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UNIQUENESS THEOREMS IN THE EQUILIBRIUM THEORY OF THERMOELASTICITY WITH MICROTEMPERATURES FOR MICROSTRETCH SOLIDS

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In this paper the linear equilibrium theory of thermoelasticity with microtemperatures for isotropic microstretch solids is considered and some basic results of the classical theories of elasticity and thermoelasticity are generalized. Green's formulas in the theory are obtained. A wide class of internal and external boundary value problems are formulated, and uniqueness theorems are proved.

1. Introduction

In the last years the theory of thermoelasticity for bodies with microstructure has been intensively studied. A thermodynamic theory for elastic materials with inner structure whose particles, in addition to microdeformations, possess microtemperatures was proposed in [Grot 1969]. Riha [1975; 1976] developed a theory of micromorphic fluids with microtemperatures.

The linear theory of thermoelasticity with microtemperatures for materials with inner structure whose particles, in addition to the classical displacement and temperature fields, possess microtemperatures was presented in [Ieşan and Quintanilla 2000], where an existence theorem was proved and the continuous dependence of solutions of the initial data and body loads was established. The exponential stability of solutions of equations in this theory was established in [Casas and Quintanilla 2005]. The fundamental solutions of equations in the theory of thermoelasticity with microtemperatures were constructed in [Svanadze 2004b]. Representations of Galerkin type and general solutions of equations of dynamic and steady vibrations in this theory were obtained in [Scalia and Svanadze 2006]. In [Scalia and Svanadze 2009b; Svanadze 2003], the basic boundary value problems (BVPs) of steady vibrations were investigated using the potential method and the theory of singular integral equations. In [Scalia and Svanadze 2009a; Scalia et al. 2010; Ieşan and Scalia 2010], basic theorems in the equilibrium and steady vibrations theories of thermoelasticity with microtemperatures were proved.

The theory of micromorphic elastic solids with microtemperatures, in which microelements possess microtemperatures and can stretch and contract independently of their translations, was presented in [Ieşan 2001]. The fundamental solutions of equations in this theory were constructed in [Svanadze 2004a]. Uniqueness theorems in the dynamical theory thermoelasticity of porous media with microtemperatures were proved in [Quintanilla 2009]. The existence and uniqueness of solutions in the linear theory of heat conduction in micromorphic continua were established in [Ieşan 2002]. Recently, the representations of solutions in the theory of thermoelasticity with microtemperatures for microstretch solids were obtained in [Svanadze and Tracinà 2011].

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The theory of micropolar thermoelasticity with microtemperatures was presented in [Ieşan 2007]. The existence and asymptotic behavior of the solutions in this theory were proved in [Aouadi 2008]. A linear theory of thermoelastic bodies with microstructure and microtemperatures which permits the transmission of heat as thermal waves at finite speed was constructed in [Ieşan and Quintanilla 2009], and existence and uniqueness results in the context of the dynamic theory were established. An extensive review and the basic results in the microcontinuum field theories are given in [Eringen 1999; Ieşan 2004].

In this paper the linear equilibrium theory of thermoelasticity with microtemperatures for isotropic microstretch solids [Ieşan 2001] is considered and some basic results of the classical theories of elasticity and thermoelasticity (see [Kupradze et al. 1979; Knops and Payne 1971]) are generalized. Green's formulae are obtained for the theory. A wide class of internal and external BVPs are formulated, and uniqueness theorems are proved.

2. Basic equations

We consider an isotropic elastic material with microstructure that occupies a region Ω of Euclidean three-dimensional space E^3. Let $\boldsymbol{x} = (x_1, x_2, x_3)$ be a point of E^3 and set $\boldsymbol{D_x} = (\partial/\partial x_1, \partial/\partial x_2, \partial/\partial x_3)$.

The fundamental system of field equations in the linear equilibrium theory of thermoelasticity with microtemperatures for isotropic microstretch solids consists of the equations of equilibrium [Ieşan 2001]

$$t_{jl,j} + \rho F_l^{(1)} = 0, \tag{2-1}$$

the first moment of energy

$$q_{jl,j} + q_l - Q_l + \rho F_l^{(2)} = 0, \tag{2-2}$$

the balance of energy

$$q_{l,l} + \rho s_1 = 0, \tag{2-3}$$

the balance of first stress moment

$$h_{l,l} - s + \rho\, s_2 = 0, \tag{2-4}$$

the constitutive equations

$$\begin{aligned} t_{jl} &= (\lambda\, e_{rr} - \beta\,\theta + b\,\varphi)\delta_{jl} + 2\mu e_{jl}, \\ q_l &= k\,\theta_{,l} + k_1\, w_l, \\ q_{jl} &= -k_4\, w_{r,r}\delta_{jl} - k_5\, w_{j,l} - k_6\, w_{l,j}, \\ Q_l &= (k_1 - k_2) w_l + (k - k_3)\,\theta_{,l}, \\ h_l &= \gamma\varphi_{,l} - d\, w_l, \\ s &= b\, e_{rr} - m\,\theta + \xi\,\varphi, \end{aligned} \tag{2-5}$$

and the geometric equations

$$e_{lj} = \tfrac{1}{2}(u_{l,j} + u_{j,l}), \tag{2-6}$$

where $\boldsymbol{u} = (u_1, u_2, u_3)$ is the displacement vector, $\boldsymbol{w} = (w_1, w_2, w_3)$ is the microtemperature vector, θ is the temperature measured from the constant absolute temperature T_0 ($T_0 > 0$), φ is the microdilatation function, t_{jl} are the components of stress tensor, ρ is the reference mass density ($\rho > 0$), h_l is the microstretch, $\boldsymbol{F}^{(1)} = (F_1^{(1)}, F_2^{(1)}, F_3^{(1)})$ is the body force, $\boldsymbol{q} = (q_1, q_2, q_3)$ is the heat flux vector, s is the intrinsic body load, s_1 is the heat supply, s_2 is the general external body load, q_{jl} are the components of first heat flux moment tensor, $\boldsymbol{Q} = (Q_1, Q_2, Q_3)$ is the mean heat flux vector, $\boldsymbol{F}^{(2)} = (F_1^{(2)}, F_2^{(2)}, F_3^{(2)})$ is first heat source moment vector, $\lambda, \mu, \beta, \gamma, \xi, b, d, m, k, k_1, k_2, \ldots, k_6$ are constitutive coefficients, δ_{lj} is the Kronecker delta, e_{lj} are the components of strain tensor, the subscripts preceded by a comma denote partial differentiation with respect to the corresponding Cartesian coordinate, $j, l = 1, 2, 3$, and repeated indices are summed over the range $\{1, 2, 3\}$.

