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STEEL SHIM STRESSES IN MULTILAYER BEARINGS UNDER COMPRESSION AND BENDING

JAMES M. KELLY AND DIMITRIOS KONSTANTINIDIS

Solutions are given for the tensile stresses in the steel reinforcing shims of elastomeric isolators. The method makes use of generalized plane stress and uses a stress function approach, treating the shim as a thin plate with body forces generated by surface shears on the top and bottom of the plate. It is shown that the pressure in the rubber acts as a potential for these body forces. The solutions are applicable to low and moderate shape factor bearings where it is acceptable to assume that the elastomer is incompressible, and also to the more common current situation when the shape factor is so large that the compressibility of the rubber must be included. The stress state in the steel reinforcing plates is calculated for both pure compression of the bearing and for when the bearing is loaded by a bending moment. These two cases, separately and in combination, are the typical situation in current practice. While a solution for the stress state in the shims of a circular isolator, assuming incompressibility and under pure compression, has been available using an analogy with the stresses in a rotating circular plate, the use of the stress function method is new and suggests a method to extend the solutions to other shapes of isolator other than circular. The solutions for pure compression, including compressibility of the rubber, and the solutions for bending, both incompressible and compressible, are entirely new.

1. Introduction

The essential characteristic of the elastomeric isolator is the very large ratio of the vertical stiffness relative to the horizontal stiffness. This is produced by the reinforcing plates, which in current industry standard are thin steel plates. These plates prevent lateral bulging of the rubber but allow the rubber to shear freely. The vertical stiffness can be several hundred times the horizontal stiffness. The steel reinforcement has the effect of generating shear stresses in the rubber in the isolator, and these stresses act on the steel plates to cause tension stresses which, if they were to become large enough, could result in failure of the steel shims through yielding or fracture. The external pressure on the isolator at which this might happen is an important design quantity for an isolator, and it is therefore necessary to be able to estimate these tensile stresses under the applied external load. An analysis is given for the stresses in a circular isolator under the assumption that the rubber is incompressible, generally in the low to moderate shape factor case. The analysis is also extended to the large shape factor case where the assumption of incompressibility cannot be made, and the effect of the bulk modulus of the rubber must be included. In current practise, most seismic isolation bearings use very large shape factors, where bulk compressibility must be included. These analyses make use of a stress function based on the observation that the pressure in the rubber acts as a potential for the internal stresses in the steel plates. Solutions are given for pure compression loading on the bearing and for bending in the bearing. Both compression and bending are

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important in this analysis since, in use, the lateral displacement of a bearing superposes on the bearing a bending moment that can add to or reduce the stresses due to compression, and could in principle generate increased tension stresses or, if large enough, cause compression stresses in the thin plates and lead to their failure due to buckling.

The basis for the requirements for the steel shim stresses in all the codes governing the design of elastomeric bearings for bridges or for vibration isolation is a very simplistic formula [Spitz 1978; Stanton and Roeder 1982] in which the rubber is assumed to be a liquid under pressure and only equilibrium is used. The geometry of the bearing is a thin strip, and the shim is inextensible. This result is almost certainly wrong and clearly cannot be applied to circular shims with holes or external loading other than pure compression. Finite element analysis has been used to verify the accuracy of the approximate solution for the pressure distribution in the rubber, for example by Billings [1992] and Imbimbo and De Luca [1998], and in principle could be used to compute the stresses in the shims for a specific design under a variety of loading conditions, but this does not appear to have been done; and, in any case, finite element analysis is unlikely to be a useful design tool for these fairly low-cost items. The method given in this paper, based on the use of a stress function approach, can be extended to different cross-sectional shapes and other types of external loading.

2. Compression and bending of a rubber pad within incompressibility theory

2.1. ***Pure compression.*** A linear elastic theory is the most common method used to predict the compression stiffness of a thin elastomeric pad. The first analysis of the compression stiffness was done using an energy approach by Rocard [1937]; further developments were made by Gent and Lindley [1959] and Gent and Meinecke [1970]. An approximate strength-of-materials-type theory, applicable to bearings with low to moderate shape factors, was developed in [Kelly 1996], based on the assumption that the material is incompressible. The solutions that are the starting point for the present analysis are developed in detail there and will only be given here in their final form.

The analysis is based on the kinematic assumptions that points on a vertical line before deformation lie on a parabola after loading, and that horizontal planes remain horizontal. We consider an arbitrarily shaped pad of thickness t and locate a rectangular Cartesian coordinate system, (x, y, z), in the middle surface of the pad, as shown in Figure 1, left. The right half of the same figure shows the displacement pattern under the kinematic assumptions above. This displacement field satisfies the constraint that the top and bottom surfaces of the pad are bonded to rigid substrates, and takes the form

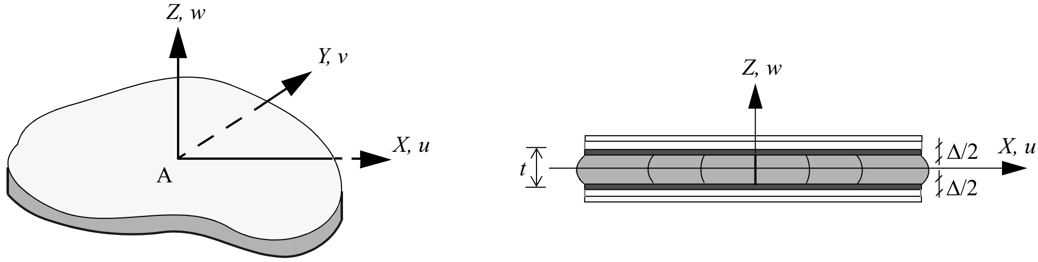

Figure 1. Constrained rubber pad and coordinate system.

$$u(x,y,z) = u_0(x,y)\Big(1 - \frac{4z^2}{t^2}\Big), \quad v(x,y,z) = v_0(x,y)\Big(1 - \frac{4z^2}{t^2}\Big), \quad w(x,y,z) = w(z). \tag{1}$$

The assumption of incompressibility produces a further constraint on the three components of strain, ϵ_{xx}, ϵ_{yy}, ϵ_{zz}, in the form

$$\epsilon_{xx} + \epsilon_{yy} + \epsilon_{zz} = 0, \tag{2}$$

which leads to

$$(u_{0,x} + v_{0,y})\Big(1 - \frac{4z^2}{t^2}\Big) + w_{,z} = 0,$$

where the commas imply partial differentiation with respect to the indicated coordinate. Integration through the thickness provides the two-dimensional form of the incompressibility constraint as

$$u_{0,x} + v_{0,y} = \frac{3\epsilon_c}{2}, \tag{3}$$

where the compression strain ϵ_c is defined by

$$\epsilon_c = -\frac{w(t/2) - w(-t/2)}{t} = \frac{\Delta}{t} \quad (\epsilon_c > 0 \text{ in compression}).$$

