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FUNCTIONALLY GRADED BARS WITH ENHANCED DYNAMIC PERFORMANCE

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To Charles and Marie-Louise Steele, with gratitude.

This paper presents analytical solutions for enhancing the dynamic performance of functionally graded material bars in axial motion. Optimized designs with maximized natural frequencies under mass equality constraint are given and discussed. The composition of the construction material is optimized by defining the spatial distribution of volume fractions of the material constituents using either continuous or discrete variations along the bar length. Three cases of boundary conditions have been examined: fixed-fixed, fixed-free and free-free bars. The major aim is to tailor the mass and stiffness distributions in the axial direction so as to maximize the frequencies and place them at their target values to avoid the occurrence of large amplitudes of vibration without the penalty of increasing total structural mass. The resulting optimization problem has been formulated as a nonlinear mathematical programming problem solved by invoking the Matlab optimization toolbox routines, which implement the method of feasible directions interacting with the associated eigenvalue problem routines. The proposed mathematical models have shown that the use of material grading can be promising in optimizing natural frequencies without mass penalty.

1. Introduction

Functionally graded materials (FGMs) are advanced composite materials fabricated to have graded variation of the relative fractions of the constituent materials. Commonly, these materials are made from a mixture of ceramic and metal or a combination of different metals. Ceramic provides high temperature resistance because of its low thermal conductivity, while metal secures the necessary strength and stiffness. FGMs show promise in many applications, such as spacecraft heat shields, high performance structural elements and critical engine components.

A few studies have addressed the dynamics of FGM structures. Loy et al. [1999] considered vibration analysis of FGM cylindrical shells composed of stainless steel and nickel, which were graded in thickness direction according to a volume fraction power-law distribution. Love's shell theory was implemented and the resulting eigenvalue problem was solved by Rayleigh–Ritz method. The behavior of the natural frequencies as well as the influence of the volume fractions were investigated and presented in detail. Cheng and Batra [2000] studied the steady-state vibrations of a simply supported FGM plate on an elastic foundation and subjected to uniform in-plane hydrostatic loads. They applied Reddy's third-order theory, taking the effect of rotary inertia into consideration. The Young's modulus and the Poisson's ratio were assumed to vary only in the plate thickness direction.

Keywords: functionally graded material, structural optimization, longitudinal vibration of bars, composite structures.

Closed-form solutions for calculating the natural frequencies of an axially graded beam were derived and given by [Elishakoff and Guédé 2004]. The modulus of elasticity was taken as a polynomial in the axial coordinate along the beam's length, and an inverse problem was solved to find the stiffness and mass distributions so that the chosen polynomial could lead to an exact mode shape. Three-dimensional exact solutions for free and forced vibrations of FGM plates were introduced in [Vel and Batra 2004] and [Elishakoff et al. 2005]. The material properties of composite plates were also allowed to vary in the thickness direction. Exact natural frequencies were obtained which can be used to assess the accuracy of the different plate theories. Tylikowski [2005] presented a study of parametric vibrations of functionally graded plates subjected to in-plane time-dependent forces. The material properties were graded in the thickness direction of the plate according to a volume fraction power-law distribution. Effects of power-law exponent on the stability behavior were also investigated.

Considering, next, the optimal frequency design problem, several methods have been developed and applied in solving both the frequency and mass optimization problems. They are broadly classified as mathematical programming techniques, optimality criteria methods and optimal control theory.

Mathematical programming has dealt successfully with a wide range of structural optimization problems, while control theory has been applied with limited success because of the great analytical complexity. Niordson [1965] and Brach [1968] initiated the formulation of the extremal problem of finding the maximum fundamental frequencies of vibrating beams under mass constraint. They applied variational methods and investigated the effect of beam taper on the resulting optimum solutions. However, their attained designs were characterized by zero cross-sectional area at the end supports, which results in infinite stresses near the boundaries of the beam. Following their work, several other researchers applied optimization techniques for solving one-dimensional structural members. Warner and Vavrick [1975] utilized a variational continuous optimization procedure to find the minimal mass of bars in axial motion under several frequency constraints. They indicated that the piece-wise uniform design can be economically efficient as compared to the continuous design. In [Maalawi and Warner 1984] we obtained global optimality solutions for practical rod constructions made of piecewise uniform sections. Their objective function was measured by maximization of the fundamental frequency under mass constraint, where conspicuous trends were deduced under specified boundary conditions. Other optimum shapes of one-dimensional structural models for maximum fundamental frequency were studied in [Masad 1997], who presented an efficient numerical approach for computing the rate of change of the associated eigenvalues. In [Maalawi 1999] we presented an exact method for the problem of frequency maximization of rods in axial motion under mass constraint. The dual problem of mass minimization under frequency constraint was also addressed, and results were presented and discussed for both continuous and discrete multi-element structural models.

In the field of the optimum design of FGM-structures, Qian and Batra [2005] considered frequency optimization of a cantilevered plate with variable volume fraction according to simple power-laws. They implemented genetic algorithms to find the optimum values of the power exponents, which maximize the natural frequencies, and concluded that the volume fraction needs to be varied in the longitudinal direction of the plate rather than in the thickness direction. Goupee and Vel [2006] proposed a methodology to optimize the natural frequencies of functionally graded beam with variable volume fraction of the constituent materials in the beam's length and height directions. They used a piecewise bi-cubic interpolation of volume fraction values specified at a finite number of grid points, and applied a genetic algorithm code to find the needed optimum designs. Considering aeroelastic optimization FGM structures, Librescu and Maalawi [2007] introduced the underlying concepts of using material grading in optimizing subsonic wings against torsional instability. They developed exact mathematical models allowing the material physical and mechanical properties to change in the wing spanwise direction, where both continuous and piecewise structural models were successfully implemented.

Little is found in the literature that deals with frequency optimization of FGM bars subject to a *mass equality constraint*. The aim of the present study is, therefore, to incorporate the effect of changing the volume fractions of the constituent materials for enhancing the dynamic performance of an axially vibrating bar. Following [Librescu and Maalawi 2007; Warner 2001; Maalawi and Warner 1984], a useful optimization tool has been built for designing efficient patterns of FGM bars having maximized frequencies while maintaining the total structural mass at constraint value. This allows the search for the required optimal volume fractions which satisfy both design criteria. The corresponding increases in the frequencies calculated with respect to a baseline design have been evaluated for several configurations, including cases of fixed-fixed, fixed-free and free-free bars fabricated from certain types of composite materials. Actually, substantial improvement in the overall dynamic performance has been attained showing the usefulness of the proposed model in using the concept of material grading to arrive at the needed optimum designs for one dimensional slender bars having arbitrary cross-sectional type.