By virtue of (2-5) and (2-6), the system (2-1)–(2-4) can be expressed in terms of the displacement vector $\boldsymbol{u}$, the microtemperature vector $\boldsymbol{w}$, the temperature θ and the microdilatation function φ. We obtain a system of eight partial differential equations of the linear equilibrium theory of thermoelasticity with microtemperatures for isotropic and homogeneous microstretch solids [Ieşan 2001]:

$$\begin{aligned} \mu\,\Delta\boldsymbol{u} + (\lambda+\mu)\,\mathrm{grad}\,\mathrm{div}\,\boldsymbol{u} - \beta\,\mathrm{grad}\,\theta + b\,\mathrm{grad}\,\varphi &= -\rho\,\boldsymbol{F}^{(1)}, \\ k_6\,\Delta\boldsymbol{w} + (k_4+k_5)\,\mathrm{grad}\,\mathrm{div}\,\boldsymbol{w} - k_3\,\mathrm{grad}\,\boldsymbol{\theta} - k_2\boldsymbol{w} &= \rho\,\boldsymbol{F}^{(2)}, \\ k\,\Delta\,\theta + k_1\,\mathrm{div}\,\boldsymbol{w} &= -\rho\,s_1, \\ \gamma\,\Delta\,\varphi - b\,\mathrm{div}\,\boldsymbol{u} - d\,\mathrm{div}\,\boldsymbol{w} + m\,\theta - \xi\,\varphi &= -\rho\,s_2. \end{aligned} \tag{2-7}$$

We introduce the matrix differential operator

$$\boldsymbol{A}(\boldsymbol{D_x}) = \big(A_{pq}(\boldsymbol{D_x})\big)_{8\times 8},$$

where, for $j, l = 1, 2, 3$, we have

$$\begin{aligned} &A_{lj}(\boldsymbol{D_x}) = \mu\Delta\,\delta_{lj} + (\lambda+\mu)\frac{\partial^2}{\partial x_l \partial x_j}, \qquad A_{l7}(\boldsymbol{D_x}) = -\beta\frac{\partial}{\partial x_l}, \qquad A_{l8}(\boldsymbol{D_x}) = b\frac{\partial}{\partial x_l}, \\ &A_{l;j+3}(\boldsymbol{D_x}) = A_{l+3;j}(\boldsymbol{D_x}) = A_{l+3;8}(\boldsymbol{D_x}) = A_{7l}(\boldsymbol{D_x}) = A_{78}(\boldsymbol{D_x}) = 0, \\ &A_{l+3;j+3}(\boldsymbol{D_x}) = (k_6\Delta - k_2)\delta_{lj} + (k_4+k_5)\,\frac{\partial^2}{\partial x_l \partial x_j}, \qquad A_{l+3;7}(\boldsymbol{D_x}) = -k_3\frac{\partial}{\partial x_l}, \\ &A_{7;l+3}(\boldsymbol{D_x}) = k_1\frac{\partial}{\partial x_l}, \qquad A_{77}(\boldsymbol{D_x}) = k\Delta, \qquad A_{8l}(\boldsymbol{D_x}) = -b\frac{\partial}{\partial x_l}, \\ &A_{8;l+3}(\boldsymbol{D_x}) = -d\frac{\partial}{\partial x_l}, \qquad A_{87}(\boldsymbol{D_x}) = m, \qquad A_{88}(\boldsymbol{D_x}) = \gamma\Delta - \xi, \end{aligned} \tag{2-8}$$

Obviously, the system (2-7) can be written as

$$\boldsymbol{A}(\boldsymbol{D_x})\boldsymbol{U}(\boldsymbol{x}) = \boldsymbol{F}(\boldsymbol{x}), \tag{2-9}$$

where $\boldsymbol{U} = (\boldsymbol{u}, \boldsymbol{w}, \theta, \varphi)$, $\boldsymbol{F} = (-\rho\boldsymbol{F}^{(1)}, \rho\boldsymbol{F}^{(2)}, -\rho s_1, -\rho\,s_2)$, and $\boldsymbol{x} \in \Omega$.

3. Boundary value problems

In this section a wide class of BVPs of the linear equilibrium theory of thermoelasticity with microtemperatures for isotropic and homogeneous microstretch solids is formulated.

Let S be the closed surface surrounding the finite domain Ω^+ in E^3, $S \in C^{2,\alpha_1}$, $0 < \alpha_1 \le 1$, $\bar{\Omega}^+ = \Omega^+ \cup S$, $\Omega^- = E^3 \backslash \bar{\Omega}^+$, $\bar{\Omega}^- = \Omega^- \cup S$.

Definition 3.1. A vector function $\boldsymbol{U}=(\boldsymbol{u},\boldsymbol{w},\theta,\varphi)=(U_1,U_2,\ldots,U_8)$ is called *regular* in Ω^- (or Ω^+) if

$$U_l\in C^2(\Omega^-)\cap C^1(\bar\Omega^-)\qquad (\text{or } U_l\in C^2(\Omega^+)\cap C^1(\bar\Omega^+))$$

and

$$U_l(\boldsymbol{x})=O(|\boldsymbol{x}|^{-1}),\qquad \frac{\partial}{\partial x_j}U_l(\boldsymbol{x})=o(|\boldsymbol{x}|^{-1})\quad \text{for } |\boldsymbol{x}|\gg 1, \tag{3-1}$$

where $j=1,2,3$ and $l=1,2,\ldots,8$.

We will use the matrix differential operators

$$\boldsymbol{P}^{(m)}(\boldsymbol{D}_{\boldsymbol{x}},\boldsymbol{n})=(P_{lj}^{(m)}(\boldsymbol{D}_{\boldsymbol{x}},\boldsymbol{n}))_{3\times 3}\quad\text{and}\quad \boldsymbol{P}(\boldsymbol{D}_{\boldsymbol{x}},\boldsymbol{n})=(P_{lj}(\boldsymbol{D}_{\boldsymbol{x}},\boldsymbol{n}))_{8\times 8},$$

where

$$\begin{aligned} P_{lj}^{(1)}(\boldsymbol{D}_{\boldsymbol{x}},\boldsymbol{n})&=\mu\delta_{lj}\frac{\partial}{\partial\boldsymbol{n}}+\mu n_j\frac{\partial}{\partial x_l}+\lambda n_l\frac{\partial}{\partial x_j}=\mu\delta_{lj}\frac{\partial}{\partial\boldsymbol{n}}+(\lambda+\mu)n_l\frac{\partial}{\partial x_j}+\mu M_{lj},\\ P_{lj}^{(2)}(\boldsymbol{D}_{\boldsymbol{x}},\boldsymbol{n})&=k_6\delta_{lj}\frac{\partial}{\partial\boldsymbol{n}}+k_5 n_j\frac{\partial}{\partial x_l}+k_4 n_l\frac{\partial}{\partial x_j}=k_6\delta_{lj}\frac{\partial}{\partial\boldsymbol{n}}+(k_4+k_5)n_l\frac{\partial}{\partial x_j}+k_5 M_{lj}\end{aligned} \tag{3-2}$$