The stress state is assumed to be dominated by the internal pressure, p, such that the normal stress components, σ_{xx}, σ_{yy}, σ_{zz}, can be equated to $-p$. The shear stress components, τ_{xz} and τ_{yz}, generated by the constraints at the top and bottom of the pad, are included, but the in-plane shear stress, τ_{xy}, is neglected. The equations of equilibrium for the stresses reduce under these assumptions to

$$\tau_{xz,z} = p_{,x}, \quad \tau_{yz,z} = p_{,y}. \tag{4}$$

We assume that the material is linearly elastic and that the shear stresses τ_{xz} and τ_{yz} are related to the shear strains by G, the shear modulus of the material. The equation for the pressure from which it and all other quantities can be derived is

$$p_{,xx} + p_{,yy} = \nabla^2 p = -\frac{12G\Delta}{t^3} = -\frac{12G}{t^2}\epsilon_c, \tag{5}$$

where $\epsilon_c = \Delta/t$ is the compression strain. The boundary condition, $p = 0$, on the edge of the pad completes the system for $p(x, y)$.

The vertical stiffness of a rubber bearing is given by the formula

$$K_V = \frac{E_c A}{t_r}, \tag{6}$$

where A is the area of the bearing, t_r is the total thickness of rubber in the bearing, and E_c is the instantaneous compression modulus of the rubber-steel composite under the specified level of vertical load. The value of E_c for a single rubber layer is controlled by the shape factor, S, defined as

$$S = \frac{\text{loaded area}}{\text{free area}},$$

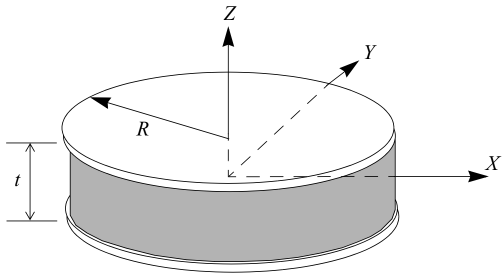

Figure 2. Coordinate system for a circular pad of radius R.

which is a dimensionless measure of the aspect ratio of the single layer of the elastomer. For example, for a circular pad of radius R and thickness t,

$$S = \frac{R}{2t}. \tag{7}$$

To determine the compression modulus, E_c, we solve for p from (5), and integrate over the area to determine the resultant normal load, P, from which E_c is given by

$$E_c = \frac{P}{A\epsilon_c}. \tag{8}$$

For example for a circular pad of radius R, as shown in Figure 2, Equation (5) reduces to

$$\nabla^2 p = \frac{d^2 p}{dr^2} + \frac{1}{r}\frac{dp}{dr} = -\frac{12G}{t^2}\epsilon_c; \quad r = \sqrt{x^2 + y^2}. \tag{9}$$

The solution is

$$p = A \ln r + B - \frac{3G}{t^2} r^2 \epsilon_c,$$

where A and B are constants of integration; because p must be bounded at $r = 0$ and $p = 0$ at $r = R$, the solution becomes

$$p = \frac{3G}{t^2}(R^2 - r^2)\epsilon_c. \tag{10}$$

It follows that

$$P = 2\pi \int_0^R p(r) r \, dr = \frac{3G\pi R^4}{2t^2}\epsilon_c,$$

and with $S = R/(2t)$ and $A = \pi R^2$, we have $E_c = 6GS^2$.

2.2. *Pure bending.* The bending stiffness of a single pad is computed using a similar approach. The displaced configuration is obtained in two stages. First visualize the deformation that would occur if the bending conformed to elementary beam theory (shown dotted in Figure 3). Because this cannot satisfy the incompressibility constraint, a further pure shear deformation is superimposed. The displacement field is given by

$$u(x, y, z) = u_0(x, y)\left(1 - \frac{4z^2}{t^2}\right) - \alpha\frac{z^2}{2t}, \quad v(x, y, z) = v_0(x, y)\left(1 - \frac{4z^2}{t^2}\right), \quad w(x, y, z) = \frac{\alpha z x}{t}. \tag{11}$$

Here, α is the angle between the rigid plates in the deformed configuration, and the bending is about the y-axis. The radius of the curvature, ρ, generated by the deformation, is related to α by $1/\rho = \alpha/t$ which, with the incompressibility condition, leads to

$$p_{,xx} + p_{,yy} = \frac{12G\alpha}{t^3}x, \tag{12}$$

with $p = 0$ on the edges.

The solution technique is to solve (12) for p and to compute the bending moment, M, from

$$M = -\int_A p(x, y)x\, dA,$$

and use an analogy with beam theory, where $M = EI(1/\rho)$, to compute the effective bending stiffness $EI_{\text{eff}} = M/(\alpha/t)$.

For a circular pad of radius R (Figure 2), the equation to be solved, using polar coordinates r and θ, is

$$p_{,rr} + \frac{1}{r}p_{,r} + \frac{1}{r^2}p_{,\theta\theta} = \frac{\alpha}{t}\frac{12G}{t^2}r\cos\theta. \tag{13}$$

The solution is

$$p(r,\theta) = \left(Ar + B\frac{1}{r} + Cr^3\right)\cos\theta,$$

where $C = (\alpha/t)(3G)/(2t^2)$. For the complete circle, $0 \le r \le R$, for p to be bounded at $r = 0$ means $B = 0$, and using the boundary condition $p = 0$ at $r = R$, gives

$$p = \frac{3G}{2t^2}\frac{\alpha}{t}(r^2 - R^2)r\cos\theta \tag{14}$$

and

$$M = \frac{G\alpha}{8t^3}\pi R^6. \tag{15}$$

The effective moment of inertia in this case, taking $E = E_c = 6GS^2$, is $\pi R^4/12$, or one third of the conventional moment of inertia for the bending of a circular cross section.