2. Structural dynamic analysis

The partial differential equation that governs the free axial motion of a vibrating bar is [Meirovitch 1997]

$$\frac{\partial}{\partial x}\left[EA(x)\frac{\partial u(x,t)}{\partial x}\right] = \rho A(x)\frac{\partial^2 u(x,t)}{\partial t^2}, \tag{1}$$

which must be satisfied over the entire bar's length L. In (1), x denotes the axial coordinate, $u(x,t)$ longitudinal displacement at any position x and time t, $E(x)$ modulus of elasticity, $A(x)$ cross-sectional area and $\rho(x)$ mass density. Three types of boundary conditions are described:

$$\begin{array}{llccc}
\text{(I)} & \text{fixed-fixed:} & u(0,t)=0 & \text{and} & u(L,t)=0 \\
\text{(II)} & \text{fixed-free:} & u(0,t)=0 & \text{and} & EA\dfrac{\partial u}{\partial x}\Big|_{x=L}=0 \\
\text{(III)} & \text{free-free:} & EA\dfrac{\partial u}{\partial x}\Big|_{x=0}=0 & \text{and} & EA\dfrac{\partial u}{\partial x}\Big|_{x=L}=0
\end{array}$$

The longitudinal displacement $u(x,t)$ is assumed to be separable in space and time, $u(x,t) = U(x)\cdot q(t)$, where the time dependence $q(t)$ is harmonic with circular frequency ω. Substituting for $d^2q/dt^2 = -\omega^2 q$, the associated eigenvalue problem can be written directly in the form

$$\frac{d}{dx}\left[EA(x)\frac{dU}{dx}\right] + \rho A(x)\omega^2 U(x) = 0 \tag{2}$$

which must be satisfied on the interval $0 < x < L$.

2.1. ***Definition of the baseline design.*** An essential phase in formulating an optimization problem is to define an appropriate baseline design to which the resulting optimal designs can be compared. The baseline design has been selected to be a composite bar made of two different materials, denoted by A and B, and has uniform mass and stiffness distributions along its length, with equal volume fractions (V) of the constituent materials, i.e., $V_A = V_B = 50\%$. It is assumed that the optimized designs shall have the same total mass, length, cross sectional area and shape, type of construction material and type of boundary conditions of those of the known baseline design. Therefore, the design variables, which are subject to change in the optimization process, shall be the distribution of the volume fractions over the bar length.

2.2. ***Material grading.*** The physical and mechanical properties are allowed to vary lengthwise, that is, the material is graded in the axial direction. The distributions of the mass density ρ and modulus of elasticity E are taken to obey the semi-empirical formulas of [Halpin and Tsai 1969]. Assuming no voids are present, we have

$$\begin{aligned} &V_A(x) + V_B(x) = 1, \\ &\rho(x) = V_A(x)\rho_A + V_B(x)\rho_B, \\ &E(x) = V_A(x)E_A + V_B(x)E_B. \end{aligned} \tag{3}$$

Normalizing with respect to the baseline design by dividing (2) by $E_0 A/L$ and differentiating once more with respect to x, we get

$$\hat{E}\frac{d^2\hat{U}}{d\hat{x}^2} + \frac{d\hat{E}}{d\hat{x}} \cdot \frac{d\hat{U}}{d\hat{x}} + \hat{\rho}\hat{\omega}^2\hat{U} = 0 \tag{4}$$

which must be satisfied on the interval $0 < \hat{x} < 1$. Dimensionless quantities are defined as follows:

Axial coordinate	$\hat{x} = x/L$
Longitudinal displacement	$\hat{U} = U/L$
Modulus of elasticity	$\hat{E} = E/E_0$
Mass density	$\hat{\rho} = \rho/\rho_0$
Circular frequency	$\hat{\omega} = \omega L\sqrt{\rho_0/E_0}$

where the baseline design parameters $E_0 = \frac{1}{2}(E_A + E_B)$ and $\rho_0 = \frac{1}{2}(\rho_A + \rho_B)$. The cross-sectional area A is assumed to be constant. The total structural mass is kept equal to that of the baseline:

$$\hat{M}_s = \int_0^1 \hat{\rho}\, d\hat{x} = 1, \tag{5}$$

implying the dimensionless mass equality constraint

$$\int_0^1 V_A(\hat{x})\, d\hat{x} = \tfrac{1}{2}. \tag{6}$$

3. Solution procedures

3.1. ***Continuous model: an exact power series solution.*** Kumar and Sujith [1997] presented exact analytical solutions for the longitudinal vibration of nonuniform rods. The equation of motion was reduced to a standard differential equation solved in terms of Bessel and Neumann functions. However, the solutions were restricted to rods having polynomial area variation and made of an isotropic material with uniform properties. A more general solution of (4), where the modulus of elasticity and mass density vary spatially, can be expressed by the power series [Edwards and Penney 2004]

$$\hat{U}(\hat{x}) = \sum_{m=1}^{2} C_m \lambda_m(\hat{x}), \tag{7}$$

where the C_m are the constants of integration and the λ_m are two linearly independent solutions that have the form:

$$\lambda_m(\hat{x}) = \sum_{n=m}^{\infty} a_{m,n}\hat{x}^{n-1}, \quad n \geq m. \tag{8}$$

The unknown coefficients $a_{m,n}$ can be determined by substitution into the differential equation (4) and equating coefficients of like powers of $\hat{x}$. The variation of the volume fractions in FGM structures is usually described by power-law distributions [Elishakoff and Guédé 2004; Qian and Batra 2005; Tylikowski 2005]. In the present study, linear and parabolic model types are considered with their corresponding derived recurrence formulas:

Linear distribution.

- Type (1): $V_A(\hat{x}) = V_A(0) - \Delta V\hat{x}$, $\Delta V = V_A(0) - V_A(1)$

$$a_{m,n} = \frac{\gamma(n-2)^2 a_{m,n-1} - \theta\hat{\omega}^2 a_{m,n-2} + \zeta\hat{\omega}^2 a_{m,n-3}}{\beta(n-1)(n-2)} \tag{9}$$

- Type (2) – Symmetric about $\hat{x} = 1/2$: $V_A(\hat{x}) = V_A(0) - 2\Delta V\hat{x}$, $\Delta V = V_A(0) - V_A(1/2)$

$$a_{m,n} = \frac{2\gamma(n-2)^2 a_{m,n-1} - \theta\hat{\omega}^2 a_{m,n-2} + 2\xi\hat{\omega}^2 a_{m,n-3}}{\beta(n-1)(n-2)} \tag{10}$$

Parabolic distribution.

