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# MEDIA WITH SEMIHOLONOMIC INTERNAL STRUCTURE

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*This article is dedicated to the memory of Marie-Louise Steele.*

The notions of semiholonomic and quasiholonomic Cosserat media are introduced and their differences outlined. Contrary to the classical holonomic and nonholonomic counterparts, the definition of semi and quasiholonomic media is not kinematic but constitutive. Possible applications include granular media embedded in a rigid matrix and colloidal suspensions in an ideal incompressible fluid.

## 1. Introduction

The publication in 1909 of the French version of the second volume of Chwolson's *Traité de physique* was generally well received by the scientific establishment of the time. Nevertheless, we find in the issue of *Nature* of July 21, 1910, a remark to the effect that a "note" by MM. E. and F. Cosserat at the end of this volume is 220 pages long and "does not in any sense harmonise" with Chwolson's work, which is "emphatically experimental in character", while the note is "strikingly mathematical". The remark concludes with the suggestion that "MM. Cosserat's note is a distinct and useful treatise, and should be able to stand on its own feet." And so it was, although the author of the review didn't seem to have taken notice. The celebrated book by the Cosserat brothers [1909], was, in fact, identical to that long note plus an additional 67 pages, which had already appeared in the previous volume of Chwolson's treatise. A similar critical remark can be found two years later in a review by Edwin B. Wilson of the Massachusetts Institute of Technology published in the *Bulletin of the American Mathematical Society* (July 1912, pp. 497–508), although six months later (February 1913, pp. 242–246) the same reviewer found it necessary to publish a separate review of the stand-alone book. Both the detailed content and the title of this review ("An advance in theoretical mechanics") demonstrate clearly that the quality of the enterprise of the brothers Cosserat did not go unnoticed.

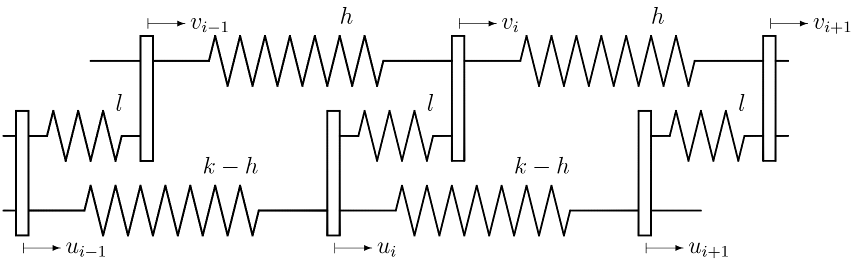
Today we mainly recognise the contribution of the Cosserats to the modelling of media with internal structure. But this does not seem to have been the main intention of the authors, who had in mind a much wider scope. Their fundamental idea was to try to encompass all physical theories (including perhaps relativity) under the umbrella of a single principle of Euclidean action, a quantity that is invariant under Euclidean transformations. They also advanced the notion of the formal equivalence between a static theory of a deformable $n$-dimensional manifold and a dynamic theory of a deformable manifold of dimension $n+1$, whereby the action is interpreted as a space-time entity. Another striking feature of the work is its error-free geometrically nonlinear formulation, particularly in view of the many mistakes made by succeeding generations, for example, in the realm of shell theory. To recover the conventional theory

*Keywords:* micromorphic nonholonomic media, Cosserat symmetries, colloids, semiholonomic jets, quasiholonomic Cosserat media, macromedium, micromedium.

of elasticity, the Cosserats propose the idea of the hidden triad ("trièdre caché"). Similarly, theories with particular constraints can be recovered by means of the concept of hidden action. Thus, rigid-body mechanics arises as a particular case of their theory. Had they not insisted on the rigidity of the triad, they could have also recovered the theory of second-grade materials by considering what we would call today second-order holonomic frames. Between these frames and the completely nonholonomic frames (corresponding to a micromorphic, rather than micropolar, continuum), there exists an intermediate type consisting of the semiholonomic frames. An intriguing question is whether the "hidden action" corresponding to this type of frames might lead to a physically meaningful interpretation. In this short article it is shown that this is indeed the case if one is willing to consider a material consisting entirely of a micromedium supported by an incoherent matrix. A distinction is drawn between strictly semiholonomic Cosserat media and a related category referred to as quasiholonomic. The presentation is intended to be as self-contained as possible.

