huber and Sánchez-Palencia 1992].

For a composite with periodic structure, these methods involve the dependence on two geometric scales through the expansion of the fields in powers of a small parameter ε, the ratio between the micro and macro scales. These techniques has been successful in providing effective quantities and methods

Keywords: composite materials, wave dispersion, asymptotics, homogenization, wave propagation, bilaminated composite, dynamic asymptotic homogenization.

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for the solution of partial differential equations for static problems in structures such as laminated, fiber-reinforced composites [Guinovart-Díaz et al. 2005], laminated piezocomposites [Castillero et al. 1998], and helical elastic and thermoelastic structures [Vivar-Pérez et al. 2005; 2006].

Approaches other than asymptotics are also available. Wang and Rokhlin [2002a; 2002b] developed a dynamic homogenization method based on Floquet wave theory for treating laminated composites in which the model was restricted to a *homogenization domain* consisting of frequencies and incident angles below certain critical values that depended on the composite. The problem of wave propagation in elastic fluid media with periodic structure is considered in [Santosa and Symes 1991] for cases in which the ratio between cell size and the shortest wavelength of the initial disturbance is small. Within this regime, an effective dispersive medium is obtained using the Bloch expansion. A similar analysis is made in [Sjöberg et al. 2005], in which the solutions to Maxwell's equations in periodic media are expanded in Bloch waves under the limiting condition that the unit cell is small compared to the wavelength.

The classical method of asymptotic homogenization describes the effect of wave dispersion by accounting for the influence of the first and second-order terms on the asymptotic expansion for relatively long wavelengths in fiber reinforced composites [Parnell and Abrahams 2006]. This approach fails when the observation time is relatively long or when the characteristic size of the perturbation is small, i.e., comparable to the representative volume element.

The classical method fails because of nonuniformity that results from the existence of unbounded higher-order terms in the asymptotic expansion. It was shown in [Fish and Chen 2001] that in an initial boundary value problem, whereas higher-order terms are capable of capturing dispersion effects, they introduce secular terms which grow unboundedly with time. Chen and Fish [2001] reported a recent attempt to solve this problem successfully by introducing one or more slow temporal scales, eliminating the problem of nonuniformity that could not be addressed by classical homogenization.

The main objective of this paper is to describe the dispersive behavior of periodic composites by means of time variable asymptotic rescaling, which is a necessary condition for the accurate description of a composite's global behavior. For this purpose, a reformulation of the problem is made in which the slow temporal scale is replaced by a single-frequency time-dependence, and an asymptotic expansion for the main frequencies is assumed to exist. This shows that the time rescaling needed to find the effective law of movement in composites is strongly frequency dependent. As an advantage, there is no need to study the selection of temporal scales, because the model only treats the fast spatial variable and yields closed form general expressions for the coefficients in the global model.

This treatment shows good agreement with the model presented in [Chen and Fish 2001]. We also present an analytical solution for the averaged problem for certain cases, including the situation in which a failure (defect due to the presence of fissures, voids, cracks, etc.) is present in the composite. Results for the asymptotic expansion of eigenfrequencies show that the range of validity of the method is restricted to low frequency wave propagation. The effective model is therefore not accurate for cases in which the initial disturbance has significant high frequency components. In asymptotic language, high frequencies are of order $O(1/\varepsilon)$, where ε is the ratio between the size of the periodic cell and characteristic length of the composite.

This work is the start of a study of wave propagation in composite materials with applications to damage detection and health monitoring for periodic laminated composites. The dispersive method is only considered here for one spatial dimension.

2. Statement of the problem

Our study reduces to a periodic laminated composite of length L, that is, a specimen consisting of a linear periodic repetition of a representative volume element (RVE) (or periodic cell) with characteristic length ε, Figure 1. Our analysis is independent of the number of phases embedded in the RVE, although ε is required to be small compared to the composite length, $\varepsilon \ll L$.

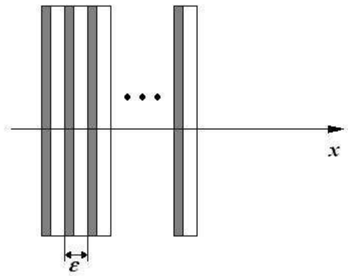

Figure 1. A laminated two-phase periodic composite.

The direction of wave propagation is assumed to be parallel to the x axes, normal to the lamination. If the laminate is considered to be isotropic, the elastodynamic equation is

$$(E_\varepsilon(x)u_x)_x - \rho_\varepsilon(x)u_{tt} = 0. \tag{2.1}$$

Here, $u = u(x,t)$ gives the longitudinal displacement from the equilibrium position at point x and time t, while $E_\varepsilon = E_\varepsilon(x)$ and $\rho_\varepsilon = \rho_\varepsilon(x)$ are the elastic modulus and the mass density at each position. The subscript ε stands for the thickness of the RVE (implying that E_ε and ρ_ε are periodic with period ε), and subscript x, t denote the respective partial derivatives.

If, for this laminated composite, we also consider a displacement $\mu(t)$ at one end $x = 0$, a load $F(t)$ at the other end $x = L$, an initial displacement $U(x)$ from the equilibrium position, and an initial velocity $V(x)$ at each point x, then the initial and boundary conditions for (2.1) are

$$u(0,t) = \mu(t), \quad E_\varepsilon(L)u_x(L,t) = F(t), \quad u(x,0) = U(x), \quad u_t(x,0) = V(x). \tag{2.2}$$

Finally, it is necessary to include the contact conditions between the faces of the laminate components. At such interfaces, the coupling conditions must be well determined. Here we will consider ideal contact conditions, where there is no discontinuity in displacement or traction at the interface. If we introduce the notation $\|f\|_\nu = \lim_{x\to\nu^+} f(x) - \lim_{x\to\nu^-} f(x)$, the ideal contact conditions are

$$\|u\|_\nu = 0, \qquad \|E_\varepsilon u_x\|_\nu = 0, \tag{2.3}$$

for every point $x = \nu$ on the interface. Under these assumptions, we would like to obtain an effective homogeneous model with constant coefficients that can approximate the response of the heterogeneous material under study. This avoids the difficulties of treating rapid variation in the coefficients due to heterogeneities and, at the same time, gives information about the dispersive nature of the laminated composite. This is achieved by first considering a single arbitrary frequency-dependence and then applying asymptotic techniques for multiple scales, which allow us to find a regular asymptotic expansion for the single arbitrary frequency and the displacement function.

3. Frequency dependence and asymptotic analysis

Following the classical methods of separation of variables or Fourier's method, a solution to (2.1) in the form $u(x,t) = X(x)T(t)$ is sought. After substitution of this product into (2.1), we obtain

$$\frac{(E_\varepsilon(x)X_x(x))_x}{\rho_\varepsilon(x)X(x)} = \frac{T_{tt}(t)}{T(t)} = -\omega_\varepsilon^2, \tag{3.1}$$

where ω_ε is the circular frequency of the longitudinal wave. This is equivalent to a pair of ordinary differential equations for $X(x)$ and $T(t)$ (Sturm–Liouville equations),

$$(E_\varepsilon X_x)_x + \omega_\varepsilon^2 \rho_\varepsilon X = 0, \qquad T_{tt} + \omega_\varepsilon^2 T = 0. \tag{3.2}$$

$X(x)$ inherits the interface conditions given in (2.3):

$$\|X\|_\nu = 0, \qquad \|E_\varepsilon X_x\|_\nu = 0. \tag{3.3}$$

Initial and boundary conditions can be derived from (2.2). Having assumed the periodicity conditions on $E_\varepsilon(x)$ and $\rho_\varepsilon(x)$ stated in the previous section, and considering that the size of the periodic cell ε is small compared to the characteristic length of the composite L, it is convenient to introduce the dependence on a new scale

$$\xi = x/\varepsilon. \tag{3.4}$$

This is the "fast spatial scale", widely used for asymptotic analysis in periodic structures [Bensoussan et al. 1978].

We can now express the elastic modulus as $E(\xi) = E(x/\varepsilon) = E_\varepsilon(x)$ and the mass density as $\rho(\xi) = \rho(x/\varepsilon) = \rho_\varepsilon(x)$. Note that $E(\xi)$ and $\rho(\xi)$ are 1-periodic (periodic with period 1), regardless of the value of ε, due to the periodic structure of the composite under consideration. The dependence of X on ξ, $X = X(x,\xi)$, now yields, from $(3.2)_1$,

$$(E(\xi)X_x(x,\xi))_x + \omega_\varepsilon^2 \rho(\xi) X(x,\xi) = 0. \tag{3.5}$$

Taking regular asymptotic expansions of the principal frequency of the perturbation and $X(x,\xi)$ gives[1]

$$\omega_\varepsilon = \omega_0 + \varepsilon\omega_1 + \varepsilon^2\omega_2 + \cdots = \sum_{n\ge 0} \varepsilon^n \omega_n, \tag{3.6}$$

$$X(x,\xi) = X_0(x,\xi) + \varepsilon X_1(x,\xi) + \varepsilon^2 X_2(x,\xi) + \cdots = \sum_{n\ge 0} \varepsilon^n X_n(x,\xi), \tag{3.7}$$

where ω_n are constant and X_n are 1-periodic with respect to the variable ξ. Introducing a comma notation for the derivative with respect to the variable indicated, $X_{,x} = \partial X/\partial x$, the chain rule and (3.4) give $X_x = \varepsilon^{-1} X_{,\xi} + X_{,x}$. Then we have for $X(x,\xi)$ and ω_ε:

$$(EX_x)_x = \frac{1}{\varepsilon^2}(EX_{,\xi})_{,\xi} + \frac{1}{\varepsilon}\left[(EX_{,\xi})_{,x} + (EX_{,x})_{,\xi}\right] + (EX_{,x})_{,x}\,, \tag{3.8}$$

$$\omega_\varepsilon^2 = \sum_{n\ge 0} \varepsilon^n \alpha_n. \tag{3.9}$$

[1] The equalities in (3.6) and (3.7) are defined in the asymptotic sense, and do not imply convergence of the series.

The numbers α_n are related to ω_n through the chain of equations:

$$\alpha_0 = \omega_0^2, \quad \alpha_1 = 2\omega_1\omega_0, \quad \alpha_2 = 2\omega_2\omega_0 + \omega_1^2, \quad \dots \quad \alpha_n = \sum_{k=0}^{n} \omega_k \omega_{n-k}. \tag{3.10}$$

With the aid of (3.8) and (3.9), it is possible to substitute the asymptotic expansions (3.6) and (3.7) into (3.5) and reorder the result by powers of ε,

$$\sum_{n\geq -2} \varepsilon^n H_n(x, \xi) = 0. \tag{3.11}$$

The coefficients $H_n(x, \xi)$, for $n \geq -2$, are given by

$$H_{-2} = (EX_{0,\xi})_{,\xi}\,, \tag{3.12}$$

$$H_{-1} = (EX_{1,\xi})_{,\xi} + (EX_{0,\xi})_{,x} + (EX_{0,x})_{,\xi}\,, \tag{3.13}$$

$$\vdots$$

$$H_n = (EX_{n+2,\xi})_{,\xi} + (EX_{n+1,\xi})_{,x} + (EX_{n+1,x})_{,\xi} + (EX_{n,x})_{,x} + \rho \sum_{k=0}^{n} \alpha_k X_{n-k}, \tag{3.14}$$

The asymptotic sum in (3.11) vanishes, yielding

$$H_n(x, \xi) = 0. \tag{3.15}$$

Bearing in mind (3.12)–(3.14), this constitutes a recurrent system of partial differential equations with unknown functions $X_n(x, \xi)$ in which the solutions X_n and X_{n+1} for the n-th and $(n+1)$-th equations are inserted into the next $(n+2)$-th equation. Once the functions X_n are found, they can be used in (3.7) to approximate $X(x, \xi)$. Observe that the numbers α_n must also be found. This is accomplished by imposing conditions of boundedness over the functions X_n discussed in the next section.

