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## APPLICATION OF THE KIRCHHOFF HYPOTHESIS TO BENDING THIN PLATES WITH DIFFERENT MODULI IN TENSION AND COMPRESSION

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When materials that exhibit different mechanical behaviors in tension and compression must be analyzed, Ambartsumyan's bimodular model for isotropic materials can be adopted. It deals with the principal stress state in a point, which is particularly important in the analysis and design of reinforced concrete structures. However, due to the inherent complexity of the constitutive relation, it is difficult to solve analytically for bending components with bimoduli except in particular simple problems. Here we propose a simplified mechanical model, based on the classical Kirchhoff hypothesis, used for the solution of the bimodular thin plates in bending. We first use the Kirchhoff hypothesis to judge the existence of the elastic neutral layers of bimodular thin plates in small-deflection bending. Based on the existent neutral layers, we extend the solution from the case of pure bending into the case of lateral force bending. We use the displacement variation method to illustrate the application of the proposed model, and compare it with FEM results strictly based on Ambartsumyan's materials model. The comparisons show that the proposed mechanical model is valid and helpful for analyzing bending structures with bimodularity.

### 1. Introduction

Many studies have indicated that most materials, including concrete, ceramics, graphite, and some composites, exhibit different tensile and compressive strains given the same stress applied in tension and compression. These materials are known as bimodular materials [Jones 1976; 1977]. Overall, there are two basic material models widely used in theoretical analysis within the engineering profession. One of these models is the criterion of positive-negative signs in the longitudinal strain of fibers proposed in [Bert 1977]. This model is mainly applicable to orthotropic materials, and is therefore widely used for research on laminated composites [Bert and Gordaninejad 1983; Reddy and Chao 1983; Srinivasan and Ramachandra 1989; Ganapathi et al. 2004; Patel et al. 2004; 2005a; 2005b; Baykara et al. 2005; Khan et al. 2009; Maceri and Vairo 2009]. Another model is the criterion of positive-negative signs of principal stress proposed by Ambartsumyan [1965; 1969; 1982]. This model is mainly applicable to isotropic materials [Kamiya 1974; 1975a; 1975b; El-Tahan et al. 1989]. In civil engineering, the principal stress state is a key point in the analysis and design of some bending components like beams, columns, plates, and shells. Shear stresses and the resulting diagonal tension must also be carefully considered in the design of reinforced concrete. This paper will focus on discussion of the latter model based on principal direction.

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*Keywords:* bimodulus, tension and compression, Kirchhoff hypothesis, plates, neutral layer.

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The bimodular materials model proposed by Ambartsumyan asserts that Young's modulus of elasticity depends not only on material properties, but also on the stress state of that point. There are only a few applications of the constitutive equation to stress analyses of components because of the inherent complexity in analysis of bimodular materials, that is, the elastic constants involved in the governing equations, which depend on the stress state of that point, are not correctly indicated beforehand. In other words, except in particularly simple problems it is not easy to estimate a priori the stress state in a point in the deformed body. In some complex problems, it is necessary to resort to FEM based on an iterative strategy [Zhang and Wang 1989; Ye et al. 2004]. Because the stress state of the point in question is unknown in advance, we have to begin with a single modulus problem, thus gaining the initial stress state to form a corresponding elasticity matrix for each element. Generally, direct iterative methods based on an incrementally evolving stiffness have been adopted by many researchers; they include an improved algorithm in which the shear stress and shear strain are set to zero to formulate the elastic matrix [Zhang and Wang 1989; Liu and Zhang 2000; He et al. 2009], an improved algorithm keeping Poisson's ratio constant while modifying the elastic matrix [Ye 1997], the initial stress technique [Yang et al. 1999], and the smoothing function technique [Yang and Zhu 2006; Yang and Wang 2008].

Analytical solutions are available in a few cases, all involving beams and columns. By taking the isolated body and then considering its static equilibrium, Yao and Ye derived the analytical solution of a bending-compression column with different moduli in tension and compression [2004a], and then the analytical solution of the bimodular retaining wall [2004b]. In the context of a bimodular beam in lateral force bending, they proposed the assumption that shear stress makes no contribution to the position of the neutral axis [Yao and Ye 2004c], deriving from it the analytical solution of a lateral force bending beams with different moduli in tension and compression.

In [He et al. 2007a], to simplify the derivation, we proposed that the bimodular beams may be turned into classical beams by the equivalent section method. Under the condition that the stress varies continuously along the direction perpendicular to neutral axis, we obtained the approximate analytical solution of a bimodular deep beam under uniformly-distributed loads [2007b], and the approximate elasticity solution of a bimodular bending-compression column [2008].

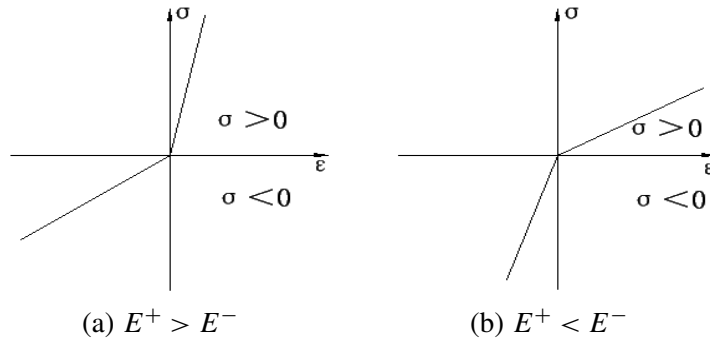
All these solutions involve beams and columns; up to now, analytical solutions based on bimodular plates have not been found. The key problem lies in how to construct a simplified mechanical model to solve this problem analytically.