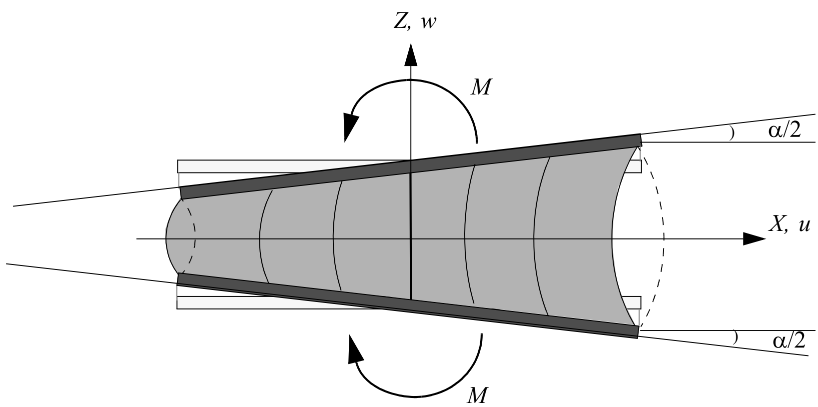

Figure 3. Pad between rigid constraint layers in pure bending.

It is useful to define a bending strain ε_b analogous to the compression strain ε_c through

$$\varepsilon_b = \frac{\alpha}{t} R = \frac{M}{E I_{\text{eff}}} R, \tag{16}$$

in terms of which we have

$$\nabla^2 p = p_{,rr} + \frac{1}{r} p_{,r} + \frac{1}{r^2} p_{,\theta\theta} = \frac{12G}{t^2 R} \varepsilon_b r \cos\theta \tag{17}$$

and

$$p = \frac{3G\varepsilon_b}{2Rt^2} (r^2 - R^2) r \cos\theta. \tag{18}$$

3. Steel stresses in circular bearings with incompressible rubber

The state of stress in a rubber layer, within a multilayer bearing under compression or bending, or a combination of the two, is assumed to be a state of pressure that would induce a bulging of the rubber were it not restrained by the thin steel reinforcing plates (often referred to as shims) that are bonded to the rubber. The restraint of the rubber by the steel causes shear stresses in the rubber which act on each side of the steel plate to induce tensile or compression stresses in the plane of the plate. It is possible to solve for these in-plane stresses using two-dimensional elasticity theory, assuming that the plate is in a state of generalized plane stress and that the surface shear stresses are equivalent to in-plane body forces. While it is possible to formulate the plane stress problem for an arbitrarily shaped plate, we consider only the circular bearing, and we will also use the notations for stresses and stress functions from [Timoshenko and Goodier 1970].

In polar coordinates (r, θ), the equations of equilibrium for the stresses in the rubber are, using the Timoshenko notation,

$$\begin{aligned}
\frac{\partial \sigma_r}{\partial r} + \frac{1}{r}\frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{\partial \tau_{rz}}{\partial z} + \frac{\sigma_r - \sigma_\theta}{r} &= 0, \\
\frac{\partial \tau_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_\theta}{\partial \theta} + \frac{\partial \tau_{\theta z}}{\partial z} + \frac{2\tau_{r\theta}}{r} &= 0, \\
\frac{\partial \tau_{rz}}{\partial r} + \frac{1}{r}\frac{\partial \tau_{\theta z}}{\partial \theta} + \frac{\partial \sigma_z}{\partial z} + \frac{\tau_{rz}}{r} &= 0.
\end{aligned} \tag{19}$$

It is assumed that $\sigma_r = \sigma_\theta = -p$, and are independent of z; and also that the in-plane shear stress $\tau_{r\theta}$ is negligible. Hence the first of these equations yields

$$\frac{\partial \tau_{rz}}{\partial z} = \frac{\partial p}{\partial r}, \tag{20}$$

and the second

$$\frac{\partial \tau_{\theta z}}{\partial z} = \frac{1}{r}\frac{\partial p}{\partial \theta}. \tag{21}$$

Inserting these equilibrium equations into the third equation in (19), differentiating with respect to z and interchanging the order of differentiation changes it to

$$p_{,rr} + \frac{1}{r} p_{,r} + \frac{1}{r^2} p_{,\theta\theta} + \frac{\partial^2 \sigma_z}{\partial z^2} = 0.$$

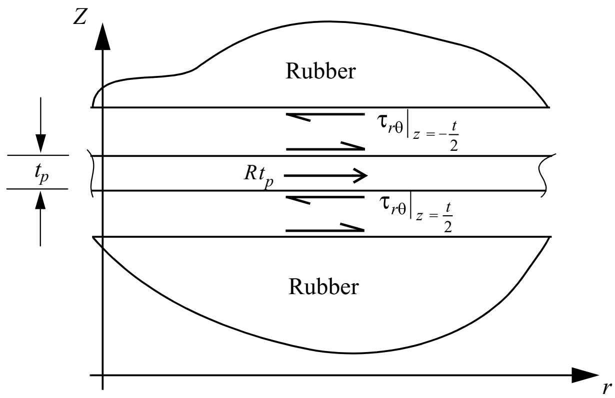

Figure 4. Shear stresses producing equivalent body forces in the plate.

This equation, with substitution from either (20) or (21), allow us, if necessary, to calculate the distribution of σ_z through the thickness of the pad. Within the stress assumptions above we have $p = p(r, \theta)$ which means that we can write

$$\tau_{rz} = \frac{\partial p}{\partial r} z, \quad \tau_{\theta z} = \frac{1}{r} \frac{\partial p}{\partial \theta} z. \tag{22}$$

The shear stresses at the bottom of the rubber layer $\tau_{rz}|_{z=-t/2}$ and $\tau_{\theta z}|_{z=-t/2}$ become the shear stresses on the top surface of the plate, and the shear stresses at the top of the rubber layer $\tau_{rz}|_{z=t/2}$ and $\tau_{\theta z}|_{z=t/2}$ are the shear stresses on the lower surface of the plate (Figure 4).