The substitution process for the asymptotic expansion (3.7) must be made in the expressions for the interface coupling conditions, (3.3), to find the conditions that X_n should satisfy at the interfaces:

$$\|X_0\|_\nu = 0, \quad \|EX_{0,\xi}\|_\nu = 0; \qquad \|X_{n+1}\|_\nu = 0, \quad \|EX_{n+1,\xi} + EX_{n,x}\|_\nu = 0 \quad \text{for } n \geq 0. \tag{3.16}$$

4. Asymptotic homogenization up to $O(\varepsilon^0)$

In this section, we describe a method for solving the system resulting from imposing (3.15) onto (3.12)–(3.14), to find the approximating functions $X_n(x, \xi)$ and the numbers α_n for each power of ε in the asymptotic expansion for X and the square of the frequency ω_ε, respectively. For this purpose it will be helpful to state the following lemma.

Lemma 1. *Consider positive functions $E(\xi)$, $f(\xi)$, and $F(\xi)$, all periodic of period 1, defined over the interval $[0, 1]$, and continuously differentiable except, perhaps, at finitely many points $0 \leq \nu_1 < \nu_2 < \cdots < \nu_m \leq 1$ where they might be discontinuous. The equation*

$$(Ev_{,\xi})_{,\xi} = f \tag{4.1}$$

in the function $v(\xi)$, defined for all points $\xi \in (0, 1)$ apart from the ν_i and satisfying the conditions

$$\|v(\xi)\|_{\nu_i} = 0, \qquad \|Ev_{,\xi} + F\|_{\nu_i} = 0, \tag{4.2}$$

has a 1*-periodic solution, unique up to an additive constant, if and only if* $\langle f\rangle = \sum_{i=1}^{m} \|F\|_{\nu_i}$, *where*

$$\langle\ \rangle = \int_0^1 d\xi \tag{4.3}$$

is the averaging operator over the RVE.

A proof can be found in the first chapter of [Bensoussan et al. 1978].

Considering (3.15) for $n=-2$ leads to the equation, of order $O(\varepsilon^{-2})$,

$$(EX_{0},_{\xi}),_{\xi} = 0. \tag{4.4}$$

Here, X_0 is restricted to 1-periodicity conditions, $X_0(x,0) = X_0(x,1)$, and to the conditions given in $(3.16)_{1,2}$. Lemma 1 supports the conclusion that, since E is a positive function, the general solution for X_0 in (4.4) is

$$X_0(x,\xi) = \hat{X}_0(x). \tag{4.5}$$

Having solved the equation for the order corresponding to $O(\varepsilon^{-2})$, the equation for next order $O(\varepsilon^{-1})$ is recalled by considering again (3.15), this time with $n=-1$,

$$(EX_{1},_{\xi}),_{\xi} + (EX_{0},_{\xi}),_{x} + (EX_{0},_{x}),_{\xi} = 0, \tag{4.6}$$

and the conditions $(3.16)_{3,4}$ for $n=0$,

$$\|X_1\|_\nu = 0, \qquad \|EX_{1},_{\xi} + EX_{0},_{x}\|_\nu = 0. \tag{4.7}$$

From (4.6) and the fact that X_0 does not depend on the fast variable ξ, it follows that

$$(EX_{1},_{\xi}),_{\xi} + E,_{\xi}\hat{X}_{0},_{x} = 0. \tag{4.8}$$

Due to the linear nature of this equation, its general solution is a sum of two terms,

$$X_1(x,\xi) = N_1(\xi)\hat{X}_{0},_{x}(x) + \hat{X}_1(x). \tag{4.9}$$

Here, $\hat{X}_1(x)$ only depends on the slow scale x. By substituting (4.9) into (4.7)–(4.8), we find an expression for the 1-periodic function $N_1(\xi)$,

$$(EN_{1},_{\xi} + E),_{\xi} = 0, \tag{4.10}$$

and the continuity conditions

$$\|N_1\| = 0, \qquad \|EN_{1},_{\xi} + E\| = 0. \tag{4.11}$$

This is the *first local problem*. Lemma 1 guarantees the existence of the local function N_1 up to an additive constant. To avoid nonuniqueness, we will take N_1 so that $\langle N_1\rangle = 0$.

Before solving for X_2, which corresponds to the next order in the asymptotic expansion of X, we note that $N_1(\xi)$ does not need to be found explicitly to obtain a homogenized model. (4.10) and $(4.11)_2$ imply that it is sufficient that $EN_{1},_{\xi} + E = C$, where C is a constant that does not depend on ξ. The average $\langle N_{1},_{\xi}\rangle = 0$ vanishes because N_1 is a 1-periodic continuous function. We have $N_{1},_{\xi} + 1 = C/E(\xi)$ and, applying the averaging operator $\langle\rangle$ to both sides of the equality, we obtain $1 = C\langle 1/E\rangle$. Finally, the equality $C = \langle 1/E\rangle^{-1}$ holds, and consequently

$$EN_1,_\xi + E = \left\langle \frac{1}{E} \right\rangle^{-1} = \hat{E}. \tag{4.12}$$

For the analysis of the $O(\varepsilon^0)$ equation, consider (3.15) and (3.14), for $n = 0$,

$$(EX_2,_\xi),_\xi + (EX_1,_\xi),_x + (EX_1,_x),_\xi + (EX_0,_x),_x + \rho\omega_0^2 X_0 = 0. \tag{4.13}$$

Here, we substitute the expressions found for X_0 and X_1 into (4.5) and (4.9), respectively,

$$(EX_2,_\xi),_\xi + \big[(EN_1),_\xi + EN_1,_\xi + E\big]\,\hat{X}_0,_{xx} + E,_\xi\,\hat{X}_1,_x + \rho\omega_0^2\hat{X}_0 = 0. \tag{4.14}$$

Averaging both sides of the equation over one period and considering that $EX_2,_\xi$ satisfies the condition $(3.16)_{3,4}$ and is therefore a 1-periodic continuous function in ξ, we have

$$\langle EN_1,_\xi + E\rangle\hat{X}_0,_{xx} + \omega_0^2\langle\rho\rangle\hat{X}_0 = 0. \tag{4.15}$$

The coefficients $\langle EN_1,_\xi + E\rangle$ and $\langle\rho\rangle$ are the effective coefficients given in previous discussions of homogenization [Pobedria 1984; Bakhvalov and Panasenko 1989]. They are well known, and for one-dimensional periodic structured composites, they can be found explicitly. Finally, we write

$$\hat{E}\hat{X}_0,_{xx} + \omega_0^2\hat{\rho}\hat{X}_0 = 0, \tag{4.16}$$

where $\hat{E}$ is given in (4.12) and $\hat{\rho} = \langle\rho\rangle$.

As we can see, (4.16) contains $\hat{X}_0$ by itself, and does not show dispersive wave propagation behavior in the composite. This result is obtained if we set $\varepsilon = 0$ in our model. In this case, the structure is effectively homogeneous and nondispersive if the component materials are nondispersive. Applying the normalization condition $\langle N_1\rangle = 0$, dropping the approximation $\langle X\rangle \approx \hat{X}_0$, and applying the principle of superposition, we are led from (4.16) and $(3.2)_2$ with $\omega_\varepsilon \approx \omega_0$ to the averaged model for the function $\langle u\rangle = \hat{u}$,

$$\hat{E}\hat{u},_{xx} - \hat{\rho}\hat{u}_{tt} = 0. \tag{4.17}$$

The classical method of asymptotic homogenization yields the same result, although this result is not expected if the wavelength is comparable to the size of the periodic cell. To describe the dispersive behavior, more terms must be considered in (3.15).

From (4.16), we have

$$\hat{X}_0 = -\frac{1}{\omega_0^2}\frac{\hat{E}}{\hat{\rho}}\hat{X}_0,_{xx}\,, \tag{4.18}$$

which, in combination with (4.14), leads to

$$(EX_2,_\xi),_x + \Big[(EN_1),_\xi + EN_1,_\xi + E - \frac{\rho}{\hat{\rho}}\langle EN_1,_\xi + E\rangle\Big]\hat{X}_0,_{xx} + + E,_\xi\,\hat{X}_1,_x = 0. \tag{4.19}$$

Because this equation is linear, the general solution, X_2, is

$$X_2(x,\xi) = N_2(\xi)\hat{X}_0,_{xx} + N_1\hat{X}_1,_x + \hat{X}_2(x). \tag{4.20}$$

Analogously to previous cases, $\hat{X}_2(x)$ only depends on x, and $N_2(\xi)$ is the 1-periodic function called the second local function. This function yields a null average, $\langle N_2\rangle = 0$, and must satisfy the *second local problem*,

$$(EN_{2},_{\xi} + EN_1),_{\xi} + EN_{1},_{\xi} + E - \frac{\rho}{\hat{\rho}}\langle EN_{1},_{\xi} + E\rangle = 0, \tag{4.21}$$

with conditions

$$\|N_2\| = 0, \qquad \|EN_{2},_{\xi} + EN_1\| = 0. \tag{4.22}$$

5. Higher-order homogenization

In this section, we continue with higher-order approximations in the asymptotic expansion, (3.11). The objective is to relate the terms of the asymptotic expansions of ω_ε and $X(x,\xi)$, given in (3.6) and (3.7), to the periodicity of the composite laminated structure.

From the equation corresponding to $O(\varepsilon)$, we have

$$(EX_{3},_{\xi}),_{\xi} + (EX_{2},_{\xi}),_{x} + (EX_{2},_{x}),_{\xi} + (EX_{1},_{x}),_{x} + \omega_0^2\rho X_1 + 2\omega_1\omega_0\rho X_0 = 0. \tag{5.1}$$

Combining the formulas for X_0, X_1, and X_2 given in (4.5), (4.9), and (4.20), respectively, and taking $\hat{c}^2 = \hat{E}/\hat{\rho}$, we have

$$\begin{aligned}(EX_{3},_{\xi}),_{\xi} &+ \left[(EN_2),_{\xi} + EN_{2},_{\xi} + EN_1 - \hat{c}^2\rho N_1\right]\hat{X}_{0},_{xxx} \\ &+ \left[(EN_1),_{\xi} + EN_{1},_{\xi} + E\right]\hat{X}_{1},_{xx} + E,_{\xi}\hat{X}_{2},_{x} + + \omega_0^2\rho\hat{X}_1 + 2\omega_0\omega_1\rho\hat{X}_0 = 0.\end{aligned} \tag{5.2}$$

Averaging this equation, and using (4.12), we have

$$\langle EN_{2},_{\xi} + EN_1 - \hat{c}^2\rho N_1\rangle\hat{X}_{0},_{xxx} + \hat{E}\hat{X}_{1},_{xx} + \omega_0^2\hat{\rho}\hat{X}_1 + 2\omega_0\omega_1\hat{\rho}\hat{X}_0 = 0. \tag{5.3}$$

It can be shown that

$$\langle EN_{2},_{\xi} + EN_1 - \hat{c}^2\rho N_1\rangle = 0. \tag{5.4}$$

The functions N_1 and N_2 are continuous because they satisfy (4.11) and $(4.22)_1$. This is also true for the functions $EN_{1},_{\xi} + E$ and $EN_{2},_{\xi} + EN_1$ due to $(4.11)_2$ and $(4.22)_2$. Then,

$$\left\langle\left[N_2(EN_{1},_{\xi} + E) - N_1(EN_{2},_{\xi} + EN_1)\right],_{\xi}\right\rangle = 0, \tag{5.5}$$

because the bracketed function is continuous and 1-periodic. Applying the rule for the derivation of the product, we have

$$\langle N_{2},_{\xi}(EN_{1},_{\xi} + E) - N_{1},_{\xi}(EN_{2},_{\xi} + EN_1)\rangle + \langle N_2(EN_{1},_{\xi} + E),_{\xi} - N_1(EN_{2},_{\xi} + EN_1),_{\xi}\rangle = 0. \tag{5.6}$$

Substituting the first and second local problems (4.10) and (4.21) yields

$$\langle N_{2},_{\xi}(EN_{1},_{\xi} + E) - N_{1},_{\xi}(EN_{2},_{\xi} + EN_1) + N_1(EN_{1},_{\xi} + E) - \hat{c}^2\rho N_1\rangle = 0. \tag{5.7}$$

