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ELASTIC SH WAVE PROPAGATION IN A LAYERED ANISOTROPIC PLATE WITH PERIODIC INTERFACE CRACKS: EXACT VERSUS SPRING BOUNDARY CONDITIONS

ANDERS BOSTRÖM AND OLEG V. KVASHA

The propagating antiplane (SH) modes in a symmetrically three-layered, anisotropic, thick plate with a periodic array of interface cracks are investigated. The exact dispersion relation can be derived with the help of a hypersingular integral equation approach and Floquet's theorem. The interface cracks can be a model for interface damage, but a much simpler model is a recently developed spring boundary condition. This boundary condition is used both for the thick plate and in the derivation of plate equations with the help of power series expansions in the thickness coordinate. For low frequencies (cracks small compared to the wavelength) the three models are shown to give the same results and this is a confirmation that the spring boundary condition is a valid approximation at low frequencies.

1. Introduction

A common failure mode in laminated composite plates is the occurrence of damage at the interface between the plies. This damage is more or less in the form of microcracks at the interface, and affects the propagation of ultrasound in the composite. Thus, ultrasonic nondestructive testing can be used to detect the damage, and in this connection it is very valuable to have a good model of the ultrasonic wave propagation in the damaged plate; see [Datta and Shah 2009] for elastic wave propagation in composite media in general. Recently, Boström and Golub [2009] proposed a (distributed) spring boundary condition to model a distribution of interface microcracks and derived a plate equation including this spring boundary condition. Another way to treat the interface microcracks is to model them as a periodic array of interface cracks; this model is expected to be good at low frequencies (cracks much smaller than the wavelength). As a model case the two-dimensional antiplane (SH) modes in a symmetrically layered, anisotropic, thick plate with a periodic array of interface cracks are investigated in this paper, and the results are compared to the case when the interface cracks are replaced by the spring boundary condition and also to the case when plate theory with the spring boundary condition is used. At low frequencies all three theories give the same result for the solution of the dispersion relation.

Zhang and Gross [1998] consider the elastic wave propagation and scattering in the presence of cracks, and investigate, in particular, the scattering by a periodic array of cracks, also for the two-dimensional SH anisotropic case. The SH wave propagation in isotropic space with a periodic array of interface cracks is considered by Angel and Achenbach [1985].

Keywords: elastic waves, periodic cracks, anisotropy, spring boundary conditions, plate equations.
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The plan of the paper is as follows. In Section 2 the exact dispersion relation is derived for the SH modes in a symmetrically three-layered, thick plate with periodic arrays of interface cracks. In Section 3 the spring boundary condition from [Boström and Golub 2009] is stated, and the derivation of symmetric and antisymmetric plate equations is sketched. The antisymmetric plate equation was not given in [Boström and Golub 2009], and the two plate equations are, furthermore, given to higher order in the plate thickness. Section 4 contains a few numerical result comparing the different theories and Section 5 offers a few concluding remarks.

2. Formulation of the problem

Consider a symmetrically three-layered, thick plate according to Figure 1, which also shows the xz coordinate system used. The middle layer has thickness d_1 and the two outer layers have thickness $d_2 - d_1$ each. The material parameters in the layers are denoted by an upper index $j = 1, 2$ for the middle and outer layers, respectively; see Figure 1. Thus the densities are ρ^1 and ρ^2 in the middle and outer layers, respectively. The principal axes of the anisotropic materials are assumed to be parallel with the coordinate axes. The only relevant stiffness constants are thus c^j_{44} and c^j_{66}. All the cracks are of equal width $2l$ and are periodically located with centers at $x = ma$, $z = \pm d_1$, $m = 0, \pm 1, \pm 2, \dots$.

Only shear horizontal (SH) waves with the polarization vector perpendicular to the plane of propagation are considered. Time harmonic waves are assumed, with the time factor $e^{-i\omega t}$ suppressed throughout. The displacement has only a y component u^j_y, which obeys the equation

$$c^j_{66} \frac{\partial^2 u^j_y}{\partial x^2} + c^j_{44} \frac{\partial^2 u^j_y}{\partial z^2} + \rho^j \omega^2 u^j_y = 0, \qquad j = 1, 2. \tag{2-1}$$

The two relevant stress components are

$$\sigma^j_{yx} = c^j_{66} \frac{\partial u^j_y}{\partial x}, \qquad \sigma^j_{yz} = c^j_{44} \frac{\partial u^j_y}{\partial z}, \qquad j = 1, 2. \tag{2-2}$$

