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DYNAMIC BUCKLING OF A BEAM ON A NONLINEAR ELASTIC FOUNDATION UNDER STEP LOADING

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An analytical model is presented for the nonlinear behavior of a beam on a nonlinear elastic foundation, subjected to sudden axial compression. Two dynamic buckling criteria, one based on full dynamic analysis (Budiansky–Roth) and the other on static analysis only (Hoff–Simitses), were applied. The effectiveness of the Hoff–Simitses criterion for structures characterized by a limit point was shown.

1. Introduction

Structures on a nonlinear elastic foundation are commonly used in engineering applications. Specifically, beams on such a foundation occupy a prominent place in structural mechanics, and can serve as a simplified model for complex nonlinear systems. The foundation can be characterized as either hardening or softening, the latter type being associated with a limit point behavior instead of bifurcation one.

Most research works on this subject were devoted to static stability, and much less (to the best of the authors' knowledge) to dynamic buckling, in spite of its practical importance. Specifically, Weitsman [1969] and Kamiya [1977] studied beams on a bimodulus and no-tension elastic foundation. The bifurcation-type behavior and the initial postbuckling one were addressed in [Fraser and Budiansky 1969; Amazigo et al. 1970; Keener 1974; Lee and Wass 1996; Kounadis et al. 2006]. Sheinman and Adan [1991] investigated the imperfection sensitivity of a beam on a nonlinear elastic foundation under static loading, including the effect of transverse shear deformation.

The term "dynamic buckling" refers to stability of a structure under time-dependent loads. It can also be used in a broader sense, covering stability analysis via the equations of motion, irrespective of the type of load. Accordingly, different dynamic buckling/stability criteria have been suggested [Budiansky and Roth 1962; Hsu 1966; 1967; Hoff and Bruce 1954; Simitses 1967; 1990], mainly based on the concept of bounded motion as proof of dynamic buckling/stability.

The theory of dynamic buckling of systems with a single degree of freedom subjected to step loading was developed in [Budiansky and Hutchinson 1966; Hutchinson and Budiansky 1966; Budiansky 1967; Elishakoff 1980]. These authors derived the relationships between the critical step load and the amplitude of the initial imperfection for structures with quadratic, cubic and quadratic-cubic nonlinearities. The effect of Rayleigh's dissipative forces was included in [Kounadis and Raftoyiannis 1990]. The extension of the aforementioned studies to potential and nonpotential systems with multiple degrees of freedom was developed in [Kounadis et al. 1991; Kounadis 1997; Kounadis et al. 1997; 1999; Raftoyiannis and Kounadis 2000; Gantes et al. 2001; Kounadis et al. 2001]. A comprehensive review on the dynamic

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buckling of elastic structures such as frames, arches, and shells can be found in [Simitses 1990]. Specifically, Birman [1989] studied the dynamic buckling of antisymmetrically laminated angle-ply rectangular plates due to axial step loads. Dube et al. [2000] studied the dynamic buckling of laminated thick shallow spherical cap.

The present study deals jointly with dynamic buckling and imperfection sensitivity. The dynamic buckling load was obtained and examined using the Budiansky–Roth criterion [1962], for which a complex full dynamic analysis is needed, and also for the total potential energy criterion [Hoff and Bruce 1954; Simitses 1967], where static nonlinear analysis suffices. The purpose of the comparison is to show the advantage in treating dynamic stability problems via the second, purely static, criterion.

The dynamic nonlinear partial differential equations are derived for a general beam on a nonlinear elastic foundation. These partial differential equation were reduced to ordinary nonlinear equations by introducing the Bathe composite method [Bathe and Baig 2005; Bathe 2007]. Then, the Newton–Raphson linearization, and finite difference scheme were used for solving the resulting nonlinear system of ordinary equations. An example of a beam on a softening foundation was considered to study the dynamic buckling and imperfection sensitivity under static and dynamic step loading.