- Type (3) – Concave: $V_A(\hat{x}) = V_A(0) + \Delta V(\hat{x}^2 - 2\hat{x})$, $\Delta V = V_A(0) - V_A(1)$

$$a_{m,n} = \frac{2\gamma(n-2)^2 a_{m,n-1} - [\gamma(n-2)(n-3) + \theta\hat{\omega}^2]a_{m,n-2} + \xi\hat{\omega}^2[2a_{m,n-3} - a_{m,n-4}]}{\beta(n-1)(n-2)} \tag{11}$$

- Type (4) – Convex: $V_A(\hat{x}) = V_A(0) - \Delta V\hat{x}^2$, $\Delta V = V_A(0) - V_A(1)$

$$a_{m,n} = \frac{[\gamma(n-2)(n-3) - \theta\hat{\omega}^2]a_{m,n-2} + \xi\hat{\omega}^2 a_{m,n-4}}{\beta(n-1)(n-2)} \tag{12}$$

- Type (5) – Symmetric about $\hat{x} = 1/2$: $V_A(\hat{x}) = V_A(0) + 4\Delta V(\hat{x}^2 - \hat{x})$, $\Delta V = V_A(0) - V_A(1/2)$

$$a_{m,n} = \frac{4\gamma(n-2)^2 a_{m,n-1} - \left[4\gamma(n-2)(n-3) + \theta\hat{\omega}^2\right]a_{m,n-2} + 4\xi\hat{\omega}^2(a_{m,n-3} - a_{m,n-4})}{\beta(n-1)(n-2)} \tag{13}$$

The Greek symbols in (9)–(13) are defined as follows, with $\Delta\hat{E} = \hat{E}_A - \hat{E}_B$ and $\Delta\hat{\rho} = \hat{\rho}_A - \hat{\rho}_B$:

$$\beta = \hat{E}_B + \Delta\hat{E}V_A(0), \quad \gamma = \Delta\hat{E}\Delta V, \quad \theta = \hat{\rho}_B + \Delta\hat{\rho}V_A(0), \quad \xi = \Delta\hat{\rho}\Delta V.$$

boundary conditions		frequency equation	$\hat{\omega}_{0,i}$
fixed-fixed, symmetric modes	$\hat{U}(0)=\hat{U}'\left(\frac{1}{2}\right)=0$	$\sum_{n=3}^{\infty} a_{2n}(n-1)/2^{n-2}=-1$	$(\pi, 3\pi, 5\pi)$
fixed-fixed, asymmetric modes	$\hat{U}(0)=\hat{U}\left(\frac{1}{2}\right)=0$	$\sum_{n=2}^{\infty} a_{2n}/2^{n-1}=0$	$(2\pi, 4\pi, 6\pi)$
fixed-free	$\hat{U}(0)=\hat{U}'(1)=0$	$\sum_{n=3}^{\infty} a_{2n}(n-1)=-1$	$(\pi, 3\pi, 5\pi)/2$
free-free, symmetric modes	$\hat{U}'(0)=\hat{U}'\left(\frac{1}{2}\right)=0$	$\sum_{n=3}^{\infty} a_{1n}(n-1)/2^{n-2}=0$	$(2\pi, 4\pi, 6\pi)$
free-free, asymmetric modes	$\hat{U}'(0)=\hat{U}\left(\frac{1}{2}\right)=0$	$\sum_{n=3}^{\infty} a_{1n}/2^{n-1}=-1$	$(\pi, 3\pi, 5\pi)$

Table 1. Frequency equations for different types of boundary conditions. The column $\hat{\omega}_{0,i}$ gives the dimensionless natural frequencies of the baseline design, with $\hat{\omega}_0=0$ corresponding to the first (rigid-body) mode of a free-free bar. Differentiation is with respect to $\hat{x}$.

A coefficient $a_{m,n}$ is set equal to zero whenever n is less than m, and the leading coefficients $a_{m,m}$ in each series are arbitrary and can be set equal to one. This method was successfully applied by [Librescu and Maalawi 2007] to determine the exact critical flight speed of a FGM subsonic wing. Table 1 summarizes the appropriate mathematical expressions of the frequency equation for any desired case, which can be obtained by application of the associated boundary conditions and consideration of nontrivial solutions.

3.2. *Piecewise model: The transfer matrix method.* A piecewise model concept was introduced in [Maalawi and Warner 1984] to obtain global optimal frequency designs of isotropic bars with piecewise uniform sections. They showed that the use of piecewise models in structural optimization gives excellent results and can be promising for similar applications. Li [2000] presented solutions of the differential equation of longitudinal vibration of bars with stiffness and mass distributions, which were described by specific power functions. He applied the transfer matrix method of [Pestel and Leckie 1963] to derive the frequency equation of multi-step bars with several boundary conditions, where analytical solutions were verified with full scale measured data of a multi-story tall building. Figure 1 shows an elastic, slender bar

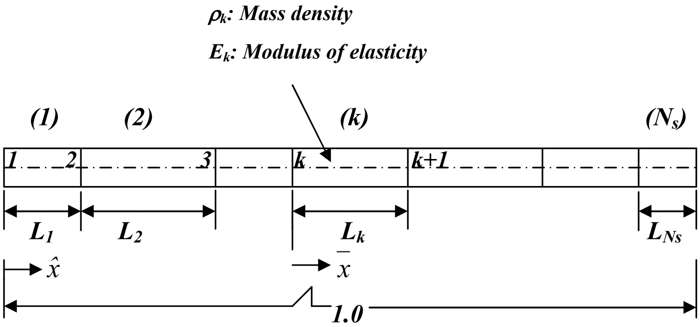

Figure 1. General configuration of a piecewise axially graded bar.

with total dimensionless length of unity constructed from any arbitrary number of uniform segments (N_s), each of which has the same cross-sectional area but different properties of the construction material. Such a configuration results in a piecewise axial grading of the material in the direction of the bar axis. Before determining the exact natural frequencies and performing the necessary mathematics, it is important to bear in mind that design optimization is only as meaningful as its core structural analysis model. Any deficiencies therein will certainly be reflected in the optimization process.