The internal stresses in the steel shims satisfy the equilibrium equations

$$\frac{\partial \sigma_r}{\partial r} + \frac{1}{r} \frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{\sigma_r - \sigma_\theta}{r} + R = 0, \quad \frac{\partial \tau_{r\theta}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_\theta}{\partial \theta} + \frac{2\tau_{r\theta}}{r} + S = 0, \tag{23}$$

where R, S are the equivalent body forces per unit volume created by the surface shear stresses and are given by

$$t_p R = \tau_{rz}\big|_{z=-t/2} - \tau_{rz}\big|_{z=t/2}, \quad t_p S = \tau_{\theta z}\big|_{z=-t/2} - \tau_{\theta z}\big|_{z=t/2},$$

where t_p is the thickness of the steel shim, leading to

$$R = -\frac{\partial p}{\partial r} \frac{t}{t_p}, \quad S = -\frac{1}{r} \frac{\partial p}{\partial \theta} \frac{t}{t_p}. \tag{24}$$

It follows that the pressure plays the role of a potential, $V(r, \theta)$, for the body forces in the form $V = (t/t_p) p(r, \theta)$ with $R = -\partial V/\partial r$ and $S = -(1/r)\partial V/\partial \theta$. The equations of equilibrium for the plate then become

$$\frac{\partial(\sigma_r - V)}{\partial r} + \frac{1}{r} \frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{(\sigma_r - V) - (\sigma_\theta - V)}{r} = 0, \quad \frac{\partial \tau_{r\theta}}{\partial r} + \frac{1}{r} \frac{\partial(\sigma_\theta - V)}{\partial \theta} + \frac{2\tau_{r\theta}}{r} = 0. \tag{25}$$

These equations are satisfied by a stress function $\Phi(r, \theta)$ for the stresses, such that

$$\sigma_r - V = \frac{1}{r} \frac{\partial \Phi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \Phi}{\partial \theta^2}, \quad \sigma_\theta - V = \frac{\partial^2 \Phi}{\partial r^2}, \quad \tau_{r\theta} = -\frac{\partial}{\partial r}\Big(\frac{1}{r} \frac{\partial \Phi}{\partial \theta}\Big); \tag{26}$$

and under the assumption of plane stress the equation for Φ is

$$\nabla^2\nabla^2\Phi + (1-\nu)\nabla^2 V = 0, \tag{27}$$

where ν is the Poisson ratio of the steel. In the case of pure compression the pressure satisfies (4):

$$\nabla^2 p = -\frac{12G}{t^2}\varepsilon_c, \tag{28}$$

and, for bending, (12):

$$\nabla^2 p = \frac{12Gr\cos\theta}{t^2 R}\varepsilon_b. \tag{29}$$

3.1. ***Stress function solution for pure compression.*** The stress function Φ for pure compression is given by the solution of

$$\nabla^2\nabla^2\Phi + \overline{C} = 0, \tag{30}$$

where $\overline{C}$ is a constant having the value

$$\overline{C} = -(1-\nu)\frac{t}{t_p}\frac{12G}{t^2}\varepsilon_c. \tag{31}$$

In this case the stress function depends only on r, and we look for a solution of that form. The result is [Timoshenko and Goodier 1970]

$$\Phi = A\ln r + Br^2\ln r + Cr^2 + D - \frac{\overline{C}r^4}{64}.$$

The resulting stresses are given by

$$\sigma_r - \frac{t}{t_p}p(r) = \frac{A}{r^2} + B(1+2\ln r) + 2C - \frac{\overline{C}r^2}{16}, \quad \sigma_\theta - \frac{t}{t_p}p(r) = -\frac{A}{r^2} + B(3+2\ln r) + 2C - \frac{3}{16}\overline{C}r^2.$$

It is clear that both A and B must vanish for the completely circular plate, and using $\sigma_r(R) = 0$ and the fact that the pressure is also zero at $r = R$, we have $2C = \overline{C}R^2/16$. Using the pressure from (10) and $\overline{C}$ from (31), the solution for the tensile stresses becomes

$$\begin{aligned}\sigma_r &= \frac{3G}{tt_p}\frac{3+\nu}{4}(R^2-r^2)\varepsilon_c = 3GS^2\varepsilon_c\frac{t}{t_p}(3+\nu)\Big(1-\frac{r^2}{R^2}\Big),\\ \sigma_\theta &= \frac{3}{4}\frac{G\varepsilon_c}{tt_p}\big((3+\nu)R^2-(1+3\nu)r^2\big) = 3GS^2\varepsilon_c\frac{t}{t_p}\Big(3+\nu-(1+3\nu)\frac{r^2}{R^2}\Big).\end{aligned} \tag{32}$$

The distribution of the stresses in the plate under pure compression is shown in Figure 5.

At the center of the plate we have

$$\sigma_{\max} = \sigma_r = \sigma_\theta = \frac{6GS^2}{2}\frac{t}{t_p}\varepsilon_c(3+\nu). \tag{33}$$

By expressing the maximum value of the stresses in terms of the average pressure over the plate, p_{ave}, given by

$$p_{\text{ave}} = E_c\varepsilon_c = 6GS^2\varepsilon_c, \tag{34}$$

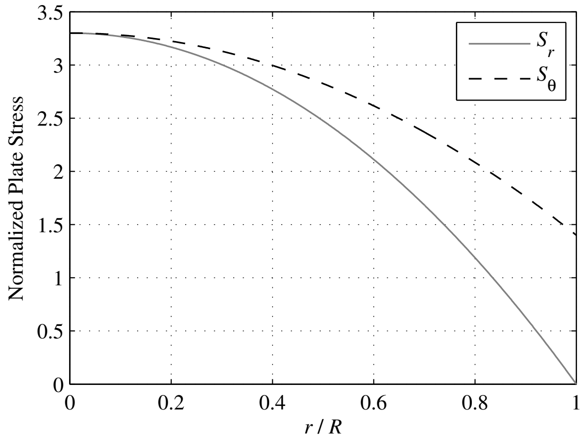

Figure 5. Plate stresses in compression, assuming incompressibility.

then

$$\frac{\sigma_{\max}}{p_{\text{ave}}} = \frac{3+\nu}{2}\frac{t}{t_p}, \tag{35}$$

which can be used to determine the maximum pressure needed to cause yield in the shim at the center. This shows why, under normal circumstances, the stresses in the shims due to the pressure is not considered important. For example, if we have steel shims 3.0 mm thick and rubber layers 15.0 mm thick, the stresses in the steel due to a pressure of 7.0 MPa (which is standard) are only 58 MPa, well below the yield level of the plate material. If the tension stress at the center is in fact at the level of the yield stress of the material, σ_o, then the average pressure to initiate yield is $p_{\text{ave}} = 2\sigma_o(t_p/t)/(3+\nu)$. This is only the start of yield, but the plate will experience further yielding as the pressure increases. The zone of yielding will spread from the center to the outer radius so that the pressure can increase further until the entire plate has yielded, that is, until the region $0 \le r \le R$ is fully plastic.