From the symmetry of the problem it follows that only one half of the plate ($-d_2 < z < 0$) needs to be considered. The boundary conditions are then

$$u^1_y\big|_{z=0} = 0, \quad \sigma^2_{yz}\big|_{z=-d_2} = 0, \qquad \text{or} \qquad \sigma^1_{yz}\big|_{z=0} = 0, \quad \sigma^2_{yz}\big|_{z=-d_2} = 0, \tag{2-3}$$

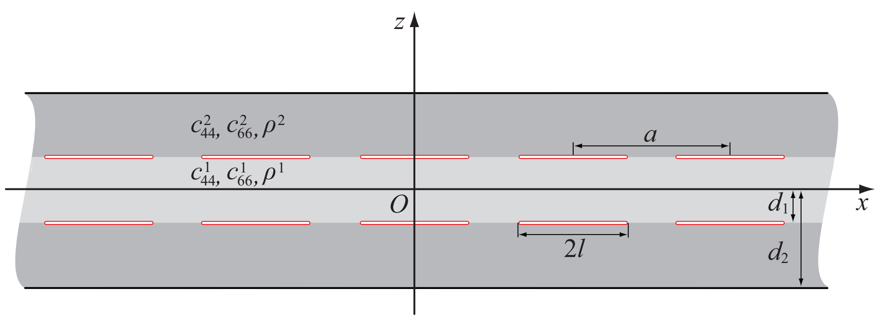

Figure 1. Elastic plate with periodic arrays of cracks.

depending on whether the antisymmetric or the symmetric waves in the plate are considered. On the interface $z = -d_1$ the displacement and traction are continuous except on the cracks where the boundary conditions are

$$\sigma^1_{yz}\big|_{z=-d_1} = \sigma^2_{yz}\big|_{z=-d_1} = 0, \qquad |x - ma| < l, \quad m = 0, \pm 1, \pm 2, \dots, \tag{2-4}$$

and this gives rise to a displacement discontinuity on the cracks. The crack-opening displacement on a crack is defined as

$$v_m(x) = u^1_y(x, -d_1) - u^2_y(x, -d_1), \qquad |x - ma| < l, \quad m = 0, \pm 1, \pm 2, \dots. \tag{2-5}$$

As the plate with the cracks is a periodic system Floquet's (or Bloch's) theorem [Yariv and Yeh 1984] states that the modes in the plate have the property

$$u^j_y(x, z) = e^{ikx} w^j(x, z), \qquad j = 1, 2, \tag{2-6}$$

where $w^j(x, z)$ are periodic functions with period a:

$$w^j(x + ma, z) = w^j(x, z), \qquad m = 0, \pm 1, \pm 2, \dots, \quad j = 1, 2. \tag{2-7}$$

The main goal here is to find the dispersion relation $k = k(\omega)$.

3. Exact solution

The wave fields u^j_y can be represented in the form of the inverse Fourier transform:

$$u^j_y(x, z) = \frac{1}{2\pi} \int_\Gamma U^j_y(\alpha, z) e^{-i\alpha x}\, d\alpha, \qquad j = 1, 2, \tag{3-1}$$

where the integration paths are going in the complex α plane along the real axis $\operatorname{Im}\alpha = 0$, deviating from it only for bypassing the real singularities of the integrands in accordance with the principle of limiting absorption [Babeshko et al. 1989]. The Fourier transforms of the wave fields $U^j_y(\alpha)$ are expressed as the products of the Fourier transforms $K^1(\alpha, z)$ and $K^2(\alpha, z)$ of the Green's functions $k^1(x, z)$ and $k^2(x, z)$ of the layers $-d_1 < z < 0$ and $-d_2 < z < -d_1$, and the Fourier transform $Q(\alpha)$ of the unknown stress $q(x)$ arising on the contact interface:

$$U^j_y(\alpha, z) = K^j(\alpha, z) Q(\alpha), \qquad j = 1, 2. \tag{3-2}$$

Mathematically, the function $k^1(x, z)$ is the solution to (2-1) with $j = 1$ obeying the boundary conditions

$$u^1_y\big|_{z=0} = 0, \quad \sigma^1_{yz}\big|_{z=-d_1} = \delta(x), \qquad \text{or} \qquad \sigma^1_{yz}\big|_{z=0} = 0, \quad \sigma^1_{yz}\big|_{z=-d_1} = \delta(x), \tag{3-3}$$

where $\delta(x)$ is the delta function. As in (2-3), the choice of the boundary conditions depends on whether the antisymmetric or the symmetric waves in the plate are considered. Analogously, the function $k^2(x, z)$ obeys (2-1) with $j = 2$ and the boundary conditions

$$\sigma^2_{yz}\big|_{z=-d_1} = \delta(x), \qquad \sigma^2_{yz}\big|_{z=-d_2} = 0. \tag{3-4}$$