## 2. A one-dimensional picture

The variety of possible Cosserat media[1] resists any attempt at a simple pictorial representation, and this deficiency is exacerbated when the picture is limited to the one-dimensional realm. Thus, for example, rigid-body motions in one dimension are unable to convey the possibility of rotations. Nevertheless, a picture may be of help in providing some insight and motivating further rigorous investigations:




The lower row of material points (represented by the lower three elongated rectangles) symbolizes the macromedium, while the upper counterpart is the micromedium. The corresponding degrees of freedom are indicated, respectively, with the letters $u_i$ and $v_i$, where the subscript runs over the number of particles. An eventual passage to the continuous limit is suggested, but not directly described. For simplicity, we represent the constitutive equations by means of linear elastic springs, so that, measured from an assumed stress-free reference configuration (which may or may not exist), the elastic energy is given, up to an irrelevant additive constant, by the expression

$$W = \tfrac{1}{2}\textstyle\sum_i (k-h)(u_{i+1}-u_i)^2 + l(v_i-u_i)^2 + h(v_{i+1}-v_i)^2. \tag{2-1}$$

The following particular values of the stiffness constants $k$, $h$, and $l$ are of interest:

- *Arbitrary $k$, $l$, and $h$*: This is the case of the *micromorphic medium*, also called the *nonholonomic* Cosserat medium.

[1] Although the terminology "Cosserat medium" is usually reserved for the particular case of a rigid triad (or micropolar continuum), in this paper we use it to denote the general case.

• $h = 0$: This case can be designated as a first-grade macromedium carrying a *zero-grade* micromedium. The grains, so to speak, do not interact elastically with each other. Note that the same physical result is obtained by specifying $h = k$ instead of $h = 0$.

• $l \to \infty$: In this case we must necessarily have $u_i = v_i$, for all $i$. This condition spells the disappearance of the micromedium. The energy expression reduces to the standard form,

$$W = \tfrac{1}{2} \textstyle\sum_i k(u_{i+1} - u_i)^2, \tag{2-2}$$

of an elastic *first-grade* material. In the one-dimensional case, the specification $l \to \infty$, by making it rigid, renders the micromedium superfluous. Clearly, in a two- or three-dimensional situation, to achieve the same effect one would need to also specify $h = 0$. Otherwise, the mere rigidity of the micromedium would still allow for an interaction between rigidly rotating microparticles (or *grains*). This is, in fact, the definition of a *micropolar medium*, which is the material originally conceived by the Cosserat brothers in their magnum opus.

• $l = 0$: For the system to remain connected, we attach (in the unstressed state) the (upper) grain to the midpoint of the corresponding lower spring. The energy expression is given by

$$W = \tfrac{1}{2} \textstyle\sum_i 2(k - h)\big((v_{i+1} - u_{i+1})^2 + (u_{i+1} - v_i)^2\big) + h(v_{i+1} - v_i)^2. \tag{2-3}$$

The lower springs (now double in number) connect between contiguous particles, while the upper springs connect between every second particle. This is the standard representation of a *second-grade* material. Notice that in this case the elastic energy (2-3) can also be written more suggestively as

$$W = \tfrac{1}{2} \textstyle\sum_i 2k\big((v_{i+1} - u_{i+1})^2 + (u_{i+1} - v_i)^2\big) - h(v_{i+1} - 2u_{+1} + v_i)^2. \tag{2-4}$$

• $k \to \infty$: This is the case of a *rigid macromedium*. We will soon demonstrate that this situation may correspond mathematically to a genuinely *semiholonomic* Cosserat medium.

• $k = 0$: Physically, this case corresponds to an *incoherent matrix* within which the micromedium provides the only degree of elastic coherence. We will use also the terminology *quasiholonomic* medium to refer to this type of material.

