Journal of
Mechanics of
Materials and Structures

A PLANE STRESS PERFECTLY PLASTIC MODE I CRACK PROBLEM FOR A YIELD CRITERION BASED ON THE SECOND AND THIRD INVARIANTS OF THE DEVIATORIC STRESS TENSOR, II

David J. Unger

Volume 4, Nº 9 November 2009

mathematical sciences publishers

JOURNAL OF MECHANICS OF MATERIALS AND STRUCTURES
Vol. 4, No. 9, 2009

A PLANE STRESS PERFECTLY PLASTIC MODE I CRACK PROBLEM FOR A YIELD CRITERION BASED ON THE SECOND AND THIRD INVARIANTS OF THE DEVIATORIC STRESS TENSOR, II

DAVID J. UNGER

In the first paper with this title (*J. Mech. Mater. Struct.* **3**:4 (2008), 795–807), the solution of a stress function of a mode I perfectly plastic crack problem was found analytically for two of the three sectors that comprise the solution of the problem for a yield condition based on the second and third invariants of the deviatoric stress tensor. Here an exact solution is derived for the remaining sector of this crack problem, which comprises the singular solution of the governing differential equation.

1. Introduction

In [Unger 2008] we obtained a statically admissible solution for the opening mode of fracture under plane stress loading conditions for a yield condition containing both the second and third invariants of the deviatoric stress tensor. The second-order nonlinear differential equation of the singular solution of this particular crack problem was reduced to a first-order differential equation of the thirtieth degree in the previous analysis. At that time an analytical solution was believed to be intractable and an approximation was used to solve the problem. Here the singular solution is reduced to quadrature by introducing a parametric formulation of the yield condition. This process allows the exact solution to be evaluated in implicit form with the aid of incomplete elliptic integrals of the first and third kinds.

In terms of the deviatoric stress invariants the yield condition employed in [Unger 2008] assumes the algebraic form

$$J_2^3 - \left(\tfrac{3}{2}J_3\right)^2 = \tfrac{2}{81}\sigma_0^6, \tag{1}$$

where

$$J_2 \equiv \tfrac{1}{3}(\sigma_1+\sigma_2)^2 - \sigma_1\sigma_2 \quad \text{and} \quad J_3 \equiv \tfrac{1}{3}(\sigma_1+\sigma_2)\left(\tfrac{2}{9}(\sigma_1+\sigma_2)^2 - \sigma_1\sigma_2\right) \tag{2}$$

are the second and third invariants [Chakrabarty 1987] of the deviatoric stress tensor, σ_1 and σ_2 being the principal stresses and σ_0 the yield stress in tension. A representation of this yield condition is shown in Figure 1 in the principal stress plane (σ_1, σ_2) along with a comparison with the Mises and Tresca yield conditions. A rotation of the coordinate axes to (ξ, η) in the principal stress plane, given by

$$\sigma_1 = \frac{\xi+\eta}{\sqrt{2}}, \qquad \sigma_2 = \frac{\xi-\eta}{\sqrt{2}}, \tag{3}$$

Keywords: plane stress, mode I crack, perfectly plastic, yield condition, second and third invariants of deviatoric stress tensor, differential algebraic equation, DAE.