The Fourier transform of both functions is easily calculated analytically:

$$K^1(\alpha, z) = \frac{\sinh \sigma^1 z}{c_{44}^1 \sigma^1 \cosh \sigma^1 d_1}, \qquad \text{or} \qquad K^1(\alpha, z) = -\frac{\cosh \sigma^1 z}{c_{44}^1 \sigma^1 \sinh \sigma^1 d_1}, \tag{3-5}$$

and

$$K^2(\alpha, z) = \frac{\cosh \sigma^2 (z + d_2)}{c_{44}^2 \sigma^2 \sinh \sigma^2 (d_2 - d_1)}, \tag{3-6}$$

where

$$\sigma^j = \left(\frac{c_{66}^j}{c_{44}^j \alpha^2} - (\kappa_{44}^j)^2 \right)^{1/2}, \qquad \kappa_{44}^j = \sqrt{\frac{\rho^j \omega^2}{c_{44}^j}}, \qquad j = 1, 2. \tag{3-7}$$

($K^2(\alpha, z)$ is the same for the symmetric and antisymmetric problems.)

To solve for $Q(\alpha)$, take the difference between the two equations Equation (3-2), which introduces the sum of all the Fourier transforms $V_m(\alpha)$ of the crack-opening displacements $\upsilon_m(x)$:

$$Q(\alpha) = L(\alpha) \sum_{m=-\infty}^{\infty} V_m(\alpha), \qquad L(\alpha) = \big(K^1(\alpha, -d_1) - K^2(\alpha, -d_1) \big)^{-1}. \tag{3-8}$$

Substitution of this expression into (3-2) implies analogous representations of the Fourier transforms of the displacements fields

$$U_y^j(\alpha, z) = N^j(\alpha, z) \sum_{m=-\infty}^{\infty} V_m(\alpha), \qquad N^j(\alpha, z) = K^j(\alpha, z) L(\alpha), \qquad j = 1, 2, \tag{3-9}$$

which leads to the integral representations

$$u_y^j(x, z) = \frac{1}{2\pi} \int_\Gamma N^j(\alpha, z) \sum_{m=-\infty}^{\infty} V_m(\alpha) e^{-i\alpha x} \, d\alpha, \qquad j = 1, 2. \tag{3-10}$$

Due to the periodicity property of the displacements fields (2-6), the Fourier transforms of the unknown crack-opening displacements are connected by

$$V_m(\alpha) = \int_{ma-l}^{ma+l} \upsilon_m(x) e^{i\alpha x} \, dx = \int_{-l}^{l} \upsilon_0(x) e^{i(\alpha x + (\alpha + k) m a)} \, dx = V_0(\alpha) e^{i(\alpha + k) m a}, \tag{3-11}$$

where $m = 0, \pm 1, \pm 2, \ldots$. Substitution of the functions $V_m(\alpha)$ in this form into the representation (3-10) and use of the formula

$$\sum_{m=-\infty}^{\infty} e^{i(\alpha + k) m a} = \sum_{m=-\infty}^{\infty} \delta\Big(\frac{\alpha a + k a}{2\pi} - m \Big) \tag{3-12}$$

lead to the representation of the wave fields in series form:

$$u_y^j(x, y) = \frac{1}{a} \sum_{m=-\infty}^{\infty} N^j(\alpha_m, z) V_0(\alpha_m) e^{-i\alpha_m x}, \qquad j = 1, 2, \tag{3-13}$$

where $\alpha_m = 2\pi m / a - k$.

Use of the displacements in the form Equation (3-13) to compute the stress and insertion into the stress-free boundary condition (2-4) gives the integral equation for the crack-opening displacement $v_0(x)$:

$$\frac{1}{a}\sum_{m=-\infty}^{\infty} L(\alpha_m)V_0(\alpha_m)e^{-i\alpha_m x} = 0, \qquad |x| < l, \tag{3-14}$$

where the Fourier transform of the crack-opening displacement is

$$V_0(\alpha) = \int_{-l}^{l} v_0(x)e^{i\alpha x}\,dx. \tag{3-15}$$

Note that every solution to (3-14) obtained for $|x| < l$ obeys it also for $|x - ma| < l$, $m = 0, \pm 1, \pm 2, \dots$ due to the periodicity of the left-hand side of the equation. The values of k, for which nontrivial solutions to (3-14) exist, are exactly the wave numbers of the modes propagating in the plate at the given frequency ω.