Finally, after eliminating parentheses and reducing terms, we obtain (5.4). Therefore, from (5.3), we conclude that

$$\hat{E}\hat{X}_{1},_{xx} + \omega_0^2\hat{\rho}\hat{X}_1 = -2\omega_0\omega_1\hat{\rho}\hat{X}_0. \tag{5.8}$$

This equation is a second-order ordinary differential equation in $\hat{X}_1$ with constant coefficients. The right-hand side satisfies the corresponding homogeneous second-order equation from (4.16). To obtain bounded solutions for (5.8), we must set $\omega_1 = 0$ because ω_0 and $\hat{X}_0$ are arbitrary. This yields

$$\hat{E}\hat{X}_{1,xx} + \omega_0^2 \hat{\rho} \hat{X}_1 = 0, \qquad \omega_1 = 0. \tag{5.9}$$

Combining (5.9) and (5.2), we have

$$(EX_{3,\xi})_{,\xi} + \big[(EN_2)_{,\xi} + EN_{2,\xi} + EN_1 - \hat{c}^2 \rho N_1\big] \hat{X}_{0,xxx} + \big[(EN_1)_{,\xi} + EN_{1,\xi} + E - \hat{c}^2 \rho\big] \hat{X}_{1,xx} + E_{,\xi} \hat{X}_{2,x} = 0. \tag{5.10}$$

The general solution, X_3, to (5.10) is

$$X_3(x,\xi) = N_3(\xi)\hat{X}_{0,xxx} + N_2 \hat{X}_{1,xx} + N_1 \hat{X}_{2,x} + \hat{X}_3(x). \tag{5.11}$$

The third 1-periodic local function N_3, for which $\langle N_3 \rangle = 0$, is the solution to the *third local problem*,

$$(EN_{3,\xi} + EN_2)_{,\xi} + EN_{2,\xi} + EN_1 - \hat{c}^2 \rho N_1 = 0, \tag{5.12}$$

with continuity conditions

$$\|N_3\| = 0, \qquad \|EN_{3,\xi} + EN_2\| = 0, \tag{5.13}$$

obtained by substituting (5.11) and (4.20) into the ideal contact conditions given in $(3.16)_{3,4}$ for $n = 2$. ω_1 does not change the result obtained thus far for ω_ε. Improvements on this value must be made at subsequent levels of approximation.

Continuing with the term of order $O(\varepsilon^2)$, we have

$$(EX_{4,\xi})_{,\xi} + (EX_{3,\xi})_{,x} + (EX_{3,x})_{,\xi} + (EX_{2,x})_{,x} + \omega_0^2 \rho X_2 + 2\omega_2 \omega_0 \rho X_0 = 0. \tag{5.14}$$

Substituting in (5.14) the values of X_2 and X_3 from (4.20) and (5.11), and the constraints (4.16) and $(5.9)_1$ satisfied by $\hat{X}_0$ and $\hat{X}_1$, we get

$$(EX_{4,\xi})_{,\xi} + \big[(EN_3)_{,\xi} + EN_{3,\xi} + EN_2 - \hat{c}^2 \rho N_2\big] \hat{X}_{0,xxxx} + \big[(EN_2)_{,\xi} + EN_{2,\xi} + EN_1 - \hat{c}^2 \rho N_1\big] \hat{X}_{1,xxx} + (EN_{1,\xi} + E)\hat{X}_{2,xx} + \omega_0^2 \rho \hat{X}_2 + 2\omega_2 \omega_0 \rho \hat{X}_0 = 0. \tag{5.15}$$

Averaging over this last equality yields

$$\langle EN_{3,\xi} + EN_2 - \hat{c}^2 \rho N_2 \rangle \hat{X}_{0,xxxx} + \hat{E}\hat{X}_{2,xx} + \omega_0^2 \hat{\rho} \hat{X}_2 + 2\omega_2 \omega_0 \hat{\rho} \hat{X}_0 = 0. \tag{5.16}$$

Considering the second-order homogeneous equation (4.16), we have

$$\hat{X}_0 = -\frac{\hat{c}^2}{\omega_0^2} \hat{X}_{0,xx} = \frac{\hat{c}^4}{\omega_0^4} \hat{X}_{0,xxxx}. \tag{5.17}$$

Consequently, we can rewrite (5.16) as

$$\hat{E}\hat{X}_{2,xx} + \omega_0^2 \hat{\rho} \hat{X}_2 = -\Big[\langle EN_{3,\xi} + EN_2 - \hat{c}^2 \rho N_2 \rangle \frac{\omega_0^4}{\hat{c}^4} + 2\omega_2 \omega_0 \hat{\rho}\Big] \hat{X}_0. \tag{5.18}$$

Once again, we have obtained a second-order differential equation, this time for the function $\hat{X}_2$. Because $\hat{X}_0$ satisfies the corresponding second order homogeneous equation, the right-hand side of (5.18) does also. To avoid unbounded solutions for $\hat{X}_2$, we must select ω_2 so that the coefficient of $\hat{X}_0$ in the right-hand side is equal to zero:

$$\omega_2 = -\frac{\omega_0^3}{2\hat{c}^4\hat{\rho}}\langle EN_{3,\xi} + EN_2 - \hat{c}^2\rho N_2\rangle. \tag{5.19}$$

The equation for $\hat{X}_2$ is

$$\hat{E}\hat{X}_{2,xx} + \omega_0^2\hat{\rho}\hat{X}_2 = 0. \tag{5.20}$$

With this result, an averaged expression for (2.1), up to $O(\varepsilon^2)$, can be obtained. Combining

$$\hat{X} = \hat{X}_0 + \varepsilon\hat{X}_1 + \varepsilon^2\hat{X}_2 \tag{5.21}$$

and the normalization condition, $\langle N_n\rangle = 0$ for $n = 1, 2, \ldots$, it can be seen that $\langle X\rangle \approx \hat{X}$, and we have

$$\hat{E}\hat{X}_{,xx} + \omega_0^2\hat{\rho}\hat{X} = 0. \tag{5.22}$$

If $u = X(x,\xi)T(t)$, then $\hat{u} = \langle u\rangle = \langle X(x,\xi)T(t)\rangle \approx \hat{X}T$, and

$$\hat{\rho}\hat{u}_{tt} = \hat{\rho}\hat{X}T_{tt} = -\hat{\rho}\omega_\varepsilon^2\hat{X}T, \tag{5.23}$$

considering $(3.2)_2$. Taking only the terms up to the second-order of approximation in the second equality of (5.23), we obtain

$$\begin{aligned}
\hat{\rho}\omega_\varepsilon^2\hat{X}T &\approx \hat{\rho}(\omega_0 + \varepsilon^2\omega_2)^2\hat{X}T = \hat{\rho}\omega_0^2\hat{X}T + \varepsilon^2\hat{\rho}2\omega_0\omega_2\hat{X}T + \varepsilon^4\hat{\rho}\omega_2^2\hat{X}T \\
&= \hat{\rho}\omega_0^2\hat{X}T + \varepsilon^2 2\hat{\rho}\omega_2(\omega_0 + \varepsilon^2 2\omega_2)\hat{X}T - \varepsilon^4\hat{\rho}3\omega_2^2\hat{X}T \\
&= \hat{\rho}\omega_0^2\hat{X}T + \varepsilon^2 2\hat{\rho}\frac{\omega_2}{\omega_0}\omega_\varepsilon^2\hat{X}T - \varepsilon^4\hat{\rho}3\omega_2^2\hat{X}T.
\end{aligned} \tag{5.24}$$

Neglecting terms of order $O(\varepsilon^4)$ and substituting the value for ω_2 from (5.19), this reduces to

$$\hat{\rho}\hat{X}T_{tt} = -\hat{\rho}\omega_0^2\hat{X}T - \varepsilon^2\frac{\omega_0^2}{\hat{c}^4}\kappa\omega_\varepsilon^2\hat{X}T, \tag{5.25}$$

where we have set $\kappa = \langle EN_{3,\xi} + EN_2 - \hat{c}^2\rho N_2\rangle$. In view of (5.22) and $(3.2)_2$, we can substitute $\dfrac{\omega_0^2}{\hat{c}^2}\hat{X} = -\hat{X}_{xx}$ and $\omega_\varepsilon^2 T = -T_{tt}$ to obtain

$$\hat{\rho}\hat{X}T_{tt} = \hat{E}\hat{X}_{xx}T - \frac{\varepsilon^2\kappa}{\hat{c}^2}\hat{X}_{xx}T_{tt}. \tag{5.26}$$

Finally, we have, for $\hat{u}$,

$$\hat{\rho}\hat{u}_{tt} = \hat{E}\hat{u}_{xx} - \frac{\varepsilon^2\kappa}{\hat{c}^2}\hat{u}_{xxtt}. \tag{5.27}$$

Applying the principle of superposition, we find that this equation is valid for more general functions $\hat{u}$ which are sums of stationary modes $\hat{u}(x,t) = \hat{X}(x)T(t)$ multiplied by a constant amplitude. This result demonstrates the dispersive nature of wave propagation in the composite under study.

One-dimensional homogenization yields a closed-form expression for κ, which depends on the coefficients in the original equation, $E(\xi)$ and $\rho(\xi)$. This procedure is presented in the Appendix. If we define

$$R = \int_0^\xi\Big(\frac{\rho}{\hat{\rho}} - 1\Big)ds, \qquad B = \int_0^\xi\Big(\frac{\hat{E}}{E} - 1\Big)ds, \tag{5.28}$$

then

$$\kappa = \hat{E}\left[\left\langle\Big(\frac{\hat{E}}{E}R-B\Big)\Big(R-\Big\langle\frac{\hat{E}}{E}R\Big\rangle\Big)\right\rangle+\left\langle\Big(\frac{\rho}{\hat{\rho}}B-R\Big)(B-\langle B\rangle)\right\rangle\right]. \tag{5.29}$$

If $E(\xi)\rho(\xi)$ is a constant, then $\kappa=0$. Under such conditions, $\rho(\xi)/\hat{\rho}=\hat{E}/E(\xi)$, $R=B$, and

$$\kappa = \hat{E}\left[\left\langle 2\Big(\frac{\hat{E}}{E}-1\Big)B^2\right\rangle-\left\langle\Big(\frac{\hat{E}}{E}+1\Big)B\right\rangle\left\langle\Big(\frac{\hat{E}}{E}-1\Big)B\right\rangle\right]. \tag{5.30}$$

This expression vanishes if we consider $dB/d\xi=\hat{E}/E-1$: indeed, this condition implies

$$\left\langle\Big(\frac{\hat{E}}{E}-1\Big)B^n\right\rangle=\frac{1}{n+1}\left\langle\frac{d}{d\xi}(B^{n+1})\right\rangle=0, \tag{5.31}$$

because B^n is a 1-periodic function. This fact can be used to verify that the quantity in brackets in (5.30) is zero. From a physical standpoint, this demonstrates that for constant acoustic impedance $E\rho$ in a periodic composite, dispersion is not observed in the global model because that would imply that $\kappa=0$.

6. Arbitrary orders of approximation

The results obtained in the last section can be extended to arbitrary orders of approximation for the functions $X(x,\xi)$ and the angular frequency ω_ε. To achieve this goal, we require the following result.