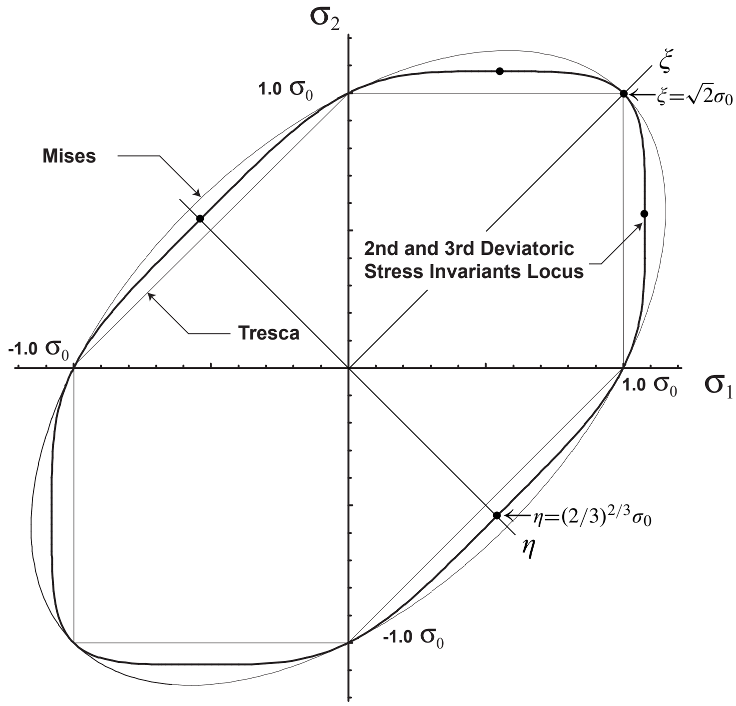

Figure 1. Three yield loci in the principal stress plane. The black dots on the darker curve correspond, clockwise from second quadrant, to $\mu = -\infty, -\frac{1}{3}, 0, \frac{1}{3}, \infty$,

further allows the yield condition to be expressed as a comparatively simple algebraic equation in the form of the sextic

$$2\xi^6 + 45\xi^4\eta^2 + 81\eta^6 = 16\sigma_0^6. \tag{4}$$

The homogeneous structure of the left-hand side of (4) suggests the introduction of the ratio μ:

$$\eta = \mu\xi. \tag{5}$$

By substituting η from (5) into (4) and solving for ξ in terms of μ, one obtains the following parametric representation of the yield locus:

$$\xi = \frac{2^{2/3}\sigma_0}{(2+45\mu^2+81\mu^6)^{1/6}}, \qquad \eta = \frac{2^{2/3}\sigma_0\mu}{(2+45\mu^2+81\mu^6)^{1/6}}. \tag{6}$$

Equations (3) and (5) allow the identification of the parameter μ in terms of the principal stresses as

$$\mu = \frac{\sigma_1-\sigma_2}{\sigma_1+\sigma_2}. \tag{7}$$

The locations of several specific values of μ are indicated on the yield locus in Figure 1.

2. Singular solution

In order to satisfy the equations of equilibrium in the plane, a stress function in polar coordinates (r, θ) was introduced in [Unger 2008]. The stresses, which are independent of the coordinate r, assume the following relationships with the stress function $f(\theta)$ and its first two derivatives with respect to θ, $f'(\theta)$ and $f''(\theta)$:

$$\sigma_\theta = 2f(\theta), \qquad \tau_{r\theta} = -f'(\theta) = -p, \qquad \sigma_r = f''(\theta) + 2f(\theta) = p\frac{dp}{df} + 2f, \tag{8}$$

where σ_r and σ_θ are the normal stresses in the radial and transverse directions, respectively, and $\tau_{r\theta}$ is the in-plane shear stress. In terms of the previously defined stress coordinates (ξ, η), the following relationships exist among the in-plane stresses:

$$\xi = \frac{\sigma_1 + \sigma_2}{\sqrt{2}} = \frac{\sigma_r + \sigma_\theta}{\sqrt{2}}, \qquad \eta = \frac{\sigma_1 - \sigma_2}{\sqrt{2}} = \frac{\sqrt{(\sigma_r - \sigma_\theta)^2 + 4\tau_{r\theta}^2}}{\sqrt{2}}. \tag{9}$$

We next define a function Q and its first derivative q with respect to f:

$$Q \equiv \frac{p^2}{2} + 2f^2 = \frac{\tau_{r\theta}^2 + \sigma_\theta^2}{2}, \tag{10}$$

$$q \equiv \frac{dQ}{df} = p\frac{dp}{df} + 4f = \sigma_r + \sigma_\theta = \sqrt{2}\xi = \frac{2^{7/6}\sigma_0}{(2 + 45\mu^2 + 81\mu^6)^{1/6}}. \tag{11}$$