In this paper, we use the Kirchhoff hypothesis to judge the existence of the elastic neutral layers of bimodular thin plates in small-deflection bending. Based on the existent neutral layers, we extend the solution from the case of pure bending, which strictly satisfies the constitutive model proposed by Ambartsumyan, to the case of lateral force bending. We use the displacement variation method to illustrate the application of the proposed model, and compare with FEM results.

## 2. Mechanical model

**2.1. Bimodular material model.** Ambartsumyan [1982] linearized the bimodular materials model by two straight lines whose tangents at the origin are discontinuous, as shown in Figure 1. The basic assumptions of this model are as follows:

- The body under study is continuous, homogeneous, and isotropic.



**Figure 1.** Constitutive model of bimodulus materials proposed by Ambartsumyan.

- Small deformation is assumed.
- The Young’s modulus and Poisson’s ratio of the materials are  $E^+$  and  $\mu^+$  when the materials are in tension along a certain direction and  $E^-$  and  $\mu^-$  when they are in compression.
- When the three principal stresses are uniformly positive or uniformly negative, the three basic equations are essentially the same as those of classical theory; when the signs of the three principal stresses are different, the differential equations of equilibrium and the geometrical equations are the same as those of classical materials theory, with the exception of the physical equations.
- $\mu^+/E^+ = \mu^-/E^-$ , to ensure that the flexibility matrix is symmetric.

In a spatial problem, let the stress and strain components in the principal coordinates  $\alpha, \beta,$  and  $\gamma$  be, respectively,  $\{\sigma_I\} = (\sigma_\alpha \ \sigma_\beta \ \sigma_\gamma)^T$  and  $\{\varepsilon_I\} = (\varepsilon_\alpha \ \varepsilon_\beta \ \varepsilon_\gamma)^T$ . The constitutive model proposed by Ambartsumyan is

$$\begin{Bmatrix} \varepsilon_\alpha \\ \varepsilon_\beta \\ \varepsilon_\gamma \end{Bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{Bmatrix} \sigma_\alpha \\ \sigma_\beta \\ \sigma_\gamma \end{Bmatrix},$$

where  $a_{ij}$  ( $i, j = 1, 2, 3$ ) denote the flexibility coefficients determined by the polarity of the signs of the principal stresses. For instance, if  $\sigma_\alpha > 0, \sigma_\beta < 0,$  and  $\sigma_\gamma > 0$ , the flexibility coefficients in the physical relation should be  $a_{11} = a_{33} = 1/E^+, a_{22} = 1/E^-, a_{21} = a_{31} = a_{13} = a_{23} = -\mu^+/E^+,$  and  $a_{12} = a_{32} = -\mu^-/E^-$ . The rest of the physical equations may be deduced analogously.

Because the stress state of the point in question is unknown in advance, we have to begin with a single modulus problem, thus gaining the initial stress state to form a corresponding elasticity matrix for each element. This method is only available for the numerical iterative technology based on FEM. For the analytical solution of such a problem, however, we will find that, since the physical equations originally built on the principal stress direction  $\alpha, \beta, \gamma$  are rebuilt on general coordinates  $x, y, z$  via coordinate conversion, many nonlinear items concerning the principal stress and its direction cosine are generated in the physical equations. Therefore, solving becomes very difficult and it is necessary to simplify the mechanical model to obtain the approximate analytical solution.

**2.2. Existence of neutral layers.** In the small-deflection bending of thin plates, let the stress components be  $\sigma_x, \sigma_y, \sigma_z, \tau_{xy}, \tau_{yz},$  and  $\tau_{zx}$  and the strain components be  $\varepsilon_x, \varepsilon_y, \varepsilon_z, \gamma_{xy}, \gamma_{yz},$  and  $\gamma_{zx}$ . The components

of displacement at a point, in the  $x$  and  $y$  directions, are denoted by  $u$  and  $v$ , respectively. Due to lateral loading, deformation takes place; the midsurface at any point has deflection  $w$ . From the Kirchhoff hypothesis we deduce that

$$\varepsilon_x = \frac{1}{E}(\sigma_x - \mu\sigma_y), \quad \varepsilon_y = \frac{1}{E}(\sigma_y - \mu\sigma_x), \quad \gamma_{xy} = \frac{2(1 + \mu)}{E}\tau_{xy}, \quad (1)$$

$$(u)_{z=0} = 0, \quad (v)_{z=0} = 0, \quad (2)$$

$$(\varepsilon_x)_{z=0} = 0, \quad (\varepsilon_y)_{z=0} = 0, \quad (\gamma_{xy})_{z=0} = 0. \quad (3)$$

The last line says that the midsurface remains unstrained under bending: there is no normal strain in tension and compression and no shear strain. Therefore, we may think of the midsurface as the neutral layer of thin plates in bending. From (1) and (3), we have

$$(\sigma_x)_{z=0} = \mu(\sigma_y)_{z=0}, \quad (\sigma_y)_{z=0} = \mu(\sigma_x)_{z=0}, \quad (\tau_{xy})_{z=0} = 0. \quad (4)$$

Substituting the second expression in (4) into the first one, we have

$$(\sigma_x)_{z=0} = \mu^2(\sigma_x)_{z=0}. \quad (5)$$

Under the condition  $(\sigma_x)_{z=0} \neq 0$ ,  $\mu^2 = 1$  may be obtained. However, it is obvious that  $\mu^2 = 1$  fails to satisfy the physical meaning which isotropic materials should have. Only under the condition  $(\sigma_x)_{z=0} = 0$ , (5) holds. Similarly, we also obtain  $(\sigma_y)_{z=0} = 0$ . Therefore,  $(\sigma_x)_{z=0} = 0$  and  $(\sigma_y)_{z=0} = 0$  are the conditions which the stress components should satisfy in the neutral layer.

