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DYNAMICS OF A LIGHT HOOP WITH AN ATTACHED HEAVY DISK: INSIDE AN INTERACTION PULSE

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An analysis of motion of a light hoop with a heavy circular disk attached to the inner side of its rim over a rough horizontal surface is presented. When the radius of the disk is small relative to the radius of the hoop, the motion is characterized by the sequence of sharp interaction pulses between the hoop and the horizontal surface, during which a complex transition between stick, slip, and reversed slip takes place. This is accompanied by the corresponding changes in the frictional force, which include abrupt reversal of its direction, and rapid increase and discontinuous changes of the normal component of the reactive force. The intensity of the interaction rapidly diminishes with an increase of the radius of the disk or the mass of the hoop. The effects of the coefficient of friction on the dynamic interactions, stick and slip transitions, and the energy dissipation are discussed.

1. Introduction

This paper is an analysis of motion of a light hoop with a heavy circular disk attached to the inner side of its rim released to move on a rough horizontal surface. This seemingly simple problem in dynamics is characterized by a surprising complexity of the involved kinematics and kinetics of motion, which reveals a variety of information of a possible interest for the mechanics of frictional impact. When the radius of the disk is small relative to the radius of a light hoop, the motion is characterized by the sequence of sharp interaction pulses between the hoop and the horizontal surface during which a complex transition between stick, slip, and reversed slip takes place. The sharp interaction pulses arise in response to large angular velocity and angular acceleration of the hoop built in a short interval of time during which the disk passes above the horizontal surface. The motion is also characterized by the corresponding sudden changes in the velocity components of the center of the disk. The intensity of the sharp interactions between the hoop and the horizontal surface as well as the overall complexity of motion (transitions from one type of motion to another) diminish with the increasing size of the disk or the mass of the hoop. For sufficiently large disks, the rolling without transition to sliding prevails throughout the motion. The analysis demonstrates that for light hoops and very small disks, the time integral of the tangential component of the interaction pulse is in general different from the product of the coefficient of kinetic friction and the time integral of the normal component of the interaction pulse (referred to in the mechanics of frictional impact as the Whittaker [1961] or Kane and Levinson [1985] assumption for the impulse components).

The presented analysis resulted from the study of a puzzling dynamics problem of motion of a massless hoop with an attached point mass, which is mentioned in the books by Littlewood [1986] and Kilmister

Keywords: friction, interaction pulse, light hoop, rolling, sliding, stick-slip transition.
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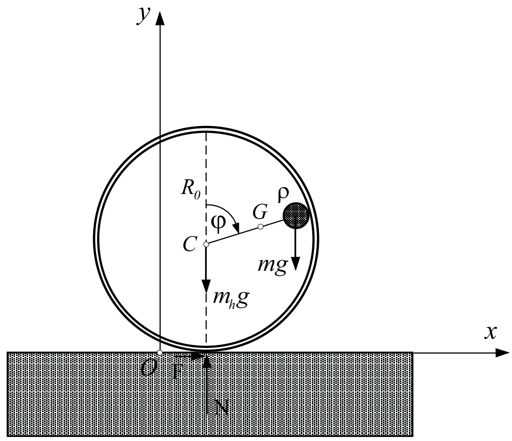

Figure 1. Rolling/sliding motion of a thin light hoop of mass m_{h} with the inner and outer radii R_0 and $R = rR_0$, respectively. The attached disk has radius ρ and mass m. The mass center of the system is at the point G such that $\overline{CG} = \vartheta R_0$, where $\vartheta = (1-k)m/M$, $k = \rho/R_0$, and $M = m + m_{\mathrm{h}}$ is the total mass of the system.

and Reeve [1966], and which has been a subject of several recent publications addressing the question of whether the hoop lifts off the horizontal surface during its motion [Tokieda 1997; Butler 1999; Pritchett 1999; Theron 2000; 2002; Theron and du Plessis 2001; Liu and Yun 2004]. This hopping of the hoop dilemma was associated with the uncertainties in the description of motion of a hypothetical dynamic system which engages in rotation but does not possess any rotational inertia, which can become dynamically indeterminate. In the process of elaborating on this dynamic indeterminacy, the analysis unveiled another interesting aspect of the problem, namely the description of sharp interaction pulses between the hoop and the horizontal surface as the disk passes just above the horizontal surface. This resembles some features of the interactions occurring during the frictional impact of colliding bodies, which can be treated only by additional considerations or assumptions about the deformation in the contact region [Johnson 1985; Brogliato 1999; Stronge 2000]. The analysis presented in this paper may thus help to better understand the impulsive interactions and the stick/slip transitions in the latter problems.

2. Motion of a light hoop with an attached circular disk

The dynamics of motion of a light hoop with an attached disk was studied in detail in [Pritchett 1999; Theron 2000; Theron and du Plessis 2001], with the goal of addressing the question of possible hopping of the hoop and the dynamic indeterminacy in some stages of motion of an idealized system of a massless hoop with an attached point mass. In this and the subsequent section we extend this analysis to describe the interaction pulses and the stick/slip transitions throughout the motion of the hoop, for various mass ratios and the geometric and friction parameters. We adopt the model in which the mass m is uniformly distributed within a circular disk of radius ρ, which is attached to the inner side of a thin hoop with inner radius R_0, outer radius R, and a uniformly distributed mass m_{h} (Figure 1).

2.1. ***Rolling motion.*** If the coefficient of friction between the hoop and the horizontal surface is sufficiently high, as specified in the sequel by (9), the hoop begins its motion by rolling, such that the coordinates of the mass center of the system are

$$x = x_C + \vartheta R_0 \sin\varphi, \qquad y = R + \vartheta R_0 \cos\varphi. \tag{1}$$

Here, $x_C = R(\varphi - \varphi_0)$ is the horizontal coordinate of the center of the hoop, $\vartheta = (1-k)m/M$, $k = \rho/R_0$, and $M = m + m_\mathrm{h}$ is the mass of the moving system. With this notation, the distance between the center of the hoop and the mass center of the system is $\overline{CG} = \vartheta R_0$. The corresponding equations of motion (ignoring the coefficient of rolling resistance and the associated rolling resistance moment) are

$$M\ddot{x} = F, \qquad M\ddot{y} = N - Mg, \qquad J\ddot{\varphi} = N\vartheta R_0 \sin\varphi - F(R + \vartheta R_0 \cos\varphi). \tag{2}$$