2. Dynamic stability criteria

In the Budiansky and Roth criterion, the dynamic buckling load is defined as the level at which a large increase occurs in the displacement amplitude. In the Hoff–Simitses criterion, the critical load is defined as the static postlimit load level at which the modified total potential energy is zero. The latter is obtained by subtracting the total potential energy of the unbuckled state from the total potential energy, thus eliminating all trajectories nested in the total potential energy but not leading to buckling. This criterion corresponds to the lower bound of the critical conditions; for example, for an external applied step load, and for autonomous mechanical systems in general.

3. Problem formulation and solution procedure

Figure 1 illustrates a beam on a nonlinear elastic foundation subjected to axial step loading. By Bernoulli–Euler beam theory, the equations of motion (with rotary inertia neglected) read

$$\begin{aligned} -\rho A\ddot{u} + N_{xx,x} + q_u = 0, \\ -\rho A\ddot{w} + M_{xx,xx} + \big(N_{xx}(w_{,x} + \overline{w}_{,x})\big)_{,x} - R(w) + q_w = 0, \end{aligned} \tag{3-1}$$

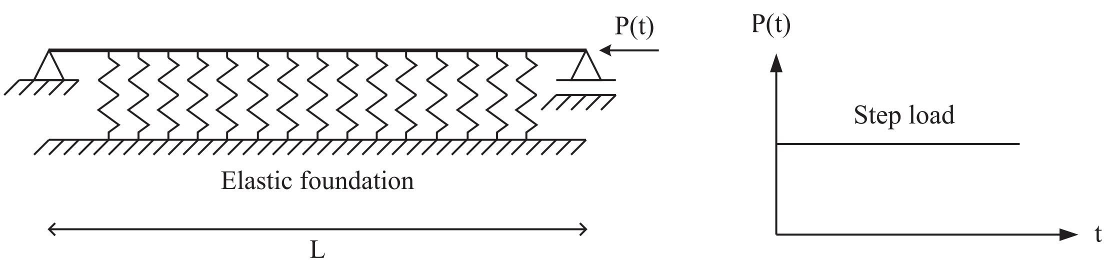

Figure 1. Beam on elastic foundation subjected to axial step load.

where $u = u(x, t)$ and $w = w(x, t)$ are the axial and normal displacements of the beam, respectively; $\bar{w}_x$ is the initial geometrical imperfection; A and I are the cross-section area and moment of inertia; N_{xx} and M_{xx} the resultant axial force and bending moment; q_u and q_w the external applied loads in the axial and normal directions, respectively. The superior dot $(\dot{\ })$, denotes the derivative with respect to time, and $(\ )_{,x}$ the derivative with respect to the axial coordinate. The response of the foundation is characterized by two parameters $\{K_1,\ K_3\}$ (see [Sheinman and Adan 1991]) and described by the function

$$R(w) = K_1 w + K_3 w^3. \tag{3-2}$$

The nonlinear kinematic relations for the beam entail the assumption of large displacements, moderate rotations, and small strains. Thus the constitutive relations (resultant axial force and bending moment) and the kinematic relations (axial strain and change of curvature) read:

$$\begin{Bmatrix} N_{xx} \\ M_{xx} \end{Bmatrix} = \begin{bmatrix} A_{11} & 0 \\ 0 & D_{11} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \kappa_{xx} \end{Bmatrix}, \qquad \begin{Bmatrix} \varepsilon_{xx} \\ \kappa_{xx} \end{Bmatrix} = \begin{Bmatrix} u_{,x} + \frac{1}{2}(w_{,x} + 2\bar{w}_{,x})w_{,x} \\ -w_{,xx} \end{Bmatrix}, \tag{3-3}$$

A_{11} and D_{11} being the axial and flexural rigidities. Specifically, for isotropic materials the rigidities are given by $A_{11} = EA$, and $D_{11} = EI$.