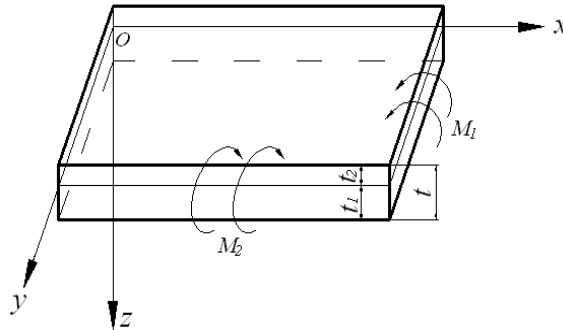
The above conclusion is obtained based on the Kirchhoff hypothesis, that is, it is a classical single modulus problem. But here we consider different moduli in tension and compression of materials. However, from the viewpoint of phenomenalism, a bending thin plate, either with a single modulus or with a bimodulus, will uniformly generate a deflected shape under a lateral force, where the lower layer of the plate is in tension and the upper layer is in compression. Therefore, the neutral layer in tension free and compression does exist, like the case of classical plate, and is determined by the conditions

$$\sigma_x = 0, \quad \sigma_y = 0. \quad (6)$$

For materials without consideration of the bimodulus, the neutral layer is exactly the midsurface of the plate. When considering the bimodularity of materials, however, the neutral layer is no longer located in the midsurface of the plate. It will be located in a certain layer depending on tensile and compressive Young's modulus of elasticity and Poisson's ratio of the materials. Next, we will determine the unknown neutral layer.

### 3. Bimodular plates in bending

**3.1. Pure bending.** Having demonstrated that a neutral layer exists when the bimodular plate is in pure bending, we now consider such a plate in which the  $xy$  plane parametrizes the unknown neutral layer. Bounded by the unknown neutral layer, the full cross sections in the  $x$  and  $y$  directions are divided into tensile and compressive sections, as shown in Figure 2, where  $t$  is the thickness of the plate,  $t_1$  and  $t_2$  are the thicknesses in tension and compression, respectively, and  $M_x$  and  $M_y$  are the bending moments in the  $x$  and  $y$  directions, respectively.



**Figure 2.** Bimodular thin plate under pure bending.

Because the normal stresses in the  $x$  and  $y$  directions coincide with the principal stresses in the  $\alpha$  and  $\beta$  directions, according to the materials model proposed by Ambartsumyan, the constitutive relations of the plate in tension and compression should be, respectively,

$$\sigma_x^+ = \frac{E^+}{1 - (\mu^+)^2}(\varepsilon_x + \mu^+ \varepsilon_y), \quad \sigma_y^+ = \frac{E^+}{1 - (\mu^+)^2}(\varepsilon_y + \mu^+ \varepsilon_x), \tag{7}$$

and

$$\sigma_x^- = \frac{E^-}{1 - (\mu^-)^2}(\varepsilon_x + \mu^- \varepsilon_y), \quad \sigma_y^- = \frac{E^-}{1 - (\mu^-)^2}(\varepsilon_y + \mu^- \varepsilon_x), \tag{8}$$

where  $E^\pm$  and  $\mu^\pm$  are defined on page 757. Considering the conclusions of Section 2.2, we may express the stress components in terms of deflection  $w$  as

$$\sigma_x^+ = -\frac{E^+ z}{1 - (\mu^+)^2} \left( \frac{\partial^2 w}{\partial x^2} + \mu^+ \frac{\partial^2 w}{\partial y^2} \right), \quad \sigma_y^+ = -\frac{E^+ z}{1 - (\mu^+)^2} \left( \frac{\partial^2 w}{\partial y^2} + \mu^+ \frac{\partial^2 w}{\partial x^2} \right), \quad 0 \leq z \leq t_1, \tag{9}$$

and

$$\sigma_x^- = -\frac{E^- z}{1 - (\mu^-)^2} \left( \frac{\partial^2 w}{\partial x^2} + \mu^- \frac{\partial^2 w}{\partial y^2} \right), \quad \sigma_y^- = -\frac{E^- z}{1 - (\mu^-)^2} \left( \frac{\partial^2 w}{\partial y^2} + \mu^- \frac{\partial^2 w}{\partial x^2} \right), \quad -t_2 \leq z \leq 0. \tag{10}$$

Since the stress formulas in the whole thickness of the plate are different, it is necessary to integrate in subsection to obtain the bending moments and normal forces along the  $x$  and  $y$  directions. Using (9) and (10) we may compute these quantities as follows:

$$N_x = \int_0^{t_1} \sigma_x^+ dz + \int_{-t_2}^0 \sigma_x^- dz = -\frac{E^+ t_1^2}{2[1 - (\mu^+)^2]} \left( \frac{\partial^2 w}{\partial x^2} + \mu^+ \frac{\partial^2 w}{\partial y^2} \right) + \frac{E^- t_2^2}{2[1 - (\mu^-)^2]} \left( \frac{\partial^2 w}{\partial x^2} + \mu^- \frac{\partial^2 w}{\partial y^2} \right) = 0, \tag{11}$$