The chain rule of differentiation and (11) imply that

$$\frac{dq}{df} = \frac{dq}{d\mu}\frac{d\mu}{df} = -\frac{2^{1/6}6\mu(27\mu^4 + 5)\sigma_0}{(81\mu^6 + 45\mu^2 + 2)^{7/6}}\frac{d\mu}{df}. \tag{12}$$

The Clairaut operator U [Zwillinger 1989, pp. 158–160], associated with a certain class of differential equations to which the governing equation of this problem belongs, is defined by

$$U \equiv f\frac{dQ}{df} - Q = fq - Q. \tag{13}$$

The operational procedure to find the singular solution of a Clairaut equation is to first solve the differential equation for the operator U. Next, differentiate this expression with respect to the independent variable to generate a second equation. The elimination of the first derivative of the dependent variable with respect to the independent variable between the original equation and the second equation constitutes the singular solution to the problem [Zwillinger 1989, pp. 158–160]. In our case the dependent variable is Q and the independent variable is f. The first derivative of Q with respect to f is defined as q.

It is readily determined that the following relationships hold true from (6), (8), (9)–(11), and (13):

$$\sqrt{(\sigma_r - \sigma_\theta)^2 + 4\tau_{r\theta}^2} = \sqrt{q^2 - 8U} = \sqrt{2}\eta = \frac{2^{7/6}\sigma_0\mu}{(2 + 45\mu^2 + 81\mu^6)^{1/6}}. \tag{14}$$

By substituting q from (11) into (14) and solving for U one finds

$$U = \frac{\sigma_0^2(1 - \mu^2)}{2^{2/3}(2 + 45\mu^2 + 81\mu^6)^{1/3}}. \tag{15}$$

Differentiating U from (13) with respect to f and using the definition of q from (11), one establishes that

$$\frac{dU}{df} = q + f\frac{dq}{df} - \frac{dQ}{df} = f\frac{dq}{df}. \tag{16}$$

Introducing the parametric relationship between U and μ from Equation (15) and differentiating it with respect to the function f produces

$$f\frac{dq}{df} = \frac{dU}{df} = \frac{dU}{d\mu}\frac{d\mu}{df} = -\frac{2^{1/3}\mu(81\mu^4+30\mu^2+17)\sigma_0^2}{(81\mu^6+45\mu^2+2)^{4/3}}\frac{d\mu}{df}. \tag{17}$$

Dividing (17) by (12), a parametric representation of the singular solution f is found in terms of the parameter μ as

$$f = \left(f\frac{dq}{df}\right)\Big/\frac{dq}{df} = \frac{(81\mu^4+30\mu^2+17)\sigma_0}{2^{5/6}3(27\mu^4+5)(81\mu^6+45\mu^2+2)^{1/6}}. \tag{18}$$

Solving for Q from (13) and substituting the parametric relationships for variables defined in (11), (15), and (18), one finds that

$$Q = fq - U = \frac{\sigma_0^2(81\mu^6+81\mu^4+75\mu^2+19)}{2^{2/3}3(27\mu^4+5)(81\mu^6+45\mu^2+2)^{1/3}}. \tag{19}$$

Solving (10) for p and then substituting the parametric relationships for f and Q, established in (18) and (19), produces the parametric relationship

$$p = \sqrt{2Q-4f^2} = \frac{2^{1/6}\sigma_0(9\mu^2-1)^{3/2}(9\mu^4+3\mu^2+4)^{1/2}}{3(27\mu^4+5)(81\mu^6+45\mu^2+2)^{1/6}}. \tag{20}$$

Differentiating f from (18) with respect to μ produces

$$\frac{df}{d\mu} = -\frac{3\sigma_0\mu(9\mu^2-1)^2(243\mu^8+324\mu^6+234\mu^4+148\mu^2+75)}{2^{5/6}(27\mu^4+5)^2(81\mu^6+45\mu^2+2)^{7/6}}. \tag{21}$$

Dividing (21) by (20), defines the function $h(\mu)$ as

$$h(\mu) \equiv \frac{1}{p}\frac{df}{d\mu} = \frac{d\theta}{d\mu} = -\frac{9\mu(9\mu^2-1)^{1/2}(243\mu^8+324\mu^6+234\mu^4+148\mu^2+75)}{2(9\mu^4+3\mu^2+4)^{1/2}(27\mu^4+5)(81\mu^6+45\mu^2+2)}, \tag{22}$$

which is the first derivative of the polar coordinate θ with respect to the parameter μ.