For the k-th segment, (4) reduces to

$$\frac{d^2\hat{U}}{d\bar{x}^2} + \alpha_k^2 \hat{U} = 0, \qquad \alpha_k = \hat{\omega}\sqrt{\hat{\rho}_k/\hat{E}_k}, \tag{14}$$

where $\hat{E}_k = E_k/E_0$ and $\hat{\rho}_k = \rho_k/\rho_0$ are the dimensionless modulus of elasticity and mass density of the k-th segment. Equation (14) must be satisfied in the interval $0 \le \bar{x} \le \hat{L}_k$, where $\bar{x} = \hat{x} - \hat{x}_k$ and $\hat{L}_k = L_k/L$. Its general solution is:

$$\hat{U}(\bar{x}) = C_1 \sin(\alpha_k \bar{x}) + C_2 \cos(\alpha_k \bar{x}). \tag{15}$$

Expressing the constants of integration C_1 and C_2 in terms of the state variables $[\hat{U}, \hat{F} = \hat{E}\, d\hat{U}/d\bar{x}]$ at both ends of the k-th segment, we get

$$\begin{Bmatrix} \hat{U}_{k+1} \\ \hat{F}_{k+1} \end{Bmatrix} = [T^{(k)}] \begin{Bmatrix} \hat{U}_k \\ \hat{F}_k \end{Bmatrix},$$

where

$$[T^{(k)}] = \begin{bmatrix} \cos \alpha_k \hat{L}_k & \sin \alpha_k \hat{L}_k/(\alpha_k \hat{E} k) \\ -\alpha_k \hat{E}_k \sin \alpha_k \hat{L}_k & \cos \alpha_k \hat{L}_k \end{bmatrix} \tag{16}$$

is the transfer matrix of the k-th segment. It is now possible to compute the state variables progressively along the bar length by applying continuity requirements among the interconnecting boundaries of the various bar segments. Therefore, the state variables at both ends of the bar can be related to each other through an overall transfer matrix denoted by $[T]$:

$$[T] = [T^{(N_s)}][T^{(N_s-1)}]\dots[T^{(2)}][T^{(1)}]. \tag{17}$$

The required frequency equation for determining the natural frequencies can then be obtained by applying the associated boundary conditions (refer to Table 1) and considering only the nontrivial solution of the resulting matrix equation.

4. Optimization problem formulation

4.1. *General.* Attractive goals of designing efficient structural members include minimization of structural weight, maximization of the fundamental frequencies [Masad 1997], and minimization of total cost of production. Global optimization models of elastic beam-type structures were presented in [Maalawi and El-Chazly 2002], where both stability and dynamic optimization problems were formulated in a standard mathematical programming coupled with finite element analysis routines. Another important consideration is the reduction or control of the vibration level [Maalawi and Negm 2002]. Vibration can greatly influence the performance of several structural systems because of its adverse effects on stability, fatigue life and noise. The reduction of vibration may be attained either by a direct maximization of the natural frequencies or by separating the natural frequencies of the structure from the harmonics of the exciting applied loads. This would avoid resonance and large amplitudes of vibration, which may cause severe damage of the structure. Direct maximization of the natural frequencies can ensure a simultaneous balanced improvement in both of the overall stiffness and structural mass distributions of the vibrating bar. The associated optimization problems are usually cast in nonlinear mathematical programming form [Vanderplaats 1999] where the objective is to minimize a function $F(\underline{X})$ of a vector $\underline{X}$ of design variables, subject to certain number of constraints $G_j(\underline{X}) \leq 0$, $j = 1, 2, \ldots, m$. Iterative techniques are usually used for solving such optimization problems in which a series of directed design changes are made between successive points in the design space. The method of feasible directions is one of the most powerful methods in finding the required constrained optimum solution. The search direction $\underline{S}_j$ must satisfy the conditions $\underline{S}_j \underline{\nabla} F < 0$ and $\underline{S}_j \underline{\nabla} G_j < 0$, where $\underline{\nabla} F$ and $\underline{\nabla} G_j$ are the gradient vectors of the objective and constraint functions, respectively. For checking the constrained minima, the Kuhn–Tucker test [Vanderplaats 1999] is applied at the design point $\underline{X}_D$, which lies on one or more set of active constraints. The Kuhn–Tucker equations are necessary conditions for optimality for a constrained optimization problem and their solution forms the basis to the method of feasible directions.

4.2. ***Optimal frequency model.*** In the present optimization problem, two alternatives of the objective function form have been implemented and examined. The first one is represented by a direct maximization of a weighted sum of the natural frequencies, which is expressed mathematically as follows:

$$\text{Minimize } F(\underline{X}) = -\sum_i \alpha_i \hat{\omega}_i, \qquad \sum_i \alpha_i = 1, \quad 0 \leq \alpha_i \leq 1, \tag{18}$$

where $\hat{\omega}_i$ are the normalized frequencies, α_i weighting factors measuring the relative importance of each frequency [Kasprzak and Lewis 2001] and $\underline{X}$ is the chosen design variable vector. For continuous models with known power-law distribution, $\underline{X}$ represents the volume fractions at the ends of the bar, while for discrete model $\underline{X} = (V_{A,k}, \hat{L}_k)_{k,1,2,\ldots N_s}$. The second alternative is to minimize a weighted sum of the squares of the difference between each frequency $\hat{\omega}_i$ and its target or desired value $\hat{\omega}_i^*$:

$$\text{Minimize } F(\underline{X}) = \sum_i \alpha_i (\hat{\omega}_i - \hat{\omega}_i^*)^2. \tag{19}$$

Both objectives are subject to the constraints

$$\text{Mass constraint:} \quad \hat{M}_s = 1, \tag{20}$$

$$\text{Side constraints:} \quad \underline{X}_L \leq \underline{X} \leq \underline{X}_U, \tag{21}$$

where $\underline{X}_L$ and $\underline{X}_U$ are the lower and upper limiting values imposed on the design variables vector $\underline{X}$ in order not to obtain unrealistic odd-shaped designs in the final optimum solutions. Approximate values of the target frequencies are usually chosen to be within close ranges; sometimes called frequency-windows; of those corresponding to an initial baseline design, which are adjusted to be far away from the critical exciting frequencies. The proper choice of the best form of the objective function has to wait for actual computer experimentation. Several computer program packages are available now for solving the above design optimization model, which can be coded to interact with structural and eigenvalue

	Material A (carbon fibers)	Material B (epoxy matrix)
Mass density (g/cm^3)	$\rho_f = 1.81$	$\rho_m = 1.27$
Young's modulus (GPa)	$E_{1f} = 235$	$E_m = 4.3$
Shear modulus (GPa)	$G_{12f} = 27$	$G_m = 1.6$
Poisson's ratio	$\nu_{12f} = 0.2$	$\nu_m = 0.35$

Table 2. Material properties of carbon-AS4 / epoxy-3501-6 composite.

analysis software. The Matlab optimization toolbox is a powerful tool that includes many routines for different types of optimization encompassing both unconstrained and constrained minimization algorithms [Venkataraman 2002]. One of its useful routines, fmincon, implements the method of feasible directions in finding the constrained minimum of an objective function of several variables.