## 3. Cosserat bodies

In continuum mechanics, a *material body* $\mathcal{B}$ is defined as a three-dimensional differentiable manifold that can be covered with a single coordinate chart. A *configuration* $\kappa$ is defined as an embedding of $\mathcal{B}$ into the three-dimensional Euclidean space $\mathbb{E}^3$:

$$\kappa : \mathcal{B} \longrightarrow \mathbb{E}^3. \tag{3-1}$$

In terms of coordinate charts $X^I$ $(I = 1, 2, 3)$ and $x^i$ $(i = 1, 2, 3)$ in the body and in space, respectively, the configuration $\kappa$ is given by three smooth functions:

$$X^I \mapsto x^i = \kappa^i(X^I). \tag{3-2}$$

To convey the presence of extra kinematic degrees of freedom, however, these definitions need to be expanded so that the differential geometry can properly reflect the existence of the microstructure and its possible deformability. We recall that, given an $m$-dimensional differentiable manifold $\mathcal{M}$, its *principal frame bundle* $F\mathcal{M}$ is obtained by adjoining at each point $x \in \mathcal{M}$ the collection $F_x\mathcal{M}$ of all the possible bases of its tangent space $T_x\mathcal{M}$. The set thus obtained has a canonical structure of a differentiable manifold of dimension $m + m^2$. It is endowed with the *natural projection* map

$$\pi_M : F\mathcal{M} \longrightarrow \mathcal{M}, \tag{3-3}$$

which assigns to each point $p \in F\mathcal{M}$ the point $\pi(p) \in \mathcal{M}$ to which it is attached. If $x^i$ $(i = 1, \dots, m)$ is a coordinate chart on $U \subset \mathcal{M}$ with natural basis $\boldsymbol{e}_i = \partial/\partial x^i$, we can construct an associated chart in $F\mathcal{M}$ by assigning to each point $p \in \pi_M^{-1}(U)$ the numbers $\{x^i, p^i_j\}$, where $p^i_j$ is the $i$-th component of the $j$-th vector of the frame $p$ in the natural basis $\{\boldsymbol{e}_1, \dots, \boldsymbol{e}_m\}$. Expressed in terms of coordinates, the natural projection is given by

$$x^i, p^i_j \mapsto x^i. \tag{3-4}$$

We define a *Cosserat body* as the principal frame bundle $F\mathcal{B}$ of an ordinary material body $\mathcal{B}$. The physical intent is that, while the underlying body $\mathcal{B}$ represents the *macromedium*, each *fibre* $F_x\mathcal{B}$ represents the *microparticle* or *grain* at $x \in \mathcal{B}$.

Concomitantly with the enlargement of the scope of material bodies, we need to introduce a more general definition of the notion of configuration. To this end, we consider *fibre-preserving* maps

$$K : F\mathcal{B} \longrightarrow F\mathbb{E}^3 \tag{3-5}$$

such that $K$ is a *fibre-bundle morphism* between $F\mathcal{B}$ and its image. By fibre preservation, we mean the commutativity of the diagram:

$$\begin{tikzcd} F\mathcal{B} \arrow[r, "K"] \arrow[d, "\pi_B"] & F\mathbb{E}^3 \arrow[d, "\pi_E"] \\ \mathcal{B} \arrow[r, "\kappa"] & \mathbb{R}^3, \end{tikzcd} \tag{3-6}$$

where $\kappa$ is a well-defined map between the base manifolds. Thus, a *Cosserat deformation* $K$ automatically implies the existence of an ordinary deformation $\kappa$, representing the deformation of the macrostructure. By fibre-bundle morphism we imply that, fibre by fibre, each of the restrictions $K|_X$ $(X \in \mathcal{B})$ commutes with the multiplicative right action of the general linear group $\mathrm{GL}(3; \mathbb{R})$. In terms of coordinates, this means that there exists an $X$-dependent matrix $K^i_I$ such that any Cosserat configuration is completely defined by twelve smooth functions

$$x^i = \kappa^i(X^J) \tag{3-7}$$

and

$$K^i_I = K^i_I(X^J). \tag{3-8}$$

The physical meaning of these assumptions is that each grain can undergo only homogeneous deformations, as represented by the local matrix $K^i_I$. In other words, each grain behaves as a pseudorigid body. A more detailed treatment can be found in [Epstein and de León 1996; 1998; Epstein and Elżanowski 2007].