The moment of inertia of the hoop/disk system for its mass center G is

$$J = \eta^2 M R_0^2, \qquad \eta^2 = \frac{m_\mathrm{h}}{2M}(1+r^2) + \frac{m_\mathrm{h}}{m}\vartheta^2 + \frac{m}{2M}k^2, \tag{3}$$

where $r = R/R_0$ (so that ηR_0 is the radius of gyration of the hoop/disk system). When (1) is substituted into the first two equations of (2), the normal and friction force can be expressed as

$$N = M\big[g - \vartheta R_0(\cos\varphi\dot{\varphi}^2 + \sin\varphi\ddot{\varphi})\big], \qquad F = M\big[R\ddot{\varphi} - \vartheta R_0(\sin\varphi\dot{\varphi}^2 - \cos\varphi\ddot{\varphi})\big]. \tag{4}$$

The differential equation for φ then follows from the third equation in (2) as

$$[\varsigma + 2(1+\cos\varphi)]\ddot{\varphi} - \sin\varphi\dot{\varphi}^2 = \frac{g}{R}\sin\varphi, \tag{5}$$

with the initial conditions $\varphi(0) = \varphi_0$ and $\dot{\varphi}(0) = 0$ at time $t = 0$. The nondimensional parameter ς is defined by

$$\varsigma = \frac{3k^2 + 4k(r-1) + 2(r-1)^2 + (1+3r^2)m_\mathrm{h}/m}{2r(1-k)}. \tag{6}$$

Since $2\ddot{\varphi} = \mathrm{d}(\dot{\varphi}^2)/\mathrm{d}\varphi$, (5) can be solved for $\dot{\varphi}^2$ to obtain

$$\dot{\varphi}^2 = \frac{2g}{R}\,\frac{\cos\varphi_0 - \cos\varphi}{\varsigma + 2(1+\cos\varphi)}. \tag{7}$$

Alternatively, the expression for the square of the angular velocity follows directly from the energy conservation during the rolling motion, which requires that $2Mg(y_0 - y) = M(\dot{x}^2 + \dot{y}^2) + J\dot{\varphi}^2$. Upon the time-differentiation of (7), or by substituting (7) back into (5), the angular acceleration is found to be

$$\ddot{\varphi} = \frac{g\sin\varphi}{R}\,\frac{\varsigma + 2(1+\cos\varphi_0)}{\big[\varsigma + 2(1+\cos\varphi)\big]^2}. \tag{8}$$

Having determined $\dot{\varphi}^2$ and $\ddot{\varphi}$ as the functions of φ, the normal and friction forces can be calculated from (4). The rolling motion proceeds as long as $|F| \le \mu_\mathrm{s} N$ and $N > 0$. The minimum coefficient of static friction needed for rolling to begin at the inception of motion is

$$\mu_\mathrm{s}^{\min} = \frac{F(\varphi_0)}{N(\varphi_0)} = \frac{[r + \vartheta\cos\varphi_0]\sin\varphi_0}{r\big[\varsigma + 2(1+\cos\varphi_0)\big] - \vartheta\sin^2\varphi_0}. \tag{9}$$

The larger the disk radius, the smaller the minimum coefficient of static friction required for rolling.

If $k=0$, $m_{\mathrm{h}}=0$, and $r=1$ then $\mu_{\mathrm{s}}^{\min}=\tan(\varphi_0/2)$. For $\mu_{\mathrm{s}}^{\min}<\tan(\varphi_0/2)$, a massless hoop with an attached point mass released from rest with a point mass at an angle φ_0 skims the surface without interaction with it ($N=F=0$, $\ddot{x}=0$, $\ddot{y}=-g$) until the impact of the point mass with the surface at $\varphi=\pi$ [Theron and du Plessis 2001]. For $\mu_{\mathrm{s}}^{\min}>\tan(\varphi_0/2)$, the rolling stage of the motion of a massless hoop with an attached point mass is specified by (5) with $\varsigma=0$, which has a closed form solution $\sin(\varphi/2)=\sin(\varphi_0/2)\cosh\big[(g/4R)^{1/2}t\big]$, with the corresponding normal force $N=mg\big[2\cos^2(\varphi/2)-\cos^2(\varphi_0/2)\big]$.

2.2. ***Combined rolling/sliding motion.*** If φ_1 is the angle at which $|F|=\mu_{\mathrm{s}}N$, the hoop will engage in a combined rolling/sliding motion, which continues in alternating cycles of decreasing amplitude, until the hoop comes to rest with the disk at its bottom. The analysis has to be performed sequentially following each stage of motion by checking the direction of the relative slip between the hoop and the horizontal surface, which is forward or backward depending on the sign of $(\dot{x}_C-R\dot{\varphi})$. During a forward slip ($\dot{x}_C-R\dot{\varphi}>0$), the velocity of the contact point of the hoop with the horizontal surface is directed toward the right, and the friction force toward the left. The opposite is true in the case of backward slip ($\dot{x}_C-R\dot{\varphi}<0$). Theron [2000] refers to a forward slip as skidding and to a backward slip as spinning. This terminology will be used for the rest of the paper. Each time the relative slip between the hoop and the horizontal surface changes from spinning to skidding, or vice versa, the force of friction reverses its direction. There could also be intervals of resumed rolling and depending on the ratios ρ/R_0 and m/M the hoop may settle in an oscillating rolling motion around $\varphi=\pi$.

Assuming that $N\geq 0$ throughout the motion, so that there is no hopping of the hoop ($y_C=R$), which is verified and confirmed by calculations a posteriori, the coordinates of the center of the disk during the rolling/sliding stage of motion are given by (1) with $x_C\neq R(\varphi-\varphi_0)$. Denoting by $\mu<\mu_{\mathrm{s}}$ the coefficient of kinetic friction, the frictional force is

$$F=-\hat{\mu}N,\qquad \hat{\mu}=\mu\,\mathrm{sign}(\dot{x}_C-R\dot{\varphi}),\tag{10}$$

and the equations of motion become

$$\begin{aligned}\ddot{x}_C+\vartheta R_0(-\sin\varphi\dot{\varphi}^2+\cos\varphi\ddot{\varphi})&=-\hat{\mu}(N/M),\\ -\vartheta R_0(\cos\varphi\dot{\varphi}^2+\sin\varphi\ddot{\varphi})&=(N/M)-g,\\ \eta^2R_0\ddot{\varphi}&=[\vartheta\sin\varphi+\hat{\mu}(r+\vartheta\cos\varphi)](N/M).\end{aligned}\tag{11}$$

When the second equation in (11) is substituted into the third equation to eliminate N/M, the differential equation for $\dot{\varphi}^2$ is found to be

$$\tfrac{1}{2}\big[\sin\varphi+\alpha z(\varphi)\big]\frac{\mathrm{d}\dot{\varphi}^2}{\mathrm{d}\varphi}+\cos\varphi\dot{\varphi}^2=\frac{g}{\vartheta R_0},\tag{12}$$

where

$$\alpha=\frac{\eta^2}{\vartheta}=\frac{m_{\mathrm{h}}}{2m}\frac{r^2+1}{1-k}+\frac{m_{\mathrm{h}}}{m+m_{\mathrm{h}}}(1-k)+\frac{k^2}{2(1-k)},\tag{13}$$

and

$$z(\varphi)=\big[\vartheta(\sin\varphi+\hat{\mu}\cos\varphi)+r\hat{\mu}\big]^{-1}.\tag{14}$$

The associated initial condition is specified by the continuity of $\dot{\varphi}^2$ at the transition from rolling to rolling/sliding at $\varphi=\varphi_1$.