5. Optimization results and discussions

The mathematical models developed above have been applied to obtain the required optimal solutions of FGM bars with different boundary conditions (see Table 1). The selected construction material is composed of carbon-AS4 (material A) and epoxy3501-6 (material B), which has favorable characteristics and is highly desirable in several mechanical, civil and aerospace engineering applications [Daniel and Ishai 2006]; see Table 2.

Two different forms of the objective function measuring frequency optimization have been defined in (18) and (19). Whatever is the approach taken, it can be useful to investigate first the behavior of the natural frequencies for the different boundary conditions and see how they are changed with the selected design variables while maintaining the total structural mass constant. Computer experimentation for optimizing fixed-fixed and free-free bars with either discrete or continuous material grading have indicated that good patterns with improved dynamic characteristics would have symmetric grading about the mid-span point of the bar ($\hat{x} = 0.5$). Therefore, half of the bar can only be analyzed with the benefit of reducing the total number of the design variables, and consequently, the computational time.

5.1. *Piecewise models.* Considering first the case of fixed-fixed bar constructed from four segments with symmetry about the midspan, half of the bar can be analyzed using only four variables denoted by $(V_A, \hat{L})_{k=1,2}$. Furthermore, one of the segment lengths can be eliminated because of the equality constraint imposed on the total length. Another variable can also be discarded by applying the mass equality constraint (6), which reduces the number of design variables to only two, say $(V_A, \hat{L})_1$.

Figure 2 depicts the functional behavior of the dimensionless first and second frequencies augmented with the mass equality constraint. It is seen that the functions are well behaved and continuous everywhere in the design space $(V_A - \hat{L})_1$, except in the empty regions located at the upper right of the whole domain, where the mass equality constraint is violated. Such empty regions are bounded from below by the curve $(V_A \hat{L})_1 = 0.5$; that is the volume fraction of material (A) is equal to zero in the second segment: $V_{A,2} = 0$.

The level curves of the lowest fundamental frequency associated with the first symmetric mode are shown in on the left in Figure 2. The feasible domain is seen to be split by the baseline contours $(\hat{\omega}_1 = \pi)$

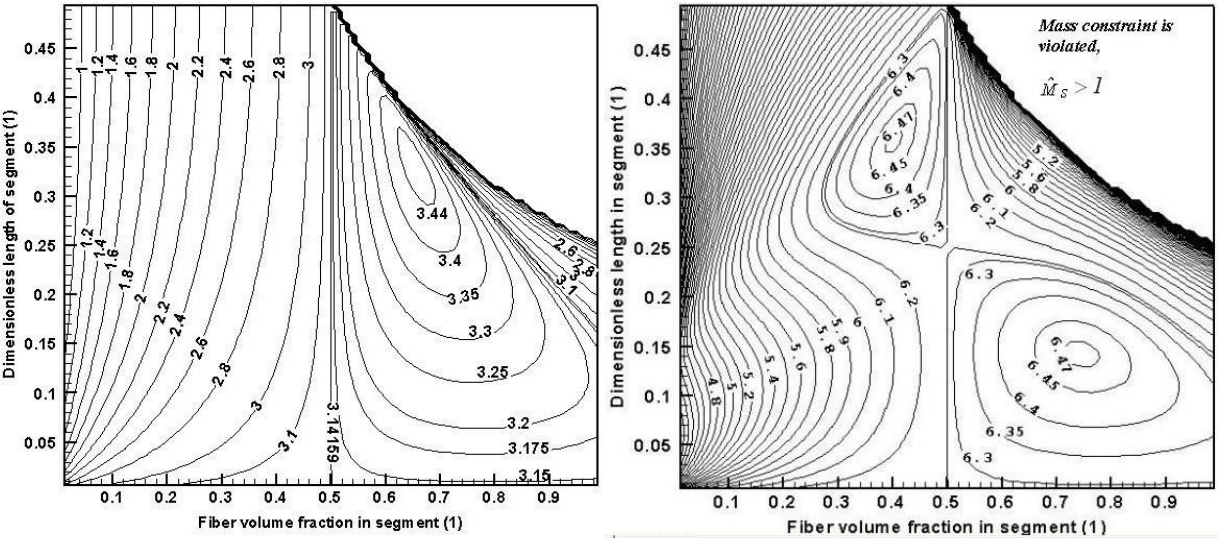

Figure 2. Isomerits of the dimensionless frequency under mass equality constraint of a 4-segment fixed-fixed bar, symmetric about midspan. Left: fundamental frequency (symmetric mode). Right: second frequency (asymmetric mode). Construction material is carbon fibers-AS4 / epoxy-3501-6.

into two distinct zones. The one to the right encompasses the constrained global maxima, which is calculated to be $\hat{\omega}_{1,\max} = 3.45406$ at the optimal design point $(V_A, \hat{L})_{k=1,2} = (0.650, 0.3375), (0.1885, 0.1625)$. Actually, each design point inside the feasible domain corresponds to different material properties as well as different stiffness and mass distributions, while maintaining the total structural mass of the vibrating rod constant. Figure 2, right, shows the developed isomerits of the second frequency associated with the first asymmetric mode. Two global maxima can be observed having the calculated values $\hat{\omega}_{2,\max} = 6.4732$ and 6.4711, which correspond, respectively, to the design points $(V_A, \hat{L})_{k=1,2} =$ (0.405, 0.365), (0.757, 0.135) located at the upper left region, and (0.750, 0.145), (0.3975, 0.355) at the lower right region of the feasible domain. The developed isomerits for other cases of fixed-free and free-free bars made of carbon/epoxy composites are shown in Figures 3 and 4, respectively. It is seen that a bar with free-free boundary conditions behaves in the opposite trend to the case of fixed-fixed bar. The global maximum of the fundamental frequency in the former occurs at the lower region to the left of the design space (Figure 4) having the same value of $\hat{\omega}_{1,\max} = 3.45406$ with altering segment locations, namely, $(V_A, \hat{L})_{k=1,2} = (0.1885, 0.1625), (0.650, 0.3375)$.