$$N_y = \int_0^{t_1} \sigma_y^+ dz + \int_{-t_2}^0 \sigma_y^- dz = -\frac{E^+ t_1^2}{2[1 - (\mu^+)^2]} \left( \frac{\partial^2 w}{\partial y^2} + \mu^+ \frac{\partial^2 w}{\partial x^2} \right) + \frac{E^- t_2^2}{2[1 - (\mu^-)^2]} \left( \frac{\partial^2 w}{\partial y^2} + \mu^- \frac{\partial^2 w}{\partial x^2} \right) = 0,$$

$$M_x = \int_0^{t_1} \sigma_x^+ z dz + \int_{-t_2}^0 \sigma_x^- z dz = -\frac{E^+ t_1^3}{3[1 - (\mu^+)^2]} \left( \frac{\partial^2 w}{\partial x^2} + \mu^+ \frac{\partial^2 w}{\partial y^2} \right) - \frac{E^- t_2^3}{3[1 - (\mu^-)^2]} \left( \frac{\partial^2 w}{\partial x^2} + \mu^- \frac{\partial^2 w}{\partial y^2} \right), \tag{12}$$

$$M_y = \int_0^{t_1} \sigma_y^+ z dz + \int_{-t_2}^0 \sigma_y^- z dz = -\frac{E^+ t_1^3}{3[1 - (\mu^+)^2]} \left( \frac{\partial^2 w}{\partial y^2} + \mu^+ \frac{\partial^2 w}{\partial x^2} \right) - \frac{E^- t_2^3}{3[1 - (\mu^-)^2]} \left( \frac{\partial^2 w}{\partial y^2} + \mu^- \frac{\partial^2 w}{\partial x^2} \right).$$

From (11), we obtain

$$\frac{E^+t_1^2}{1-\mu^+} = \frac{E^-t_2^2}{1-\mu^-}. \tag{13}$$

Combining  $t_1 + t_2 = t$ , we solve for the thicknesses of the plate in tension and compression as follows:

$$t_1 = \frac{\sqrt{E^-(1-\mu^+)}}{\sqrt{E^+(1-\mu^-)} + \sqrt{E^-(1-\mu^+)}}t, \quad t_2 = \frac{\sqrt{E^+(1-\mu^-)}}{\sqrt{E^+(1-\mu^-)} + \sqrt{E^-(1-\mu^+)}}t. \tag{14}$$

Thus, the position of the unknown neutral layer of the plate in pure bending is finally determined analytically.

**3.2. Lateral force bending.** While the plate is in lateral force bending, for example, under the uniformly distributed loads,  $q$ , as shown in Figure 3, not only the torsional stress,  $\tau_{xy}$ , but also the transverse shear stresses,  $\tau_{zx}$  and  $\tau_{zy}$ , as well as the extrusion stress,  $\sigma_z$ , exist in the plate. However, according to the conclusion in Section 2.2, the neutral layer does exist if the thickness of the plate is small compared with the deflection of the plate. Therefore, the torsional stresses in the plate in tension and compression may be expressed in terms of the deflection  $w$  as, respectively,

$$\tau_{xy}^+ = -\frac{E^+z}{1+\mu^+} \frac{\partial^2 w}{\partial x \partial y}, \quad 0 \leq z \leq t_1, \tag{15a}$$

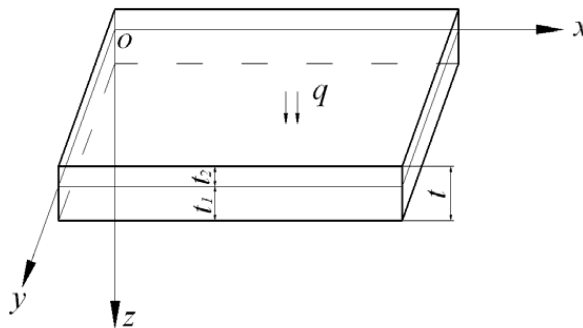
$$\tau_{xy}^- = -\frac{E^-z}{1+\mu^-} \frac{\partial^2 w}{\partial x \partial y}, \quad -t_2 \leq z \leq 0. \tag{15b}$$

The twist moment  $M_{xy}$  may be computed as

$$M_{xy} = \int_0^{t_1} \tau_{xy}^+ z dz + \int_{-t_2}^0 \tau_{xy}^- z dz = -\frac{1}{3} \left( \frac{E^+t_1^3}{1+\mu^+} + \frac{E^-t_2^3}{1+\mu^-} \right) \frac{\partial^2 w}{\partial x \partial y}. \tag{16}$$

Under uniformly distributed loads,  $q$ , the differential equation of equilibrium for bending of thin plates is

$$\frac{\partial^2 M_x}{\partial x^2} + 2 \frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} + q = 0. \tag{17}$$



**Figure 3.** Bimodular thin plate under lateral force bending.

Substituting (12) and (16) into (17), we obtain

$$\left\{ \frac{E^+ t_1^3}{3[1 - (\mu^+)^2]} + \frac{E^- t_2^3}{3[1 - (\mu^-)^2]} \right\} \nabla^4 w = q. \quad (18)$$