and, for $m=1,2$ and $j,l=1,2,3$,

$$\begin{aligned} P_{lj}(\boldsymbol{D}_{\boldsymbol{x}},\boldsymbol{n})&=P_{lj}^{(1)}(\boldsymbol{D}_{\boldsymbol{x}},\boldsymbol{n}), & P_{l7}(\boldsymbol{D}_{\boldsymbol{x}},\boldsymbol{n})&=-\beta n_l,\\ P_{l+3;j+3}(\boldsymbol{D}_{\boldsymbol{x}},\boldsymbol{n})&=P_{lj}^{(2)}(\boldsymbol{D}_{\boldsymbol{x}},\boldsymbol{n}), & P_{l8}(\boldsymbol{D}_{\boldsymbol{x}},\boldsymbol{n})&=b\,n_l,\\ P_{7;l+3}(\boldsymbol{D}_{\boldsymbol{x}},\boldsymbol{n})&=k_1\,n_l, & P_{77}(\boldsymbol{D}_{\boldsymbol{x}},\boldsymbol{n})&=k\frac{\partial}{\partial\boldsymbol{n}},\\ P_{8;l+3}(\boldsymbol{D}_{\boldsymbol{x}},\boldsymbol{n})&=-d\,n_l, & P_{88}(\boldsymbol{D}_{\boldsymbol{x}},\boldsymbol{n})&=\gamma\frac{\partial}{\partial\boldsymbol{n}},\end{aligned}$$
$$P_{l;j+3}=P_{l+3;j}=P_{l+3;7}=P_{l+3;8}=P_{7l}=P_{78}=P_{8l}=P_{87}=0, \tag{3-3}$$

where $\boldsymbol{n}=(n_1,n_2,n_3)$, $\boldsymbol{n}(z)$ is the external unit normal vector to S at z, $\partial/\partial\boldsymbol{n}$ is the derivative along the vector $\boldsymbol{n}$, and

$$M_{lj}=n_j\frac{\partial}{\partial x_l}-n_l\frac{\partial}{\partial x_j}. \tag{3-4}$$

$\boldsymbol{P}(\boldsymbol{D}_{\boldsymbol{x}},\boldsymbol{n})$ and $\boldsymbol{P}(\boldsymbol{D}_{\boldsymbol{x}},\boldsymbol{n})\boldsymbol{U}(\boldsymbol{x})$ are the stress operator and stress vector, respectively, in the linear theory of thermoelasticity with microtemperatures for microstretch solids; see [Ieşan 2001].

The internal BVPs of the linear equilibrium theory of thermoelasticity with microtemperatures for isotropic and homogeneous microstretch solids are formulated as follows:

Problem $(I)^+_{\boldsymbol{F},\boldsymbol{f}}$**.** Find a regular (classical) solution to system (2-9) for $\boldsymbol{x}\in\Omega^+$ satisfying the boundary conditions

$$\lim_{\boldsymbol{x}\to z\in S}\boldsymbol{u}(\boldsymbol{x})\equiv\{\boldsymbol{u}(z)\}^+=\boldsymbol{f}^{(1)}(z), \tag{3-5}$$

$$\{\boldsymbol{w}(z)\}^+=\boldsymbol{f}^{(2)}(z), \tag{3-6}$$

$$\{\theta(z)\}^+=f_7(z), \tag{3-7}$$

$$\{\varphi(z)\}^+=f_8(z) \tag{3-8}$$

with $\boldsymbol{f}^{(1)} = (f_1, f_2, f_3)$, $\boldsymbol{f}^{(2)} = (f_4, f_5, f_6)$; here $f_1, f_2, \dots, f_8$ are known functions and $\boldsymbol{F}$ is a known eight-component vector function. Obviously, we can rewrite the boundary condition (3-5)–(3-8) in the form

$$\{\boldsymbol{U}(z)\}^+ = \boldsymbol{f}(z),$$

where $\boldsymbol{f} = (f_1, f_2, \dots, f_8)$.

Problem $(II)^+_{\boldsymbol{F},\boldsymbol{f}}$**.** Find a regular solution to system (2-9) for $\boldsymbol{x} \in \Omega^+$ satisfying the boundary conditions

$$\left\{\boldsymbol{P}^{(1)}(\boldsymbol{D}_z, \boldsymbol{n}(z))\,\boldsymbol{u}(z) - \beta\,\theta(z)\,\boldsymbol{n}(z) + b\,\varphi(z)\,\boldsymbol{n}(z)\right\}^+ = \boldsymbol{f}^{(1)}(z), \tag{3-9}$$

$$\left\{\boldsymbol{P}^{(2)}(\boldsymbol{D}_z, \boldsymbol{n}(z))\,\boldsymbol{w}(z)\right\}^+ = \boldsymbol{f}^{(2)}(z), \tag{3-10}$$

$$\left\{k\frac{\partial\theta(z)}{\partial\boldsymbol{n}(z)} + k_1\,\boldsymbol{w}(z)\boldsymbol{n}(z)\right\}^+ = f_7(z), \tag{3-11}$$

$$\left\{\gamma\frac{\partial\varphi(z)}{\partial\boldsymbol{n}(z)} - d\,\boldsymbol{w}(z)\boldsymbol{n}(z)\right\}^+ = f_8(z). \tag{3-12}$$

Obviously, by virtue of (3-3) we can rewrite the boundary conditions (3-9)–(3-12) in the form

$$\{\boldsymbol{P}(\boldsymbol{D}_z, \boldsymbol{n}(z))\,\boldsymbol{U}(z)\}^+ = \boldsymbol{f}(z).$$

Problem $(III)^+_{\boldsymbol{F},\boldsymbol{f}}$**.** Find a regular solution to (2-9) for $\boldsymbol{x} \in \Omega^+$ satisfying (3-5), (3-10), (3-7), (3-8).

Problem $(IV)^+_{\boldsymbol{F},\boldsymbol{f}}$**.** Find a regular solution to (2-9) for $\boldsymbol{x} \in \Omega^+$ satisfying (3-5), (3-6), (3-7), (3-12).

Problem $(V)^+_{\boldsymbol{F},\boldsymbol{f}}$**.** Find a regular solution to (2-9) for $\boldsymbol{x} \in \Omega^+$ satisfying (3-5), (3-10), (3-7), (3-12).

Problem $(VI)^+_{\boldsymbol{F},\boldsymbol{f}}$**.** Find a regular solution to (2-9) for $\boldsymbol{x} \in \Omega^+$ satisfying (3-9), (3-6), (3-7), (3-8).