Note that the problem in this case is the same as that for the stresses in a thin disk due to centrifugal forces. That solution is given in Timoshenko and Goodier [1970] using an entirely different approach based on displacements, and, except for constants, is identical to this. The same reference also provides the solution for a circular plate of radius b with a central hole of radius a, which for $a \le r \le b$ is

$$\sigma_r = \frac{3}{4}\frac{G\varepsilon_c}{tt_p}(3+\nu)\Big(b^2 + a^2 - \frac{a^2b^2}{r^2} - r^2\Big), \quad \sigma_\theta = \frac{3}{4}\frac{G\varepsilon_c}{tt_p}(3+\nu)\Big(b^2 + a^2 + \frac{a^2b^2}{r^2} - \frac{1+3\nu}{3+\nu}r^2\Big). \tag{36}$$

The maximum radial stress is at $R = \sqrt{ab}$, where

$$\sigma_r = \frac{3}{4}\frac{G\varepsilon_c}{tt_p}(3+\nu)(b-a)^2, \tag{37}$$

and the maximum tangential stress is at the inner boundary, where

$$\sigma_\theta = \frac{3}{4}\frac{G\varepsilon_c}{tt_p}(3+\nu)\Big(b^2 + \frac{1-\nu}{3+\nu}a^2\Big). \tag{38}$$

This is always larger than the maximum radial stress. When the radius a of the hole becomes very small, the maximum tangential stress approaches a value twice as large as that for the complete circular plate. This another example of a stress concentration at a small hole.

3.2. ***Stress function solution for pure bending.*** For the case of pure bending on the pad, the equation for the stress function takes the form

$$\nabla^2\nabla^2\Phi + \bar{C} r\cos\theta = 0, \tag{39}$$

where now

$$\bar{C} = (1-\nu)\frac{12G}{tt_p R}\varepsilon_b. \tag{40}$$

We look for a solution of the form

$$\Phi(r,\theta) = f(r)\cos\theta.$$

From [Timoshenko and Goodier 1970], the ordinary differential equation for f is

$$\frac{d^4 f}{dr^4} + \frac{2}{r}\frac{d^3 f}{dr^3} - \frac{3}{r^2}\frac{d^2 f}{dr^2} + \frac{3}{r^3}\frac{df}{dr} - \frac{3}{r^4} f + \bar{C} r = 0. \tag{41}$$

The solution is

$$f(r) = Ar^3 + \frac{B}{r} + Cr + Dr\ln r - \frac{\bar{C} r^5}{192}. \tag{42}$$

The resulting stresses are

$$\begin{aligned}
\sigma_r - \frac{t}{t_p} p &= \Bigl(2Ar - \frac{2B}{r^3} + \frac{D}{r}\Bigr)\cos\theta + \bar{R},\\
\sigma_\theta - \frac{t}{t_p} p &= \Bigl(6Ar + \frac{2B}{r^3} + \frac{D}{r}\Bigr)\cos\theta + \bar{S},\\
\tau_{r\theta} &= \Bigl(2Ar - \frac{2B}{r^3} + \frac{D}{r}\Bigr)\sin\theta + \bar{T},
\end{aligned}$$

where

$$\begin{aligned}
\bar{R} &= \Bigl(\frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\Bigr)\Bigl(-\frac{\bar{C} r^5}{192}\cos\theta\Bigr) = -\frac{\bar{C} r^3}{48}\cos\theta,\\
\bar{S} &= \frac{\partial^2}{\partial r^2}\Bigl(-\frac{\bar{C} r^5}{192}\cos\theta\Bigr) = -\frac{5\bar{C} r^3}{48}\cos\theta,\\
\bar{T} &= -\frac{\partial}{\partial r}\Bigl(\frac{1}{r}\frac{\partial}{\partial\theta}\Bigl(-\frac{\bar{C} r^5}{192}\cos\theta\Bigr)\Bigr) = -\frac{\bar{C} r^3}{48}\sin\theta.
\end{aligned}$$

For the complete plate $0 \le r \le R$, both B and D must vanish, giving

$$\sigma_r = \frac{t}{t_p} p(r,\theta) + 2Ar\cos\theta - \frac{\bar{C} r^3}{48}\cos\theta;$$

and since $p(r,\theta)$ is zero on the boundary, the requirement that $\sigma_r(R,\theta) = 0$ gives $2A = \bar{C} R^2/48$, leading to the final results

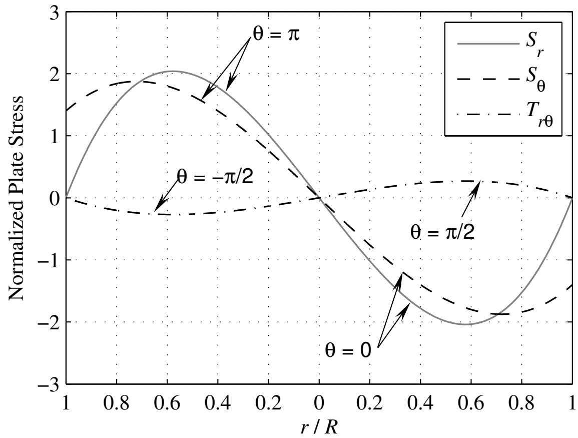

Figure 6. Plate stresses under bending moment for the incompressible case.

$$
\begin{aligned}
\sigma_r(r,\theta) &= -\frac{G\varepsilon_b}{4tt_pR}(5+\nu)(R^2-r^2)r\cos\theta,\\
\sigma_\theta(r,\theta) &= \frac{G\varepsilon_b}{4tt_pR}\big((1+5\nu)r^2-3(1+\nu)R^2\big)r\cos\theta,\\
\tau_{r\theta}(r,\theta) &= -\frac{G\varepsilon_b}{4tt_pR}(1-\nu)(R^2-r^2)r\sin\theta.
\end{aligned}
\tag{43}
$$

These results are plotted as nondimensional stresses in Figure 6.

$$
\begin{aligned}
S_r &= \frac{\sigma_r(r,\theta)}{GS^2\varepsilon_b(t/t_p)} = -(5+\nu)\left(1-\left(\frac{r}{R}\right)^2\right)\frac{r}{R}\cos\theta,\\
S_\theta &= \frac{\sigma_\theta(r,\theta)}{GS^2\varepsilon_b(t/t_p)} = \left((1+5\nu)\left(\frac{r}{R}\right)^2-3(1+\nu)\right)\frac{r}{R}\cos\theta,\\
T_{r\theta} &= \frac{\tau_{r\theta}(r,\theta)}{GS^2\varepsilon_b(t/t_p)} = (1-\nu)\left(1-\left(\frac{r}{R}\right)^2\right)\frac{r}{R}\sin\theta.
\end{aligned}
\tag{44}
$$

4. Steel stresses due to compression in circular pads with large shape factors

The theory for the compression of a rubber pad given in the preceding section is based on two assumptions: first, that the displacement pattern is defined by (1); second, that the normal stress components in all three directions are equal to the pressure, p, in the material. The equation that is solved for p results from the integration of the incompressibility constraint, that is, Equation (2), through the thickness of the pad, leading to an equation for $p(x, y)$ of the form given in (5). To include the influence of bulk compressibility, we need only replace the incompressibility constraint by

$$
\varepsilon_{xx}+\varepsilon_{yy}+\varepsilon_{zz} = -\frac{p}{K}, \tag{45}
$$

where K is the bulk modulus. Integration through the thickness leads to an equation for $p(x, y)$ of the form

$$\nabla^2 p - \frac{12p}{t^2}\frac{G}{K} = -\frac{12G}{t^2}\varepsilon_c, \tag{46}$$

that is solved as before, with $p = 0$ on the edge of the pad.