Equation (18) is the governing differential equation of the neutral layer. If we let

$$D^* = \frac{E^+ t_1^3}{3[1 - (\mu^+)^2]} + \frac{E^- t_2^3}{3[1 - (\mu^-)^2]}, \quad (19)$$

where  $D^*$  is the flexural rigidity of the bimodular plate, (18) may be written in a familiar form:

$$D^* \nabla^4 w = q. \quad (20)$$

Note that the transverse shear stresses  $\tau_{zx}$  and  $\tau_{zy}$  and the extrusion stress  $\sigma_z$  acting on the sections in tension and compression have not been determined. Due to the lack of longitudinal loads and the body force being zero, the first two expressions of the differential equations of equilibrium in tension may be written as

$$\frac{\partial \tau_{zx}^+}{\partial z} = -\frac{\partial \sigma_x^+}{\partial x} - \frac{\partial \tau_{yx}^+}{\partial y}, \quad \frac{\partial \tau_{zy}^+}{\partial z} = -\frac{\partial \sigma_y^+}{\partial y} - \frac{\partial \tau_{xy}^+}{\partial x}. \quad (21a)$$

Substituting (9) and (15a) into (21a) and considering  $\tau_{yx}^+ = \tau_{xy}^+$  as well as the stress boundary conditions at the bottom of the thin plate,

$$(\tau_{zx}^+)_{z=t_1} = 0, \quad (\tau_{zy}^+)_{z=t_1} = 0, \quad (22a)$$

we obtain

$$\tau_{zx}^+ = \frac{E^+}{2[1 - (\mu^+)^2]} (z^2 - t_1^2) \frac{\partial}{\partial x} \nabla^2 w, \quad \tau_{zy}^+ = \frac{E^+}{2[1 - (\mu^+)^2]} (z^2 - t_1^2) \frac{\partial}{\partial y} \nabla^2 w, \quad 0 \leq z \leq t_1. \quad (23a)$$

The third expression of the differential equations of equilibrium in tension is

$$\frac{\partial \sigma_z^+}{\partial z} = -\frac{\partial \tau_{xz}^+}{\partial x} - \frac{\partial \tau_{yz}^+}{\partial y}. \quad (24a)$$

Substituting (23a) into (24a) and considering  $\tau_{xz}^+ = \tau_{zx}^+$  and  $\tau_{yz}^+ = \tau_{zy}^+$  as well as the stress boundary conditions at the bottom of the plate,

$$(\sigma_z^+)_{z=t_1} = 0, \quad (25a)$$

we obtain

$$\sigma_z^+ = \frac{E^+}{2[1 - (\mu^+)^2]} \left( t_1^2 z - \frac{z^3}{3} - \frac{2}{3} t_1^3 \right) \nabla^4 w, \quad 0 \leq z \leq t_1. \quad (26a)$$

Similarly, the first two expressions of the differential equations of equilibrium in compression may be written as

$$\frac{\partial \tau_{zx}^-}{\partial z} = -\frac{\partial \sigma_x^-}{\partial x} - \frac{\partial \tau_{yx}^-}{\partial y}, \quad \frac{\partial \tau_{zy}^-}{\partial z} = -\frac{\partial \sigma_y^-}{\partial y} - \frac{\partial \tau_{xy}^-}{\partial x}. \quad (21b)$$

Substituting (10) and (15b) into (21b) and considering  $\tau_{yx}^- = \tau_{xy}^-$  as well as the stress boundary conditions at the top of the thin plate,

$$(\tau_{zx}^-)_{z=-t_2} = 0, \quad (\tau_{zy}^-)_{z=-t_2} = 0, \quad (22b)$$



we obtain

$$\tau_{zx}^- = \frac{E^-}{2[1 - (\mu^-)^2]}(z^2 - t_2^2) \frac{\partial}{\partial x} \nabla^2 w, \quad \tau_{zy}^- = \frac{E^-}{2[1 - (\mu^-)^2]}(z^2 - t_2^2) \frac{\partial}{\partial y} \nabla^2 w. \quad -t_2 \leq z \leq 0. \quad (23b)$$

The third expression of the differential equations of equilibrium in compression is

$$\frac{\partial \sigma_z^-}{\partial z} = -\frac{\partial \tau_{xz}^-}{\partial x} - \frac{\partial \tau_{yz}^-}{\partial y}. \quad (24b)$$

Substituting (23b) into (24b) and considering  $\tau_{xz}^- = \tau_{zx}^-$  and  $\tau_{yz}^- = \tau_{zy}^-$  as well as the stress boundary conditions at the top of the plate,

$$(\sigma_z^-)_{z=-t_2} = -q, \quad (25b)$$

we obtain

$$\sigma_z^- = \frac{E^-}{2[1 - (\mu^-)^2]} \left( t_2^2 z - \frac{z^3}{3} - \frac{2}{3} t_2^3 \right) \nabla^4 w, \quad -t_2 \leq z \leq 0. \quad (26b)$$

Thus, all stress components have been expressed in terms of  $w$ , as shown in (9), (10), (15), (23a), (23b), (26a), and (26b).

Using (23a) and (23b), we may compute the transverse shear forces  $Q_x$  and  $Q_y$  as follows:

$$Q_x = \int_0^{t_1} \tau_{xz}^+ dz + \int_{-t_2}^0 \tau_{xz}^- dz = -D^* \frac{\partial}{\partial x} \nabla^2 w, \quad Q_y = \int_0^{t_1} \tau_{yz}^+ dz + \int_{-t_2}^0 \tau_{yz}^- dz = -D^* \frac{\partial}{\partial y} \nabla^2 w. \quad (27)$$

Thus, the bending moments, the torsional moment and the transverse shear forces have been expressed in terms of  $w$ , as shown in (12), (16), and (27).