Lemma 2. *For all $n\geq 0$, we have*

$$X_n(x,\xi)=\sum_{m=0}^{n}N_{n-m}(\xi)\frac{d^{n-m}\hat{X}_m}{dx^{n-m}}(x), \tag{6.1}$$

and the expressions for α_n become

$$\alpha_n=-\frac{1}{\hat{\rho}}\Big(-\frac{\omega_0^2}{\hat{c}^2}\Big)^{n/2+1}\left\langle EN_{n+1},_{\xi}+EN_n+\sum_{k=0}^{n/2-1}\alpha_{2k}\Big(-\frac{\hat{c}^2}{\omega_0^2}\Big)^{k+1}\rho N_{n-2k}\right\rangle, \tag{6.2}$$

for n even, or

$$\alpha_n=0, \tag{6.3}$$

otherwise. By convention, we take $N_0\equiv 1$, and d^0/dx^0 is the identity operator. The local functions N_n are 1-periodic, of null average, and must satisfy the recurrent set of local problems given by

$$(EN_{n+2},_{\xi}+EN_{n+1}),_{\xi}+EN_{n+1},_{\xi}+EN_n+\sum_{k=0}^{[n/2]}\alpha_{2k}\Big(-\frac{\hat{c}^2}{\omega_0^2}\Big)^{k+1}\rho N_{n-2k}=0. \tag{6.4}$$

$[n/2]$ is the largest integer less than or equal to $n/2$, and the 1-periodic solution N_{n+2} to this equation is restricted to the continuity conditions

$$\|N_{n+2}\|=0,\qquad \|EN_{n+2},_{\xi}+EN_{n+1}\|=0. \tag{6.5}$$

All functions $\hat{X}_n$ satisfy the equation

$$\hat{c}^2\hat{X}_{n},_{xx}+\omega_0^2\hat{X}_n=0. \tag{6.6}$$

Proof. (The reader may prefer to skip to the end of the proof, on page 964.) We proceed by induction. Suppose n_0 is even, (6.1)–(6.3) are valid for $n < n_0+2$, and (6.4)–(6.6) hold for every $n < n_0$. From the expression for the order $O(\varepsilon^{n_0})$, we have

$$(EX_{n_0+2},_{\xi}),_{\xi} + (EX_{n_0+1},_{\xi}),_{x} + (EX_{n_0+1},_{x}),_{\xi} + (EX_{n_0},_{x}),_{x} + \rho \sum_{k=0}^{n_0/2} \alpha_k X_{n_0-k} = 0. \tag{6.7}$$

A necessary and sufficient condition for the existence of a 1-periodic solution X_{n_0+2} for this equation is

$$\Big\langle (EX_{n_0+1},_{x}),_{\xi} + (EX_{n_0},_{x}),_{x} + \rho \sum_{k=0}^{n_0} \alpha_k X_{n_0-k} \Big\rangle = 0. \tag{6.8}$$

Once the expressions for X_{n_0} and X_{n_0+1} from (6.3) are substituted into this equality, we have

$$\sum_{m=0}^{n_0-1} \Big\langle EN_{n_0-m+1},_{\xi} + EN_{n_0-m} + \sum_{k=0}^{\left[\frac{n_0-m}{2}\right]} \alpha_{2k} \Big(-\frac{\hat{c}^2}{\omega_0^2}\Big)^{k+1} \rho N_{n_0-m-2k} \Big\rangle \frac{d^{n_0-m+2}\hat{X}_m}{dx^{n_0-m+2}} \\ + \langle EN_1,\xi + E\rangle \frac{d^2\hat{X}_{n_0}}{dx^2} + \omega_0^2 \langle\rho\rangle \hat{X}_{n_0} = 0. \tag{6.9}$$

As long as $(6.5)_1$ is valid for $n < n_0$, we have

$$\Big\langle EN_{n+1},_{\xi} + EN_n + \sum_{k=0}^{[n/2]} \alpha_{2k} \Big(-\frac{\hat{c}^2}{\omega_0^2}\Big)^{k+1} \rho N_{n-2k} \Big\rangle = 0, \tag{6.10}$$

for $n < n_0$. All terms in the sum from $m = 0$ to $m = n_0 - 1$ in (6.9) vanish except for the one corresponding to $m = 0$, and (6.9) is equivalent to

$$\langle EN_1,\xi + E\rangle \frac{d^2\hat{X}_{n_0}}{dx^2} + \omega_0^2\langle\rho\rangle \hat{X}_{n_0} = -\Big\langle EN_{n_0+1},_{\xi} + EN_{n_0} + \sum_{k=0}^{\frac{n_0}{2}} \alpha_{2k} \Big(-\frac{\hat{c}^2}{\omega_0^2}\Big)^{k+1} \rho N_{n_0-2k} \Big\rangle \frac{d^{n_0+2}\hat{X}_0}{dx^{n_0+2}}. \tag{6.11}$$

At the same time, we have

$$\frac{d^{n_0+2}\hat{X}_0}{dx^{n_0+2}} = \Big(-\frac{\omega_0^2}{\hat{c}^2}\Big)^{n_0/2+1} \hat{X}_0. \tag{6.12}$$

(6.11) is a second-order differential equation, with constant coefficients, in the unknown functions $\hat{X}_{n_0+1}$. To obtain a bounded solution, we must set the right-hand side equal to zero because $\hat{X}_0$ satisfies the corresponding homogeneous equation. Then,

$$\Big\langle EN_{n_0+1},_{\xi} + EN_{n_0} + \sum_{k=0}^{n_0/2} \alpha_{2k} \Big(-\frac{\hat{c}^2}{\omega_0^2}\Big)^{k+1} \rho N_{n_0-2k} \Big\rangle = 0, \tag{6.13}$$

and solving for α_{n_0}, we obtain precisely (6.2), for $n = n_0$, and for $\hat{X}_{n_0}$, we get (6.6). This yields

$$\hat{X}_{n_0} = -\frac{\hat{c}^2}{\omega_0^2} \frac{d^2\hat{X}_{n_0}}{dx^2}. \tag{6.14}$$

We can use this fact and (6.2) to obtain, from (6.7)

$$(EX_{n_0+2},_{\xi}),_{\xi} + \sum_{m=0}^{n_0+1} \Bigg[(EN_{n_0-m+1}),_{\xi} + EN_{n_0-m+1},_{\xi} + EN_{n_0-m} + \sum_{k=0}^{\left[\frac{n_0-m}{2}\right]} \alpha_{2k}\rho N_{n_0-m-2k} \Bigg] \frac{d^{n_0-m+2}\hat{X}_m}{dx^{n_0-m+2}} = 0.$$

Because this equation is linear, the general solution X_{n_0+2}, for (6.15), is the expression given in (6.1) for $n = n_0+2$, where the local functions $N_{n_0+2}(\xi)$ satisfy (6.4)–(6.5) when $n = n_0$.

Next, we consider the equation corresponding to the order $O(\varepsilon^{n_0+1})$:

$$(EX_{n_0+3},_{\xi}),_{\xi} + (EX_{n_0+2},_{\xi}),_{x} + (EX_{n_0+2},_{x}),_{\xi} + (EX_{n_0+1},_{x}),_{x} + \rho \sum_{k=0}^{n_0/2} \alpha_k X_{n_0-k+1} = 0. \tag{6.15}$$

If we average over a period, we have

$$\Big\langle (EX_{n_0+2},_x),_\xi + (EX_{n_0+1},_x),_x + \rho \sum_{k=0}^{n_0/2} \alpha_k X_{n_0-k+1} \Big\rangle = 0. \tag{6.16}$$

Analogously to the previous case, this is equivalent to

$$\langle EN_1, \xi + E\rangle \frac{d^2 \hat{X}_{n_0+1}}{dx^2} + \omega_0^2 \langle\rho\rangle \hat{X}_{n_0+1} = \\ -\Big\langle EN_{n_0+2},_\xi + EN_{n_0+1} + \sum_{k=0}^{\frac{n_0}{2}} \alpha_{2k} \Big(-\frac{\hat{c}^2}{\omega_0^2}\Big)^{k+1} \rho N_{n_0-2k+1} \Big\rangle \frac{d^{n_0+3} \hat{X}_0}{dx^{n_0+3}} - \alpha_{n_0+1} \langle\rho\rangle \hat{X}_0. \tag{6.17}$$

Here it can be proved that

$$\Big\langle EN_{n_0+2},_\xi + EN_{n_0+1} + \sum_{k=0}^{\frac{n_0}{2}} \alpha_{2k} \Big(-\frac{\hat{c}^2}{\omega_0^2}\Big)^{k+1} \rho N_{n_0-2k+1} \Big\rangle = 0. \tag{6.18}$$

Consider, for that purpose, the identity

$$\Big\langle \sum_{n=0}^{n_0+1} (-1)^n \left[N_{n_0-n+2}(EN_{n+1},_\xi + EN_n)\right],_\xi \Big\rangle = 0. \tag{6.19}$$

Applying here the rule for the derivative of the product, we arrive at

$$\Big\langle \sum_{n=0}^{n_0+1} (-1)^n N_{n_0-n+2},_\xi (EN_{n+1},_\xi + EN_n) \Big\rangle + \Big\langle \sum_{n=0}^{n_0+1} (-1)^n N_{n_0-n+2}(EN_{n+1},_\xi + EN_n),_\xi \Big\rangle = 0. \tag{6.20}$$

We can substitute the expressions for the local problems in (6.4) into the second term of the left-hand side of (6.20) to obtain, after some algebra,

$$\Big\langle \sum_{n=0}^{n_0+1} (-1)^n EN_{n_0-n+2},_\xi N_{n+1},_\xi \Big\rangle + \langle EN_{n_0+2},_\xi \rangle + \Big\langle \sum_{n=1}^{n_0+1} (-1)^n EN_{n_0-n+2},_\xi N_n \Big\rangle \\ + \Big\langle \sum_{n=1}^{n_0+1} (-1)^{n+1} EN_{n_0-n+2} N_n,_\xi \Big\rangle + \langle EN_{n_0+1}\rangle + \Big\langle \sum_{n=2}^{n_0+1} (-1)^{n+1} EN_{n_0-n+2} N_n,_\xi \Big\rangle \\ + \Big\langle \sum_{q=0}^{n_0/2} \alpha_{2q} \Big(-\frac{\hat{c}}{\omega_0^2}\Big)^{q+1} \rho N_{n_0-2q+1} \Big\rangle + \Big\langle \sum_{n=2}^{n_0+1} (-1)^{n+1} N_{n_0-n+2} \sum_{k=0}^{[n/2-1]} \alpha_{2k} \Big(-\frac{\hat{c}}{\omega_0^2}\Big)^{k+1} \rho N_{n-2k+1} \Big\rangle = 0. \tag{6.21}$$

The first term in this expression is equal to zero. To verify this, is sufficient to change the summation index to $n = n_0 - m + 1$. Recalling that n_0 is an even number, we have

$$\sum_{n=0}^{n_0+1} (-1)^n EN_{n_0-n+2},_\xi N_{n+1},_\xi = \sum_{n_0-m+1=0}^{n_0-m+1=n_0+1} (-1)^{n_0-m+1} EN_{m+1},_\xi N_{n_0-m+2},_\xi = -\sum_{m=0}^{n_0+1} (-1)^m EN_{m+1},_\xi N_{n_0-m+2},_\xi,$$

That is, the sum is equal to its negative and hence vanishes. A similar procedure can be used to verify that the third and fourth terms in left-hand side of (6.21) cancel, the sixth and eighth terms are zero as well, and (6.21) gives (6.18). Finally, (6.17) reduces to

$$\Big\langle EN_1, \xi + E \Big\rangle \frac{d^2 \hat{X}_{n_0+1}}{dx^2} + \omega_0^2 \langle\rho\rangle \hat{X}_{n_0+1} = -\alpha_{n_0+1} \langle\rho\rangle \hat{X}_0. \tag{6.22}$$

Here, we must take $\alpha_{n_0+1} = 0$ to obtain bounded solutions for the unknown $\hat{X}_{n_0+1}$ in (6.22), consistent with (6.3), which is the goal of this proof. This leaves, for $\hat{X}_{n_0+1}$, the equation given in (6.6) for $n = n_0 + 1$. This can be used to restate (6.15):

$$(EX_{n_0+3},_\xi),_\xi + \sum_{m=0}^{n_0+2}\Big[(EN_{n_0-m+2}),_\xi + EN_{n_0-m+2},_\xi + EN_{n_0-m+1} + \sum_{k=0}^{\left[\frac{n_0-m+1}{2}\right]} \alpha_{2k}\rho N_{n_0-m-2k+1}\Big]\frac{d^{n_0-m+3}\hat{X}_m}{dx^{n_0-m+3}} = 0. \quad (6.23)$$

Then, the general solution, X_{n_0+3}, to this equation is given by (6.1), for $n = n_0+3$, and by substitution it can be seen that the local function N_{n_0+3} must satisfy (6.4)–(6.5) for $n = n_0+1$ which is the goal of the proof.