To obtain solutions to (3-14), the crack-opening displacement is expanded into a series:

$$v_0(x) = \sum_{n=0}^{\infty} c_n p_n(x), \qquad p_n(x) = U_n\Big(\frac{x}{l}\Big)\sqrt{1-\Big(\frac{x}{l}\Big)^2}, \tag{3-16}$$

where $U_n(x)$, $n = 0, 1, \dots$ are Chebyshev polynomials of the second kind. Note the important fact that this expansion takes care of the stress singularity at the crack tips. Substitution into (3-14) yields

$$\sum_{n=0}^{\infty} c_n\Bigg[\frac{1}{a}\sum_{m=-\infty}^{\infty} L(\alpha_m)P_n(\alpha_m)e^{-i\alpha_m x}\Bigg] = 0, \qquad |x| < l, \tag{3-17}$$

where

$$P_n(\alpha) = \int_{-l}^{l} p_n(x)e^{i\alpha x}\,dx = \pi l i^n (n+1)\frac{J_{n+1}(\alpha l)}{\alpha}, \tag{3-18}$$

where $J_{n+1}(\alpha l)$ is a Bessel function. Following the Galerkin scheme, (3-17) is multiplied by $p_j(x)$ for $j = 0, 1, 2, \dots$ and then integrated over $-l < x < l$ to obtain the homogeneous system of linear algebraic equations

$$\sum_{n=0}^{\infty} A_{jn}(k,\omega)c_n = 0, \qquad j = 0, 1, 2, \dots, \tag{3-19}$$

where the matrix

$$A_{jn}(k,\omega) = \frac{1}{a}\sum_{m=-\infty}^{\infty} L(\alpha_m)P_n(\alpha_m)P_j(-\alpha_m). \tag{3-20}$$

Nontrivial solutions to this system exist when the matrix $A_{jn}(k,\omega)$ becomes singular, that is

$$\det\big(A_{jn}(k,\omega)\big) = 0. \tag{3-21}$$

This is the desired dispersion relation. It implicitly relates the wave numbers k to the given frequency ω.

4. Approximate plate equations

If the cracks are small as compared to the characteristic wavelengths in the plate, it is possible to consider the system in a simple fashion using an approximate spring boundary condition. First it is noted that the exact distribution of the cracks is expected to be unimportant for small cracks. In [Boström and Golub 2009] an approximate spring boundary condition was derived to model an interface with a random distribution of small cracks of the same size, and this boundary condition should therefore also be applicable to a periodic distribution of interface cracks. Thus, the exact periodic boundary condition at the interface $z=-d_1$ is replaced by the simplified spring boundary condition (*ibid.*):

$$\sigma_{yz}^1=\sigma_{yz}^2=\kappa(u^1-u^2),\qquad z=-d_1, \tag{4-1}$$

where the spring constant is

$$\kappa=\frac{1}{Cl\pi}\cdot\frac{4c_{44}^1c_{44}^2k_{44}^1k_{44}^2}{c_{44}^1k_{44}^1k_{66}^2+c_{44}^2k_{44}^2k_{66}^1}. \tag{4-2}$$

Here $k_{mm}^j=\omega/v_{mm}^j$ are the wave numbers and $v_{mm}^j=\sqrt{c_{mm}^j/\rho^j}$ are SH-wave velocities along the z and x axes (for $m=4,6$, respectively) in the middle and outer layers ($j=1,2$), and $C=2l/a$ is the relative density of cracks. It is noted that the spring constant κ must be large for the approximations in deriving the spring boundary condition to be valid [Boström and Golub 2009]. The dispersion relation for the plate with this boundary condition is easily derived and is not given here.

If also the thickness of the plate, as well as the cracks, is small as compared to the characteristic wavelengths in the plate, the problem can be further simplified by using plate theory. Here the approach of Boström et al. [2001] and Boström and Golub [2009] is followed. Thus, the assumption of time harmonic waves is lifted, and it is noted that the spring constant in (4-2) is frequency independent. The displacement fields are expanded as

$$u_y^1(x,z)=\sum_{n=0}^{\infty}u_n^1(x)z^n,\qquad u_y^2(x,z)=\sum_{n=0}^{\infty}u_n^2(x)(z+d_1)^n, \tag{4-3}$$

using Taylor power series expansions. Substitution of these formulas into (2-1) leads to the recurrence relations

$$u_n^j=\frac{1}{n!}\cdot\frac{1}{(c_{44}^j)^{n/2}}\cdot D_j^{n/2}u_0^j,\quad n=2,4,\dots,\qquad u_n^j=\frac{1}{n!}\cdot\frac{1}{(c_{44}^j)^{(n-1)/2}}\cdot D_j^{(n-1)/2}u_1^j,\quad n=3,5,\dots, \tag{4-4}$$

where the wave operator is

$$D_ju_n^j=\left(\rho^j\frac{\partial^2u_n^j}{\partial t^2}-c_{66}^j\frac{\partial^2u_n^j}{\partial x^2}\right),\qquad j=1,2. \tag{4-5}$$