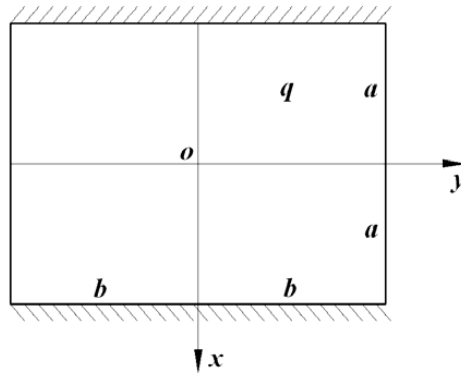
**3.3. Application of the displacement variation method.** For a variety of boundary conditions, the displacement of plates along the thickness direction,  $w$ , may be taken as different expressions to satisfy the given boundary conditions and then be determined via the differential (20). For example, for a rectangular thin plate with four simply supported sides, the Navier solution may be adopted; for such a plate with two opposite simply supported sides, the Levy solution may be adopted. However, in some cases, it is convenient to use the displacement variation method to solve such a problem. For example, for a rectangular thin plate with four sides fixed, the Galerkin approach may be adopted; for such a plate with two opposite sides fixed, the Ritz approach may be adopted. In this paper, for the purpose of comparison with the existent FEM results [Gao et al. 1998], we take a bimodular rectangular thin plate with two long sides fixed as our object of study, as shown in Figure 4, where  $2a$  and  $2b$  are the short and long sides, respectively, and the plate is under the action of normal uniformly distributed loads,  $q$ . Due to the existence of free sides, we adopt the Ritz approach to solve this problem.

The displacement boundary conditions at the fixed sides is

$$(w)_{x=\pm a} = 0, \quad \left( \frac{\partial w}{\partial x} \right)_{x=\pm a} = 0, \quad (28)$$

and the boundary condition at the free sides should satisfy

$$(w)_{y=\pm b, x \neq \pm a} \neq 0, \quad \left( \frac{\partial w}{\partial y} \right)_{y=\pm b, x \neq \pm a} \neq 0. \quad (29)$$



**Figure 4.** A bimodular plate under normal uniformly distributed loads.

Therefore, after considering the symmetry of this problem, we take the formula of  $w$  as

$$w = C_1 w_m = C_1(x^2 - a^2)^2 \left[ \left( \frac{y}{4b} \right)^2 + 1 \right], \tag{30}$$

where  $C_1$  is an undetermined coefficient and it is obvious that the above formula can satisfy boundary conditions (28) and (29). If we let the strain potential energy be  $U$ , from the Ritz approach, we have the following formula:

$$\frac{\partial U}{\partial C_1} = \iint q w_m dx dy. \tag{31}$$

Next, we will derive the formula for  $U$  in the case of different moduli in tension and compression.

In the small-deflection bending problem of a bimodular thin plate, according to the computational hypotheses, the strain components  $\varepsilon_z, \gamma_{yz}, \gamma_{zx}$  may be neglected; therefore, the strain potential energy  $U$  may be simplified as

$$U = \frac{1}{2} \int_{-a}^a \int_{-b}^b \int_0^{t_1} (\sigma_x^+ \varepsilon_x + \sigma_y^+ \varepsilon_y + \tau_{xy}^+ \gamma_{xy}) dx dy dz + \frac{1}{2} \int_{-a}^a \int_{-b}^b \int_{-t_2}^0 (\sigma_x^- \varepsilon_x + \sigma_y^- \varepsilon_y + \tau_{xy}^- \gamma_{xy}) dx dy dz, \tag{32}$$

where  $t_1$  and  $t_2$  are the thickness of the plate in tension and compression, respectively, and may be obtained from (14). The strain components  $\varepsilon_x, \varepsilon_y, \gamma_{xy}$  are

$$\varepsilon_x = -\frac{\partial^2 w}{\partial x^2} z, \quad \varepsilon_y = -\frac{\partial^2 w}{\partial y^2} z, \quad \gamma_{xy} = -2 \frac{\partial^2 w}{\partial x \partial y} z. \tag{33}$$

Substituting (9), (10), (15), and (33) into (32), after integrating over  $z$ , we have

$$U = \frac{E^+ t_1^3}{6[1 - (\mu^+)^2]} \int_{-a}^a \int_{-b}^b \left\{ (\nabla^2 w)^2 - 2(1 - \mu^+) \left[ \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2} - \left( \frac{\partial^2 w}{\partial x \partial y} \right)^2 \right] \right\} dx dy + \frac{E^- t_2^3}{6[1 - (\mu^-)^2]} \int_{-a}^a \int_{-b}^b \left\{ (\nabla^2 w)^2 - 2(1 - \mu^-) \left[ \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2} - \left( \frac{\partial^2 w}{\partial x \partial y} \right)^2 \right] \right\} dx dy. \tag{34}$$

From (30), we may have

$$\frac{\partial^2 w}{\partial x^2} = 4C_1(3x^2 - a^2) \left[ \left( \frac{y}{4b} \right)^2 + 1 \right], \quad \frac{\partial^2 w}{\partial y^2} = \frac{C_1}{8b^2}(x^2 - a^2)^2, \quad \frac{\partial^2 w}{\partial x \partial y} = \frac{C_1}{2b^2}(x^3 - a^2x)y. \quad (35)$$

Substituting (35) into (34), after integrating we obtain

$$U = \left\{ \frac{E^+ t_1^3}{6[1 - (\mu^+)^2]} F^+ + \frac{E^- t_2^3}{6[1 - (\mu^-)^2]} F^- \right\} C_1^2, \quad (36)$$

where

$$F^+ = \frac{8}{315} \frac{a^9}{b^3} + \left( \frac{16}{315} - \mu^+ \frac{136}{105} \right) \frac{a^7}{b} + \frac{4003}{75} a^5 b, \quad F^- = \frac{8}{315} \frac{a^9}{b^3} + \left( \frac{16}{315} - \mu^- \frac{136}{105} \right) \frac{a^7}{b} + \frac{4003}{75} a^5 b. \quad (37)$$