Equation (12) does not have an explicit closed form solution but can be solved numerically. Once $\dot{\varphi}^2$ is determined, the angular acceleration follows from

$$\ddot{\varphi} = \left[\sin\varphi + \alpha z(\varphi)\right]^{-1}\left(\frac{g}{\vartheta R_0} - \cos\varphi\dot{\varphi}^2\right). \tag{15}$$

The force N is then calculated from the second equation in (11). The velocity of the center of the disk $\dot{x}_C$ is determined by integrating the first equation in (11). Since $\ddot{x}_C = \dot{\varphi}\mathrm{d}\dot{x}_C/\mathrm{d}\varphi$, this becomes

$$\frac{\mathrm{d}\dot{x}_C}{\mathrm{d}\varphi} = \left\{-g\hat{\mu} + \vartheta R_0\left[(\sin\varphi + \hat{\mu}\cos\varphi)\dot{\varphi}^2 - (\cos\varphi - \hat{\mu}\sin\varphi)\ddot{\varphi}\right]\right\}/\dot{\varphi}, \tag{16}$$

with the initial condition $\dot{x}_C(\varphi_1) = R\dot{\varphi}(\varphi_1)$. The integration of the differential equations (12) and (16) proceeds in parallel to check the nature of the relative slip determined by the sign$(\dot{x}_C - R\dot{\varphi})$. Care must also be taken to calculate $\dot{\varphi}$ from the square root of $\dot{\varphi}^2$ with the appropriate sign before it is used in (16) depending whether the rotation of the hoop is clockwise or counterclockwise. Furthermore, as the small disk passes above the horizontal surface, the angular acceleration $\ddot{\varphi}$, and thus $\mathrm{d}\dot{\varphi}^2/\mathrm{d}\varphi$, experiences a rapid (nearly discontinuous) change, while $\dot{\varphi}^2$ attains its maximum value. Hence there is a change of the slip direction near $\varphi = \pi$ (unless rolling takes the hoop through the state of the maximum angular velocity).

2.3. ***Resumption of rolling.*** If the rolling resumes after the rolling/sliding phase of the motion then $|F| < \mu_s N$, $N > 0$, and $R\dot{\varphi} = \dot{x}_C$. Denote by φ_3 the angle at the end of the rolling/sliding phase and let $\dot{\varphi}_3$ and $\dot{x}_{C_3} = R\dot{\varphi}_3$ denote the corresponding angular velocity of the hoop and the velocity of its center. During the subsequent pure rolling motion, one has

$$x = x_{C_3} + R(\varphi - \varphi_3) + \vartheta R_0\sin\varphi, \qquad y = R + \vartheta R_0\cos\varphi, \tag{17}$$

and it readily follows that

$$\dot{\varphi}^2 = \frac{2g}{R}\frac{\cos\varphi_3 - \cos\varphi + c_3}{\varsigma + 2(1+\cos\varphi)}, \qquad c_3 = \frac{R}{2g}\left[\varsigma + 2(1+\cos\varphi_3)\right]\dot{\varphi}_3^2. \tag{18}$$

The angular acceleration is

$$\ddot{\varphi} = \frac{\sin\varphi}{\varsigma + 2(1+\cos\varphi)}\left(\frac{g}{R} + \dot{\varphi}^2\right). \tag{19}$$

The corresponding normal and friction force follow from (4). One must verify a posteriori that $|F| < \mu_s N$ and $N > 0$. If the hoop came momentarily to rest at $\varphi = \varphi_3$ so that $\dot{\varphi}_3 = 0$ and $\dot{x}_C(\varphi_3) = 0$ then $c_3 = 0$ in (18). If $\dot{\varphi}_3 = 0$ but $\dot{x}_C(\varphi_3) \neq 0$ then the hoop slips back rather than role forward. In this case, the equations from Section 2.2 hold with the negative values of $\dot{\varphi}$ in (16).

2.4. ***The effects of the mass of the hoop.*** If γ_h is a uniform mass density of the hoop, its mass is $m_h = (r^2 - 1)\pi R_0^2\gamma_h$ and its centroidal moment of inertia $J_h = 0.5(1 + r^2)m_h R_0^2$. If γ_m is the mass density of the disk, its mass is $m = k^2\pi R_0^2\gamma_m$ with its centroidal moment of inertia $J_m = 0.5k^2 m R_0^2$. Thus, the mass and inertia ratios are

$$\frac{m_h}{m} = \frac{r^2 - 1}{k^2}\frac{\gamma_h}{\gamma_m}, \qquad \frac{J_h}{J_m} = \frac{r^2 + 1}{k^2}\frac{m_h}{m}. \tag{20}$$

The ratio of the contributions from the hoop and the disk to the moments of inertia with respect to the mass center G of the hoop/disk system are

$$\frac{J_{\mathrm{h}}^{G}}{J_{\mathrm{m}}^{G}}=\frac{(r^2+1)(1+m_{\mathrm{h}}/m)^2+2(r-1)^2}{k^2(1+m_{\mathrm{h}}/m)^2+2(m_{\mathrm{h}}/m)^2(r-1)^2}\,\frac{m_{\mathrm{h}}}{m}. \tag{21}$$

For a very thin hoop, the two inertia ratios are nearly the same, that is,

$$\frac{J_{\mathrm{h}}}{J_{\mathrm{m}}}\approx\frac{J_{\mathrm{h}}^{G}}{J_{\mathrm{m}}^{G}}\approx\frac{2}{k^2}\frac{m_{\mathrm{h}}}{m},\qquad \frac{m_{\mathrm{h}}}{m}\approx\frac{2(r-1)}{k^2}\frac{\gamma_{\mathrm{h}}}{\gamma_{\mathrm{m}}}. \tag{22}$$

For a very low density of the hoop material and a high density of the disk material (such as a hoop of some foamed plastic or a porous cellular material and a disk made of tungsten), the ratio $\gamma_{\mathrm{h}}/\gamma_{\mathrm{m}}$ would be of the order of 10^{-2}. Then, if $r=1.02$ and $k=0.1$, the mass ratio is $m_{\mathrm{h}}/m\approx 0.04$ while the moment of inertia ratios are $J_{\mathrm{h}}/J_{\mathrm{m}}\approx J_{\mathrm{h}}^{G}/J_{\mathrm{m}}^{G}\approx 8.24$. This illustrates the importance of the rotational inertia of the hoop even for very light hoops whenever the attached disk is sufficiently small. In order for $J_{\mathrm{h}}/J_{\mathrm{m}}$ to be equal to $1/10$, which would imply that the rotational inertia of the hoop can be ignored, the thickness of the hoop would have to be exceedingly small: $r=1.00025$ in the case where $k=0.1$ and $\gamma_{\mathrm{h}}/\gamma_{\mathrm{m}}\approx 10^{-2}$. Consequently, in Section 3 we present the results of calculations for both vanishing and nonvanishing mass ratios m_{h}/m.