Table 3 summarizes the attained optimal solutions with increasing the number of segments for the different types of boundary conditions. It is seen that the attained optimization gain (i.e. the percentage increase in the fundamental frequency above its baseline value) increases with the number of segments. However, it should be kept in mind that the cost of manufacturing will also be increased. Therefore, a compromise has to be made between the reduction of vibration and the cost of manufacturing. In all, the bars can be economically built from a fewer number of segments having different length and material properties, since each pattern with a specified number of segments has its own exact global

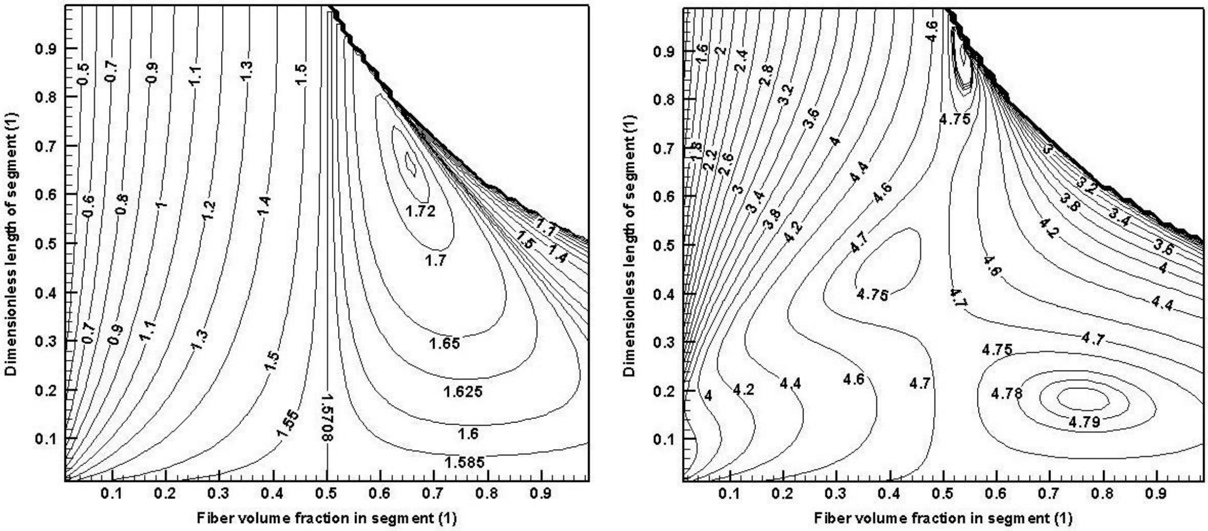

Figure 3. Frequency isomerits of a 2-segment, fixed-free bar under mass constraint ($\hat{M}_s = 1.0$): fundamental frequency (left) and second frequency (right). Material is carbon fibers-AS4 / epoxy-3501-6.

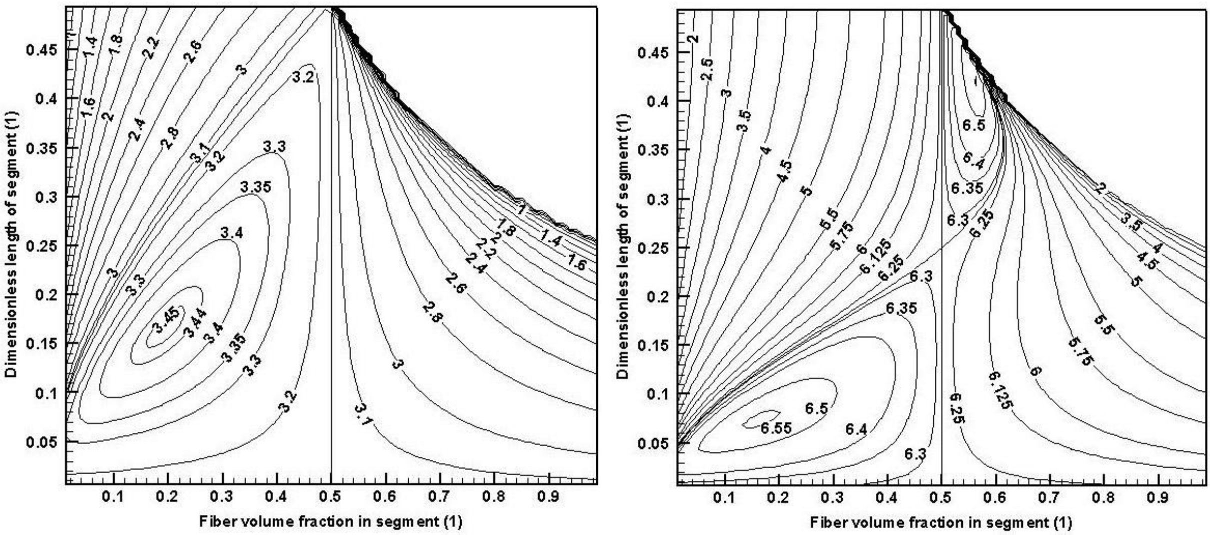

Figure 4. Frequency isomerits of a symmetric, 4-segment, free-free bar under mass constraint ($\hat{M}_s = 1.0$). Left: fundamental frequency (asymmetric mode). Right: second frequency (symmetric mode). Material is carbon fibers-AS4 / epoxy-3501-6.

optimal solution. One does not have to use more segments in order to increase the accuracy of the resulting solutions. This is one of the major outcomes of the present model formulation, which result in exact solutions no matter the number of segments is. Another remarkable observation is that the attained optimal designs based on maximization of a single frequency do not guarantee maximization of other

$N_s =$	1	2	3	4
fixed-fixed (half-span)	$(0.50, 0.50)_1$ (baseline)	$(0.650, 0.338)_1$ $(0.189, 0.163)_2$	$(0.719, 0.244)_1$ $(0.413, 0.144)_2$ $(0.138, 0.112)_3$	$(0.750, 0.23125)_1$ $(0.425, 0.13125)_2$ $(0.225, 0.07500)_3$ $(0.0625, 0.0625)_4$
$(\hat{\omega}_1, \hat{\omega}_2, \hat{\omega}_3)$	(3.142, 6.283, 9.425)	(3.454, 5.537, 8.081)	(3.531, 5.487, 8.287)	(3.562, 5.133, 8.231)
increase in $\hat{\omega}_1$	0%	9.93%	12.38%	13.37%
fixed-free	$(0.50, 1.00)_1$ (baseline)	$(0.650, 0.675)_1$ $(0.189, 0.325)_2$	$(0.725, 0.500)_1$ $(0.381, 0.306)_2$ $(0.107, 0.194)_3$	$(0.750, 0.4625)_1$ $(0.425, 0.2625)_2$ $(0.225, 0.1500)_3$ $(0.0625, 0.125)_4$
$(\hat{\omega}_1, \hat{\omega}_2, \hat{\omega}_3)$	(1.571, 4.712, 7.854)	(1.727, 4.044, 7.354)	(1.766, 4.105, 6.552)	(1.781, 4.116, 6.454)
increase in $\hat{\omega}_1$	0%	9.93%	12.41%	13.37%
free-free (half-span)	$(0.50, 0.50)_1$ (baseline)	$(0.200, 0.165)_1$ $(0.648, 0.335)_2$	$(0.138, 0.112)_1$ $(0.413, 0.144)_2$ $(0.719, 0.244)_3$	$(0.0625, 0.0625)_1$ $(0.2250, 0.0750)_2$ $(0.425, 0.13125)_3$ $(0.750, 0.23125)_4$
$(\hat{\omega}_1, \hat{\omega}_2, \hat{\omega}_3)$	(3.142, 6.283, 9.425)	(3.454, 5.990, 8.170)	(3.531, 5.918, 8.287)	(3.562, 5.874, 8.231)
increase in $\hat{\omega}_1$	0%	9.93%	12.38%	13.37%