Problem $(VII)^+_{\boldsymbol{F},\boldsymbol{f}}$**.** Find a regular solution to (2-9) for $\boldsymbol{x} \in \Omega^+$ satisfying (3-9), (3-10), (3-7), (3-8).

Problem $(VIII)^+_{\boldsymbol{F},\boldsymbol{f}}$**.** Find a regular solution to (2-9) for $\boldsymbol{x} \in \Omega^+$ satisfying (3-9), (3-6), (3-7), (3-12).

Problem $(IX)^+_{\boldsymbol{F},\boldsymbol{f}}$**.** Find a regular solution to (2-9) for $\boldsymbol{x} \in \Omega^+$ satisfying (3-9), (3-10), (3-7), (3-12).

Problem $(X)^+_{\boldsymbol{F},\boldsymbol{f}}$**.** Find a regular solution to (2-9) for $\boldsymbol{x} \in \Omega^+$ satisfying (3-9), (3-6), (3-11), (3-12).

Problem $(XI)^+_{\boldsymbol{F},\boldsymbol{f}}$**.** Find a regular solution to (2-9) for $\boldsymbol{x} \in \Omega^+$ satisfying (3-5), (3-6), (3-11), (3-12).

Problem $(XII)^+_{\boldsymbol{F},\boldsymbol{f}}$**.** Find a regular solution to (2-9) for $\boldsymbol{x} \in \Omega^+$ satisfying (3-5), (3-10), (3-11), (3-12).

We now turn to the external BVPs, spelling out only the first two (the external BVPs $(III)^-_{\boldsymbol{F},\boldsymbol{f}}$ through $(XII)^-_{\boldsymbol{F},\boldsymbol{f}}$ are formulated similarly).

Problem $(I)^-_{\boldsymbol{F},\boldsymbol{f}}$**.** Find a regular solution to system (2-9) for $\boldsymbol{x} \in \Omega^-$ satisfying the boundary condition

$$\lim_{\boldsymbol{x}\to z\in S} \boldsymbol{U}(\boldsymbol{x}) \equiv \{\boldsymbol{U}(z)\}^- = \boldsymbol{f}(z),$$

where $\boldsymbol{F}$ and $\boldsymbol{f}$ are known eight-component vector functions, and supp $\boldsymbol{F}$ is a finite domain in Ω^-.

Problem $(II)^-_{\boldsymbol{F},\boldsymbol{f}}$**.** Find a regular solution to system (2-9) for $\boldsymbol{x} \in \Omega^-$ satisfying the boundary condition

$$\{\boldsymbol{P}(\boldsymbol{D}_z, \boldsymbol{n}(z))\boldsymbol{U}(z)\}^- = \boldsymbol{f}(z).$$

4. Green's formulae

In this section the Green's formulae in the linear equilibrium theory of thermoelasticity with microtemperatures for isotropic and homogeneous microstretch solids are obtained.

We introduce the notation

$$W(\boldsymbol{U},\boldsymbol{U}')=W^{(1)}(\boldsymbol{u},\boldsymbol{u}')+W^{(2)}(\boldsymbol{w},\boldsymbol{w}')+(b\,\varphi-\beta\,\theta)\operatorname{div}\boldsymbol{u}'+(k_2\boldsymbol{w}+k_3\operatorname{grad}\theta)\,\boldsymbol{w}'$$
$$+(k_1\,\boldsymbol{w}+k\operatorname{grad}\theta)\operatorname{grad}\theta'+(\gamma\operatorname{grad}\varphi-d\,\boldsymbol{w})\,\operatorname{grad}\varphi'+(b\operatorname{div}\boldsymbol{u}-m\,\theta+\xi\,\varphi)\,\varphi',\quad\text{(4-1)}$$

where $\boldsymbol{u}'=(u_1',u_2',u_3')$ and $\boldsymbol{w}'=(w_1',w_2',w_3')$ are three-component vector functions, θ' and φ' are scalar functions, $\boldsymbol{U}'=(\boldsymbol{u}',\boldsymbol{w}',\theta',\varphi')$ and

$$W^{(1)}(\boldsymbol{u},\boldsymbol{u}')=\tfrac{1}{3}(3\lambda+2\mu)\operatorname{div}\boldsymbol{u}\operatorname{div}\boldsymbol{u}'$$
$$+\mu\Bigg[\frac{1}{2}\sum_{\substack{l,j=1\\ l\neq j}}^{3}\left(\frac{\partial u_j}{\partial x_l}+\frac{\partial u_l}{\partial x_j}\right)\left(\frac{\partial u_j'}{\partial x_l}+\frac{\partial u_l'}{\partial x_j}\right)+\frac{1}{3}\sum_{l,j=1}^{3}\left(\frac{\partial u_l}{\partial x_l}-\frac{\partial u_j}{\partial x_j}\right)\left(\frac{\partial u_l'}{\partial x_l}-\frac{\partial u_j'}{\partial x_j}\right)\Bigg],$$

$$W^{(2)}(\boldsymbol{w},\boldsymbol{w}')=\tfrac{1}{3}(3k_4+k_5+k_6)\operatorname{div}\boldsymbol{w}\operatorname{div}\boldsymbol{w}'+\tfrac{1}{2}(k_6-k_5)\operatorname{curl}\boldsymbol{w}\operatorname{curl}\boldsymbol{w}'$$
$$+\frac{k_6+k_5}{2}\Bigg[\frac{1}{2}\sum_{\substack{l,j=1\\ l\neq j}}^{3}\left(\frac{\partial w_j}{\partial x_l}+\frac{\partial w_l}{\partial x_j}\right)\left(\frac{\partial w_j'}{\partial x_l}+\frac{\partial w_l'}{\partial x_j}\right)+\frac{1}{3}\sum_{l,j=1}^{3}\left(\frac{\partial w_l}{\partial x_l}-\frac{\partial w_j}{\partial x_j}\right)\left(\frac{\partial w_l'}{\partial x_l}-\frac{\partial w_j'}{\partial x_j}\right)\Bigg].\quad\text{(4-2)}$$

We are now in a position to prove Green's theorem in the linear equilibrium theory of thermoelasticity with microtemperatures for the domains Ω^+ and Ω^-.