The recurrence relations can be used to eliminate all the expansion functions except u_0^1 and u_1^1; and u_0^2 and u_1^2. Substitution into the boundary conditions (2-3) and (4-1) and keeping terms up to a certain order in thickness lead to the equations of motion of the plate. In the case of symmetric waves in the plate the equation becomes

$$\left[d_1 D_1 + (d_2 - d_1) D_2 + \frac{d_1(d_2 - d_1)}{\kappa} D_1 D_2 + + \frac{d_1(d_2 - d_1)^2}{2c_{44}^2} D_1 D_2 + \frac{d_1^2(d_2 - d_1)}{2c_{44}^1} D_1 D_2\right] u_0^1 = 0, \quad (4\text{-}6)$$

while in the antisymmetric case it becomes

$$\left[1 + \frac{(d_2 - d_1)}{\kappa} D_2 + \frac{d_1^2}{2c_{44}^1} D_1 + \frac{d_1(d_2 - d_1)}{c_{44}^1} D_2 + \frac{(d_2 - d_1)^2}{2c_{44}^2} D_2\right] u_1^1 = 0. \quad (4\text{-}7)$$

The first equation is written to third order in thickness, the second to second order (but it also contains a zero order term), but it is straightforward to go to higher orders.

The effects of the spring boundary conditions, manifesting themselves in terms that depend on the spring constant κ, enter the two plate equations in a slightly different manner. In the symmetric equation the effects enter to second order in plate thickness, without any effects to the lowest order (where the equation just becomes a thickness weighted wave equation). The antisymmetric equation is a little strange in that the spring term is of linear order, while the lowest meaningful order without the springs ($\kappa \to \infty$) is of second order. This might have something to do with the fact that the antisymmetric equation has no low frequency mode, see the numerical results in Section 5.

5. Numerical examples

In this section some numerical results are given to compare the different approximations for a layered plate with interface damage. The first approximation is the thick plate with a periodic array of interface cracks as treated in Section 2. The second is the thick plate with the spring boundary condition (4-1). The corresponding dispersion relation is very easily derived and is not given. The third approach is to use the spring boundary condition together with the plate equations (4-6) or (4-7) (taken to the order given), with the dispersion relations following directly from the plate equations. In this way both the spring boundary condition and the plate equations are checked. Another matter, which can only be checked against experiments, is how well these theories really model a real plate with interface damage.

A fiber-reinforced graphite-epoxy composite which has density $\rho = 1578\,\mathrm{kg/m^3}$ and stiffness constants $c_{44}^1 = 3.50\,\mathrm{GPa}$ and $c_{66}^1 = 7.07\,\mathrm{GPa}$ is considered. The lay-up is assumed to be $0°\backslash 90°\backslash 0°$, which means that the material in the middle layer is rotated 90° relative to the top and bottom layers and the stiffness constants are connected by the relations $c_{44}^2 = c_{66}^1$ and $c_{66}^2 = c_{44}^1$. The thicknesses of the layers are assumed to be equal for all examples, so $2d_1 = (d_2 - d_1) = d/3$, where $d = 2d_2$ is the total thickness of the plate.

Figures 2–7 show the dispersion curves, that is, dimensionless frequency $\omega d\sqrt{\rho/c_{44}^1}$ as a function of dimensionless wave number kd. Figures 2, 4, and 6 show results for symmetric modes and Figures 3, 5, and 7 show results for antisymmetric modes. Figures 2 and 3 show the modes for $l/d = 0.05$ and $a/d = 0.5$, so the cracks are small and the damage parameter is $C = 2l/a = 0.2$, that is, the cracks occupy 20% of the interface. In Figures 4 and 5 the crack size and the spacing is doubled, that is, $l/d = 0.1$ and $a/d = 1$. Finally, in Figures 6 and 7 the values are $l/d = 0.5$ and $a/d = 2$. The solid lines show results for the thick plate with periodic interface cracks, the dashed lines results for the thick plate with spring boundary conditions, and the dash-dotted lines results for the plate equations. Note that the plate