This sixth-order set of nonlinear partial differential equations is converted into an equivalent set of six first-order ones by recourse to the following variables:

$$z = \{u,\ w,\ \phi_x,\ N_{xx},\ Q_{xz},\ M_{xx}\}^T \tag{3-4}$$

The equivalent set, $\boldsymbol{\psi} = \{\psi_1,\ \psi_2,\ \psi_3,\ \psi_4,\ \psi_5,\ \psi_6\}^T$, reads

$$\boldsymbol{\psi} = \begin{Bmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \\ \psi_5 \\ \psi_6 \end{Bmatrix} = \begin{Bmatrix} -\rho A\ddot{u} + N_{xx,x} + q_u \\ -\rho A\ddot{w} + Q_{xz,x} - R(w) + q_w \\ M_{xx,x} - Q_{xz} - N_{xx}(\phi_x - \bar{w}_{,x}) \\ N_{xx} - A_{11}u_{,x} - \frac{1}{2}A_{11}(\phi_x - 2\bar{w}_{,x})\phi_x \\ \phi_x + w_{,x} \\ M_{xx} - D_{11}\phi_{x,x} \end{Bmatrix} = \mathbf{0}, \tag{3-5}$$

with the following boundary conditions at $x = 0$ and $x = L$:

$$\begin{aligned} N_{xx} &= \overline{N}_{xx} \quad \text{or} \quad u = \bar{u}, \\ Q_{xz} &= \overline{Q}_{xz} \quad \text{or} \quad u = \bar{w}, \\ M_{xx} &= \overline{M}_{xx} \quad \text{or} \quad \phi_x = \bar{\phi}_x, \end{aligned} \tag{3-6}$$

where the bar denotes an applied force or displacement at the boundaries.

No unconditionally stable time integration procedure exists for the dynamic solution of nonlinear equations. Here, the composite implicit time integration procedure [Bathe and Baig 2005; Bathe 2007] was chosen for solving (3-5) in the time space. Assuming that the dynamic solution at time t (i.e. z_t, $\dot{z}_t$ and $\ddot{z}_t$) is completely known, the solution at time $t + \Delta t$ is computed by introducing the substep at time $t + \theta\Delta t$, where $\theta \in \{0, 1\}$ (for instance, $\theta = 1/2$). Specifically, using Newmark's scheme [1959], the velocity and acceleration fields are explicitly given in terms of the displacement field at time $t+\theta\Delta t$, and the displacement, velocity and acceleration fields at t:

$$\begin{aligned} \dot{z}^{t+\theta\Delta t} &= \frac{\gamma}{\beta(\theta\Delta t)}(z^{t+\theta\Delta t}-z^t)+\Big(1-\frac{\gamma}{\beta}\Big)\dot{z}^t+\Big(1-\frac{\gamma}{2\beta}\Big)(\theta\Delta t)\ddot{z}^t \\ \ddot{z}^{t+\theta\Delta t} &= \frac{1}{\beta(\theta\Delta t)^2}(z^{t+\theta\Delta t}-z^t)-\frac{1}{\beta(\theta\Delta t)}\dot{z}^t-\Big(\frac{1}{2\beta}-1\Big)\ddot{z}^t \end{aligned} \tag{3-7}$$

The solution of z at $t+\theta\Delta t$, (i.e., $z^{t+\theta\Delta t}$), was obtained, after substitution of (3-7) in (3-5), by linearizing the latter via the Newton–Raphson method:

$$\begin{aligned} &\boldsymbol{\psi}^{t+\theta\Delta t}=\boldsymbol{\psi}^{t+\theta\Delta t}(z^{t+\theta\Delta t},\ z^{t+\theta\Delta t}_{,x};\ z^t,\ \dot{z}^t,\ \ddot{z}^t)=\mathbf{0}, \\ &\frac{\partial\boldsymbol{\psi}^{t+\theta\Delta t}}{\partial z^{t+\theta\Delta t}}\Delta z^{t+\theta\Delta t}+\frac{\partial\boldsymbol{\psi}^{t+\theta\Delta t}}{\partial z^{t+\theta\Delta t}_{,x}}\Delta z^{t+\theta\Delta t}_{,x}+\boldsymbol{\psi}^{t+\theta\Delta t}=\mathbf{0}. \end{aligned} \tag{3-8}$$