3. Numerical results

In Section 3.1 we present numerical results for infinitesimally thin massless hoops ($r=1, m_{\mathrm{h}}=0$) and a small disk of mass m and radius $\rho=0.1R$. The static and kinetic friction coefficients used are $\mu_{\mathrm{s}}=0.35$ and $\mu=0.3$. In Section 3.2 we discuss the results for of a massless hoop with larger disks attached. The effects of the nonvanishing mass of the hoop on the resulting motion of the system and the interaction pulses are considered in Section 3.3, and the effects of the coefficient of friction are examined in Section 3.4. In all cases, the initial configuration of the hoop/disk system was specified to be $\varphi_0=10°$. A similar analysis can be performed for other inclination angles.

3.1. *Massless hoop and a disk with k = 0.1.* Figure 2 shows the time variation of the horizontal velocity component of the contact point of an infinitesimally thin massless hoop during its first three rotation cycles, when the radius of the attached disk is $\rho=0.1R$. The initial horizontal plateau corresponds to rolling portion of the motion, from φ_0 to $\varphi_1=0.71402$ rad. This is followed by spinning from φ_1 to $\varphi_2=3.18085$ rad, rolling from φ_2 to $\varphi_3=3.20455$ rad, and skidding from φ_3 to $\varphi_4=4.54542$ rad, which is the end of the first clockwise rotation of the hoop. The high precision is needed in calculations to accurately identify the small intervals of slip or stick near $\varphi=\pi$. The variation of the contact point velocity during the first interaction pulse (at time $t\cong 5.74\sqrt{R/g}$) is shown in Figure 2, right. The short horizontal plateau corresponds to brief rolling of the hoop. The skidding continues by a counterclockwise rotation from φ_4 to $\varphi_5=2.49277$ rad, which is the end of the first cycle of rotation of the hoop (at $t\cong 8.84\sqrt{R/g}$). The velocity of the contact point is positive during the entire second pulse. The other cycles follow and are dominated by skidding with brief intermissions by spinning and rolling during the pulses under clockwise rotation of the hoop (odd-numbered pulses). The time variations of the normal and friction force during the first three rotation cycles are shown in the left halves of Figures 3 and 4.

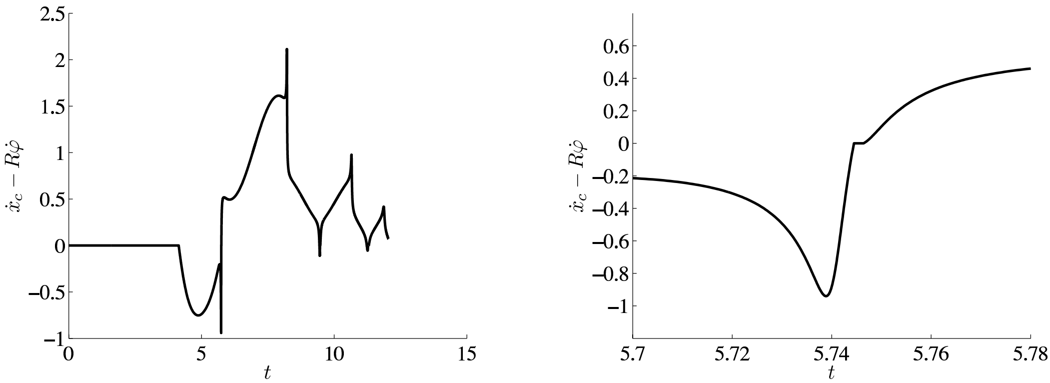

Figure 2. Left: Time variation of the velocity of the contact point of an infinitesimally thin massless hoop (scaled by $\sqrt{Rg}$) in the case $k = 0.1$. Right: The same velocity around the first interaction pulse indicating its rapid change, which includes a brief rolling period ($\dot{x}_c = R\dot{\varphi}$) in between spinning and skidding. The time scale is $\sqrt{R/g}$.

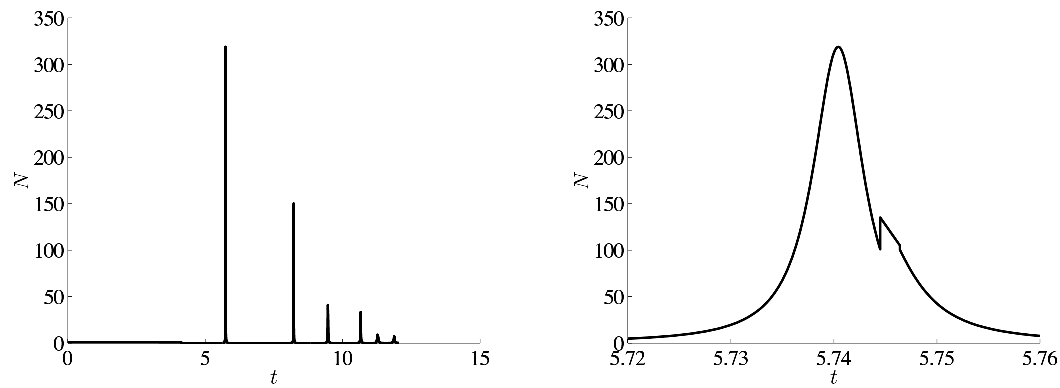

Figure 3. Left: Time variation of the normal force (scaled by mg) during the first three cycles of rotation of an infinitesimally thin massless hoop in the case $k = 0.1$. Right: The corresponding change during the first interaction pulse. The time scale is $\sqrt{R/g}$.