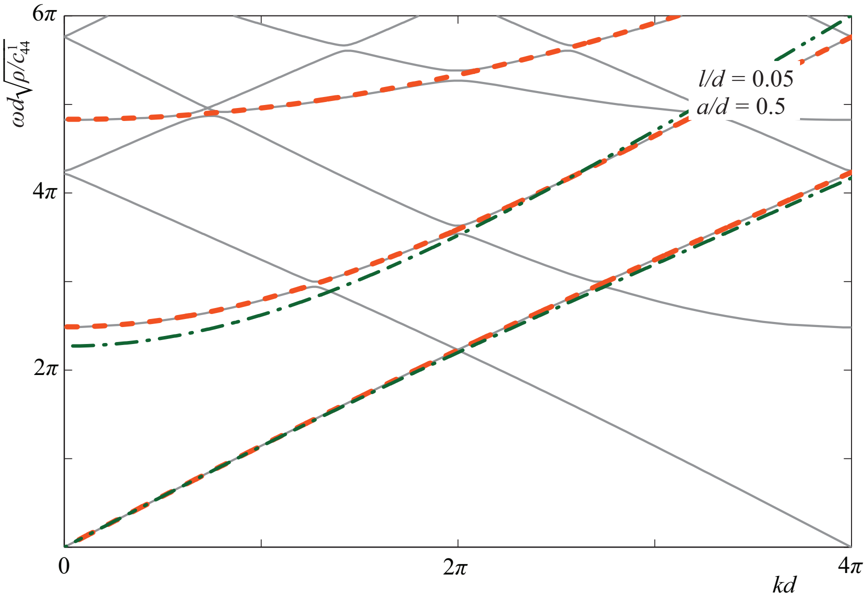

Figure 2. Dispersion curves in the wave number kd-frequency $\omega d\sqrt{\rho/c_{44}^1}$ plane for the symmetric modes and $l/d = 0.05$ and $a/d = 0.5$; solid lines: thick plate with periodic cracks; dotted lines: thick plate with spring boundary conditions; dash-dotted lines: plate equation with spring boundary conditions.

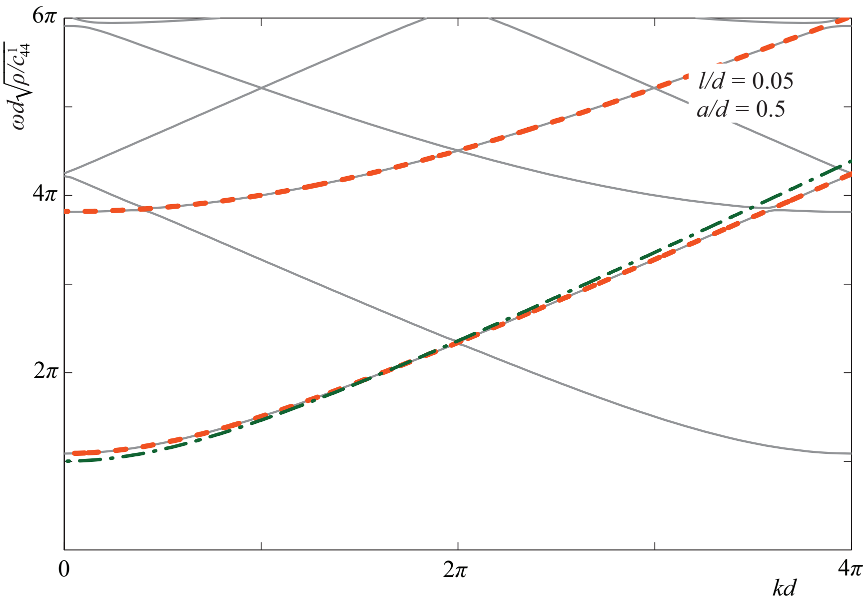

Figure 3. Dispersion curves in the wave number kd-frequency $\omega d\sqrt{\rho/c_{44}^1}$ plane for the antisymmetric modes and $l/d = 0.05$ and $a/d = 0.5$; solid lines: thick plate with periodic cracks; dotted lines: thick plate with spring boundary conditions; dash-dotted lines: plate equation with spring boundary conditions.