Finally, equations (3-8) were solved by means of a finite-difference scheme. Once convergence was reached, the complete dynamic solution (of the first substep) was obtained using (3-7). Then in the second substep, the velocity and acceleration fields were approximated according to [Collatz 1966] by

$$\begin{aligned} \dot{z}^{t+\Delta t} &= c_1 z^t+c_2 z^{t+\theta\Delta t}+c_3 z^{t+\Delta t}, \\ \ddot{z}^{t+\Delta t} &= c_1 \dot{z}^t+c_2 \dot{z}^{t+\theta\Delta t}+c_3 \dot{z}^{t+\Delta t}, \end{aligned} \tag{3-9}$$

where the constant coefficients are given by

$$c_1=\frac{1-\theta}{\theta\Delta t},\quad c_2=\frac{-1}{(1-\theta)\theta\Delta t},\quad c_3=\frac{2-\theta}{(1-\theta)\Delta t} \tag{3-10}$$

Substitution of (3-9) in (3-5) yielded the equations needed to obtain the solution for z at time $t+\Delta t$ (i.e., $z^{t+\Delta t}$), which were treated in the same manner as above:

$$\begin{aligned} &\boldsymbol{\psi}^{t+\Delta t}=\boldsymbol{\psi}^{t+\Delta t}(z^{t+\Delta t},\ z^{t+\Delta t}_{,x};\ z^t,\ \dot{z}^t,\ \ddot{z}^t,\ z^{t+\theta\Delta t},\ \dot{z}^{t+\theta\Delta t},\ \ddot{z}^{t+\theta\Delta t})=\mathbf{0}, \\ &\frac{\partial\boldsymbol{\psi}^{t+\Delta t}}{\partial z^{t+\Delta t}}\Delta z^{t+\Delta t}+\frac{\partial\boldsymbol{\psi}^{t+\Delta t}}{\partial z^{t+\Delta t}_{,x}}\Delta z^{t+\Delta t}_{,x}+\boldsymbol{\psi}^{t+\Delta t}=\mathbf{0}. \end{aligned} \tag{3-11}$$

Once convergence of the solution ($z^{t+\Delta t}$) was achieved, the complete dynamic solution could be obtained using (3-9).

4. Results and discussion

The beam and the softening foundation used as an example in demonstrating the effectiveness of the Hoff–Simitses criterion had the following properties [Sheinman and Adan 1991]: Beam-length $L=4.0\,m$; rectangular cross-section with width $b=0.04\,m$ and depth $h=0.08\,m$; mass density $\rho=7850\,\text{kg/m}^3$; modulus of elasticity $E=2.1\times10^{11}\,\text{Nm}^{-2}$. The elastic foundation parameters are: $K_1=1000\,\text{kNm}^{-2}$ and $K_3=-100\,\text{MNm}^{-4}$. The imperfection shape was taken as

$$\bar{w}(x)=\xi h\sin(\pi x/L),\quad \xi=0.01. \tag{4-1}$$

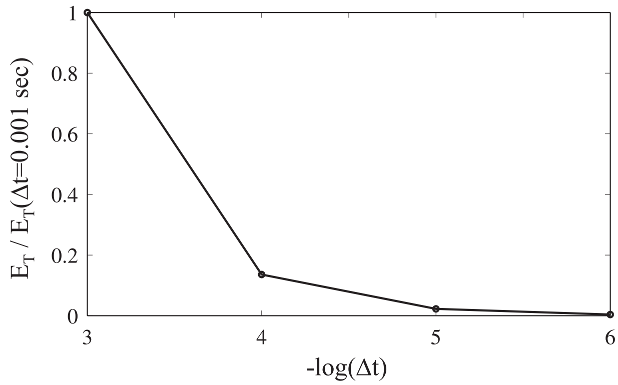

Figure 2. Convergence curve; normalized total energy versus time step Δt.

Of the examined criteria for convergence of the time-history solution with respect to the time step Δt (in seconds), the one of vanishing (beam at rest at $t = 0$) the total energy, $E_T =$ kinetic energy + total potential energy, was found to be the most representative. An example of this convergence is illustrated in Figure 2. The small time step was chosen on view of the high frequency of the characteristic behavior in the axial direction.