Finally, the expressions for X_0 and X_1 become

$$X_0(x,\xi) = \hat{X}_0(x), \qquad X_1(x,\xi) = N_1(\xi)\frac{d\hat{X}_0}{dx}(x) + \hat{X}_1(x),$$

from Section 4. Combining this with the first local problem, (4.10)–(4.11), and the relation $\alpha_0 = \omega_0^2$ we conclude the proof for the lemma. □

The equality (6.1) gives the following asymptotic expansion for the function $X(x,\xi)$,

$$X(x,\xi) = \sum_{n\geq 0}\varepsilon^n \sum_{m=0}^{n} N_{n-m}(\xi)\frac{d^{n-m}\hat{X}_m}{dx^{n-m}}(x). \quad (6.24)$$

As a consequence, if we take

$$\hat{X}(x) = \sum_{n\geq 0}\varepsilon^n \hat{X}_n(x), \quad (6.25)$$

then (6.24) and the normalization condition $\langle N_n\rangle = 0$ yield

$$X(x,\xi) = \sum_{n\geq 0}\varepsilon^n N_n(\xi)\frac{d^n\hat{X}}{dx^n}, \qquad \langle X\rangle = \hat{X}, \qquad \hat{E}\hat{X}_{xx} + \omega_0^2\hat{\rho}\hat{X} = 0. \quad (6.26)$$

Because we now have an explicit expression for $\hat{X}$ and have solved the local problems, X can be successfully approximated. The condition that $\alpha_n = 0$ if n is odd implies that $\omega_n = 0$ if n is odd, and

$$\omega_\varepsilon = \sum_{n\geq 0}\varepsilon^{2n}\omega_{2n}. \quad (6.27)$$

Note that all ω_{2n} satisfy the recurrence condition in (6.3) and can therefore be found for arbitrary n once ω_0 is obtained. Then, for the function $T(t)$,

$$T_{tt} + (\omega_0 + \varepsilon^2\omega_2 + \varepsilon^4\omega_4 + \cdots)^2 T = 0. \quad (6.28)$$

This allows us to define $\hat{u}$. The boundary conditions allow calculation of the eigenfunctions $\hat{X}^{(n)}$ and the eigenfrequencies $\omega_0^{(n)}$ from (6.26). The formula (6.2) for α_n tell us that all quantities α_n and ω_n depend recurrently on ω_0. Once ω_0 and the local functions N_n are determined up to a certain order, ω_n can be obtained which define a suitable approximation for ω_ε. Substituting ω_ε into the equation for $T(t)$ in $(3.2)_2$ and defining initial conditions, the functions $T^{(n)}(t)$ can be calculated to give $\hat{u} = \sum_n^\infty \hat{X}^{(n)}T^{(n)}$. This result will be described in the next section, in which an analytic solution for $\hat{u}$ is obtained for select cases. It should be emphasized that the procedure followed so far is equivalent to the one introduce in the original problem, (2.1)–(2.2). This procedure depends on a rescaled temporal variable $\tau = (1+\varepsilon r_1 + \varepsilon^2 r_2 + \cdots)t$, following the method of strained coordinates or the method of Linsted-Poincaré [Sánchez-Huber and Sánchez-Palencia 1992].

Note that $r_n = \omega_n/\omega_0$ depend on ω_0. From (6.2) and (6.3), we can deduce a general expression for ω_ε that depends on ω_0. Considering the expressions given in (3.10) and using induction, one obtains

$$\omega_\varepsilon = \omega_0\Big(1 - \varepsilon^2\omega_0^2\frac{K_1}{\hat{c}^2} + \cdots + (-1)^n\varepsilon^{2n}\omega_0^{2n}\frac{K_n}{\hat{c}^{2n}} + \cdots\Big), \tag{6.29}$$

where K_n depend only on the local functions N_i, for $i = 1, 2, \ldots, 2n+1$, and the coefficients in the original equation.

7. Solution for the averaged model

We next consider wave propagation problems under various initial and boundary conditions. We present analytical solutions to the propagation equations and an explicit expression for the averaged model.

7A. ***Perturbation from the steady state.*** First, consider the one-dimensional problem of wave propagation given by (2.1), with boundary conditions given in $(2.2)_{1,2}$, with $\mu(t) = 0$ and $F(t) = 0$. The initial conditions are given by $(2.2)_{3,4}$ with $U(x) = f(x)$ and $V(x) = 0$. This corresponds to an initial disturbance from the equilibrium position.

From (6.26) and the homogeneous boundary conditions introduced, we have the following equation and boundary conditions for $\hat{X}$:

$$\hat{E}\hat{X},_{xx} + \omega_0^2\hat{\rho}\hat{X} = 0, \qquad \hat{X}(0) = 0, \quad \hat{X},_x(L) = 0. \tag{7.1}$$

This is a second-order linear differential equation with constant coefficients, and the solution can be explicitly determined as $\hat{X}^{(n)}(x) = \sin(\omega_0^{(n)}/\hat{c})x$, where

$$\omega_0^{(n)} = \frac{(2n-1)\pi\hat{c}}{2L}, \tag{7.2}$$

for $n = 1, 2, 3, \ldots,$ yielding

$$\hat{X}^{(n)}(x) = \sin\frac{(2n-1)\pi x}{2L}. \tag{7.3}$$

Next, the functions $T^{(n)}(t)$, corresponding to each value of $\omega_0^{(n)}$, are solved as follows:

$$T_{tt}^{(n)} + (\omega_\varepsilon^{(n)})^2 T^{(n)} = 0, \qquad T^{(n)}(0) = f^{(n)}, \qquad T_t^{(n)}(0) = 0. \tag{7.4}$$

Here the $\omega_\varepsilon^{(n)}$, with $n = 1, 2, 3, \ldots,$ can be found from $\omega_0^{(n)}$ using (6.29), and the $f^{(n)}$ are the coefficients of the Fourier expansion of the initial condition $f(x)$, relative to the orthogonal basis $\hat{X}^{(n)}$:

$$f^{(n)} = \frac{\int_0^L f(x)\hat{X}^{(n)}(x)\,dx}{\int_0^L [\hat{X}^{(n)}(x)]^2\,dx}. \tag{7.5}$$

The approximation for $\omega_\varepsilon^{(n)}$ is easily calculated to second order. Using the expression for ω_2 in (5.19) and the expansion for ω_ε in (6.29), we derive

$$\omega_{(2)}^{(n)} = \omega_0^{(n)}\Big(1 - \frac{(\varepsilon\omega_0^{(n)})^2}{\hat{c}^2}\frac{K}{2}\Big), \tag{7.6}$$

where $K = \kappa/\hat{E}$ is a material constant based on the parameter κ of (5.29). The $f^{(n)}$ are then given by

$$f^{(n)} = \frac{2}{L}\int_0^L f(x)\sin\frac{(2n-1)\pi x}{2L}\,dx, \tag{7.7}$$

for $n = 1, 2, 3, \dots$. The solution to (7.4) is given by $T_{(2)}^{(n)}(t) = f^{(n)}\cos\omega_{(2)}^{(n)}t$, where

$$T_{(2)}^{(n)}(t) = f^{(n)}\cos\frac{(2n-1)\pi\hat{c}}{2L}\Bigl(1-\Bigl[\frac{\varepsilon(2n-1)\pi}{2L}\Bigr]^2\frac{K}{2}\Bigr)t. \tag{7.8}$$

Having found the expressions for $\hat{X}^{(n)}$ and $T^{(n)}$, with $n = 1, 2, 3, \dots$, $\hat{u}$ is given analytically as

$$\hat{u}(x,t) = \sum_{n=1}^{\infty} f^{(n)}\sin\frac{(2n-1)\pi x}{2L}\cos\frac{(2n-1)\pi\hat{c}}{2L}\Bigl(1-\Bigl[\frac{\varepsilon(2n-1)\pi}{2L}\Bigr]^2\frac{K}{2}\Bigr)t. \tag{7.9}$$

Note that the same analytical solution is obtained using the classical asymptotic homogenization method, setting $\varepsilon = 0$ in (7.9). The qualitative differences between our approach and the classical asymptotic homogenization approach become evident upon inspection of (7.9). The difference between these two approaches arises when $\bigl(\varepsilon(2n-1)\pi/(2L)\bigr)^2(K/2)t$ is comparable to unity, that is, when t is of the order

$$\Bigl(\frac{2L}{\varepsilon(2n-1)\pi}\Bigr)^2\frac{2}{K}.$$

7B. ***One moving boundary.*** Next, we evaluate the problem of wave propagation in (2.1) setting $F(t) = U(x) = V(x) = 0$ in (2.2). These conditions correspond to a regime of movement on one boundary with a free load on the other border, starting from the equilibrium position. For homogeneous boundary conditions, we consider the auxiliary function $\upsilon(x,t) = u(x,t) - \mu(t)$, which satisfies

$$(E\upsilon_x)_x - \rho\upsilon_{tt} = \rho\mu''(t), \qquad \upsilon(0,t) = 0, \quad \upsilon_x(L,t) = 0, \quad \upsilon(x,0) = -\mu(0), \quad \upsilon_t(x,0) = -\mu'(0).$$

The total derivative of μ is denoted by a prime. Following the procedure in the previous section, we find that $\omega_0^{(n)}$ and $\hat{X}^{(n)}$ are as in (7.2) and (7.3). Then,

$$T_{tt}^{(n)} + (\omega_\varepsilon^{(n)})^2 T^{(n)} = -\hat{k}^{(n)}\mu''(t), \qquad T^{(n)}(0) = -\hat{k}^{(n)}\mu(0), \qquad T_t^{(n)}(0) = -\hat{k}^{(n)}\mu'(0), \tag{7.10}$$

where

$$\hat{k}^{(n)} = \frac{\int_0^L \hat{X}^{(n)}(x)\,dx}{\int_0^L[\hat{X}^{(n)}(x)]^2\,dx} = \frac{4}{(2n-1)\pi}. \tag{7.11}$$

This nonhomogeneous second-order equation with constant coefficients can be solved the theory of distributions; see [Schwartz 1966] for details. We obtain

$$T^{(n)}(t) = -\hat{k}^{(n)}\mu(t) + \omega_\varepsilon^{(n)}\hat{k}^{(n)}\int_0^t \mu(s)\sin\omega_\varepsilon^{(n)}(t-s)ds. \tag{7.12}$$

This gives an analytic expression for $\hat{\upsilon}(x,t) = \langle\upsilon\rangle$,

$$\hat{\upsilon}(x,t) = -\mu(t) + \sum_{n=1}^{\infty}\omega_\varepsilon^{(n)}\hat{k}^{(n)}\hat{X}^{(n)}(x)\int_0^t \mu(s)\sin\omega_\varepsilon^{(n)}(t-s)ds. \tag{7.13}$$

Considering approximations only up to the second power of ε, that is, $\omega_\varepsilon^{(n)} \approx \omega_{(2)}^{(n)}$, we get

$$\hat{u}(x,t) = \frac{2\hat{c}}{L}\sum_{n=1}^{\infty}\left[1-\left(\frac{\varepsilon(2n-1)\pi}{2L}\right)^2\frac{K}{2}\right]\sin\frac{(2n-1)\pi x}{2L} \times \int_0^t \mu(s)\sin\frac{(2n-1)\pi\hat{c}}{2L}\left[1-\left(\frac{\varepsilon(2n-1)\pi}{2L}\right)^2\frac{K}{2}\right](t-s)ds. \quad (7.14)$$