By integrating $h(\mu)$ over $d\mu$, one finds the parametric relationship between the polar angle θ and the parameter μ

$$\theta = \int h(\mu)\,d\mu + \beta, \tag{23}$$

where β is a constant of integration equal to π. The symbolic mathematics computer program Mathematica® 7 was used to evaluate this indefinite integral analytically. For conciseness, the result is reproduced here in decimal form as

$$\theta = \sum_{j=1}^{6} C_j(\mu) + \pi, \tag{24}$$

$$C_1(\mu) \equiv (-0.327975 + 0.498185i) F(\psi \mid m), \tag{25}$$

$$C_2(\mu) \equiv (-0.971129 + 0.0747711i) \Pi(-1.25 + 0.968241i;\ \psi \mid m), \tag{26}$$

$$C_3(\mu) \equiv -(0.0622228 + 0.148923i) \Pi(-0.8090192 + 0.468624i;\ \psi \mid m), \tag{27}$$

$$C_4(\mu) \equiv (0.16139 - 0.00170961i) \Pi(0.896508 - 0.265318i;\ \psi \mid m), \tag{28}$$

$$C_5(\mu) \equiv (0.315179 + 0.921599i) \Pi(1.5625 - 0.242062i;\ \psi \mid m), \tag{29}$$

$$C_6(\mu) \equiv (0.884757 - 1.34392i) \Pi(1.787511 + 4.153800i;\ \psi \mid m), \tag{30}$$

where

$$\psi \equiv \sin^{-1}\left(i\sqrt{\frac{2(9\mu^2-1)}{5+3i\sqrt{15}}} \right), \qquad m \equiv \frac{5+3i\sqrt{15}}{5-3i\sqrt{15}}, \tag{31}$$

and the first $F(\psi \mid m)$ and third $\Pi(n;\ \psi \mid m)$ incomplete elliptical integrals are defined by

$$F(\psi \mid m) \equiv \int_0^{\psi} \frac{dz}{\sqrt{1 - m\sin^2 z}}, \qquad \Pi(n;\ \psi \mid m) \equiv \int_0^{\psi} \frac{dz}{(1 - n\sin^2 z)\sqrt{1 - m\sin^2 z}}. \tag{32}$$

Functions $C_1(\mu)$ and $C_6(\mu)$ in (24) are real valued functions despite the use of complex variable notation in their representation. On the other hand, functions $\{C_2(\mu), C_5(\mu)\}$ and $\{C_3(\mu), C_4(\mu)\}$ constitute complex conjugates and must be added together in pairs to obtain real valued functions. The original exact symbolic output, which involves numerous radicals, has been truncated in decimal form here to save space. Consequently, if one wishes to manipulate the truncated form of the solution, for instance, for plotting, one may need to take the real part of Equation (24) computationally to avoid receiving an error message related to a tiny but nonzero imaginary part.

3. Crack problem

The branch of the general solution to the governing differential equation appropriate for this crack problem was obtained in [Unger 2008] and will not be repeated here. The general solution applies to sectors AOB and BOC of the crack geometry shown in the insert of Figure 2. The singular solution, defined parametrically by (18) and (24), governs the leading sector of the crack problem COD as illustrated in Figure 2. An iterative computer program was developed to determine the parameters of the general solution and to find the corresponding angles which divide the three distinct regions of the half plane subject to equilibrium. The solution parameters defined in [Unger 2008] were found by this iterative procedure as

$$c = 0.15501\sigma_0, \qquad \alpha = -0.39503, \tag{33}$$

while the corresponding angles were determined as

$$\theta_{AOB} = 0.52000 = 29.79^\circ, \qquad \theta_{AOC} = 1.76886 = 101.35^\circ. \tag{34}$$