**Remark 3.1.** As already pointed out, the original formulation by the Cosserat brothers considered the case in which $K^i_I$ is orthogonal. In the terminology of [Eringen 1999], this case corresponds to the micropolar continuum. The more general case in which $K^i_I$ is an arbitrary nonsingular matrix corresponds to the micromorphic continuum of Eringen. We use the term “Cosserat body” in this more general sense.

We can see that in a Cosserat body there exist two, in principle independent, mechanisms for dragging vectors by means of a deformation: The first mechanism is the ordinary dragging of vectors by means of the deformation gradient of the macromedium, represented by the matrix with entries $F^i_I = x^i_{,I}$. The second mechanism is the one associated with the deformation of the microparticle or grain, and is represented by the matrix with entries $K^i_I$. Note that in a second-grade body these two mechanisms are identified with each other, thus suggesting that different kinds of Cosserat media may be obtained by either kinematic restrictions of this kind or by constitutive restrictions. In fact, the Cosserat brothers themselves already advanced these possibilities and introduced the outmoded terminology of “trièdre caché” (hidden triad) and “$W$ caché” (hidden strain-energy function) to refer, respectively, to these kinematic or constitutive restrictions. We will follow in their steps.

## 4. Nonholonomic, semiholonomic, and holonomic jets

Given two smooth manifolds, $\mathcal{M}$ and $\mathcal{N}$, of dimensions $m$ and $n$, respectively, we say that two maps $f, g : \mathcal{M} \longrightarrow \mathcal{N}$ have the same $k$-jet at a point $X \in \mathcal{M}$ if: (i) $f(X) = g(X)$, and (ii) in a coordinate chart in $\mathcal{M}$ containing $X$ and a coordinate chart in $\mathcal{N}$ containing the image $f(X)$, all the partial derivatives of $f$ and $g$ up to and including the order $k$ are respectively equal.

Although the above definition is formulated in terms of charts, it is not difficult to show by direct computation that the property of having the same derivatives up to and including order $k$ is in fact independent of the coordinate systems used in either manifold. Notice that, in order for this to work, it is imperative to equate *all* the lower-order derivatives. If, for example, we were to equate just the second derivatives, without regard to the first, the equality of the second derivatives would not be preserved under arbitrary coordinate transformations.

The property of having the same $k$-jet at a point is, clearly, an equivalence relation. The corresponding equivalence classes are called *$k$-jets at $X$*. Any function in a given $k$-jet is then called a *representative* of the $k$-jet. The $k$-jet at $X$ of which a given function $f : \mathcal{M} \longrightarrow \mathcal{N}$ is a representative is denoted by $j^k_X f$. The collection of all $k$-jets at $X \in \mathcal{M}$ is denoted by $J^k_X(\mathcal{M}, \mathcal{N})$. The point $X$ is called the *source* of $j^k_X f$ and the image point $f(X)$ is called its *target*.

Let a smooth map $f : \mathcal{M} \longrightarrow \mathcal{N}$ be given in terms of coordinates $X^I$ $(I = 1, \dots, m)$ and $x^i$ $(i = 1, \dots, n)$ in $\mathcal{M}$ and $\mathcal{N}$, respectively, by the functions

$$x^i = x^i(X^1, \dots, X^m), \quad i = 1, \dots, n. \tag{4-1}$$

The jet $j^2_X f$, for example, is then given by the coordinate expressions

$$x^i(X^1, \dots, X^m), \quad \left[\frac{\partial x^i}{\partial X^I}\right]_X, \quad \left[\frac{\partial^2 x^i}{\partial X^J \partial X^I}\right]_X, \tag{4-2}$$

a total of $n + mn + m^2 n$ numbers.

