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EXPONENTIAL SOLUTIONS FOR A LONGITUDINALLY VIBRATING INHOMOGENEOUS ROD

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A special class of closed form solutions for inhomogeneous rods is investigated, arising from the following problem: for a given distribution of the material density, find the axial rigidity of an inhomogeneous rod so that the *exponential* mode shape serves as the vibration mode. Specifically, for a rod clamped at one end and free at the other, the exponentially varying vibration mode is postulated and the associated semi-inverse problem is solved. This yields distributions of axial rigidity which, together with a specific law of material density, satisfy the governing eigenvalue problem. The results obtained can be used in the context of functionally graded materials for vibration tailoring, that is, for the design of a rod with a given natural frequency according to a postulated vibration mode.

1. Introduction

Recently, several closed-form solutions have been derived by the semi-inverse method [Elishakoff 2005] for the problem of eigenvalues of inhomogeneous structures. In particular Candan and Elishakoff [2001] solved the problem of construction of a bar with a specified mass density and a preselected polynomial mode shape, while Ram and Elishakoff [2004] solved the analogous problem in the discrete setting. It turns out that a bar with a tip mass [Elishakoff and Perez 2005] or with a translational spring [Elishakoff and Yost ≥ 2009] can also possess a polynomial mode shape.

In a personal communication to the second author (2007), Dr. A. R. Khvoles posed the question of whether or not an inhomogeneous rod may possess an exponential mode shape. This question is elucidated in the present study. The solution can serve as a benchmark for the validation of various approximate analyses and numerical techniques.

Formulation of problem. Let us consider an inhomogeneous rod of length L, cross-sectional area $A(x)$, varying modulus of elasticity $E(x)$, and varying material density $\rho(x)$. The governing differential equation of the dynamic behavior of such an inhomogeneous rod is given by

$$\frac{\partial}{\partial x}\left[E(x)A(x)\frac{\partial u(x,t)}{\partial x}\right]-\rho(x)A(x)\frac{\partial u^2(x,t)}{\partial t^2}=0, \tag{1}$$

where x is the axial coordinate, t the time, and $u(x,t)$ the axial displacement.

For simplicity, the nondimensional coordinate $\xi = x/L$ is introduced. Harmonic vibration is studied so that the displacement $u(x,t)$ is represented as

$$u(\xi,t)=U(\xi)e^{i\omega t}, \tag{2}$$

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where $U(\xi)$ is the postulated mode shape and ω the corresponding natural frequency which has to be determined. Upon substitution of Equation (2) into (1), the latter becomes

$$\frac{d}{d\xi}\left[E(\xi)A(\xi)\frac{dU(\xi)}{d\xi}\right]+L^2\rho(\xi)A(\xi)\omega^2U(\xi)=0. \tag{3}$$

The semi-inverse eigenvalue problem is posed as follows: Find an inhomogeneous rod that with reference to a specified exponential mode, $U(\xi)$, satisfies its boundary conditions and the governing dynamic equation of motion. This semi-inverse problem requires the determination of the distribution of axial rigidity, $D(\xi)=E(\xi)A(\xi)$, that together with a prespecified law for the mass distribution, $m(\xi)=A(\xi)\rho(\xi)$, satisfies (3).

We postulate the following form for the mode shape:

$$U(\xi)=A_0+A_1\xi\exp(\lambda\xi). \tag{4}$$

In this study, the differential equation (1) will be solved in a closed form for a rod that is clamped at one end and free at the other.

2. Clamped-free rod

We consider an inhomogeneous rod for which the following boundary conditions must be satisfied:

$$U(0)=0, \qquad U'(0)\neq 0, \tag{5}$$

$$U(1)\neq 0, \qquad N(1)=0, \tag{6}$$

where $N(1)$ is the axial force at $\xi=1$, namely $N(1)=E(1)A(1)U'(1)/L$. Therefore in order to satisfy the boundary condition the mode shape assumes the form

$$U(\xi)=A_1\xi\exp(-\xi), \qquad U'(\xi)=A_1(1-\xi)\exp(-\xi), \tag{7}$$

whose graph, for $A_1=1$, is shown in Figure 1.

Assuming that the mode shape is known, by integrating (3) we obtain

$$E(\xi)A(\xi)\frac{dU(\xi)}{d\xi}=-\omega^2L^2\int_0^{\xi}\rho(\eta)A(\eta)U(\eta)\,d\eta+N(0)L, \tag{8}$$

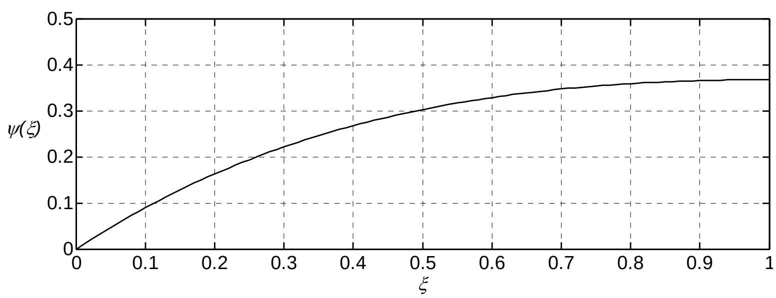

Figure 1. Postulated mode shape, (7).