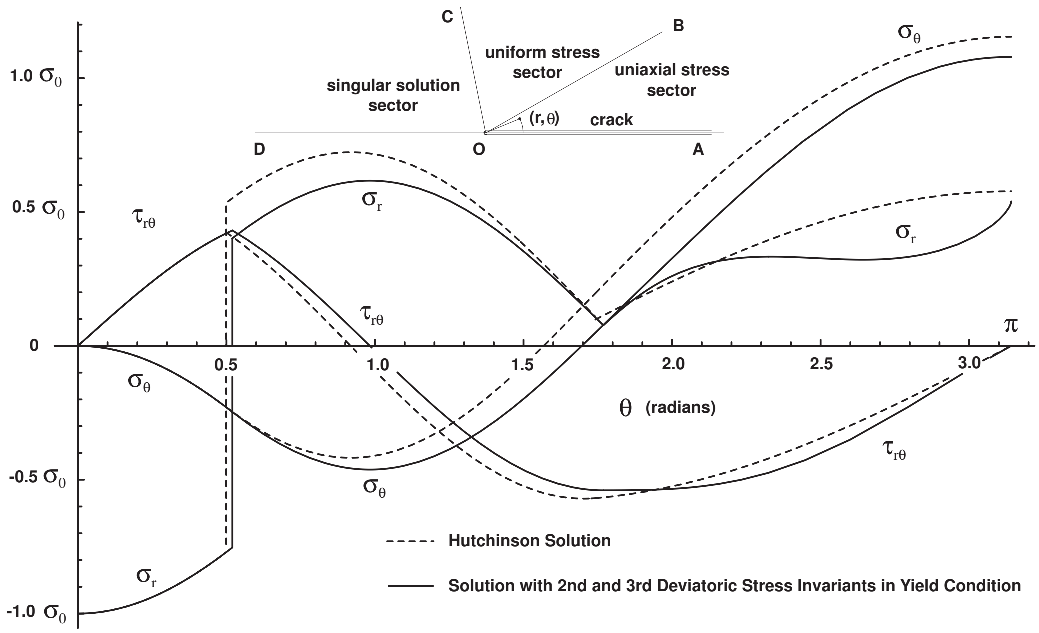

Figure 2. Comparison of perfectly plastic stress fields of the mode I fracture problem.

The stresses obtained from the stress functions are also plotted in Figure 2. For comparison the analogous solution obtained in [Hutchinson 1968] using the Mises yield condition is also shown. The results obtained here are qualitatively similar to those obtained in [Unger 2008] using an approximate singular solution. Quantitatively, the angle θ_{AOB}, determined using the approximate singular solution, differs from the value obtained using the exact singular solution by about 2.6% error. Correspondingly, the approximate value of θ_{AOC} differs from the exact value by about 1.3% error.

References

[Chakrabarty 1987] J. Chakrabarty, *Theory of plasticity*, pp. 21–23, McGraw-Hill, New York, 1987.

[Hutchinson 1968] J. W. Hutchinson, "Plastic stress and strain fields at a crack tip", *J. Mech. Phys. Solids* **16**:5 (1968), 337–347.

[Unger 2008] D. J. Unger, "A plane stress perfectly plastic mode I crack problem for a yield condition based on the second and third invariants of the deviatoric stress tensor", *J. Mech. Mater. Struct.* **3**:4 (2008), 795–807.

[Zwillinger 1989] D. Zwillinger, *Handbook of differential equations*, Academic Press, Boston, 1989.

Received 25 May 2009. Accepted 13 Jun 2009.

DAVID J. UNGER: du2@evansville.edu
Department of Mechanical and Civil Engineering, University of Evansville, 1800 Lincoln Avenue, Evansville, IN 47722, United States
http://mece.evansville.edu/faculty/unger.asp