The right end of (31) may be computed as

$$\int_{-a}^a \int_{-b}^b q w_m dx dy = \int_{-a}^a \int_{-b}^b q (x^2 - a^2)^2 \left[ \left( \frac{y}{4b} \right)^2 + 1 \right] dx dy = \frac{98}{45} q a^5 b. \quad (38)$$

Substituting (36) and (38) into (31), we may obtain the formula for  $C_1$ . After substituting it into (30), we finally obtain the formula for  $w$ :

$$w = \frac{\frac{98}{15} q a^5 b}{\frac{E^+ t_1^3}{1 - (\mu^+)^2} F^+ + \frac{E^- t_2^3}{1 - (\mu^-)^2} F^-} (x^2 - a^2)^2 \left[ \left( \frac{y}{4b} \right)^2 + 1 \right], \quad (39)$$

where  $t_1$  and  $t_2$  are known and may be obtained from (14). The maximum deflection takes place at the midpoints of the two free sides, that is, while  $x = 0$  and  $y = \pm b$ , we have

$$w_{\max} = \frac{\frac{833}{120} q a^9 b}{\frac{E^+ t_1^3}{1 - (\mu^+)^2} F^+ + \frac{E^- t_2^3}{1 - (\mu^-)^2} F^-}. \quad (40)$$

## 4. Comparisons and discussions

**4.1. Comparisons with FEM based on Ambartsumyan's model.** The bimodular problem attributes non-linearity to materials. Since existent commercial FEM programs do not include the bimodular materials model proposed by Ambartsumyan, it is necessary to work out a numerical program based on an iterative technique. Because the stress state of the point in question is unknown in advance, we have to begin with a single modulus problem, thus gaining the initial stress state to form a corresponding elasticity matrix for each element. Generally, direct iterative methods based on an incrementally evolving stiffness have been adopted by many researchers.

Based on an incrementally evolving stiffness, Gao et al. [1998] analyzed a bimodular thin plate by the direct iterative method. In the FEM analysis, the elements of the plate should be layered along the direction of the plate thickness and each layer may be considered as being in the same stress state.

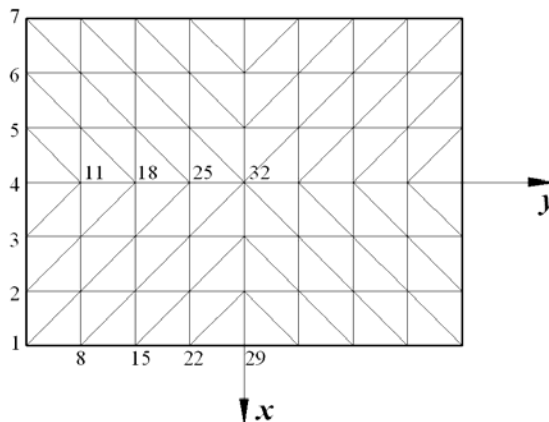
A rectangular thin plate fixed at its two longer sides is under the action of normal uniformly-distributed loads, as shown in Figure 5. For convenience, all the quantities are taken as dimensionless, specifically, the plate thickness is 0.24, the length and width of the plate are 8 and 6, respectively, and the intensity of

	$E^+/E^-$ :	1.4	1.25	1.0	0.85	0.75	0.5
Node 2	(a)	0.0330	0.0553	0.0628	0.0681	0.0738	0.0839
	(b)	0.0450	0.0574	0.0646	0.0706	0.0757	0.0957
Node 4	(a)	0.1731	0.1863	0.2107	0.2281	0.2440	0.2856
	(b)	0.1749	0.1858	0.2095	0.2288	0.2453	0.3001
Node 18	(a)	0.1665	0.1786	0.2034	0.2222	0.2431	0.2763
	(b)	0.1672	0.1776	0.2002	0.2188	0.2345	0.2964
Node 32	(a)	0.1601	0.1778	0.2028	0.2205	0.2387	0.2737
	(b)	0.1646	0.1749	0.1971	0.2154	0.2309	0.2918

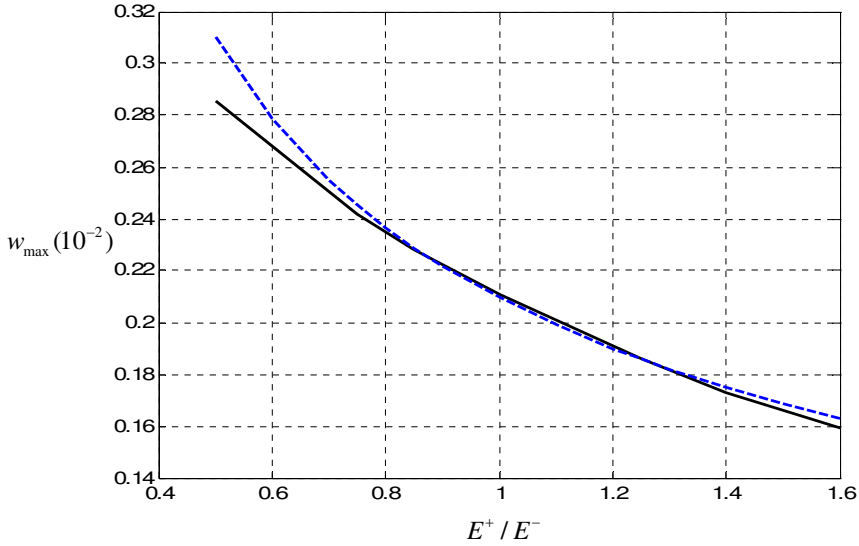
**Table 1.** The deflection results of FEM and analytical formulas ( $\times 10^{-2}$ ). Results from (a) [Gao et al. 1998] and (b) this paper.