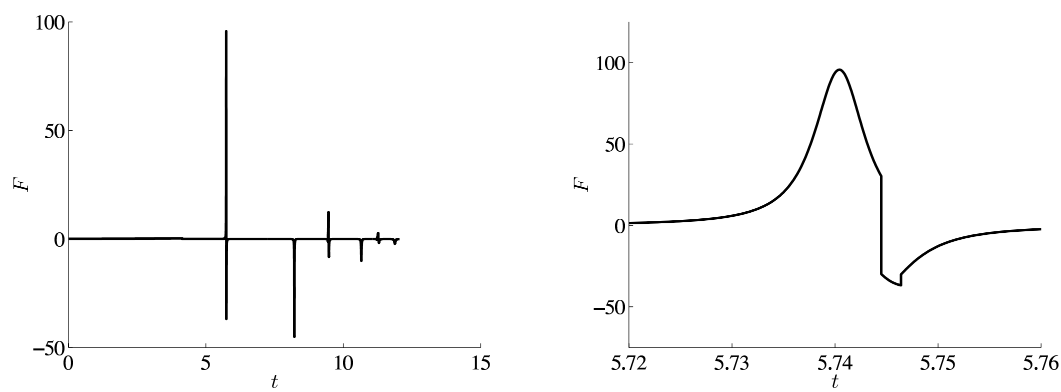

Figure 4. Left: Time variation of the friction force (scaled by mg) during the first three cycles of rotation of an infinitesimally thin massless hoop in the case $k = 0.1$. Right: The corresponding change during the first interaction pulse. The time scale is $\sqrt{R/g}$.

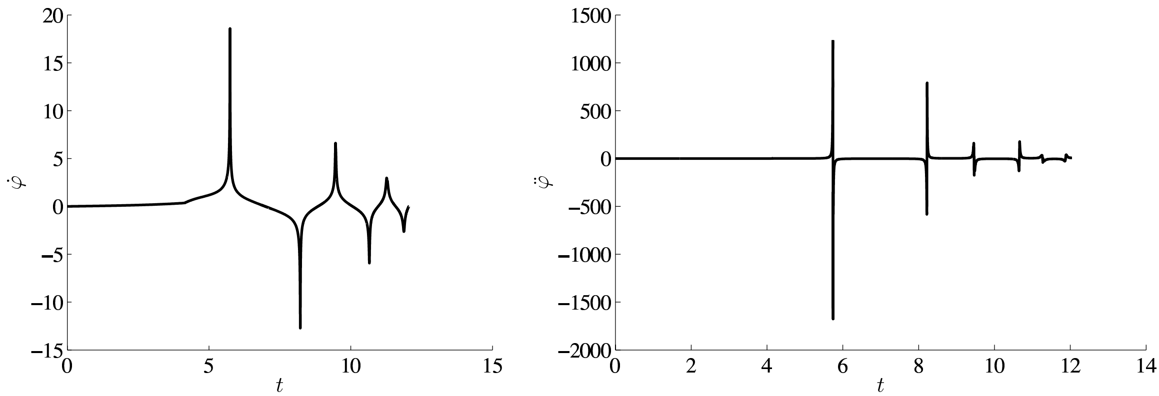

Figure 5. Left: Time variation of the angular velocity (scaled by $\sqrt{g/R}$) during the first three rotation cycles of an infinitesimally thin massless hoop in the case $k = 0.1$. Right: Time variation of the angular acceleration (scaled by g/R). The time scale is $\sqrt{R/g}$.

The jumps in the normal and friction force due to transitions from spinning to rolling and from rolling to skidding during the first pulse are shown in the right halves of Figures 3 and 4, consistent with the variation of the contact point velocity shown in Figure 2, right.

The normal force is finite during the initial rolling interval and is almost equal to zero afterwards except for its rapid buildup near $\varphi = \pi$ when the small disk passes just above the horizontal surface. The large values of N are due to large values of the angular velocity $\dot{\varphi}$ near $\varphi = \pi$ and correspondingly large accelerations $\ddot{x}$ and $\ddot{y}$ of the center of the disk.[1] Note a discontinuous change of the normal force at the transition from rolling to spinning. The friction force around $\varphi = \pi$ shows a rapid change from positive values (spinning) to negative values (skidding) consistent with the variation of the contact point velocity, shown in Figure 2. It is due to this change in the direction of the friction force during a brief period of the sharp interaction pulse that the time integral of the friction force is not equal to μ times the time integral of the normal force.[2] The time variation of the angular velocity and the angular acceleration during the first three rotation cycles of the hoop are shown in Figure 5.

3.2. ***Massless hoop and larger disks.*** As the radius of the disk increases, and with it its rotational inertia, the intensity of the interaction pulse around $\varphi = \pi$ decreases. This is illustrated by considering an infinitesimally thin massless hoop with a disk of radius $\rho = 0.25R$. For brevity, only the first clockwise

[1] Under a large normal force, the hoop and the supporting surface would deform if their deformability was included in the analysis, and the elastic energy stored during this deformation would provide a propulsion for a possible hopping of the hoop at $\varphi = \pi$. The dynamics of an elastic hoop was studied by Theron [2002]. The inclusion of the mass of the hoop, which significantly decreases $N_{\max}$, is considered in Section 3.3.

[2] A similar situation arises in the mechanics of frictional impact of colliding bodies. Modeling of such impacts was greatly stimulated by Kane's [1984] double pendulum *dynamics puzzle* where the frictional component of an impulse is assumed to be equal to the normal component multiplied by the coefficient of kinetic friction. For some values of the coefficients of friction and normal restitution and for some kinematic parameters, this leads to an increase of kinetic energy when the pendulum impacts on a rough horizontal surface. Keller [1986] explained this by noting that there was a reversal of the slip direction during the impact process, and thus the coefficient of proportionality between the tangential and normal impulse was not equal to the coefficient of kinetic friction. Since then numerous papers has been published proposing different models of frictional impact as evidenced by comprehensive treatments of the subject in [Brach 1991; Brogliato 1999; Stronge 2000; Stewart 2000].

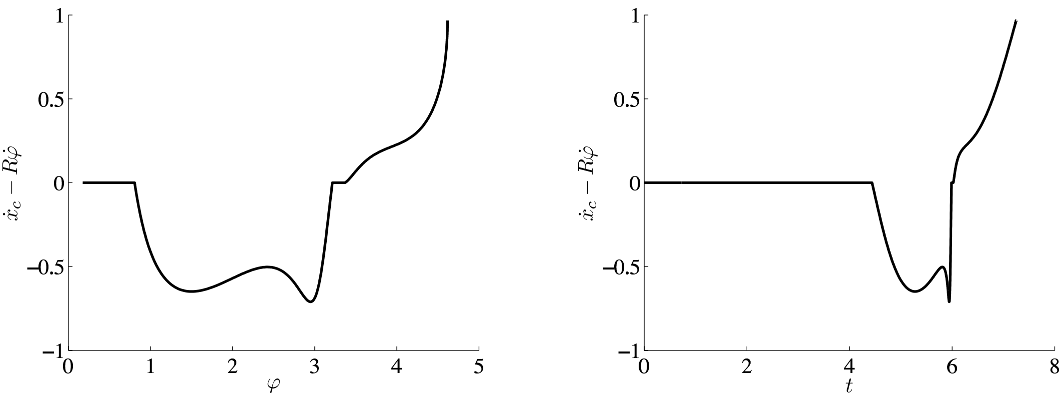

Figure 6. Left: Angle variation of the velocity of the contact point of the hoop (scaled by $\sqrt{Rg}$) during the first clockwise rotation of the hoop in the case $k = 0.25$. Right: The corresponding time variation of the same velocity. The time scale is $\sqrt{R/g}$.