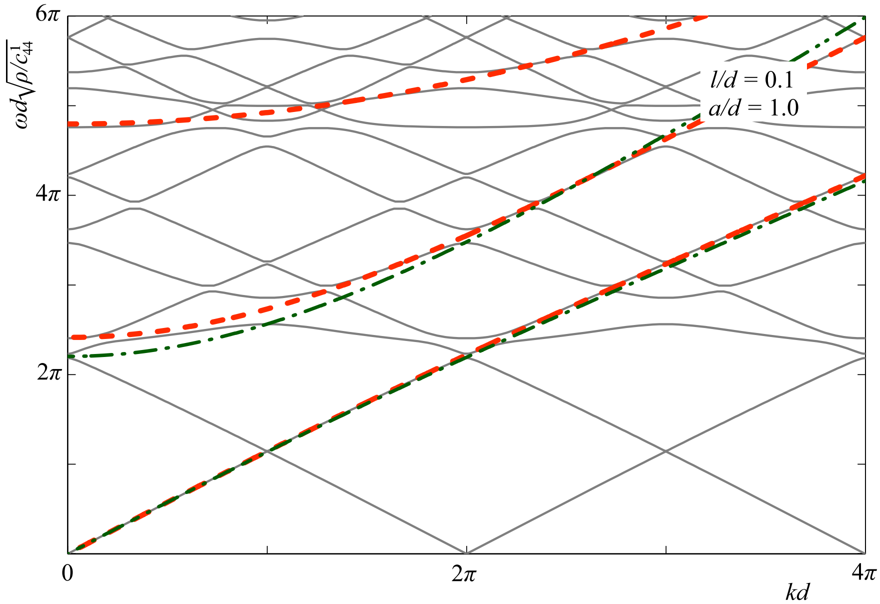

Figure 4. Dispersion curves in the wave number kd-frequency $\omega d\sqrt{\rho/c^1_{44}}$ plane for the symmetric modes and $l/d = 0.1$ and $a/d = 1.0$; solid lines: thick plate with periodic cracks; dotted lines: thick plate with spring boundary conditions; dash-dotted lines: plate equation with spring boundary conditions.

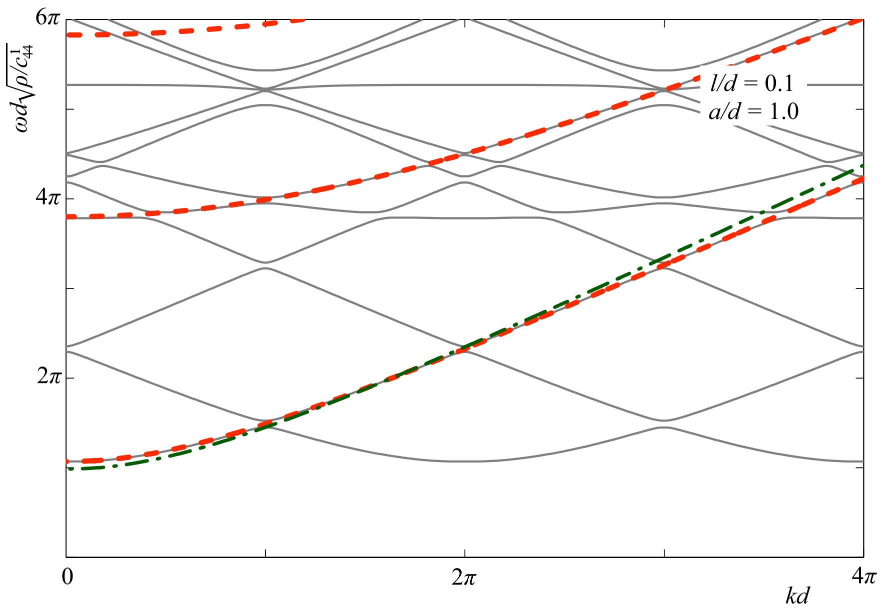

Figure 5. Dispersion curves in the wave number kd-frequency $\omega d\sqrt{\rho/c^1_{44}}$ plane for the antisymmetric modes and $l/d = 0.1$ and $a/d = 1.0$; solid lines: thick plate with periodic cracks; dotted lines: thick plate with spring boundary conditions; dash-dotted lines: plate equation with spring boundary conditions.

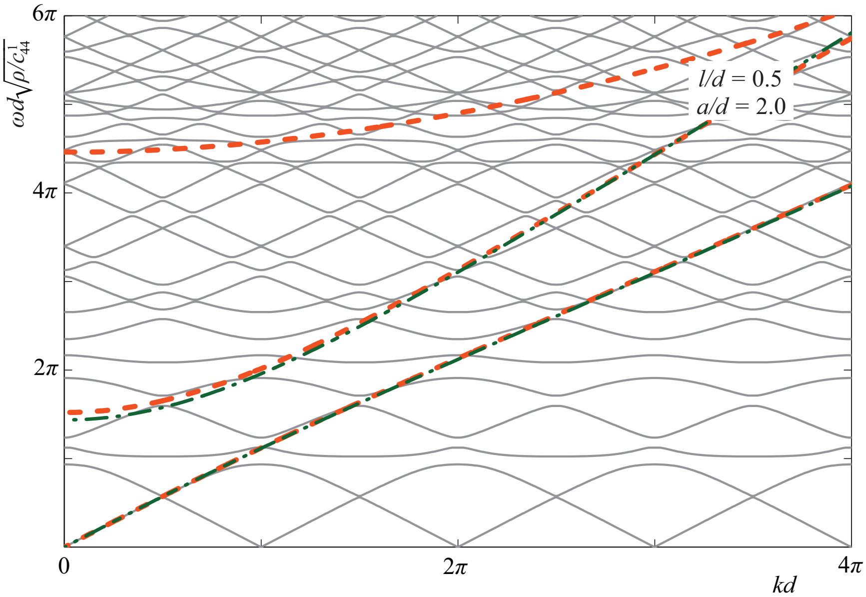

Figure 6. Dispersion curves in the wave number kd-frequency $\omega d\sqrt{\rho/c_{44}^1}$ plane for the symmetric modes and $l/d = 0.5$ and $a/d = 2.0$; solid lines: thick plate with periodic cracks; dotted lines: thick plate with spring boundary conditions; dash-dotted lines: plate equation with spring boundary conditions.