Table 3. Optimal patterns $(V_A, \hat{L})_{k=1,2,\ldots,N_s}$ of multisegment FGM bars with maximized fundamental frequency $\hat{\omega}_1$ under mass equality constraint. Patterns for fixed-fixed and free-free bars are symmetric about the midpoint of the bar.

frequencies. For example, in the case of fixed-fixed bar, the design based upon maximization of $\hat{\omega}_2$ alone (see Figure 2) can result in a degraded value of the fundamental frequency ($\hat{\omega}_1 = 2.820$), located in the region to the left of the design space, which is lower than that of the baseline design by about 10.24%. Therefore, if one really seeks to maximize the overall stiffness-to-mass ratio of the vibrating rod, a multi-objective design optimization ought to be implemented instead [Kasprzak and Lewis 2001]. In this regard, the proper determination of the values of the weighting factors α_i ought to be based on the fact that each frequency shall be maximized from its initial value corresponding to the uniform baseline design. The author suggests the following procedure:

- Initialize the values of the weighting factors by the reciprocal of the initial reference values; $\alpha_i = 1/\hat{\omega}_{0,i}$.
- Normalize the resulting values by dividing each one by their sum in order to make the final sum equal to 1.0, i.e. $\alpha_i \leftarrow \alpha_i / \sum_i \alpha_i$.

Considering only the first three frequencies, Table 4 gives the appropriate values of the weighting factors corresponding to the different types of boundary conditions. The attained optimal solutions for different optimization strategies are given in Table 5 for bars constructed from two segments.

	α_1	α_2	α_3
fixed-fixed	54.5%	27.3%	18.2%
fixed-free	65.2%	21.7%	13.1%
free-free	54.5%	27.3%	18.2%

Table 4. Appropriate values of the weighting factors for different boundary conditions.

	max $\hat{\omega}_1$	max $\hat{\omega}_2$	max $\sum_{i=1}^{3} \alpha_i \hat{\omega}_i$
fixed-fixed	$(0.6500, 0.3375)_1$	$(0.405, 0.365)_1$	$(0.750, 0.145)_1$
(half length)	$(0.1885, 0.1625)_2$	$(0.757, 0.135)_2$	$(0.398, 0.355)_2$
$(\hat{\omega}_1, \hat{\omega}_2, \hat{\omega}_3)$	(3.454, 5.537, 8.081)	(2.820, 6.473, 8.914)	(3.289, 6.471, 9.468)
fixed-free	$(0.6500, 0.675)_1$	$(0.538, 0.90)_1$	$(0.5315, 0.9125)_1$
	$(0.1885, 0.325)_2$	$(0.163, 0.10)_2$	$(0.1715, 0.0875)_2$
$(\hat{\omega}_1, \hat{\omega}_2, \hat{\omega}_3)$	(1.727, 4.041, 7.352)	(1.636, 4.845, 7.834)	(1.625, 4.839, 7.922)
free-free	$(0.200, 0.165)_1$	$(0.560, 0.425)_1$	$(0.163, 0.075)_1$
(half length)	$(0.648, 0.335)_2$	$(0.163, 0.075)_2$	$(0.560, 0.425)_2$
$(\hat{\omega}_1, \hat{\omega}_2, \hat{\omega}_3)$	(3.454, 5.990, 8.170)	(2.537, 6.557, 8.502)	(3.335, 6.557, 9.526)

Table 5. Optimal patterns $(V_A, \hat{L})_{k=1,2}$ of a two-segment FGM bar for different optimization strategies. Patterns for fixed-fixed and free-free bars are symmetric about the middle of bar length.

More calculations by implementation of the models described by (18) and (19) have revealed the fact that maximization of the natural frequencies is a much better design criterion than the frequency placement criterion. The latter resulted in a slightly time consuming optimization process and a slow rate of convergence towards the optimum solution. Such a quadratic formulation complicates matters by increasing nonlinearities in the design space and, therefore, difficult to minimize globally. Moreover, the determination of the target frequencies in most cases is not easy and the resulting solutions are greatly dependent on the choice of their values. Direct maximization of the natural frequencies without regard to frequency-windows was found to be more significant and best representative to the overall dynamic performance. If it happened that the maximum frequency violates frequency windows, which was found to be a rare situation, another value of the frequency can be chosen near the global optima, and the frequency equation can be solved via an inverse approach for any one of the unknown design variables instead.

5.2. ***Continuous models.*** Linear and parabolic models for material grading along the bar span (Figure 5) have been examined. The attained optimal solutions for both cases are given in Table 6 for the different types of boundary conditions. Two consecutive solutions are given for each case; the first solution is based on maximization of the fundamental frequency alone, while the second is based on maximization of a weighted sum of the first three frequencies. Upper and lower limits were imposed on the volume fraction of material $(A)(0.0 \le V_A \le 1.0)$ in order to avoid having unrealistic configurations in the final

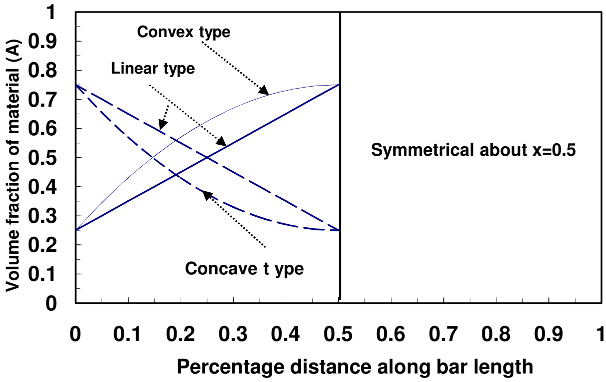

Figure 5. Symmetric shape models of volume fraction distribution along bar length.