We are particularly interested in the case of 1-jets. Let us evaluate, accordingly, the coordinate expression of $j^1_X K$, where $K$ is a Cosserat configuration, as defined in coordinates by (3-7) and (3-8). Notice that the dimension of both the source and the target manifolds in this case is 12. Following the definition, we conclude that $j^1_X K$ consists of the elements

$$x^i, \quad K^i_I, \quad \left[\frac{\partial x^i}{\partial X^I}\right]_X, \quad \left[\frac{\partial K^i_I}{\partial X^J}\right]_X, \tag{4-3}$$

which we can abbreviate as

$$x^i, \quad K^i_I, \quad F^i_I = x^i_{,I}, \quad K^i_{I,J}. \tag{4-4}$$

If no further restrictions are imposed on $K$, we speak of the components (4-4) as the representatives of a *nonholonomic* 1-jet at $X \in \mathcal{B}$. It is possible, however, to demand in an intrinsic manner, independent of the coordinates, that the functions $K$ under consideration satisfy the following compatibility requirement in a neighbourhood of $X$:

$$K^i_I \equiv x^i_{,I}. \tag{4-5}$$

In this case, the collection of 1-jets obtained is smaller. Not only are the second and third entries in (4-4) the same, but also, by virtue of the identical satisfaction of (4-5) in a neighbourhood of $X$, we must have:

$$K^i_{I,J} = x^i_{,IJ} = K^i_{J,I}. \tag{4-6}$$

In other words, the last element of the jet is symmetric with respect to its lower indices. We will indicate the coordinate expression of these *holonomic jets* as follows:

$$x^i, \quad F^i_I, \quad K^i_{I,J} = K^i_{J,I}. \tag{4-7}$$

Finally, there exists a third type of jet, somewhat intermediate between the two extremes just presented. It is obtained when the potential representatives $K$ are restricted to satisfy the condition

$$K^i_I(X) = x^i_{,I}(X). \tag{4-8}$$

Thus, we demand the satisfaction of (4-5) not identically in a neighbourhood of $X$, but just at the point $X$ itself. The 1-jets thus obtained are known as *semiholonomic jets*. The coordinate expression of a semiholonomic jet is

$$x^i, \quad K^i_I, \quad K^i_{I,J}. \tag{4-9}$$

Notice that the last entry is no longer necessarily symmetric.

**Remark 4.1.** Given an actual arbitrary configuration $K$, it will give rise automatically to point-wise nonholonomic jets. If the configuration is restricted so that condition (4-5) is satisfied over the whole base manifold $\mathcal{B}$, it will give rise to everywhere holonomic jets. In this sense, it is possible to speak of nonholonomic or holonomic configurations, respectively. On the other hand, it is not possible to define semiholonomic configurations. Indeed, if condition (4-8) were to be imposed at each point, we would immediately revert to condition (4-5), thus obtaining a holonomic configuration.

## 5. Semiholonomic Cosserat media

The last section ended in a definitely pessimistic note. Indeed, if semiholonomic configurations cannot be properly defined, there seems to be no point in attempting a definition of semiholonomic media. This kinematic impasse, however, can perhaps be resolved by means of a constitutive statement. We could say, for example, that a nonholonomic Cosserat medium is semiholonomic if its constitutive equation involves only the semiholonomic part of the 1-jet of the configuration. Physically, this would correspond to a response that is in some sense oblivious of the presence of the macromedium. In this section, we look into this and other possibilities with some care.

Since we are contemplating a particular case of nonholonomic Cosserat media, it will be useful to record the law governing the change of constitutive law of such a medium under a change of reference configuration. For specificity, we will limit ourselves to a single scalar constitutive law, such as the free-energy density per unit mass $\psi$. Let the constitutive law with respect to a reference configuration $K_0$ be given in a coordinate system $X^I$ by the expression

$$\psi = \psi_0(K^i_I, F^i_I, K^i_{I,J}; X^I), \tag{5-1}$$

and let the counterpart for a reference configuration $K_1$ with coordinates $Y^A$ be given by

$$\psi = \psi_1(K^i_A, F^i_A, K^i_{A,B}; Y^A), \tag{5-2}$$

with an obvious notational scheme. The deformation from $K_0$ to $K_1$ is given by twelve quantities, written

$$Y^A = Y^A(X^I), \quad K^A_I(X^J). \tag{5-3}$$

By the law of composition of jets (or derivatives), we obtain between the constitutive expressions the relation

$$\psi_1(K^i_A, F^i_A, K^i_{A,B}; Y^A(X^J)) = \psi_0(K^i_A K^A_I, F^i_A F^A_I, K^i_{A,B} K^A_I F^B_J + K^i_A K^A_{I,J}; X^J), \tag{5-4}$$

where $F^A_I = Y^A_{,I}$.