We now consider a circular pad with a large shape factor, an external radius R, and a thickness t. The pressure in the pad is axisymmetrical, that is, $p = p(r)$ where $0 \le r \le R$; therefore, (46) becomes

$$\frac{d^2p}{dr^2} + \frac{1}{r}\frac{dp}{dr} - \lambda^2(p - K\varepsilon_c) = 0, \tag{47}$$

where $\lambda^2 = 12G/(Kt^2)$. The boundary conditions to be satisfied are $p = 0$ at $r = R$, and p is finite at $r = 0$.

The solution involves the modified Bessel functions of the first and second kinds, I_0 and K_0. Because the solution is bounded at $r = 0$, the term in K_0 is excluded, and the general solution for $p(r)$ is

$$p(r) = K\Big(1 - \frac{I_0(\lambda r)}{I_0(\lambda R)}\Big)\varepsilon_c. \tag{48}$$

Integrating p over the area of the pad gives

$$P = K\pi R^2\Big(1 - \frac{2}{\lambda R}\frac{I_1(\lambda R)}{I_0(\lambda R)}\Big)\varepsilon_c, \tag{49}$$

where I_1 is the modified Bessel function of the first kind of order 1.

The resulting expression for the compression modulus is

$$E_c = K\Big(1 - \frac{2}{\lambda R}\frac{I_1(\lambda R)}{I_0(\lambda R)}\Big), \tag{50}$$

where

$$\lambda R = \Big(\frac{12GR^2}{Kt^2}\Big)^{1/2} = \Big(\frac{48G}{K}\Big)^{1/2} S,$$

and the shape factor, S, is $R/(2t)$.

The tension stresses in the steel reinforcing plates can be calculated in the same way as before by using the stress function method. The equation for the stress function remains the same,

$$\nabla^2\nabla^2\Phi + (1-\nu)\nabla^2 V = 0;$$

and the definition of the potential $V(r, \theta)$ is the same, $V = \dfrac{t}{t_p}p(r, \theta)$; but the pressure now satisfies the equation

$$\nabla^2 p - \frac{12p}{t^2}\frac{G}{K} = -\frac{12G}{t^2}\varepsilon_c. \tag{51}$$

Thus the stress function now must satisfy the equation

$$\nabla^2\nabla^2\Phi + (1-\nu)\frac{t}{t_p}\Big(\frac{12G}{t^2}\Big(\frac{p}{K} - \varepsilon_c\Big)\Big) = 0. \tag{52}$$

With p given by (48), we have

$$\nabla^2\nabla^2\Phi - (1-\nu)\frac{t}{t_p}\frac{12G}{t^2}\frac{I_0(\lambda r)}{I_0(\lambda R)}\varepsilon_c = 0. \tag{53}$$

The solution for pure compression is rotationally symmetric, and the complementary part of the solution remains the same, but it is necessary to determine the particular integral that corresponds to the term $I_0(\lambda r)$. To develop this, we recall that in radial symmetry,

$$\nabla^2\nabla^2(f) = \frac{1}{r}\frac{d}{dr}\Bigl(r\frac{d}{dr}\Bigl(\frac{1}{r}\frac{d}{dr}\Bigl(r\frac{df}{dr}\Bigr)\Bigr)\Bigr). \tag{54}$$

By taking $f = I_0(\lambda r)$, changing the variable to $x = \lambda r$, and recalling that x so defined is dimensionless and that λ is a reciprocal length, we have $\nabla^2\nabla^2\Phi|_r = (1/\lambda^4)\nabla^2\nabla^2\Phi|_x$. Thus we have successively

$$\begin{aligned}
&\frac{1}{x}\frac{d}{dx}\Bigl(x\frac{d}{dx}\Bigl(\frac{1}{x}\frac{d}{dx}\Bigl(x\frac{df}{dx}\Bigr)\Bigr)\Bigr) = I_0(x),\\
&x\frac{d}{dx}\Bigl(\frac{1}{x}\frac{d}{dx}\Bigl(x\frac{df}{dx}\Bigr)\Bigr) = \int xI_0(x)dx = xI_1(x),\\
&\frac{1}{x}\frac{d}{dx}\Bigl(x\frac{df}{dx}\Bigr) = \int I_1(x)dx = I_0(x),\\
&x\frac{df}{dx} = \int xI_0(x)dx = xI_1(x),\\
&\quad f = I_0(x).
\end{aligned}$$

The result for the stress function is $\Phi = A\ln r + Br^2\ln r + Cr^2 + D + \frac{1}{\lambda^4}\bar{C}I_0(\lambda r)$, where now the constant $\bar{C}$ is given by

$$\bar{C} = (1-\nu)\frac{t}{t_p}\frac{12G}{t^2}\frac{1}{I_0(\lambda R)}\varepsilon_c. \tag{55}$$

The stresses in the plate are now given by

$$\begin{aligned}
\sigma_r - \frac{t}{t_p}p(r) &= \frac{A}{r^2} + B(1+2\ln r) + 2C + \frac{\bar{C}}{\lambda^2}\frac{I_1(\lambda r)}{\lambda r},\\
\sigma_\theta - \frac{t}{t_p}p(r) &= -\frac{A}{r^2} + B(3+2\ln r) + 2C + \frac{\bar{C}}{\lambda^2}\Bigl(I_0(\lambda r) - \frac{I_1(\lambda r)}{\lambda r}\Bigr).
\end{aligned}$$