the uniformly distributed loads is 1. Considering the values of different moduli in tension and compression, we fix the values  $E^- = 1.4 \times 10^6$  and  $\mu^- = 0.2$ , and set the values of  $E^+$  and  $\mu^+$  only by changing their ratio  $E^+/E^- = \mu^+/\mu^-$ . We plot the maximum deflections of the plate, which is obtained by FEM and the analytical method presented in this paper, as shown in Figure 6. We also list some numerical results based on analytical formulas (39) derived in this paper, as shown in Table 1. It is easily seen that, if  $E^+/E^- > 0.6$ , the analytical solutions obtained in this paper agree well with the FEM results strictly based on Ambartsumyan's model.

In order to demonstrate the validity of the simplified model, we also list some results in the cases of different layer numbers while  $E^+/E^- = 0.85$ , as shown in Table 2. From the table, we see that when the whole plate is divided into two layers along the thickness direction, the numerical results are relatively accurate compared with the other cases in which the plate is divided into a greater number of layers. For the iterative results of the deflection, we see that more layering does little to improve the computational accuracy. The simplified model proposed in this paper is founded just on the fact that the whole plate is divided into two layers along the direction of plate thickness, one in tension and the other in compression.



**Figure 5.** An FEM computation of a bimodular thin plate [Gao et al. 1998].



**Figure 6.** The relation between the maximum deflections (node 4) and  $E^+/E^-$ . (The solid line shows results for FEM, the dashed line for analytical solutions.)

From the analyses above, it may be seen that our simplified model may be valid for a bimodular thin plate in bending.

**4.2. Discussions.**

*Regression and advantage of the analytical solutions.* We observe that in (14), (19), and (20), while  $E^+ = E^-$  and  $\mu^+ = \mu^-$ , we have  $t_1 = t_2 = t/2$ ,  $D = Et^3/(12(1 - \mu^2))$ , and  $D\nabla^4w = q$ . Moreover, the strain potential energy (34) derived in this paper can also be reduced to the classical formula. In fact, all the analytical formulas for bimodular plates in bending may be reduced to the classical problem of plates. These regressions show that the derivation process in this paper is reasonable.

In this paper, we introduce a new quantity, the flexural stiffnesses of bimodular thin plates,  $D^*$ . The flexural stiffness we introduce plays an important role in solving for bimodular plates. The study above indicates that the bimodularity of materials has great influence on the structural rigidity. In most cases, the influence of the bimodularity of materials may be integrated into the flexural stiffness. By simple

Number of layers	$-w_4 (\times 10^{-2})$	$-w_{32} (\times 10^{-2})$	Number of iterations
2	0.2283	0.2203	3
4	0.2280	0.2206	4
6	0.2281	0.2205	4
8	0.2281	0.2205	5
10	0.2281	0.2205	6

**Table 2.** The results in cases of different number of layers when  $E^+/E^- = 0.85$  [Gao et al. 1998].

substitution of the flexural stiffness, we can readily obtain the solutions for bimodular thin plates directly from the classical solutions of plates for a variety of boundary conditions. Since solutions of the Kirchhoff plate theories for a variety of boundary conditions are easily determined or are available in most textbooks on mechanics of materials, the simplified model presented herein makes it easier to analyze bimodular thin plates analytically.

*Essence and limitations of the simplified model.* In civil engineering, the principal stress state is a key point in the analysis and design of bending components like beams, columns, plates, and shells. Shear stresses and the resulting diagonal tensions must also be carefully considered in the design of reinforced concrete. In this paper, the application of the Kirchhoff hypothesis tests the existence of the neutral layer and realizes the subarea in tension and compression. The simplified mechanical model proposed in this paper essentially regards the whole bending plate as a laminated structure consisting of only two layers, one in tension and another in compression.

The results obtained in this paper are founded on the Kirchhoff hypothesis, which are for relatively thin plates, and thus the results are not applicable to the analysis of bimodular thick plates. In thick plates, the shearing stresses are important. Such bimodular thick plates should be treated by means of a more general theory owing to the fact that some assumptions of the Kirchhoff hypotheses are no longer appropriate.

## 5. Concluding remarks

Based on the classical Kirchhoff hypothesis, we propose a simplified mechanical model used in solving bimodular thin plates in small-deflection bending. The following main conclusions can be drawn.

- The application of the Kirchhoff hypothesis tests the existence of the elastic neutral layers of bimodular thin plates in small-deflection bending, and opens up possibilities for subareas in tension and compression.
- The application of the Kirchhoff hypothesis may extend the solution from the case of pure bending, which strictly follows the bimodular materials model proposed by Ambartsumyan, into the case of lateral force bending.
- The flexural stiffnesses derived in this paper enables us to obtain easily the analytical solutions of bimodular thin plates via the classical problems for a variety of boundary conditions.

This work will be helpful for predicting the mechanical behaviors of bimodular materials. In particular, these results may be useful in analyzing concrete-like materials and fiber-reinforced composite materials that contain cracks and are undergoing contact, whose macroscopic constitutive behavior depends on the direction of the macroscopic strain, similarly to the case of bimodular materials [Zinno and Greco 2001; Greco 2009].

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