The point of bringing this transformation equation to bear is the following result, whose proof is an immediate consequence of the transformation law (5-4).

**Proposition 5.1.** *If the constitutive law (5-1), in the reference configuration $K_0$, is independent of the second argument $(F^i_I)$, so is the expression of the same constitutive law in any other reference configuration $K_1$ independent of the second argument $(F^i_A)$.*

As a direct corollary of this proposition, we can propose the following definition.

**Definition 5.2.** A nonholonomic Cosserat medium is said to be *semiholonomic* at $X$ if its constitutive law at $X$ is independent of the deformation gradient of the macromedium.

From the mathematical standpoint, it is necessary to note that this definition does not imply the existence of a *canonical* projection of a nonholonomic jet onto a semiholonomic part. In fact, such a canonical projection does not exist. What the definition implies is that once a noncanonical choice is effected in one particular reference configuration, this choice can be convected to all other configurations by means of the correct application of the transformation (5-4). In particular, this convection involves the gradient of the change of reference configuration $(F^A_I)$. Another way to state the choice of a particular "projection" is to say that a particular parallelism (whose physical meaning may, for example, be related to the existence of some particular stress-free configuration) must be chosen as part and parcel of the constitutive law of a semiholonomic Cosserat medium.

From the physical point of view, a semiholonomic Cosserat medium may be said to consist of an incoherent matrix upon which a coherent micromedium has been installed. The interaction between the grains, however, "remembers" the existence of a particular configuration of the macromedium as the only remaining physical trace of its existence. It is interesting to remark that, since the macromedium plays no other role, the configuration space of a semiholonomic medium may be, in a possible application, assumed to be rigid.

The converse of the above statement is not true: a nonholonomic Cosserat medium with a rigid matrix is not automatically semiholonomic. Indeed, by a direct application of the principle of frame indifference, the constitutive law (5-1) can be reduced to the form

$$\psi = \psi(\boldsymbol{R}^T\boldsymbol{K}, \boldsymbol{U}, \boldsymbol{R}^T\nabla\boldsymbol{K}; X), \tag{5-5}$$

where the polar decomposition $\boldsymbol{F} = \boldsymbol{R}\boldsymbol{U}$ has been exploited and where block letters stand for the collections of homonymous indexed quantities used in previous formulas. Using now the polar decomposition

$$\boldsymbol{K} = \boldsymbol{R}'\boldsymbol{U}', \tag{5-6}$$

we may write (5-5) as

$$\psi = \psi(\boldsymbol{r}\boldsymbol{U}', \boldsymbol{U}, \boldsymbol{r}\boldsymbol{R}'^T\nabla\boldsymbol{K}; X), \tag{5-7}$$

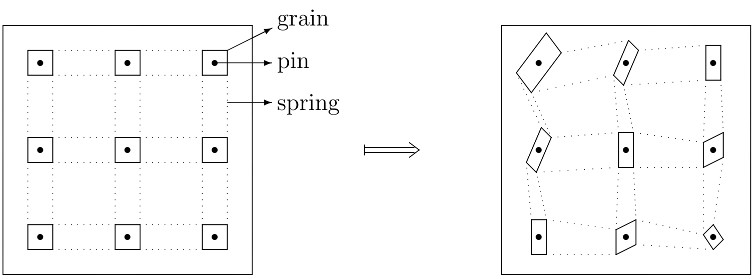
where

$$\boldsymbol{r} = \boldsymbol{R}^T\boldsymbol{R}' \tag{5-8}$$

is the (referential) *relative rotation* of the grain with respect to the macromedium. If the macromedium is rigid, we must have necessarily $\boldsymbol{U} = \boldsymbol{I}$. But for a semiholonomic body the constitutive law must be independent of *both* components $\boldsymbol{U}$ and $\boldsymbol{R}$ of the polar decomposition of $\boldsymbol{F}$. It follows, therefore, that rigidity alone does not imply semiholonomy. If, on the other hand, the constitutive law of a rigid-matrix Cosserat medium is independent of the rotation $\boldsymbol{R}$, we may choose $\boldsymbol{R} = \boldsymbol{R}'$ (or, equivalently, $\boldsymbol{r} = \boldsymbol{I}$), thereby leading to the following reduced equation of a semiholonomic Cosserat body:

$$\psi = \psi(\boldsymbol{U}', \boldsymbol{R}'^T\nabla\boldsymbol{K}; X). \tag{5-9}$$