Theorem 4.1. *If $\boldsymbol{U}=(\boldsymbol{u},\boldsymbol{w},\theta)$ is a regular vector field in Ω^+ and $\boldsymbol{U}'=(\boldsymbol{u}',\boldsymbol{w}',\theta')\in C^1(\Omega^+)$, then*

$$\int_{\Omega^+}\big[\boldsymbol{A}(\boldsymbol{D_x})\,\boldsymbol{U}(\boldsymbol{x})\,\boldsymbol{U}'(\boldsymbol{x})+W(\boldsymbol{U},\boldsymbol{U}')\big]\,d\boldsymbol{x}=\int_S\boldsymbol{P}(\boldsymbol{D_z},\boldsymbol{n}(\boldsymbol{z}))\,\boldsymbol{U}(\boldsymbol{z})\,\boldsymbol{U}'(\boldsymbol{z})\,d_{\boldsymbol{z}}S,\quad\text{(4-3)}$$

where $\boldsymbol{A}(\boldsymbol{D_x})$ and $\boldsymbol{P}(\boldsymbol{D_z},\boldsymbol{n}(\boldsymbol{z}))$ are the operators defined by (2-8) and (3-3), respectively.

Proof. We introduce the matrix differential operators

$$\boldsymbol{A}^{(1)}(\boldsymbol{D_x})=\big(A_{lj}^{(1)}(\boldsymbol{D_x})\big)_{3\times3},\quad A_{lj}^{(1)}(\boldsymbol{D_x})=A_{lj}(\boldsymbol{D_x}),$$
$$\boldsymbol{A}^{(2)}(\boldsymbol{D_x})=\big(A_{lj}^{(2)}(\boldsymbol{D_x})\big)_{3\times3},\quad A_{lj}^{(2)}(\boldsymbol{D_x})=A_{l+3;\,j+3}(\boldsymbol{D_x}).$$

From Green's formula in the classical theory of elasticity, expressed as

$$\int_{\Omega^+}\big[\boldsymbol{A}^{(1)}(\boldsymbol{D_x})\,\boldsymbol{u}\,\boldsymbol{u}'+W^{(1)}(\boldsymbol{u},\boldsymbol{u}')\big]\,d\boldsymbol{x}=\int_S\boldsymbol{P}^{(1)}(\boldsymbol{D_z}(\boldsymbol{n},\boldsymbol{z}))\,\boldsymbol{u}(\boldsymbol{z})\,\boldsymbol{u}'(\boldsymbol{z})\,d_{\boldsymbol{z}}S$$

(see [Kupradze et al. 1979]), we have

$$\int_{\Omega^+}\big[(\boldsymbol{A}^{(1)}\,\boldsymbol{u}-\beta\operatorname{grad}\theta+b\operatorname{grad}\varphi)\boldsymbol{u}'+W^{(1)}(\boldsymbol{u},\boldsymbol{u}')+(\beta\operatorname{grad}\theta-b\operatorname{grad}\varphi)\,\boldsymbol{u}'\big]\,d\boldsymbol{x}$$
$$=\int_S\boldsymbol{P}^{(1)}(\boldsymbol{D_z},\boldsymbol{n}(\boldsymbol{z}))\,\boldsymbol{u}(\boldsymbol{z})\,\boldsymbol{u}'(\boldsymbol{z})\,d_{\boldsymbol{z}}S.\quad\text{(4-4)}$$

On account of the identity $\int_{\Omega^+} (\operatorname{grad}\theta\, \boldsymbol{u}' + \theta \operatorname{div} \boldsymbol{u}')\, d\boldsymbol{x} = \int_S \theta\, \boldsymbol{n}\, \boldsymbol{u}'\, d_z S$ (see [Kupradze et al. 1979]), it follows from (4-4) that

$$\int_{\Omega^+} \big[(\boldsymbol{A}^{(1)}\, \boldsymbol{u} - \beta \operatorname{grad}\theta + b \operatorname{grad}\varphi)\boldsymbol{u}' + W^{(1)}(\boldsymbol{u}, \boldsymbol{u}') + (b\,\varphi - \beta\,\theta) \operatorname{div} \boldsymbol{u}'\big]\, d\boldsymbol{x} = \int_S \boldsymbol{R}(\boldsymbol{D}_z, \boldsymbol{n}(z))\boldsymbol{v}(z)\boldsymbol{u}'(z)\, d_z S, \tag{4-5}$$

where $\boldsymbol{v} = (\boldsymbol{u}, \theta, \varphi)$ is a five-component vector and

$$\boldsymbol{R}(\boldsymbol{D}_{\boldsymbol{x}}, \boldsymbol{n}) = (R_{lj}(\boldsymbol{D}_{\boldsymbol{x}}, \boldsymbol{n}))_{3\times 5}, \quad R_{lj}(\boldsymbol{D}_{\boldsymbol{x}}, \boldsymbol{n}) = P_{lj}(\boldsymbol{D}_{\boldsymbol{x}}, \boldsymbol{n}), \quad R_{l4}(\boldsymbol{D}_{\boldsymbol{x}}, \boldsymbol{n}) = -\beta\, n_l, \quad R_{l5}(\boldsymbol{D}_{\boldsymbol{x}}, \boldsymbol{n}) = b\, n_l,$$

for $l, j = 1, 2, 3$. It may be shown similarly that Green's formula [Kupradze et al. 1979]

$$\int_{\Omega^+} \big[\boldsymbol{A}^{(2)}(\boldsymbol{D}_{\boldsymbol{x}})\, \boldsymbol{w}\, \boldsymbol{w}' + W^{(2)}(\boldsymbol{w}, \boldsymbol{w}')\big]\, d\boldsymbol{x} = \int_S \boldsymbol{P}^{(2)}(\boldsymbol{D}_z(\boldsymbol{n}, z))\, \boldsymbol{w}(z)\, \boldsymbol{w}'(z)\, d_z S$$

may be rewritten as

$$\int_{\Omega^+} \big[(\boldsymbol{A}^{(2)}(\boldsymbol{D}_{\boldsymbol{x}})\, \boldsymbol{w} - k_2 \boldsymbol{w} - k_3 \operatorname{grad}\theta)\, \boldsymbol{w}' + W^{(2)}(\boldsymbol{w}, \boldsymbol{w}') + (k_2 \boldsymbol{w} + k_3 \operatorname{grad}\theta)\, \boldsymbol{w}'\big]\, d\boldsymbol{x} = \int_S \boldsymbol{P}^{(2)}(\boldsymbol{D}_z(\boldsymbol{n}, z))\, \boldsymbol{w}(z)\, \boldsymbol{w}'(z)\, d_z S. \tag{4-6}$$