7C. *Modeling failures.* Next we consider the problem described in Section 7B with the added presence of a failure in the composite at $x = \theta L$, where $0 < \theta < 1$. The failure will be described mathematically as a dimensionless spring at $x = \theta L$ in the domain $[0, L]$. In addition to satisfying (2.1) and the boundary and initial conditions given in Section 7B, $F(t) = U(x) = V(x) = 0$ in (2.2), the displacement functions must satisfy

$$q\|u\|_{x=\theta L} = E\frac{du}{dx}\bigg|_{x=\theta L}, \qquad \left\|E\frac{du}{dx}\right\|_{x=\theta L} = 0, \quad (7.15)$$

where q is the elastic coefficient for the dimensionless spring. In the limit as q approaches infinity, the right-hand side of (7.15) approaches zero (division by q), which corresponds to the case when no failure is present. When q approaches zero, the left-hand side of the equality approaches zero, which corresponds to the case when two faces at $x = \theta L$ are under free stress conditions, that is, the material consists of two separate pieces. The methodology used for the standard case is applied again, with the same auxiliary function $v(x,t) = u(x,t) - \mu(t)$. Thus, we are looking for an expression for $\hat{X}$, satisfying (7.1) and the conditions

$$\|\hat{X}\|_{x=\theta L} = \frac{\hat{E}}{q}\frac{d\hat{X}}{dx}\bigg|_{x=\theta L}, \qquad \left\|\frac{d\hat{X}}{dx}\right\|_{x=\theta L} = 0. \quad (7.16)$$

In this case, the function $\hat{X}$ defined by

$$\hat{X}(x) = \begin{cases} A\sin(\omega_0 x/\hat{c}) & \text{for } 0 < x < \theta L, \\ B\cos(\omega_0(L-x)/\hat{c}) & \text{for } \theta L < x < L, \end{cases} \quad (7.17)$$

automatically satisfies the conditions (7.1). Substituting (7.17) into (7.16) and introducing the quantity $\phi = \omega_0 L/\hat{c}$ for convenience, we obtain a system of linear equations in A and B:

$$\begin{cases} B\cos\big((1-\theta)\varphi\big) - A\left(\sin(\theta\varphi) + \frac{\varphi\hat{E}}{qL}\cos(\theta\varphi)\right) = 0, \\ B\sin\big((1-\theta)\varphi\big) - A\cos(\theta\varphi) = 0. \end{cases} \quad (7.18)$$

The only solution is $A = 0$ and $B = 0$ unless the determinant vanishes, leading after simplification to the condition

$$\cos\varphi - \frac{\hat{E}}{qL}\varphi\cos\varphi\theta\sin\varphi(1-\theta) = 0. \quad (7.19)$$

Once the solutions $\varphi^{(n)}$ to this equation are found, we can take

$$A = \sin\varphi^{(n)}(1-\theta), \quad B = \cos\varphi^{(n)}\theta \quad \text{if } \varphi^{(n)} \neq \frac{2n-1}{2}\pi,$$
$$A = 1, \quad B = 1 \quad \text{if } \varphi^{(n)} = \frac{2n-1}{2}\pi,$$

from which we finally determine the functions $\hat{X}^{(n)}$:

$$\hat{X}^{(n)}(x) = \begin{cases} \sin\varphi^{(n)}(1-\theta)\sin\varphi^{(n)}x/L & \text{if } 0 < x/L < \theta, \\ \cos\varphi^{(n)}\theta\cos\varphi^{(n)}(1-x/L) & \text{if } \theta < x/L < 1. \end{cases} \tag{7.20}$$

This expression holds if $\varphi^{(n)}$ is not a half-integer multiple of π; otherwise $\hat{X}^{(n)}$ takes the form given in equation (7.3). The steps for finding $T^{(n)}$ are analogous to those in Section 7B. Since $T^{(n)}$ satisfies (7.10) we write it in the for (7.12). Again, for $\hat{k}^{(n)}$ we have

$$\hat{k}^{(n)} = \frac{\int_0^L \hat{X}^{(n)}(x)\,dx}{\int_0^L [\hat{X}^{(n)}(x)]^2\,dx}. \tag{7.21}$$

This finally gives

$$\hat{k}^{(n)} = \frac{2}{\varphi^{(n)}} \frac{\sin\varphi^{(n)}(1-\theta)}{\theta\sin^2\varphi^{(n)}(1-\theta) + (1-\theta)\cos^2\varphi^{(n)}\theta + (\hat{E}/qL)\sin^2\varphi^{(n)}(1-\theta)\cos^2\varphi^{(n)}\theta}, \tag{7.22}$$

except when $\varphi^{(n)}$ is a half-integer multiple of π, in which case $\hat{k}^{(n)}$ is as in (7.11). The expression for $\hat{v}$ is exactly the same we found in (7.13), except that $\hat{X}^{(n)}$ and $\hat{k}^{(n)}$ have the values in (7.20) and (7.22). For ω_ε we have

$$\omega_\varepsilon^{(n)} \approx \omega_{(2)}^{(n)} = \frac{\hat{c}\varphi^{(n)}}{L}\left[1 - \left(\frac{\varepsilon\varphi^{(n)}}{L}\right)^2 \frac{K}{2}\right]. \tag{7.23}$$

We have now arrived at the final analytic expression for $\hat{u}(x,t)$,

$$\hat{u}(x,t) = 2\frac{\hat{c}}{L}\sum_{n=1}^{\infty} r_n \left[1 - \left(\frac{\varepsilon\varphi^{(n)}}{L}\right)^2 \frac{K}{2}\right] \hat{X}_n(x) \int_0^t \mu(s)\sin\frac{\hat{c}\varphi^{(n)}}{L}\left[1 - \left(\frac{\varepsilon\varphi^{(n)}}{L}\right)^2 \frac{K}{2}\right](t-s)ds, \tag{7.24}$$

where the r_n, for $n = 1, 2, \ldots$, are given by

$$r_n = \begin{cases} \dfrac{\sin\varphi^{(n)}(1-\theta)}{\theta\sin^2\varphi^{(n)}(1-\theta) + (1-\theta)\cos^2\varphi^{(n)}\theta + \frac{\hat{E}}{qL}\sin^2\varphi^{(n)}(1-\theta)\cos^2\varphi^{(n)}\theta} & \text{if } \varphi^{(n)} \neq \dfrac{2n-1}{2}\pi, \\ 1, & \text{if } \varphi^{(n)} = \dfrac{2n-1}{2}\pi. \end{cases} \tag{7.25}$$

8. Numerical results

We performed several numerical computations in order to illustrate these results. For this purpose, we used the example of the composite described in [Chen and Fish 2001]. For all calculations, $L = 40\,\text{m}$ and $\varepsilon = 0.2\,\text{m}$. The periodic cell is composed of two homogeneous materials with properties $E_1 = 120\,\text{GPa}$, $E_2 = 6\,\text{GPa}$, $\rho_1 = 8000\,\text{kg/m}^3$, and $\rho_2 = 3000\,\text{kg/m}^3$, distributed on the periodic cell with a volume ratio of $\nu = 0.5$. This gives $\hat{c} = (\hat{E}/\hat{\rho})^{1/2} = 1441.5\,\text{m/s}$ and $K = 0.03849\,\text{m}^2$.

8A. ***Propagation of an initial disturbance.*** To verify the efficacy of the results obtained, we compared our formulation to the method proposed in [Chen and Fish 2001]. Consider the problem of an initial disturbance from the steady state with homogeneous boundary conditions at points $x = 0$ and $x = L$,

$$(E_\varepsilon u_x)_x - \rho_\varepsilon u_t t = 0, \quad u(0,t) = 0, \quad E_\varepsilon(L) u_x(L,t) = 0, \quad u(x,0) = f(x), \quad u_t(x,0) = 0.$$

The method proposed here yields an analytic solution for $\hat{u} = \langle u \rangle$ (to a second-order approximation), as described in Section 7A,

$$\hat{u}(x,t) = \sum_{n=0}^{\infty} f^{(n)} \sin \frac{(2n-1)\pi x}{2L} \cos \left[\frac{(2n-1)\pi \hat{c}}{2L} \left(1 - \left[\frac{(2n-1)\pi \varepsilon}{2L} \right]^2 \frac{K}{2} \right) t \right], \tag{8.1}$$

where

$$f^{(n)} = \frac{2}{L} \int_0^L f(x) \sin \frac{2n-1}{2} \frac{\pi x}{L}\, dx, \qquad K = \frac{\kappa}{\hat{E}} = \frac{1}{\hat{E}} \langle E N_{3,\xi} + E N_2 - \hat{c}^2 \rho N_2 \rangle.$$

To reproduce the conditions given in [Chen and Fish 2001], we worked with following class of initial disturbances:

$$f(x) = \frac{f_0}{\delta^8} \big(x - (x_0 - \delta) \big)^4 \big(x - (x_0 + \delta) \big)^4 \big(1 - H(x - x_0 - \delta) \big) \big(1 - H(x_0 - \delta - x) \big), \tag{8.2}$$

where $H(x)$ is the Heaviside step function, and f_0, δ, and x_0 are the magnitude, half-width, and center coordinate of the pulse. For calculations, we only considered pulses of magnitude $f_0 = 1$ centered at $x_0 = 20\,\mathrm{m}$ with different values for the half-width $\delta = 1.4\,\mathrm{m}$, $\delta = 0.8\,\mathrm{m}$, and $\delta = 0.6\,\mathrm{m}$, illustrated in Figure 2. These values were selected to evaluate the effect of the typical width of the disturbance and the size of the RVE.

The results of the comparison are shown in Figure 3. They agree well with those given in [Chen and Fish 2001] and corroborate the conclusion that asymptotic homogenization does not give good results if the characteristic size of the initial perturbation is comparable to the size of the periodic cell. According to the method of Chen and Fish, this discrepancy can be seen for long observation times. Our model demonstrates that this discrepancy should appear if the length traveled by the initial perturbation is

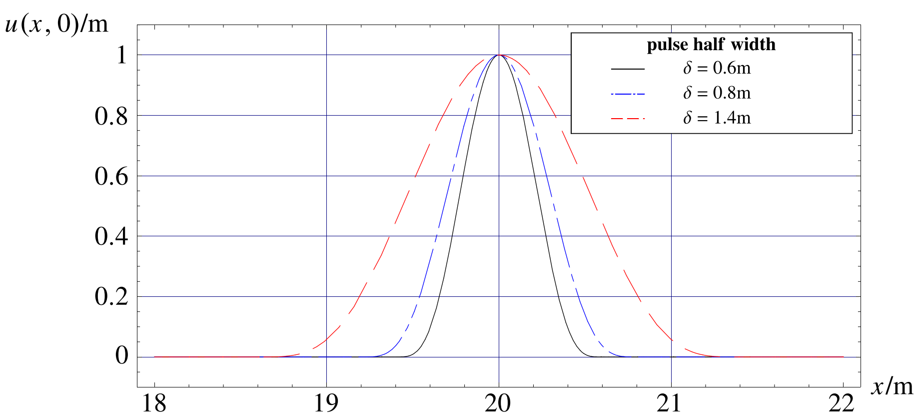

Figure 2. Shape and position of the initial pulses used for numerical illustration.