Since in the case of a complete plate A and B must vanish, and the pressure is zero at the outside radius, the condition that $\sigma_r(R) = 0$ means that

$$C = -\frac{\bar{C}}{2\lambda^2}\frac{I_1(\lambda R)}{\lambda R},$$

and, with $p(r)$ given by (48), we have

$$\begin{aligned}
\sigma_r &= \frac{t}{t_p}K\varepsilon_c\Bigl(1 - \frac{I_0(\lambda r)}{I_0(\lambda R)} - \frac{1-\nu}{I_0(\lambda R)}\Bigl(\frac{I_1(\lambda R)}{\lambda R} - \frac{I_1(\lambda r)}{\lambda r}\Bigr)\Bigr),\\
\sigma_\theta &= \frac{t}{t_p}K\varepsilon_c\Bigl(1 - \frac{I_0(\lambda r)}{I_0(\lambda R)} - \frac{1-\nu}{I_0(\lambda R)}\Bigl(\frac{I_1(\lambda R)}{\lambda R} + \frac{I_1(\lambda r)}{\lambda r} - I_0(\lambda r)\Bigr)\Bigr).
\end{aligned} \tag{56}$$

At the center of the plate, the stresses become

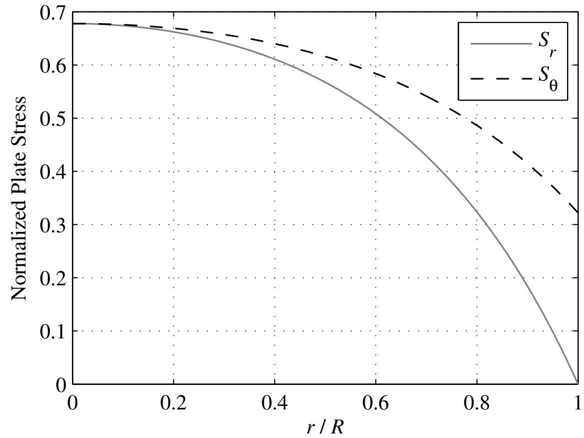

Figure 7. Plate stresses in compression for the large shape factor case.

$$\sigma_r = \sigma_\theta = \frac{t}{t_p} K \varepsilon_c \left(1 - \frac{1}{I_0(\lambda R)} - \frac{1-\nu}{I_0(\lambda R)} \left(\frac{I_1(\lambda R)}{\lambda R} - \frac{1}{2} \right) \right). \tag{57}$$

To illustrate these results, we look at the case when the shape factor is 30, the rubber shear modulus is 0.42 MPa, and the bulk modulus is 2000 MPa, giving $\lambda R = 3$. The normalized plate stresses denoted by $S_r = \sigma_r / ((t/t_p) K \varepsilon_c)$ and $S_\theta = \sigma_\theta / ((t/t_p) K \varepsilon_c)$ are shown as functions of r/R in Figure 7.

If we compare the maximum value of the stress at the center of the plate of this solution with the same result for the incompressible case, we can use (50) to determine the average pressure, and (57) to get

$$\frac{\sigma_{\max}}{p_{\text{ave}}} = \frac{t}{t_p} \frac{I_0(\lambda R) - 1 - (1-\nu)\big(I_1(\lambda R)/(\lambda R) - \frac{1}{2}\big)}{I_0(\lambda R) - 2 I_1(\lambda R)/(\lambda R)}. \tag{58}$$

Figure 8 is a graph of (58) for a range of values of λR from 0 to 5 (0 being the incompressible case). It shows that for the same value of ε_c and t/t_p, the peak stresses are smaller. This implies that the result for the incompressible computation is a conservative estimate of the stresses in the large shape factor case,

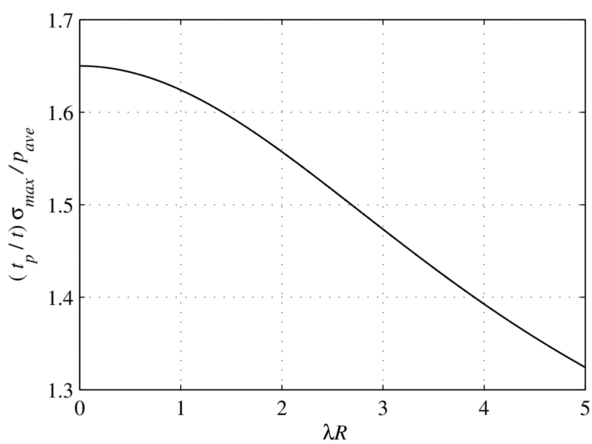

Figure 8. Ratio of maximum plate stresses for compressible and incompressible cases.

which, if it can be used, greatly simplifies the prediction of the maximum value of the tension stress in this case.

5. Steel stresses due to bending in circular pads with large shape factors

For a circular pad, the equation to be solved is

$$p_{,rr} + \frac{1}{r} p_{,r} + \frac{1}{r^2} p_{,\theta\theta} - \lambda^2 p = \frac{\alpha}{t} \lambda^2 K r \cos\theta, \tag{59}$$

with $p = 0$ at $r = R$.

The result for $p(r, \theta)$ is

$$p = \frac{\alpha K}{t}\Big(R\frac{I_1(\lambda r)}{I_1(\lambda R)} - r\Big)\cos\theta = K\varepsilon_b\Big(\frac{I_1(\lambda r)}{I_1(\lambda R)} - \frac{r}{R}\Big)\cos\theta. \tag{60}$$

It follows that the $\nabla^2 p$ reduces to

$$\nabla^2 p = \frac{12G}{t^2}\frac{I_1(\lambda r)}{I_1(\lambda R)}\cos\theta\varepsilon_b, \tag{61}$$

and that the equation to be solved for the stress function becomes

$$\nabla^2\nabla^2\Phi + (1-\nu)\frac{t}{t_p}\frac{12G}{t^2}\frac{I_1(\lambda r)}{I_1(\lambda R)}\cos\theta\varepsilon_b = 0. \tag{62}$$

We look for a solution of the form

$$\Phi(r, \theta) = f(r)\cos\theta.$$

From [Timoshenko and Goodier 1970], the ordinary differential equation for f is

$$\frac{d^4 f}{dr^4} + \frac{2}{r}\frac{d^3 f}{dr^3} - \frac{3}{r^2}\frac{d^2 f}{dr^2} + \frac{3}{r^3}\frac{df}{dr} - \frac{3}{r^4} f + \bar{C} I_1(\lambda r) = 0, \tag{63}$$

where for this case we have

$$\bar{C} = (1-\nu)\frac{t}{t_p}\frac{12G}{t^2}\frac{\varepsilon_b}{I_1(\lambda R)}. \tag{64}$$

The complementary part of the solution is

$$f(r) = Ar^3 + \frac{B}{r} + Cr + Dr\ln r,$$

and the particular integral can be shown (after considerable algebraic manipulation) to be $-(\bar{C}/\lambda^4)I_1(\lambda r)$.