Volume fraction distribution		$(\hat{\omega}_1, \hat{\omega}_2, \hat{\omega}_3)$
fixed-fixed ($0 < \hat{x} < 0.5$)		
Linear:	$V_A(\hat{x}) = 1.0 - 2\hat{x}$	(3.5922, 4.9481, 8.3353)
	$V_A(\hat{x}) = 0.75 - \hat{x}$	(3.4416, 6.0931, 9.3144)
Parabolic (concave):	$V_A(\hat{x}) = 1.0 + 3(\hat{x}^2 - \hat{x})$	(3.4706, 6.1781, 9.0588)
	$V_A(\hat{x}) = 0.75 + 1.5(\hat{x}^2 - \hat{x})$	(3.3566, 6.3231, 9.3781)
fixed-free ($0 < \hat{x} < 1.0$)		
Linear:	$V_A(\hat{x}) = 1.0 - \hat{x}$	(1.7961, 4.1677, 6.6348)
	$V_A(\hat{x}) = 0.75 - 0.5\hat{x}$	(1.7208, 4.6572, 7.6833)
Parabolic (convex):	$V_A(\hat{x}) = 0.75(1 - \hat{x}^2)$	(1.7928, 4.4189, 7.0611)
	$V_A(\hat{x}) = 0.625 - 0.375\hat{x}^2$	(1.6958, 4.6936, 7.7527)
Parabolic (concave):	$V_A(\hat{x}) = 1.0 + 0.75(\hat{x}^2 - 2\hat{x})$	(1.7353, 4.5294, 7.4584)
	$V_A(\hat{x}) = 0.75 + 0.375(\hat{x}^2 - 2\hat{x})$	(1.6781, 4.6891, 7.7692)
free-free ($0 < \hat{x} < 0.5$)		
Linear:	$V_A(\hat{x}) = 2\hat{x}$	(3.5922, 5.7091, 8.3353)
	$V_A(\hat{x}) = 0.25 + \hat{x}$	(3.4416, 6.1841, 9.3144)
Parabolic (convex):	$V_A(\hat{x}) = 3(\hat{x} - \hat{x}^2)$	(3.5856, 6.2167, 8.8378)
	$V_A(\hat{x}) = 0.25 - 1.5(\hat{x}^2 - \hat{x})$	(3.3916, 6.3588, 9.3872)

Table 6. Optimal constant-mass patterns with linear and parabolic distributions of $V_A(\hat{x})$ for different types of boundary conditions. Patterns with fixed-fixed or free-free boundary conditions are symmetric about the middle of bar length.

attained solutions. In all cases the total structural mass was maintained constant by imposing the equality constraint described by (6). As a general observation, for fixed-fixed and fixed-free boundary conditions, patterns with higher fiber volume fraction near the fixed ends are always favorable. The opposite trend is true for cases of free-free bars. Maximization of the fundamental frequency alone produces an optimization gain of about 14.33% for the linear model with 0% and 100% volume fractions at the ends of the optimized bars with different boundary conditions. However, a drastic reduction in the second and third frequencies can be observed. Better solutions have been achieved by maximizing a weighted sum of the first three frequencies, where the parabolic model was found to excel the linear one in producing balanced improvements in all frequencies. Results have also indicated that the fixed-fixed bars are recommended to have concave distribution rather than convex one. The latter produce poor patterns with degraded stiffness-to-mass ratio levels. The opposite trend was observed for the free-free bars, where the convex type is much more favorable than the concave type. Both concave and convex shapes can be accepted for a cantilevered bar. Finally, it should be pointed out here that, for certain ranges of the design variables many terms ought to be taken in the assumed power series (see Section 3.1) in order to achieve convergence. The proper number of terms to be taken in the series needs to be investigated.

6. Conclusions

In view of the importance of enhancing the dynamic performance and raising the overall (stiffness/mass) level of a FGM bar in axial motion, appropriate optimization models have been formulated for both continuous and discrete distributions of the volume fractions of the selected composite material. Several forms of the objective function have been examined by maximizing the natural frequencies while preserving the total structural mass at a constant value equals to that of a known baseline design. It was found that maximization of the fundamental frequency alone for any type of boundary conditions does not guarantee maximization of other higher frequencies. A local maximum for any one of the frequencies can correspond to a local minimum for another frequency. A weighted-sum formulation must, therefore, be considered in order to achieve a simultaneous balanced improvement among all frequencies. The weighting factors are best determined from the fact that each frequency ought to be maximized from its initial reference value of the uniform baseline design. Moreover, computer experiments have revealed that maximization of the natural frequencies is a much better design criterion than the frequency placement criterion. The attained optimal solutions in the latter were found to be too sensitive to the selected target frequencies, which are not easy to determine. If it happened that any one of the maximized frequencies violates frequency windows, which was found to be a rare situation, another value of the frequency can be chosen near the global optima, and an inverse approach can be utilized by solving the associated frequency equation for any one of the unknown design variables instead. Optimization of multisegment fixed-fixed or free-free bars has indicated that good patterns should be symmetric about the midspan of the bar. The given exact structural analysis leads to the exact frequencies no matter the number of segments is. The optimized bars can be fabricated economically from any arbitrary number of uniform segments with material grading in the axial direction. The increase in the number of segments would naturally result in an increase in the attained maximized frequencies, but care ought to be taken regarding the penalty of increasing the production and manufacturing costs. It has been confirmed that the segment length is most significant design variable in the whole optimization process. Some investigators who apply finite elements have not recognized that the length of each element can be taken as a main design variable in the whole set of optimization variables. It has also been shown that normalization of all terms results in a naturally scaled objective function, constraints and design variables, which is recommended when applying different optimization techniques. The results from the present approach reveals that piecewise grading of the material can be promising producing truly efficient bar designs with enhanced dynamic performance. In conclusion, a powerful design tool has been obtained by formulating an appropriate objective function and applying mathematical programming techniques to the resulting optimization problem. It is the author's wish that the results presented in this paper will be compared and validated through other optimization techniques such as genetic algorithms or any appropriate global optimization algorithm. Future potential areas such as optimization of FGM beams in bending and torsion shall be formulated and identified. Other secondary effects such as material and geometric nonlinearities due to large deformation shall be investigated in future studies.

A personal note

Charles and Marie-Louise Steele hold a special place in my heart for their support and encouragement of my research in structural mechanics. In 2000, as editor of the *International Journal of Solids and Structures*, Marie-Louise encouraged me to submit there my "Buckling optimization of flexible columns". I was initially discouraged, but she pushed me to improve the paper, which was eventually accepted after extensive revision and published in 2002, outlining a new approach to the problem under consideration. She was likewise supportive after she and Charles founded JoMMS, where I published papers in 2007 and 2008. Early in 2009, she encouraged me to apply for the Fulbright research program at Stanford; we had been planning to collaborate in the academic year 2009/2010 on a research project about wind turbine blades made of functionally graded material.

Marie-Louise was always helping people to an uncommon degree. I believe that her passing away has been a great loss not only to her family and friends but to the research community. God rest her soul.

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