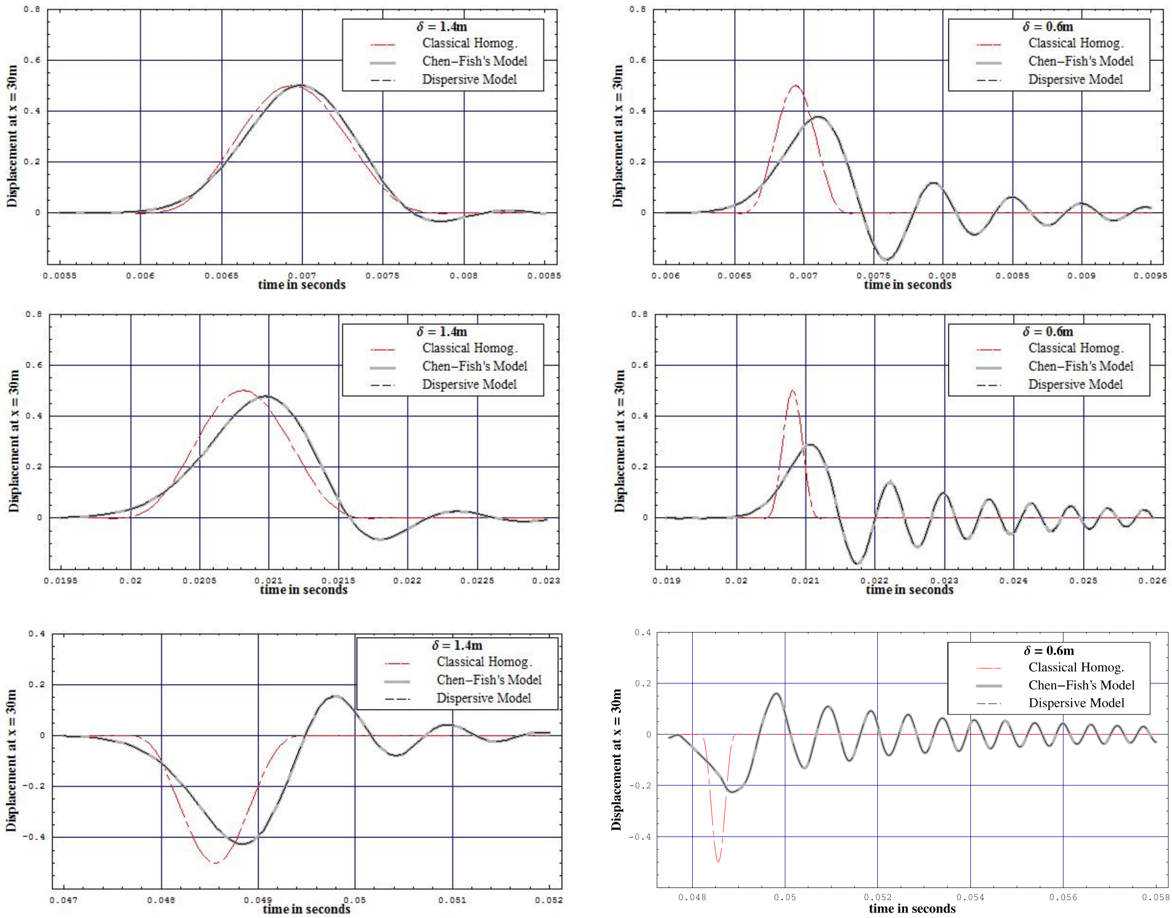

Figure 3. Value of the solution at $x = 30\,\mathrm{m}$ for $\delta = 1.4\,\mathrm{m}$ (left column) and for $\delta = 0.6\,\mathrm{m}$ (right column) as a function of time.

relatively large (of order $O(1/\varepsilon)$). In the left column of the figure, it is apparent that classical asymptotic homogenization can be applied provided that the distance traveled by the wave front is not too large. The same is not true when the width of the perturbation is 4 times the size of the periodic cell and the distance traveled is larger than 20 m, as shown in the right column.

8B. ***Traveling pulse.*** We next illustrate the results of the proposed method by describing the behavior of a traveling pulse under the dispersion effect induced by the heterogeneous periodic structure of the composite material. We consider the case of a pulse applied to one end, $x = 0$, with free load conditions on the other end, $x = L$. If the process starts from static equilibrium, the problem is described by

$$(E_\varepsilon u_x)_x - \rho_\varepsilon u_t t = 0, \qquad u(0,t) = \mu(t), \qquad E_\varepsilon(L) u_x(L,t) = 0, \qquad u(x,0) = 0, \qquad u_t(x,0) = 0. \quad (8.3)$$

The proposed method gives the following analytic solution (up to second order) for $\hat{u} = \langle u \rangle$, as seen in Section 7B:

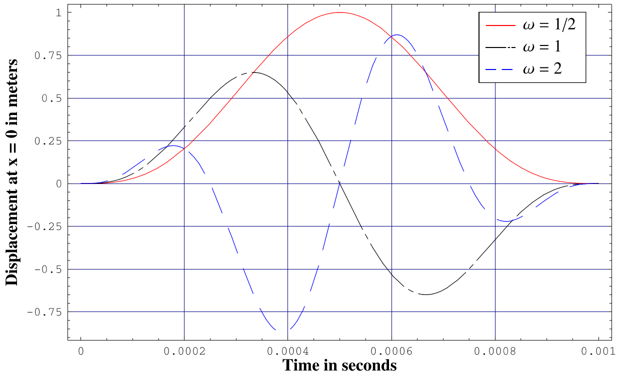

Figure 4. Different pulses used for numerical illustration.

$$\hat{u}(x,t) = \frac{2\hat{c}}{L}\sum_{n=1}^{\infty}\left[1-\left(\frac{\varepsilon(2n-1)\pi}{2L}\right)^2\frac{K}{2}\right]\sin\frac{(2n-1)\pi x}{2L} \times \int_0^t \mu(s)\sin\frac{(2n-1)\pi\hat{c}}{2L}\left[1-\left(\frac{\varepsilon(2n-1)\pi}{2L}\right)^2\frac{K}{2}\right](t-s)ds. \quad (8.4)$$

The following type of pulses will be considered:

$$\mu(t) = \frac{A}{2}\left(1-\cos\frac{2\pi t}{d}\right)\sin\frac{2\pi\omega t}{d}H\left(1-\frac{t^2}{d^2}\right), \quad (8.5)$$

where $H(x)$ is the Heaviside step function, and A, d, and ω are the magnitude, duration, and number of oscillations. The shapes of these pulses for $A = 1$ m, $d = 0.001$ s, and $\omega = \frac{1}{2}, 1, 2$ are illustrated in Figure 4. For numerical experimentation we considered only these values of A, d and ω.

The results for $\omega = \frac{1}{2}$ are shown in Figure 5, left. The pulse shapes as a function of t are indicated by a dashed line for the classical asymptotic homogenization and by a solid line for the dispersive model. A decrease of the pulse amplitude due to the dispersion effect and wiggles behind the wave front predicted by the dispersive model are apparent. For greater distances traveled by the pulse, the dispersive effect becomes more pronounced. At larger values of ω, the effect appears earlier, at smaller distances. The explanation for this is that for larger values of ω the characteristic size of pulse shape variation becomes

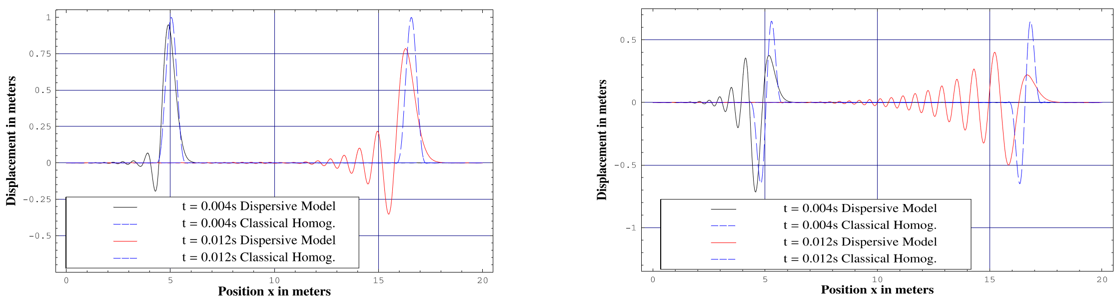

Figure 5. Prediction of the evolution of the pulse shape for $\omega = \frac{1}{2}$ (left) and $\omega = 1$ (right), and for times $t = 0.004$ s and $t = 0.012$ s, using standard asymptotic homogenization (dashed line) and the proposed dispersive model (solid line).

smaller and approaches the size of the periodic cell, producing more reflections and refractions at the interfaces separating component materials. This effect is also observed in Figure 5, right, for the evolution of the pulse shape when $\omega = 1$.

8C. ***Interaction with failures.*** Next we consider numerical descriptions of the behavior of a traveling pulse when a failure in the periodic structure composite is present. A failure in the material is modeled by a dimensionless spring with elasticity constant q. The boundary conditions and equation of motion describing wave propagation under these assumptions are given in (8.3). The failure is accounted for mathematically by including conditions (7.15) on the displacements functions $u(x, t)$ at a point θL, $0 < \theta < 1$, belonging to the interval $[0, L]$.

Using the proposed method as in Section 7C, the expression for $\hat{u} = \langle u \rangle$ becomes

$$\hat{u}(x,t) = 2\frac{\hat{c}}{L}\sum_{n=1}^{\infty} r_n \left[1 - \left(\frac{\varepsilon\varphi^{(n)}}{L}\right)^2 \frac{K}{2}\right] \hat{X}_n(x) \int_0^t \mu(s) \sin \frac{\hat{c}\varphi^{(n)}}{L}\left[1 - \left(\frac{\varepsilon\varphi^{(n)}}{L}\right)^2 \frac{K}{2}\right](t-s)\, ds, \quad (8.6)$$

where K and $\hat{c} = (\hat{E}/\hat{\rho})^{1/2}$ are material constants, the $\varphi^{(n)}$, for $n = 1, 2, \ldots$, are the roots of

$$\cos\varphi - \frac{\hat{E}}{qL}\varphi \cos\varphi\theta \sin\varphi(1-\theta) = 0, \quad (8.7)$$

the values of r_n are given in (7.25), and the functions $\hat{X}_n$, in this case, are piecewise defined as in (7.20) for $\varphi^{(n)} \neq (n - \frac{1}{2})\pi$, and

$$\hat{X}_n(x) = \sin\frac{(2n-1)\pi x}{2L} \quad \text{for } \varphi^{(n)} = (n - \tfrac{1}{2})\pi.$$

We will work with the same type of pulses as in (8.5), again with $A = 1$ m, $d = 0.001$ s and $\omega = \frac{1}{2}, 1$.

The n-th root $\varphi^{(n)}$ of (8.7) lies in the interval $(0, \pi/2]$ for $n = 1$ and in $\big((2n-3)\pi/2,\ (2n-1)\pi/2\big]$ for $n > 1$. A variant of Newton's method was used to find the roots numerically. Figure 6 shows the

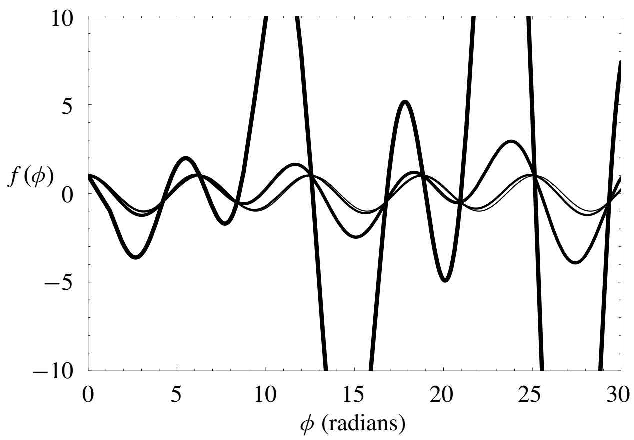

Figure 6. Plots of the function $f(\varphi) = \cos\varphi - (\hat{E}/qL)\varphi\cos\varphi\theta\sin\varphi(1-\theta)$ against φ over the interval $[0, 9\pi]$ for $\theta = 1/4$ and different choices of q. Thicker lines and wider variations correspond to low q; in order, $q = 2 \times 10^8\ \mathrm{N/m^2}$ ($\hat{E}/qL = 1.429$), $2 \times 10^9\ \mathrm{N/m^2}$ (0.143), $2 \times 10^{10}\ \mathrm{N/m^2}$ (0.014), and ∞ (0).

distribution of roots on the real axis for $f(\varphi) = \cos\varphi - (\hat{E}/qL)\varphi\cos\varphi\theta\sin\varphi(1-\theta)$, $\theta = \frac{1}{4}$, and several values of q. Solutions to (8.7) are given by the intersections of $f(\varphi)$ with the φ-axis.