rotation of the hoop is considered. Figure 6 shows the variation of the horizontal velocity component of the contact point. The initial horizontal plateau corresponds to rolling portion of the motion, from φ_0 to $\varphi_1 = 0.80625$ rad. This is followed by spinning from φ_1 to $\varphi_2 = 3.21545$ rad, rolling from φ_2 to $\varphi_3 = 3.37076$ rad, and skidding from φ_3 to $\varphi_4 = 4.61783$ rad, which is the end of the first clockwise rotation of the hoop. The normal and friction force are plotted versus the angle φ in Figure 7. The peak values of both forces are greatly reduced from their values corresponding to small disks with $k = 0.1$.

If the hoop with an attached disk with radius $\rho = 0.75R$ is released from rest at $\varphi_0 = 10°$, the rolling mode prevails throughout the motion of the hoop ($\dot{x}_C = R\dot{\varphi}$ and $|F| < \mu_s N$), which is described by the closed form solution in Section 2.1. Here, the maximum normal force is $N_{\max} = 1.3\,mg$, compared with $N_{\max} = 3.52\,mg$ in the case where $\rho = 0.5R$, $N_{\max} = 32.4\,mg$ in the case where $\rho = 0.25R$, and $N_{\max} = 318\,mg$ in the case where $\rho = 0.1R$. Numerical evaluations also reveal that for $\varphi_0 = 10°$, the smallest radius of the disk for which the rolling prevails throughout the motion of the hoop is $\rho = 0.5385R$. The corresponding maximum normal force is $N_{\max} = 2.94\,mg$.

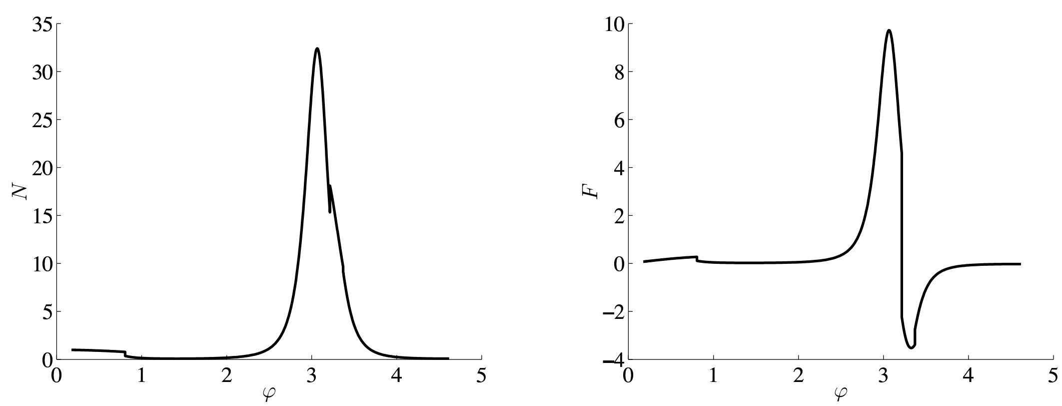

Figure 7. Left: Angle variation of the normal force (scaled by mg) during the first clockwise rotation of the hoop, in the case $k = 0.25$. Right: The corresponding friction force.

3.3. ***The effects of the mass of the hoop.*** In this subsection we present the results of calculations with the included mass of the hoop. The angle variation of the velocity of the contact point of the hoop and the horizontal surface (scaled by $\sqrt{Rg}$) during the first clockwise rotation of the hoop is shown in Figure 8, left (in the case $\varphi_0 = 10°$, $k = 0.1$, and $r = 1.02$). The dotted curve is for $m_h/m = 0$, the dashed curve is for $m_h/m = 0.01$, and the solid curve is for $m_h/m = 0.1$. Figure 8, right, shows a dramatic reduction of the magnitude of the corresponding normal force near $\varphi = \pi$ as the mass ratio m_h/m increases from 0 to 1/10. Similar decrease is observed for the friction component of the force.

With the increase of the size of the disk, the ratio J_h/J_m decreases. For example, if $r = 1.02$ and $\gamma_h/\gamma_m \approx 10^{-2}$, one finds from (20) that for $k = 0.25$ the mass and inertia ratios are $m_h/m = 0.0064$ and $J_h/J_m = 0.2048$ while for $k = 0.5$ these ratios are $m_h/m = 0.0016$ and $J_h/J_m = 0.0128$. In the latter case the rotational inertia of a light thin hoop can be reasonably ignored. The results for an infinitesimally thin massless hoop with a disk of radius $\rho = 0.75R$ were discussed in Section 3.2.

The variations of the normal and friction force in the case of equal mass densities of the hoop and the disk when $r = 1.02$ and $k = 0.1$ (giving rise to the mass ratio $m_h/m \approx 4.04$) are shown in Figure 9. In this case, the rolling prevails throughout the motion. The interaction pulse is mild with the greatly reduced magnitudes of the maximum normal and friction force ($N_{\max} \approx 5.425mg$ versus the static normal force of $5.04mg$).

3.4. ***The effects of the coefficient of friction.*** The coefficient of friction affects the extent of rolling and rolling/sliding motion of the hoop as well as the magnitude of the reactions between the hoop and the horizontal surface. Figure 10, left, shows the angle variation of the velocity of the contact point of an infinitesimally thin massless hoop in the case $\varphi_0 = 10°$ and an exceedingly small disk ($k = 10^{-5}$) simulating a point mass. The dashed curve is for $\mu_s = 0.35$ and $\mu = 0.3$, while the solid curve is for $\mu_s = 0.7$ and $\mu = 0.6$. The extent of the initial rolling increases with the increase of the coefficient of friction. Figure 10, right, shows the behavior near $\varphi = \pi$. For $\mu = 0.3$ and $\mu_s = 0.35$ the interaction