where $N(0)$ is the amplitude of the axial loading at the cross-section $\xi = 0$. For the clamped-free bar the boundary conditions (6) become

$$N(1) = \frac{E(1)A(1)}{L} \frac{dU}{d\xi}\bigg|_{\xi=1} = 0. \tag{9}$$

By evaluating (8) at $\xi = 1$ and employing the boundary condition (9) the following value of $N(0)$ is obtained:

$$N(0) = \omega^2 L \int_0^1 \rho(\alpha)A(\alpha)U(\alpha)\,d\alpha. \tag{10}$$

This condition coincides with [Elishakoff et al. 2001, equation 23]. Substitution of (10) into (8) yields

$$E(\xi)A(\xi)\frac{dU}{d\xi} = \omega^2 L^2 \int_\xi^1 \rho(\alpha)A(\alpha)U(\alpha)\,d\alpha. \tag{11}$$

In the semi-inverse formulation, the mode shape $U(\xi)$ is a postulated function, that is,

$$U(\xi) = \psi(\xi). \tag{12}$$

Substitution into (11) yields the desired axial rigidity

$$D(\xi) = E(\xi)A(\xi) = \frac{\omega^2 L^2}{\psi'(\xi)} \int_\xi^1 \rho(\alpha)A(\alpha)\psi(\alpha)\,d\alpha. \tag{13}$$

The candidate mode shape ought to satisfy the boundary conditions. Considering the candidate mode shape $\psi(\xi) = \xi \exp(-\xi)$, the following particular cases arise:

Case 1: Constant cross-sectional area and constant material density. When

$$A(\xi) = \text{const} = A_0, \qquad \rho(\xi) = \text{const} = \rho_0, \tag{14}$$

then (13) becomes

$$D(\xi) = A_0\rho_0\omega^2 L^2 \frac{e - 2e^\xi + e\xi}{e(1-\xi)}. \tag{15}$$

It is easy to verify that $D(0) > 0$ at $\xi = 0$. By applying L'Hospital's rule at $\xi = 1$, we observe that $D(1) > 0$. Therefore, assuming the distribution of axial rigidity reported in Figure 2a,

$$D(\xi) = D_0 \frac{e - 2e^\xi + e\xi}{e(1-\xi)} \tag{16}$$

in conjunction with the postulated mode shape in (7), shown in Figure 1, and the axial distribution

$$N(\xi) = D_0 \frac{e - 2e^\xi + e\xi}{e(1-\xi)} \psi'(\xi), \tag{17}$$

represented in Figure 2b, the following eigenvalue parameter is obtained: $\omega^2 = D_0/A_0\rho_0 L^2$.

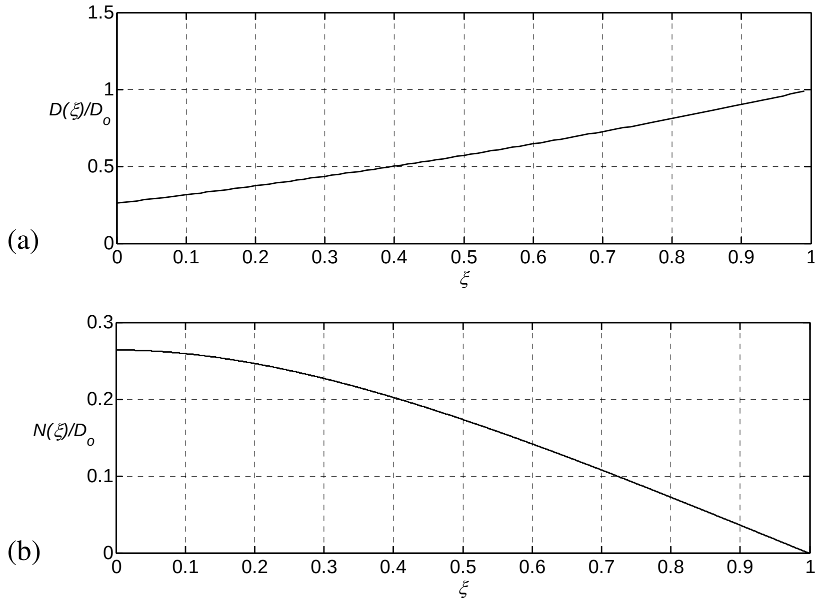

Figure 2. Variation in the axial modulus (a) and axial force (b) in an inhomogeneous bar, corresponding to the mode shape in Figure 1.