Once the quantities $\varphi^{(n)}$ and the roots of (8.7) are found, (8.6) can be evaluated. presented in the following examples. The results are illustrated in Figures 7 and 8 for $\omega = \frac{1}{2}$ and $\omega = 1$, and $q = 2\cdot 10^8$ N/m^3, $q = 2\cdot 10^9$ N/m^3, and $q = 2\cdot 10^{10}$ N/m^3. We also set $\theta = \frac{1}{4}$; that is, the failure occurs at $x = 10$ m.

The pulse shapes for different values of t are shown for the classical asymptotic homogenization (dashed line) and for the dispersive model (solid line). In these figures, the evolution of the pulse shape after reaching the point of failure $x = 10$ m is illustrated for two values of the constant q (recall that low q means severe debonding). For $q = 2\cdot 10^8$ N/m^3, the reflection of the pulse at the point of failure is almost complete for both cases. In contrast, for the larger value $q = 2\cdot 10^{10}$ N/m^3, the pulse splits and two traveling perturbations emanate from the point of failure, instead of one. Also, in contrast to the classical

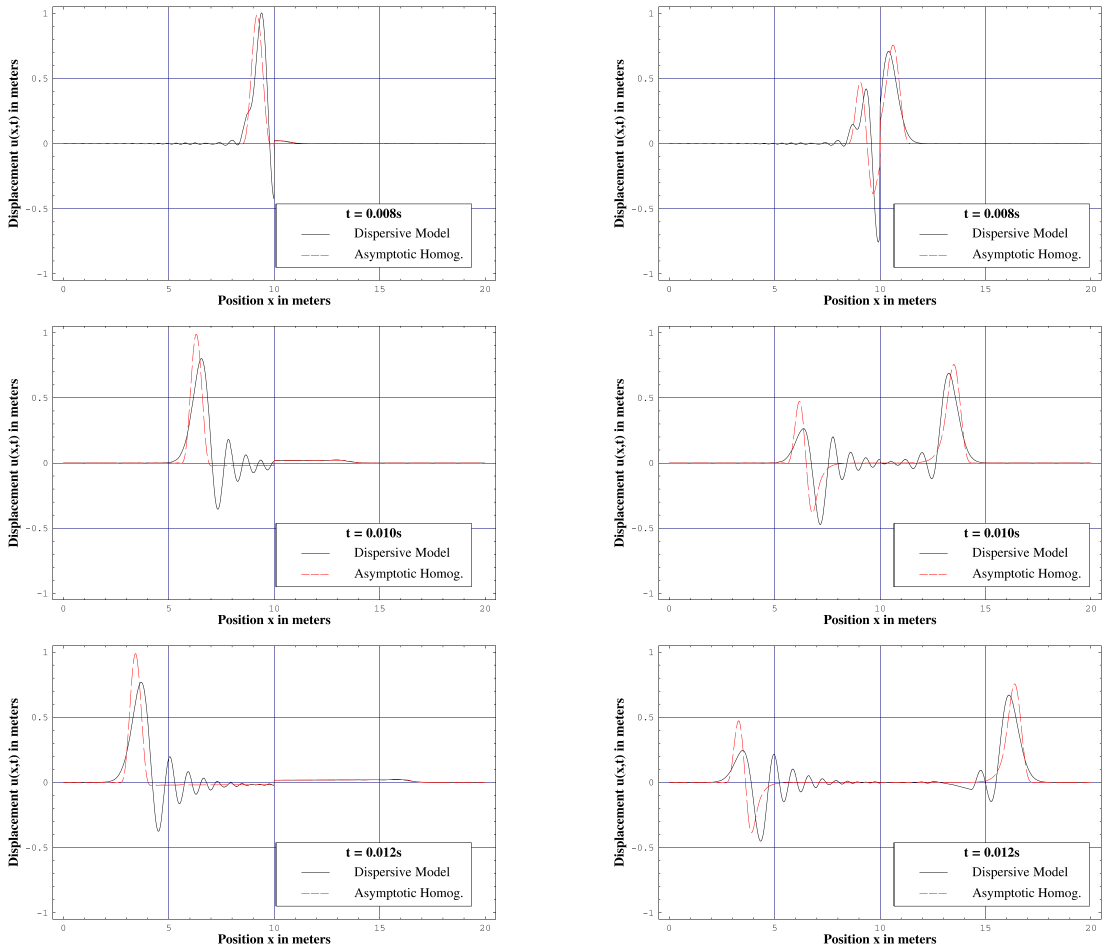

Figure 7. Prediction of the evolution of the pulse shape for $\omega = \frac{1}{2}$ and $q = 2\cdot 10^8$ N/m^3 (left) or $q = 2\cdot 10^{10}$ N/m^3 (right), at times $t = 0.008$ s, $t = 0.010$ s and $t = 0.012$ s after the pulse reaches the point of failure, $x = 10$ m, using standard asymptotic homogenization (dashed line) and the proposed dispersive model (solid line).

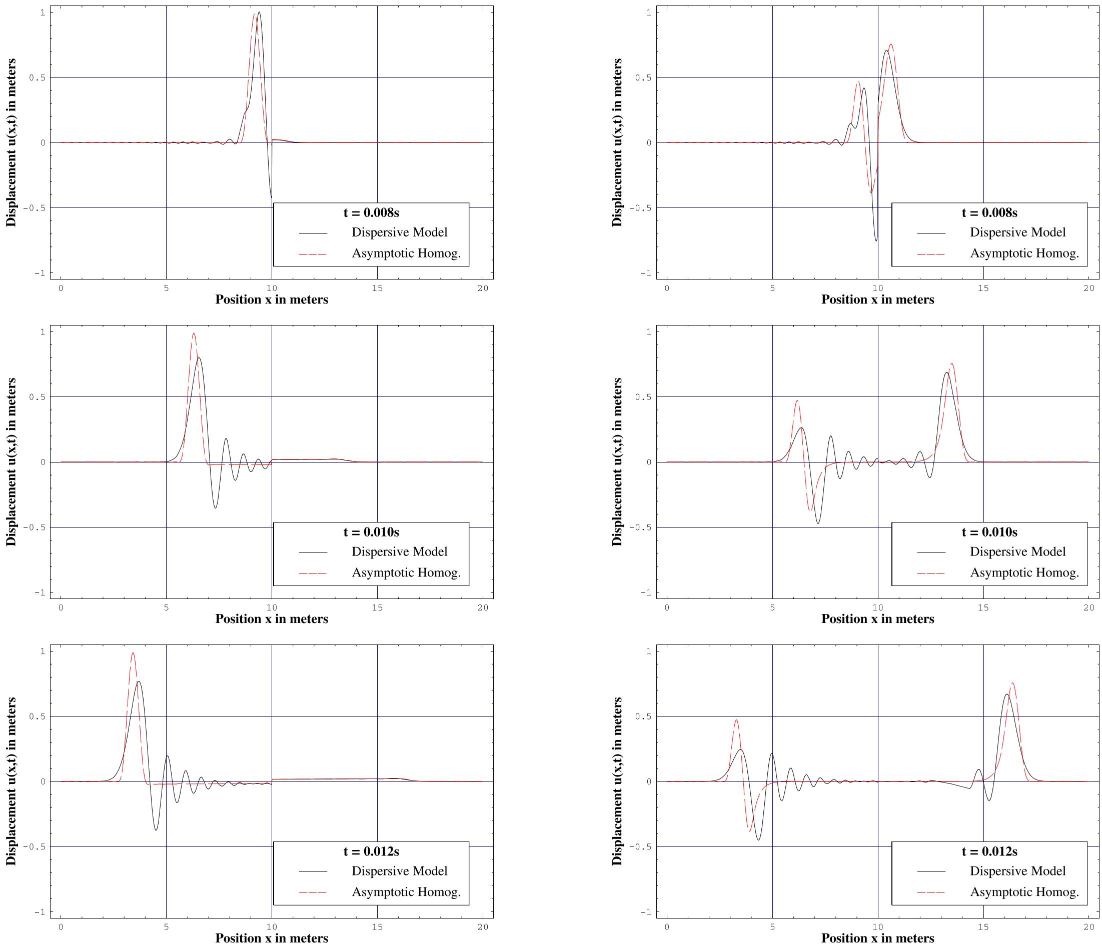

Figure 8. Prediction of the evolution of the pulse shape for $\omega = 1$ and $q = 2 \cdot 10^8$ N/m^3 (left) or $q = 2 \cdot 10^{10}$ N/m^3 (right), at times $t = 0.008$ s, $t = 0.010$ s and $t = 0.012$ s after the pulse reaches the point of failure, $x = 10$ m, using standard asymptotic homogenization (dashed line) and the proposed dispersive model (solid line).

asymptotic homogenization, the pulse shape described by the dispersive model becomes distorted. Thus the dynamical responses, translated by the reflected and transmitted perturbations after interaction with the failure, are different for each approach, and more noticeably so for larger ω.

Conclusions

In this work, an asymptotic model for describing wave propagation in periodic composites was proposed. In this approach, the heterogeneous nature of the composite introduces a perturbation in the principal frequencies relative to the homogenized problem. As a result, no new temporal scales need be considered. Instead, a regular aymptotic expansion for the eigenfrequencies is obtained from the condition of boundedness for the solution. The results are graphically illustrated for different types of boundary problems. The model is asymptotically valid for low frequency wave propagation.

This approach describes the dispersion effects in periodic composites, and we have discussed the differences between this model and the classical asymptotic homogenization. This model provides a starting point for the study of frequency perturbations in laminated composites when the angle of incidence is not perpendicular to the laminates, and the periodicity take place at small scales.

Appendix: Closed form expression of κ

We present the calculation of the constant $\kappa = \langle EN_{3,\xi} + EN_2 - \hat{c}^2 \rho N_2 \rangle$. First, (4.12) is solved to find

$$\frac{dN_1}{d\xi} = \frac{\hat{E}}{E} - 1, \tag{A.1}$$

and consequently,

$$N_1 = B - \langle B \rangle, \tag{A.2}$$

where B is given in (5.28). Considering the second local problem described in (4.21) and (4.12), we have $(EN_{2,\xi} + EN_1)_{,\xi} = \hat{E}(\rho/\hat{\rho} - 1)$. Because N_1 has a null average, it can be deduced that

$$EN_{2,\xi} + EN_1 = \hat{E}\Big(R - \Big\langle \frac{\hat{E}}{E} R \Big\rangle\Big), \tag{A.3}$$

and R is given in (5.28). Substituting the formula for N_1 gives

$$N_{2,\xi} = \frac{\hat{E}}{E}\Big(R - \Big\langle \frac{\hat{E}}{E} R \Big\rangle\Big) - B + \langle B \rangle. \tag{A.4}$$

Now, the equation (5.12) of the third local problem is multiplied by N_1 and averaged over the period to obtain

$$\big\langle N_1 (EN_{3,\xi} + EN_2)_{,\xi} \big\rangle = -\big\langle N_1 (EN_{2,\xi} + EN_1 - \hat{c}^2 \rho N_1) \big\rangle. \tag{A.5}$$

Integrating the left-hand side by parts and using (A.1) and the equality $\langle N_2 \rangle = 0$, we find that

$$\langle N_1 (EN_{3,\xi} + EN_2)_{,\xi} \rangle = -\langle N_{1,\xi} (EN_{3,\xi} + EN_2) \rangle = -\Big\langle \Big(\frac{\hat{E}}{E} - 1\Big)(EN_{3,\xi} + EN_2) \Big\rangle = \langle EN_{3,\xi} + EN_2 \rangle.$$

On the other hand, $-\langle \rho \hat{c}^2 N_2 \rangle = -\hat{E}\Big\langle \frac{\rho}{\hat{\rho}} N_2 \Big\rangle = \hat{E}\Big\langle R \frac{dN_2}{d\xi} \Big\rangle$. Together with (A.5), this leads to

$$\langle EN_{3,\xi} + EN_2 - \hat{c}^2 \rho N_2 \rangle = \Big\langle \hat{E} R \frac{dN_2}{d\xi} - N_1 (EN_{2,\xi} + EN_1 - \hat{c}^2 \rho N_1) \Big\rangle. \tag{A.6}$$

Substituting equations (A.2) and (A.3), we obtain (5.29).

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