Figure 3 shows the time history of the vertical midspan displacement of the beam, $w(x = L/2)$, under three levels of axial-load ($\overline{N}_{xx} = 0.80\, N_{xx,\mathrm{bif}}$, $\overline{N}_{xx} = 0.85\, N_{xx,\mathrm{bif}}$ and $\overline{N}_{xx} = 0.85125\, N_{xx,\mathrm{bif}}$, where $N_{xx,\mathrm{bif}} = 222.7\,\mathrm{kN}$ is the buckling load of the perfect beam). It is seen that under the first two load levels the beam undergoes simple oscillations about the near static stable equilibrium position. By contrast, the third level is associated with large oscillations and a jump to postbuckling. Since the postbuckling equilibrium solution is unstable (see Figure 6), the dynamic solution is unbounded. Figure 4 shows the phase-plane curves of the vertical midspan displacement. It is seen that the two stable dynamic solutions form closed curves, while for the unstable one the curve diverges. It should be emphasized that the fluctuations in the phase-plane curves (Figure 4) are due to the high axial frequencies.

Figure 5 illustrates the Budiansky–Roth criterion showing the maximum vertical midspan displacement versus the applied load. Again, the equations of motion were solved for several values of the

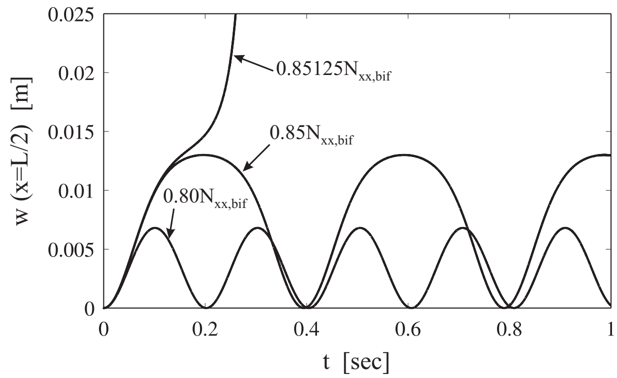

Figure 3. Vertical displacement versus time for three different load levels.

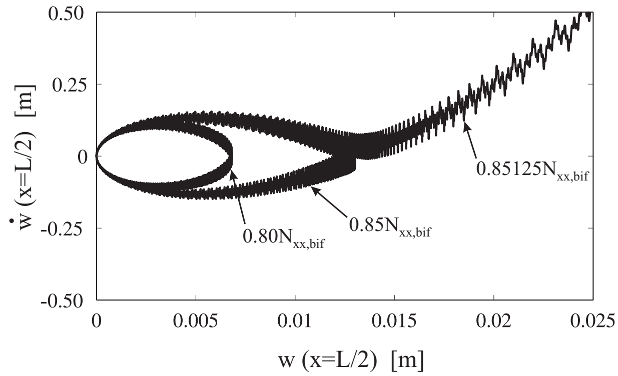

Figure 4. Phase-plane curves for three different load levels.

applied axial-load, starting from a small value and increasing it. The maximum displacement is seen to increase smoothly with the load, and culminates in a large jump (unbounded motion) at the highest level. Trail and error locates the dynamic buckling load at $N_{xx,d} = 0.85123\,N_{xx,\mathrm{bif}}$.

Figure 6 shows the nonlinear static equilibrium path used for the Hoff–Simitses criterion [Simitses 1990]. Specifically, the total potential energy, U_T, defined by

$$U_T = \frac{1}{2}\int_0^L \left(N_{xx}\varepsilon_{xx} + M_{xx}\kappa_{xx}\right) dx + \int_0^L\!\!\int R(w)dw\,dx - \int_0^L \left(q_u u + q_w w\right) dx - \left[\overline{N}_{xx}\bar{u} + \overline{Q}_{xz}\overline{w} + \overline{M}_{xx}\bar{\phi}_x\right]_0^L, \quad (4\text{-}2)$$

was modified by introducing a constant C that eliminates all trajectories that are represented in U_T but that do not lead to a buckling mode:

$$U_{T,mod} = U_T - C. \quad (4\text{-}3)$$

For the case of an axially loaded beam at $x = L$ the constant C reads

$$C = -\frac{\overline{N}^2_{xx}(L)\;L}{2A_{11}}. \quad (4\text{-}4)$$

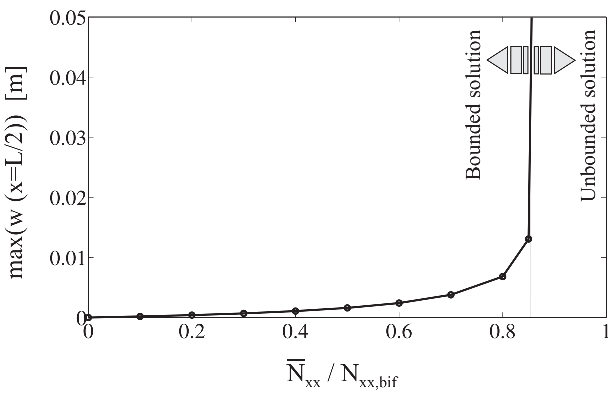

Figure 5. Maximum vertical midspan displacement versus applied load.

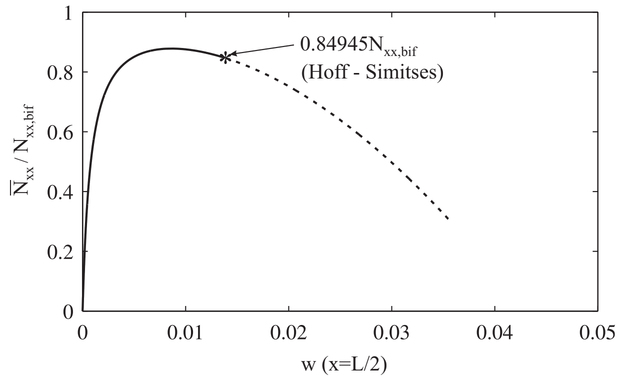

Figure 6. Applied load versus vertical midspan displacement.

Here, once the static solution is obtained (by solving (3-5) with neglecting the inertia terms), the modified total potential energy — see (4-3) — was calculated at every point of the path. The solid stretch in Figure 6 representing the path with negative modified total potential energy (bounded motion), and the dotted stretch with positive one (unbounded). It was found that this criterion (whereby the modified total potential energy is zero) yields a slightly lower dynamic buckling load ($N_{xx,d} = 0.84945\, N_{xx,\mathrm{bif}}$) than its Budiansky–Roth counterpart.

Figure 7 summarizes The dynamic sensitivity to imperfection according to the different criteria. The results are seen to be quite close (less than 1% divergence). The curve for the static buckling load (limit-point) is also plotted in this figure, and serves as an upper bound for the dynamic buckling load.

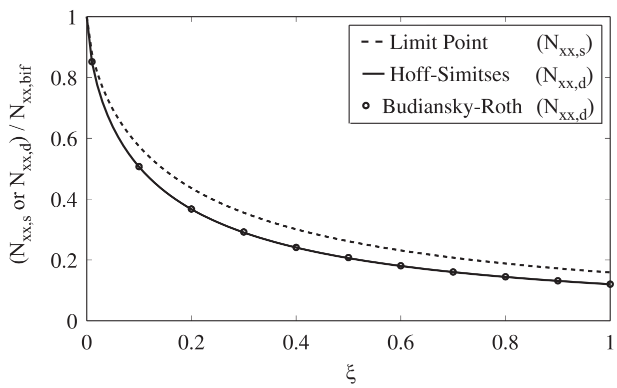

Figure 7. Imperfection sensitivity under static and dynamic step loads.

5. Summary and conclusions

A solution procedure for dynamic buckling of a beam on a nonlinear elastic foundation under dynamic step loading is presented. Two criteria (Hoff–Simitses and Budiansky–Roth) were applied and studied. It was found that the Hoff–Simitses criterion, for which static analysis suffices, is fully adequate and most effective for structures characterized by limit-point behavior. Its generalization for any dynamic loading is still a challenge.

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