In the physical interpretation, we may say that the grains are attached to the rigid macromedium by means of ideal frictionless pins, so that there is no energetic cost to produce a relative rotation between them. In the admittedly imperfect pictorial representation of the figure below, the grains in the reference configuration are depicted as squares pin-jointed at their centres to the rigid matrix and connected to their neighbours by means of springs (represented by broken lines) designed to detect differential stretches




A rigid-matrix semiholonomic Cosserat medium.

and rotations between contiguous grains. The grains themselves behave as *pseudorigid bodies*, so that their deformed versions are represented by parallelograms.

The reduced form (5-9) of the constitutive law of a semiholonomic Cosserat material applies whether or not the matrix is rigid, since in either case the response is independent of both $\boldsymbol{U}$ and $\boldsymbol{R}$.

## 6. Quasiholonomic Cosserat media

As defined, a semiholonomic Cosserat medium may not necessarily have any material symmetries. We want to contrast the above definition with the following one that, by demanding the maximum possible symmetry of the macromedium, appears to carry the same physical meaning.

**Definition 6.1.** A nonholonomic Cosserat medium is said to be *quasiholonomic* at $X$ if, for some (local) reference configuration, its symmetry group $\mathscr{H}$ at $X$ contains the subgroup given by

$$\mathscr{G} = \big\{ \{I, G, 0\} \,\big|\, G \in \mathrm{GL}(3; \mathbb{R}) \big\}, \tag{6-1}$$

where $I$ is the unit of $\mathrm{GL}(3; \mathbb{R})$.

The reason to suspect that this definition might be equivalent to the previous one is that, due to the assumed arbitrariness of $G$, it seems to imply that the deformation of the macromedium plays no role in the constitutive response. A direct application of the definition of a nonholonomic symmetry, however, leads to the conclusion that a quasiholonomic medium must have a constitutive law of the form

$$\psi = \psi(K^i_I, K^i_{I,J} F^{-J}_j; X^I) \tag{6-2}$$

in the special reference configuration used in the definition[2].

Physically, this means that the price to pay for this large symmetry group is, surprisingly, the reappearance of the deformation gradient of the macromedium in the last argument of the constitutive law so as to permit the interaction between the grains to take into account their relative spatial locations (rather than those pulled back to some putative, perhaps unstressed, reference configuration).