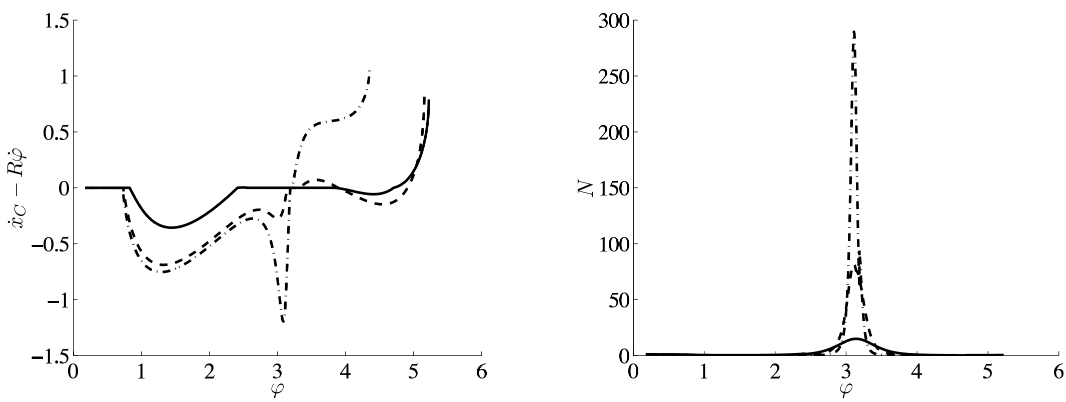

Figure 8. Left: Angle variation of the velocity of the contact point of the hoop and the horizontal surface (scaled by $\sqrt{Rg}$) in the case $\varphi_0 = 10°$, $k = 0.1$, and $r = 1.02$, during the first clockwise rotation of the hoop. The dotted curve is for $m_h/m = 0$, the dashed curve is for $m_h/m = 0.01$, and the solid curve is for $m_h/m = 0.1$. Right: The corresponding variation of the normal force N (scaled by mg) indicating a dramatic reduction of its magnitude near $\varphi = \pi$ with the increase of the mass ratio m_h/m.

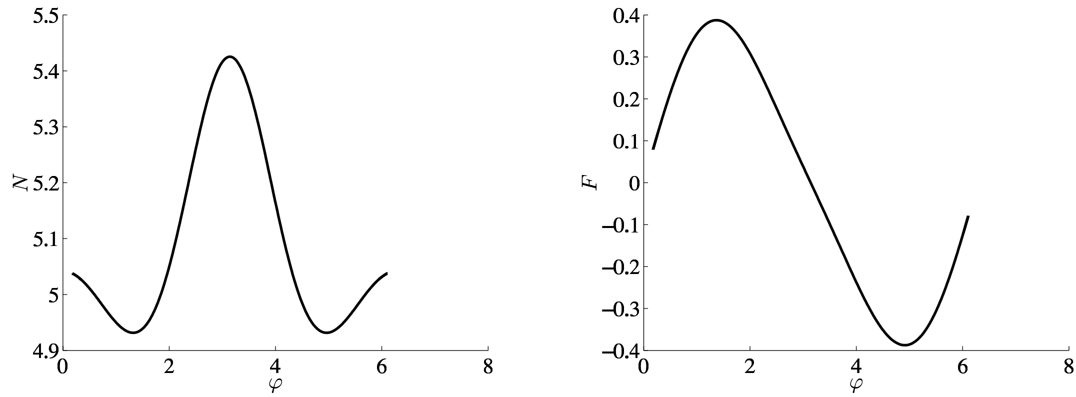

Figure 9. Left: Angle variation of the normal force (scaled by mg) during the first clockwise rotation of the hoop in the case $r = 1.02$, $k = 0.1$ and the equal mass densities so that the mass ratio $m_{\mathrm{h}}/m = 4.04$. Right: The corresponding friction force.

pulse is characterized by skidding to spinning to skidding transitions, while for $\mu = 0.6$ and $\mu_{\mathrm{s}} = 0.7$ the pulse is characterized by skidding to slipping to rolling to skidding transitions. The horizontal distance traveled by the hoop before the sharp pulse and the final coordinates of the center of the disk at the end of the first clockwise rotation of the hoop are greater for the higher coefficient of friction. The mechanical energy at the end of the first clockwise rotation is also greater for the higher coefficient of friction ($E = 1.80266\,mg$ for $\mu = 0.6$ versus $E = 1.38862\,mg$ for $\mu = 0.3$) thus illustrating that in the two considered cases there is less dissipation due to friction in the case of a higher coefficient of friction. This is because the higher coefficient of friction in general promotes more rolling, which is the energy preserving motion in the absence of reactive rolling moment.

Figure 11 shows the variations of the normal and friction force during the first clockwise rotation of the hoop, in the case $r = 1.02$, $k = 0.1$, and $m_{\mathrm{h}}/m = 0.1$, corresponding to four selected values of the friction coefficient. The maximum normal force is greatest in the case of the frictionless spinning

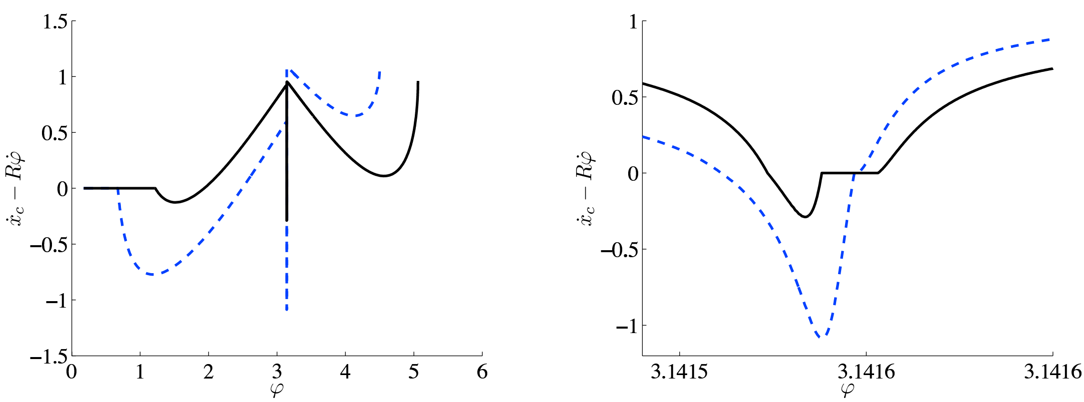

Figure 10. Left: Angle variation of the velocity of the contact point of the hoop and the horizontal surface (scaled by $\sqrt{Rg}$) in the case $k = 10^{-5}$. The dashed curve is for $\mu_{\mathrm{s}} = 0.35$ and $\mu = 0.3$ while the solid curve corresponds to $\mu_{\mathrm{s}} = 0.7$ and $\mu = 0.6$. Right: The same near $\varphi = \pi$.