By virtue of the identities [Kupradze et al. 1979]

$$\int_{\Omega^+} \left(\Delta\,\theta\,\theta' + \operatorname{grad}\theta \operatorname{grad}\theta'\right)\, d\boldsymbol{x} = \int_S \frac{\partial\theta(z)}{\partial \boldsymbol{n}(z)}\, \theta'(z)\, d_z S,$$

$$\int_{\Omega^+} \left(\operatorname{div}\boldsymbol{w}\,\theta' + \boldsymbol{w} \operatorname{grad}\theta'\right)\, d\boldsymbol{x} = \int_S \boldsymbol{w}\,\boldsymbol{n}\,\theta' d_z S,$$

we have

$$\int_{\Omega^+} \big[(k\Delta\,\theta + k_1 \operatorname{div} \boldsymbol{w})\theta' + (k \operatorname{grad}\theta + k_1 \boldsymbol{w}) \operatorname{grad}\theta'\big]\, d\boldsymbol{x} = \int_S \Big(k\frac{\partial\theta}{\partial \boldsymbol{n}} + k_1 \boldsymbol{w}\,\boldsymbol{n}\Big)\theta' d_z S. \tag{4-7}$$

It may be shown similarly that

$$\int_{\Omega^+} \big[(\gamma\,\Delta\,\varphi - b \operatorname{div} \boldsymbol{u} - d\, \operatorname{div} \boldsymbol{w} + m\,\theta - \xi\,\varphi)\,\varphi' + (\gamma \operatorname{grad}\varphi - d\, \boldsymbol{w}) \operatorname{grad}\varphi' + (b \operatorname{div} \boldsymbol{u} - m\,\theta + \xi\,\varphi)\,\varphi'\big]\, d\boldsymbol{x} = \int_S \Big(\gamma\frac{\partial\varphi}{\partial \boldsymbol{n}} - d\, \boldsymbol{w}\,\boldsymbol{n}\Big)\, \varphi' d_z S. \tag{4-8}$$

Keeping (4-1) in mind, (4-5)–(4-8) yield (4-3), and the theorem is proved. □

The following theorem holds for an infinite domain Ω^-.

Theorem 4.2. *Let $\boldsymbol{U} = (\boldsymbol{u}, \boldsymbol{w}, \theta, \varphi)$ be a regular vector field in Ω^-. Let $\boldsymbol{U}' = (\boldsymbol{u}', \boldsymbol{w}', \theta', \varphi') \in C^1(\Omega^-)$ satisfy*

$$U'(\boldsymbol{x}) = O(|\boldsymbol{x}|^{-1}) \quad \text{and} \quad \frac{\partial}{\partial x_j} U'(\boldsymbol{x}) = o(|\boldsymbol{x}|^{-1}) \quad \text{for } |\boldsymbol{x}| \gg 1,\ j = 1, 2, 3. \tag{4-9}$$

Then

$$\int_{\Omega^-} \big[\boldsymbol{A}(\boldsymbol{D}_{\boldsymbol{x}})\, \boldsymbol{U}(\boldsymbol{x})\, \boldsymbol{U}'(\boldsymbol{x}) + W(\boldsymbol{U}, \boldsymbol{U}')\big]\, d\boldsymbol{x} = -\int_S \boldsymbol{P}(\boldsymbol{D}_z, \boldsymbol{n}(z))\, \boldsymbol{U}(z)\, \boldsymbol{U}'(z)\, d_z S. \tag{4-10}$$

Proof. Let Ω_r be a sphere of sufficiently large radius r so that $\bar{\Omega}^+ \subset \Omega_r$, S_r is the boundary of the sphere Ω_r. The theorem is proved by applying Green's formula (4-3) to the finite domain $\Omega_r^- = \Omega^- \cap \Omega_r$. The positive normal to the boundary Ω_r^- is the inward one. Hence, we obtain

$$\int_{\Omega_r^-} \big[\boldsymbol{A}(\boldsymbol{D_x})\, \boldsymbol{U}(\boldsymbol{x})\, \boldsymbol{U}'(\boldsymbol{x}) + W(\boldsymbol{U}, \boldsymbol{U}')\big]\, d\boldsymbol{x} \\ = -\int_S \boldsymbol{P}(\boldsymbol{D_z}, \boldsymbol{n}(\boldsymbol{z}))\, \boldsymbol{U}(\boldsymbol{z})\, \boldsymbol{U}'(\boldsymbol{z}) d_{\boldsymbol{z}} S - \int_{S_r} \boldsymbol{P}(\boldsymbol{D_z}, \boldsymbol{n}(\boldsymbol{z}))\, \boldsymbol{U}(\boldsymbol{z})\, \boldsymbol{U}'(\boldsymbol{z})\, d_{\boldsymbol{z}} S. \tag{4-11}$$

In view of (3-1) and (4-9), the integral over S_r tends to zero when $r \to \infty$. Therefore, the limit of the right-hand side and hence the limit of the left-hand side of (4-11) exist and are equal. From (4-4) we obtain (4-10) and the theorem is proved. □

In the classical theory of elasticity one considers the *generalized stress operator* [Kupradze et al. 1979, Chapter I]. We denote this operator by $\boldsymbol{P}^{(1)}_{(\tau_1)}(\boldsymbol{D_x}, \boldsymbol{n})$ and we have (*ibid.*)

$$\begin{gathered} \boldsymbol{P}^{(1)}_{(\tau_1)}(\boldsymbol{D_x}, \boldsymbol{n}) = \big(P^{(1)}_{(\tau_1)\, lj}(\boldsymbol{D_x}, \boldsymbol{n})\big)_{3\times 3}, \\ P^{(1)}_{(\tau_1)\, lj}(\boldsymbol{D_x}, \boldsymbol{n}) = \mu \delta_{lj} \frac{\partial}{\partial \boldsymbol{n}} + (\lambda + \mu) n_l \frac{\partial}{\partial x_j} + \tau_1 M_{lj}, \end{gathered} \tag{4-12}$$

where τ_1 is an arbitrary number and M_{lj} is defined by (3-4). Obviously, the operator $\boldsymbol{P}^{(1)}$ is obtained from the operator $\boldsymbol{P}^{(1)}_{(\tau_1)}$ if we set $\tau_1 = \mu$, i.e., $\boldsymbol{P}^{(1)}_{(\mu)} = \boldsymbol{P}^{(1)}$.

In the sequel we use the matrix differential operator [Scalia et al. 2010]

$$\begin{gathered} \boldsymbol{P}^{(2)}_{(\tau_2)}(\boldsymbol{D_x}, \boldsymbol{n}) = \big(P^{(2)}_{(\tau_2)\, lj}(\boldsymbol{D_x}, \boldsymbol{n})\big)_{3\times 3}, \\ P^{(2)}_{(\tau_2)\, lj}(\boldsymbol{D_x}, \boldsymbol{n}) = k_6 \delta_{lj} \frac{\partial}{\partial \boldsymbol{n}} + (k_4 + k_5) n_l \frac{\partial}{\partial x_j} + \tau_2 M_{lj}, \end{gathered} \tag{4-13}$$

where τ_2 is an arbitrary number. Obviously, the operator $\boldsymbol{P}^{(2)}$ is obtained from operator $\boldsymbol{P}^{(2)}_{(\tau_2)}$ if we set $\tau_2 = k_5$, i.e., $\boldsymbol{P}^{(2)}_{(k_5)} = \boldsymbol{P}^{(2)}$.