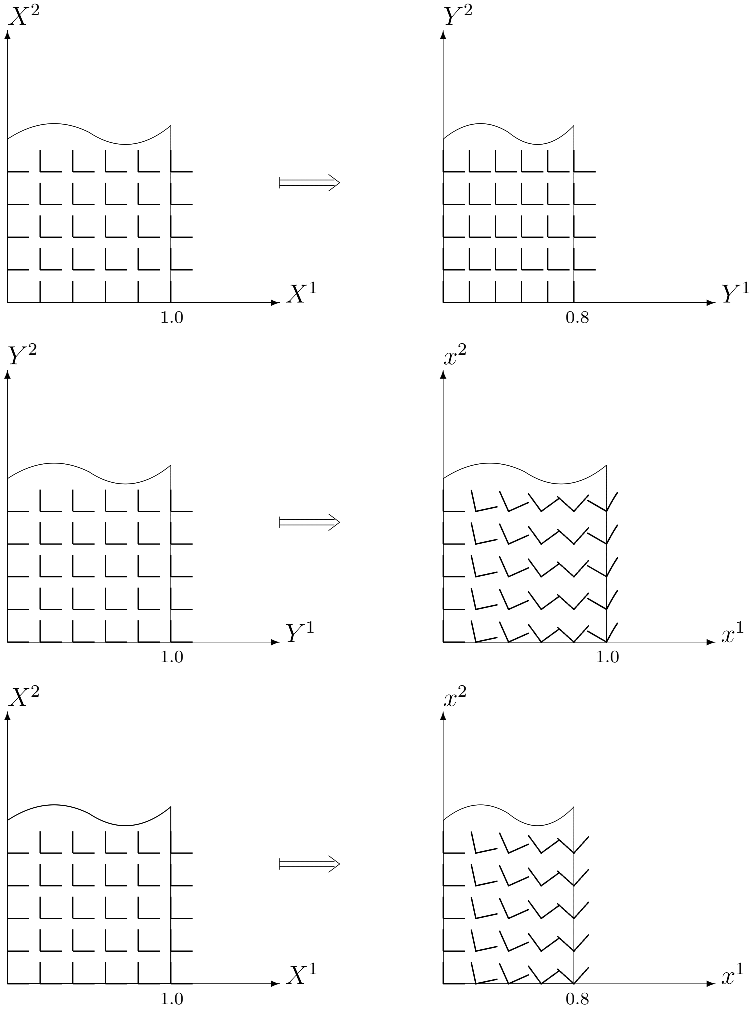
The purpose of the following simple example is to shed light on the subtle difference between semiholonomic and quasiholonomic media, as conceived in Definitions 5.2 and 6.1, respectively. To this end, we consider the successive application of two deformations, the first of which can be regarded as a change of reference configuration so as to bring the notation in line with that of the previous section. The (Cartesian) coordinate systems $X^I$, $Y^A$, and $x^i$ are assumed to coincide with each other. The first deformation is a uniaxial contraction along the $X^1$-axis, namely

$$Y^1 = 0.8X^1, \quad Y^2 = X^2, \quad Y^3 = X^3, \quad K^A_I = \delta^A_I. \tag{6-3}$$

The second deformation is a microrotation about the $Y^3$ axis that increases linearly with $Y^1$. Specifically:

$$x^1 = Y^1, \quad x^2 = Y^2, \quad x^3 = Y^3, \quad \{K^i_A\} = \begin{bmatrix} \cos\left(\frac{\pi}{3}Y^1\right) & -\sin\left(\frac{\pi}{3}Y^1\right) & 0 \\ \sin\left(\frac{\pi}{3}Y^1\right) & \cos\left(\frac{\pi}{3}Y^1\right) & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{6-4}$$

[2] In any other reference configuration, the symmetry group will contain a conjugate of the group $\mathscr{G}$ and the form of the constitutive law will be, accordingly, somewhat more involved.




First deformation (top), second deformation (middle), and their composition (bottom).

The effect of each of the two deformations on a unit-width strip in the $X^1$, $X^2$ and $Y^1$, $Y^2$ planes is shown in the figure above (top and middle parts), together with their composition (bottom part). Notice that, at the moment of composition, it is the *already contracted* strip that encounters the values of the rotation field already in place (as dictated by the second deformation), thus resulting in a maximum value for the rotation of the grain in the deformed strip of 48° rather than 60°, which was the value at the right-hand end of the strip as far as the second deformation alone was concerned. If the Cosserat body is semiholonomic, the gradient of the rotation would be obtained by dividing 48° by the original unit width. On the other hand, if the Cosserat body is quasiholonomic, it is the width measured in the final deformed configuration that matters in the calculation of the gradient. Since this width is of 0.8, we verify that the rotation gradient in the composite deformation turns out to be identical to the gradient in the second deformation. In other words, the preapplication of the first deformation (in this case a contraction of the macromedium) is irrelevant for a quasiholonomic medium.

Among various possible physical applications of both semiholonomic and quasiholonomic Cosserat media, beyond those with a rigid matrix, we mention the modelling of aggregates [Zhang et al. 2006], such as colloidal suspensions [Moosaiea and Atefia 2007], when the underlying continuum upon which the interacting particles dwell is, say, an ideal incompressible fluid. The choice of model depends on the physical nature of the interactions between the dispersed particles.

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