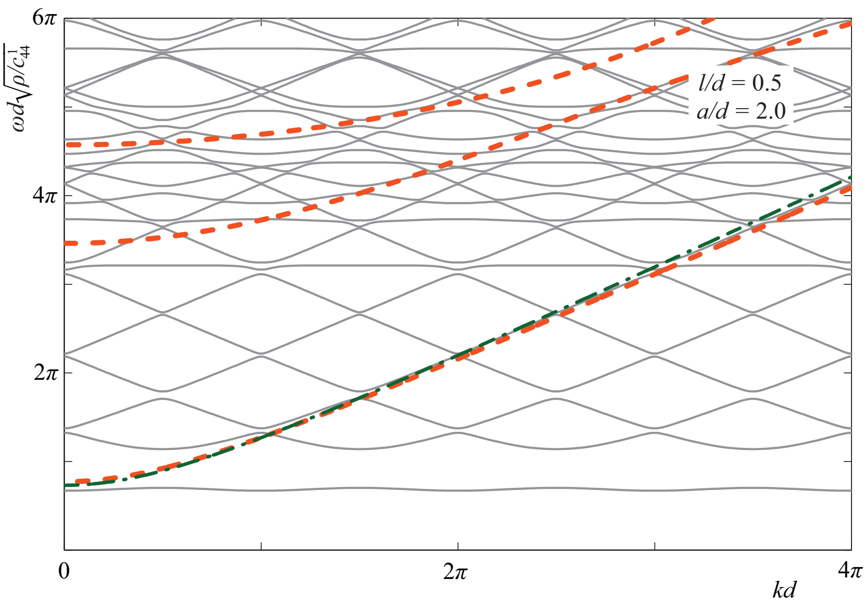

Figure 7. Dispersion curves in the wave number kd-frequency $\omega d\sqrt{\rho/c_{44}^1}$ plane for the antisymmetric modes and $l/d = 0.5$ and $a/d = 2.0$; solid lines: thick plate with periodic cracks; dotted lines: thick plate with spring boundary conditions; dash-dotted lines: plate equation with spring boundary conditions.

equations only give two and one solution to the dispersion relation for the symmetric and antisymmetric plate equations, respectively. In many cases the dashed and dash-dotted lines are on top of the solid lines; the dashed line is, in particular, exactly on top of the solid line for the first mode at low frequencies. As the interface cracks form a periodic system, this gives the usual system of Bloch modes with periodicity in the kd direction.

For the first symmetric mode at low frequencies all three theories agree very well. However, the antisymmetric problem has no mode at low frequencies, and the first antisymmetric mode, as well as the second symmetric mode, agree more or less depending on the parameters. It is seen that the thick plate with the spring boundary condition performs better than the plate theory. This indicates that the spring boundary condition can be used for higher frequencies than the plate theories.

6. Concluding remarks

The SH modes in a layered, anisotropic, thick plate with a periodic distribution of interface cracks have been investigated. The dispersion relation is compared to the case when the distribution of cracks is replaced by an equivalent spring boundary condition, and also to a plate equation with this spring boundary condition. For low frequencies the agreement is excellent between all three theories; this shows that the equivalent spring boundary condition can be used instead of the distribution of cracks. A periodic or random distribution of small cracks is a plausible model for interface damage, and the main result of the present investigations is thus a verification that the spring boundary condition should be a useful model for such damage. The great advantage with the spring boundary condition is its simplicity and that it only depends on the physical parameters of crack size and relative density of cracks (and the material parameters). Here only dispersion relations are studied but the spring boundary condition should also be applicable to more general situations, for example, for a plate with interface damage on only a part. The results presented here for periodic cracks may also have an intrinsic interest with possible applications of the passband-stopband structure.

The two-dimensional SH waves considered here can be regarded as a model case of limited interest. It is of course worthwhile to go further and investigate the more realistic two-dimensional in-plane problem and three-dimensional problems. It should be possible to follow a similar procedure in most cases. For three-dimensional problems there are a number of possibilities: isotropic or anisotropic media, and different shapes of interface cracks, most easily rectangular or (for isotropic media) circular cracks.

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