We introduce the notation

$$W_{(\boldsymbol{\tau})}(\boldsymbol{U}, \boldsymbol{U}') = W^{(1)}_{(\tau_1)}(\boldsymbol{u}, \boldsymbol{u}') + W^{(2)}_{(\tau_2)}(\boldsymbol{w}, \boldsymbol{w}') + (b\,\varphi - \beta\,\theta)\, \operatorname{div} \boldsymbol{u}' + (k_2 \boldsymbol{w} + k_3 \operatorname{grad} \theta)\, \boldsymbol{w}' \\ + (k_1\, \boldsymbol{w} + k \operatorname{grad} \theta) \operatorname{grad} \theta' + (\gamma \operatorname{grad} \varphi - d\, \boldsymbol{w})\ \operatorname{grad} \varphi' + (b \operatorname{div} \boldsymbol{u} - m\,\theta + \xi\,\varphi)\ \varphi', \tag{4-14}$$

where $\boldsymbol{\tau} = (\tau_1, \tau_2)$ and

$$W^{(1)}_{(\tau_1)}(\boldsymbol{u}, \boldsymbol{u}') = \tfrac{1}{3}(3\lambda + 4\mu - 2\tau_1) \operatorname{div} \boldsymbol{u} \operatorname{div} \boldsymbol{u}' + \tfrac{1}{2}(\mu - \tau_1) \operatorname{curl} \boldsymbol{u} \operatorname{curl} \boldsymbol{u}' \\ + \frac{1}{2}(\mu + \tau_1)\left[\frac{1}{2} \sum_{l,j=1;\, l\neq j}^{3} \left(\frac{\partial u_j}{\partial x_l} + \frac{\partial u_l}{\partial x_j}\right)\left(\frac{\partial u_j'}{\partial x_l} + \frac{\partial u_l'}{\partial x_j}\right) + \frac{1}{3} \sum_{l,j=1}^{3} \left(\frac{\partial u_l}{\partial x_l} - \frac{\partial u_j}{\partial x_j}\right)\left(\frac{\partial u_l'}{\partial x_l} - \frac{\partial u_j'}{\partial x_j}\right)\right],$$

$$W^{(2)}_{(\tau_2)}(\boldsymbol{w}, \boldsymbol{w}') = \tfrac{1}{3}(3k_4 + 3k_5 + k_6 - 2\tau_2) \operatorname{div} \boldsymbol{w} \operatorname{div} \boldsymbol{w}' + \tfrac{1}{2}(k_6 - \tau_2) \operatorname{curl} \boldsymbol{w} \operatorname{curl} \boldsymbol{w}' \\ + \frac{1}{2}(k_6 + \tau_2)\left[\frac{1}{2} \sum_{l,j=1;\, l\neq j}^{3} \left(\frac{\partial w_j}{\partial x_l} + \frac{\partial w_l}{\partial x_j}\right)\left(\frac{\partial w_j'}{\partial x_l} + \frac{\partial w_l'}{\partial x_j}\right) + \frac{1}{3} \sum_{l,j=1}^{3} \left(\frac{\partial w_l}{\partial x_l} - \frac{\partial w_j}{\partial x_j}\right)\left(\frac{\partial w_l'}{\partial x_l} - \frac{\partial w_j'}{\partial x_j}\right)\right]. \tag{4-15}$$

It is easy to see that $W^{(1)}_{(\mu)} = W^{(1)}$ and $W^{(2)}_{(k_5)} = W^{(2)}$.

Equations (4-5), (4-6), (4-12)–(4-15) and Theorems 4.1 and 4.2 have the following consequences.

Theorem 4.3. *Let $\boldsymbol{U}=(\boldsymbol{u},\boldsymbol{w},\theta,\varphi)$ be a regular vector field in Ω^+ and let $\boldsymbol{U}'=(\boldsymbol{u}',\boldsymbol{w}',\theta',\varphi')\in C^1(\Omega^+)$. Suppose $\boldsymbol{\tau}=(\tau_1,\tau_2)$ is an arbitrary vector and set, for $l,j=1,2,3$,*

$$\boldsymbol{R}_{(\tau_1)}(\boldsymbol{D_x},\boldsymbol{n})=\big(R_{(\tau_1)\,lj}(\boldsymbol{D_x},\boldsymbol{n})\big)_{3\times 5},\quad R_{(\tau_1)\,l4}(\boldsymbol{D_x},\boldsymbol{n})=-\beta\,n_l,$$
$$R_{(\tau_1)\,lj}(\boldsymbol{D_x},\boldsymbol{n})=P^{(1)}_{(\tau_1)\,lj}(\boldsymbol{D_x},\boldsymbol{n}),\qquad R_{(\tau_1)\,l5}(\boldsymbol{D_x},\boldsymbol{n})=b\,n_l.$$

Then

$$\int_{\Omega^+}\big[(\boldsymbol{A}^{(1)}\boldsymbol{u}-\beta\operatorname{grad}\theta+b\operatorname{grad}\varphi)\boldsymbol{u}'+W^{(1)}_{(\tau_1)}(\boldsymbol{u},\boldsymbol{u}')+(b\,\varphi-\beta\,\theta)\operatorname{div}\boldsymbol{u}'\big]\,d\boldsymbol{x}=\int_S\boldsymbol{R}_{(\tau_1)}(\boldsymbol{D}_z(\boldsymbol{n},z))\,\boldsymbol{v}(z)\,\boldsymbol{u}'(z)\,d_zS,$$

$$\int_{\Omega^+}\big[(\boldsymbol{A}^{(2)}(\boldsymbol{D_x})\,\boldsymbol{w}-k_2\boldsymbol{w}-k_3\operatorname{grad}\theta)\,\boldsymbol{w}'+W^{(2)}_{(\tau_2)}(\boldsymbol{w},\boldsymbol{w}')+(k_2\boldsymbol{w}+k_3\operatorname{grad}\theta)\,\boldsymbol{w}'\big]\,d\boldsymbol{x}=\int_S\boldsymbol{P}^{(2)}_{(\tau_2)}(\boldsymbol{D}_z,\boldsymbol{n}(z))\boldsymbol{w}(z)\boldsymbol{w}'(z)\,d_zS.\tag{4-16}$$