The resulting stresses are

$$\sigma_r - \frac{t}{t_p} p = \Big(2Ar - \frac{2B}{r^3} + \frac{D}{r}\Big)\cos\theta + \bar{R},$$

$$\sigma_\theta - \frac{t}{t_p} p = \Big(6Ar + \frac{2B}{r^3} + \frac{D}{r}\Big)\cos\theta + \bar{S},$$

$$\tau_{r\theta} = \Big(2Ar - \frac{2B}{r^3} + \frac{D}{r}\Big)\sin\theta + \bar{T},$$

where
$$\bar{R}=\Big(\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\Big)\Big(-\frac{\bar{C}I_1(\lambda r)\cos\theta}{\lambda^4}\Big)=\frac{2I_1(\lambda r)-I_0(\lambda r)\lambda r}{\lambda^2 r^2}\frac{\bar{C}}{\lambda^2}\cos\theta,$$
$$\bar{S}=\frac{\partial^2}{\partial r^2}\Big(-\frac{\bar{C}I_1(\lambda r)\cos\theta}{\lambda^4}\Big)=\frac{I_0(\lambda r)\lambda r-I_1(\lambda r)(2+\lambda^2 r^2)}{\lambda^2 r^2}\frac{\bar{C}}{\lambda^2}\cos\theta,$$
$$\bar{T}=-\frac{\partial}{\partial r}\bigg(\frac{1}{r}\frac{\partial}{\partial\theta}\Big(-\frac{\bar{C}I_1(\lambda r)\cos\theta}{\lambda^4}\Big)\bigg)=\frac{2I_1(\lambda r)-I_0(\lambda r)\lambda r}{\lambda^2 r^2}\frac{\bar{C}}{\lambda^2}\sin\theta.$$

For the complete plate $0\le r\le R$, both B and D must vanish and $\sigma(R,\theta)=0$, giving
$$A=\frac{1}{2R}\frac{I_0(\lambda R)\lambda R-2I_1(\lambda R)}{\lambda^2 R^2}\frac{\bar{C}}{\lambda^2}.$$

Substituting for $p(r,\theta)$ from (60) and $\bar{C}$ from (64), and making use of λ^2 from (47), leads to
$$\sigma_r=K\varepsilon_b\frac{t}{t_p}\bigg(\frac{I_1(\lambda r)}{I_1(\lambda R)}-\frac{r}{R}+\frac{1-\nu}{I_1(\lambda R)}\Big(\frac{I_0(\lambda R)\lambda R-2I_1(\lambda R)}{\lambda^2R^2}\frac{r}{R}-\frac{I_0(\lambda r)\lambda r-2I_1(\lambda r)}{\lambda^2 r^2}\Big)\bigg)\cos\theta,$$
$$\sigma_\theta=K\varepsilon_b\frac{t}{t_p}\bigg(\frac{I_1(\lambda r)}{I_1(\lambda R)}-\frac{r}{R}+\frac{1-\nu}{I_1(\lambda R)}\Big(3\frac{I_0(\lambda R)\lambda R-2I_1(\lambda R)}{\lambda^2R^2}\frac{r}{R}+\frac{I_0(\lambda r)\lambda r-I_1(\lambda r)(\lambda^2r^2+2)}{\lambda^2 r^2}\Big)\bigg)\cos\theta,$$
$$\tau_{r\theta}=K\varepsilon_b\frac{t}{t_p}\frac{1-\nu}{I_1(\lambda R)}\Big(\frac{I_0(\lambda R)\lambda R-2I_1(\lambda R)}{\lambda^2R^2}\frac{r}{R}-\frac{I_0(\lambda r)\lambda r-2I_1(\lambda r)}{\lambda^2 r^2}\Big)\sin\theta. \tag{65}$$

These results are plotted in Figure 9 again for the case where $\lambda R=3$.

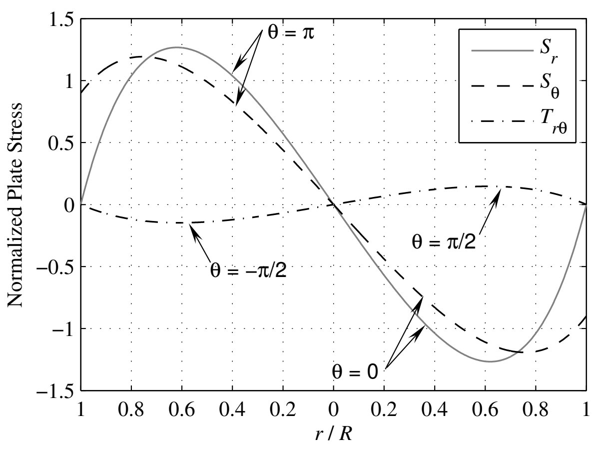

Figure 9. Plate stresses for bending in the large shape factor case.

6. Conclusion

The essential characteristic of the elastomeric isolator is the very large ratio of the vertical stiffness relative to the horizontal stiffness. This is produced by the reinforcing plates, which in current industry standard are thin steel plates. These plates prevent lateral bulging of the rubber, but allow the rubber to shear freely. The vertical stiffness can be several hundred times the horizontal stiffness. The steel reinforcement has the effect of generating shear stresses in the rubber in the isolator and these stresses act on the steel plates to cause tension stresses which, if they were to become large enough, could result in failure of the steel shims through yielding or fracture. The external pressure on the isolator at which this might happen is an important design quantity for an isolator and it is therefore necessary to be able to estimate these tensile stresses under the applied external load. When the isolator is subject to shear deformation, a bending moment is generated by the unbalanced shear forces at the top and bottom of the bearing. These bending moments also affect the stresses in the steel shims and, in contrast to the case of pure compression, the stresses in the steel plates can be compressive and could produce buckling of the shims. No previous solution for the stresses due to bending moments in a bearing has been available. Here we provide analyses for the circular isolator for the two cases: namely, when the rubber can be assumed to be incompressible, generally in the low to moderate shape factor case, and also for the more typical situation, when the shape factor is sufficiently large that compressibility of the rubber needs to be taken into account in the theory. The solution for the compressible case both in pure compression and bending is new.

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JAMES M. KELLY: jmkelly@berkeley.edu
University of California at Berkeley, Pacific Earthquake Engineering Research Center, 1301 South 46th Street, Richmond, CA 94804-4698, United States

DIMITRIOS KONSTANTINIDIS: dakon@berkeley.edu
University of California at Berkeley, Pacific Earthquake Engineering Research Center, 1301 South 46th Street, Richmond, CA 94804-4698, United States