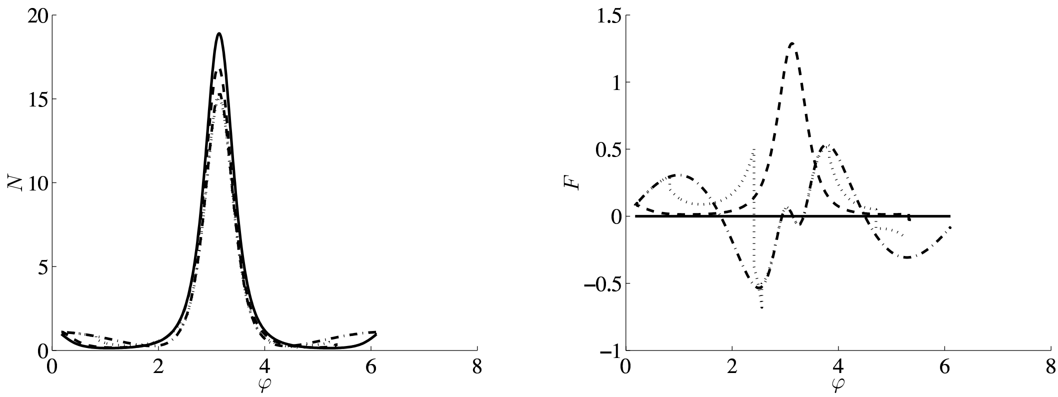

Figure 11. Angle variation of the normal force (left), and friction force (right), both scaled by mg, during the first clockwise rotation of the hoop with $k = 0.1$, $r = 1.02$, and $m_{\mathrm{h}}/m = 0.1$. The solid curve is for $\mu = 0$, the dashed curve is for $\mu_{\mathrm{s}} = 0.09$ and $\mu = 0.076$, the dotted curve is for $\mu_{\mathrm{s}} = 0.35$ and $\mu = 0.3$, and the dotted/dashed curve is for $\mu_{\mathrm{s}} = 1.2$ and $\mu = 1$.

($\mu = 0,\ F = 0$). The second selected value of the coefficient of static friction is twice greater than the minimum coefficient of static friction required for rolling at the inception of motion, calculated from (9). For the remaining two higher values of the coefficient of friction, it is found that the corresponding $N_{\max}$ is nearly the same, although for $\mu_{\mathrm{s}} = 0.35$ and $\mu = 0.3$ the motion is characterized by the stages of rolling and combined rolling/sliding, while for $\mu_{\mathrm{s}} = 1.2$ and $\mu = 1$ pure rolling prevails throughout the motion. The minimum coefficient of static friction required for the rolling to prevail throughout the motion of the hoop is about 0.78.

Figure 12 shows the corresponding variations of the square of the angular velocity $\dot{\varphi}^2$ and the angular acceleration $\ddot{\varphi}$ of the hoop, which explains the mild differences in the normal force for a sufficiently high coefficient of friction ($\mu = 0.3$ versus $\mu = 1$). The angular velocities and the angular accelerations are nearly the same in these two cases, and so are the corresponding normal forces. The friction forces are

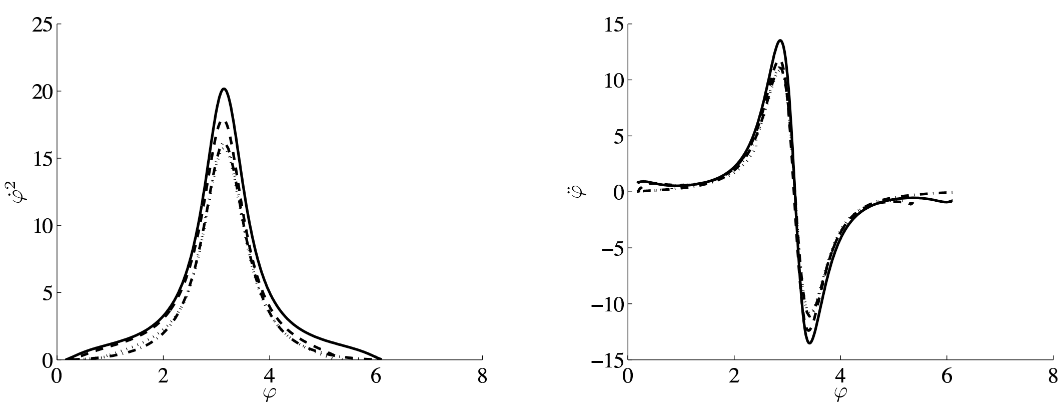

Figure 12. Angle variation of the square of the angular velocity (left), and the angular acceleration (right), both scaled by g/R, corresponding to Figure 11.

different, because they are proportional to the coefficient of friction during the combined rolling/sliding stages of motion.

There is an interval of rolling around $\varphi = \pi$ during which the friction force is almost the same for $\mu_s = 0.35$ and $\mu_s = 1.2$ being well below $0.35\,N$. This is because the angular velocities and the angular accelerations are almost the same in these two cases as are the reactive forces N and F. Note that in Figure 12, left, the angular velocity $\dot{\varphi}$ does not depend on the coefficient of friction during the initial rolling stage of motion; only the extent of this rolling depends on μ_s, so that the curves for $\dot{\varphi}^2$ depart from each other at different angles $\varphi = \varphi(\mu_s)$. Further analysis of the effect of friction on the motion of the loaded hoop can be found in [Theron and Maritz 2008].

4. Summary

In this paper we presented an analysis of motion of a light hoop with an attached heavy circular disk released to move on a rough horizontal surface. This motion involves a complex kinematics and kinetics, which reveal a variety of information of possible interest for the mechanics of frictional impact. When the radius of the disk is small relative to the radius of a light hoop, the motion is characterized by the sequence of sharp interaction pulses between the hoop and the horizontal surface during which several transitions between the stick, slip, and reversed slip can take place. This is accompanied by the corresponding changes in the friction force, which includes abrupt reversal of its direction, and rapid increase and discontinuous change of the normal component of the reactive force. The energy dissipation takes place predominantly during the short interaction pulses. The intensity of sharp interaction between the hoop and the horizontal surface, as well as the complexity of stick/slip transitions, diminish with the increasing size of the disk or the mass of the hoop. For sufficiently large disks, the rolling without transition to sliding prevails throughout the motion. The effects of the coefficient of friction on dynamic interactions, stick/slip transitions, and the energy dissipation are discussed. The analysis have demonstrated that the time integral of the tangential component of the interaction pulse is in general different from the product of the coefficient of kinetic friction and the time integral of the normal component of the interaction pulse, which is analogous to the nonproportionality of the normal and tangential impulses most commonly found in the mechanics of frictional impact. The obtained results may thus facilitate better understanding of the involved kinematics and kinetics of the latter problems in which the time variation of the normal and friction force during the impact process cannot be determined within the model of rigid body mechanics [Seifried et al. 2005; Lubarda 2009].

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