Case 2: Variable cross-sectional area and constant material density. When

$$A(\xi) \neq \text{const}, \qquad \rho(\xi) = \text{const} = \rho_0, \tag{18}$$

then (13) gives

$$E(\xi) = \frac{\omega^2 L^2}{A(\xi)\psi'(\xi)} \int_\xi^1 \rho(\eta)A(\eta)U(\eta)\,d\eta. \tag{19}$$

As an example we assume the following form for $A(\xi)$:

$$A(\xi) = A_0\big(1 + \alpha\xi \exp(-\xi)\big), \tag{20}$$

with $A_0 > 0$ and $\alpha > -1$. Integrating (19), we obtain for the Young's modulus

$$E(\xi) = \omega^2 L^2 \rho_0 \frac{4e^{1+\xi}\big(2e^\xi - e(1+\xi)\big) + \alpha\big(5e^{2\xi} - e^2(1+2\xi+2\xi^2)\big)}{4e^2(\xi-1)(e^\xi+\alpha\xi)}. \tag{21}$$

In view of (20) and (21), the axial stiffness becomes

$$D(\xi) = A_0\rho_0\omega^2 L^2 \frac{4 - 8e^{\xi-1} + 4\xi + \alpha e^{-\xi-2}\big(-5e^{2\xi} + e^2(1+2\xi+2\xi^2)\big)}{4(1-\xi)}. \tag{22}$$

In Figure 3, the area variability $A(\xi)$, the Young modulus $E(\xi)$, the axial stiffness $D(\xi)$, and the axial force $N(\xi)$ are reported for the case $\alpha = 1$.

It is worth noticing that the solution that has been derived is not reducible to the case where the variation of elastic modulus, density, and cross sectional area are constants.

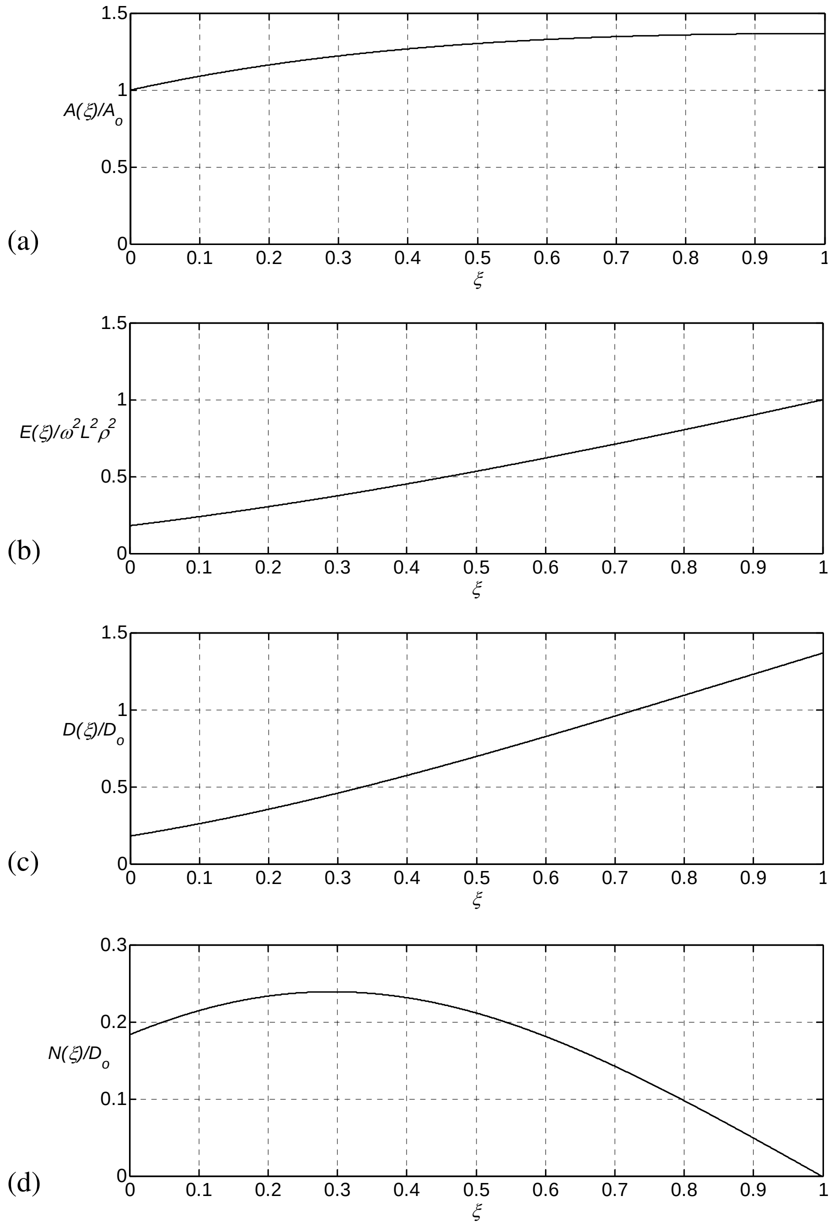

Figure 3. Variation of the cross-sectional area (a), modulus of elasticity (b), axial rigidity (b) and axial force (d) versus a nondimensional coordinate.

3. Conclusion

Apparently for the first time in the literature, it is shown that an inhomogeneous rod can possess an exponential mode shape. The derived closed-form solution can be utilized as a model solution for verification purposes.

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