Theorem 4.4. *Let $\boldsymbol{U}=(\boldsymbol{u},\boldsymbol{w},\theta,\varphi)$ be a regular vector field in Ω^- and let $\boldsymbol{U}'=(\boldsymbol{u}',\boldsymbol{w}',\theta',\varphi')\in C^1(\Omega^-)$ satisfy (4-9). Then, for $\boldsymbol{\tau}=(\tau_1,\tau_2)$ is an arbitrary vector, we have*

$$\int_{\Omega^-}\big[(\boldsymbol{A}^{(1)}\boldsymbol{u}-\beta\operatorname{grad}\theta+b\operatorname{grad}\varphi)\boldsymbol{u}'+W^{(1)}_{(\tau_1)}(\boldsymbol{u},\boldsymbol{u}')+(b\,\varphi-\beta\,\theta)\operatorname{div}\boldsymbol{u}'\big]\,d\boldsymbol{x}=-\int_S\boldsymbol{R}_{(\tau_1)}(\boldsymbol{D}_z(\boldsymbol{n},z))\,\boldsymbol{v}(z)\,\boldsymbol{u}'(z)\,d_zS,$$

$$\int_{\Omega^-}\big[(\boldsymbol{A}^{(2)}(\boldsymbol{D_x})\,\boldsymbol{w}-k_2\boldsymbol{w}-k_3\operatorname{grad}\theta)\,\boldsymbol{w}'+W^{(2)}_{(\tau_2)}(\boldsymbol{w},\boldsymbol{w}')+(k_2\boldsymbol{w}+k_3\operatorname{grad}\theta)\,\boldsymbol{w}'\big]\,d\boldsymbol{x}=-\int_S\boldsymbol{P}^{(2)}_{(\tau_2)}(\boldsymbol{D}_z(\boldsymbol{n},z))\boldsymbol{w}(z)\boldsymbol{w}'(z)\,d_zS,$$

$$\int_{\Omega^-}\big[(k\Delta\theta+k_1\operatorname{div}\boldsymbol{w})\theta'+(k\operatorname{grad}\theta+k_1\boldsymbol{w})\operatorname{grad}\theta'\big]\,d\boldsymbol{x}=-\int_S\Big(k\,\frac{\partial\theta}{\partial\boldsymbol{n}}+k_1\boldsymbol{w}\,\boldsymbol{n}\Big)\theta'd_zS,$$

$$\int_{\Omega^-}\big[(\gamma\,\Delta\varphi-b\operatorname{div}\boldsymbol{u}-d\ \operatorname{div}\boldsymbol{w}+m\,\theta-\xi\,\varphi)\,\varphi'+(\gamma\operatorname{grad}\varphi-d\,\boldsymbol{w})\operatorname{grad}\varphi'+(b\operatorname{div}\boldsymbol{u}-m\,\theta+\xi\,\varphi)\,\varphi'\big]\,d\boldsymbol{x}=-\int_S\Big(\gamma\,\frac{\partial\varphi}{\partial\boldsymbol{n}}-d\,\boldsymbol{w}\,\boldsymbol{n}\Big)\varphi'd_zS,\tag{4-17}$$

Theorem 4.5. *Let $\boldsymbol{U}=(\boldsymbol{u},\boldsymbol{w},\theta,\varphi)$ be a regular vector field in Ω^+ and let $\boldsymbol{U}'=(\boldsymbol{u}',\boldsymbol{w}',\theta',\varphi')\in C^1(\Omega^+)$. Let $\boldsymbol{\tau}=(\tau_1,\tau_2)$ be an arbitrary vector and set, for $l,j=1,2,3$,*

$$P_{(\boldsymbol{\tau})}(\boldsymbol{D_x},\boldsymbol{n})=\big(P_{(\boldsymbol{\tau})\,lj}(\boldsymbol{D_x},\boldsymbol{n})\big)_{8\times 8},\qquad P_{(\boldsymbol{\tau})\,lj}(\boldsymbol{D_x},\boldsymbol{n})=P^{(1)}_{(\tau_1)\,lj}(\boldsymbol{D_x},\boldsymbol{n}),$$
$$P_{(\boldsymbol{\tau})\,l+3;\,j+3}(\boldsymbol{D_x},\boldsymbol{n})=P^{(2)}_{(\tau_2)\,lj}(\boldsymbol{D_x},\boldsymbol{n}),\qquad P_{(\boldsymbol{\tau})\,l7}(\boldsymbol{D_x},\boldsymbol{n})=-\beta n_l,\qquad P_{(\boldsymbol{\tau})\,l8}(\boldsymbol{D_x},\boldsymbol{n})=b\,n_l,$$
$$P_{(\boldsymbol{\tau})\,7;l+3}(\boldsymbol{D_x},\boldsymbol{n})=k_1\,n_l,\quad P_{(\boldsymbol{\tau})\,77}(\boldsymbol{D_x},\boldsymbol{n})=k\frac{\partial}{\partial\boldsymbol{n}},\quad P_{(\boldsymbol{\tau})\,88}(\boldsymbol{D_x},\boldsymbol{n})=\gamma\frac{\partial}{\partial\boldsymbol{n}},\quad P_{(\boldsymbol{\tau})\,8;l+3}(\boldsymbol{D_x},\boldsymbol{n})=-d\,n_l,$$
$$P_{(\boldsymbol{\tau})\,l;\,j+3}=P_{(\boldsymbol{\tau})\,l+3;\,j}=P_{(\boldsymbol{\tau})\,l+3;7}=P_{(\boldsymbol{\tau})\,l+3;8}=P_{(\boldsymbol{\tau})\,7l}=P_{(\boldsymbol{\tau})\,78}=P_{(\boldsymbol{\tau})\,8l}=P_{(\boldsymbol{\tau})\,87}=0.\tag{4-18}$$

Then

$$\int_{\Omega^+}\big[\boldsymbol{A}(\boldsymbol{D_x})\,\boldsymbol{U}(\boldsymbol{x})\,\boldsymbol{U}'(\boldsymbol{x})+W_{(\boldsymbol{\tau})}(\boldsymbol{U},\boldsymbol{U}')\big]\,d\boldsymbol{x}=\int_S \boldsymbol{P}_{(\boldsymbol{\tau})}(\boldsymbol{D_z},\boldsymbol{n}(\boldsymbol{z}))\boldsymbol{U}(\boldsymbol{z})\boldsymbol{U}'(\boldsymbol{z})\,d_{\boldsymbol{z}}S.$$

Theorem 4.6. *Let* $\boldsymbol{U}=(\boldsymbol{u},\boldsymbol{w},\theta,\varphi)$ *be a regular vector field in* Ω^- *and let* $\boldsymbol{U}'=(\boldsymbol{u}',\boldsymbol{w}',\theta',\varphi')\in C^1(\Omega^-)$ *and* $\boldsymbol{U}'$ *satisfy* (4-9). *Then*

$$\int_{\Omega^-}\big[\boldsymbol{A}(\boldsymbol{D_x})\,\boldsymbol{U}(\boldsymbol{x})\,\boldsymbol{U}'(\boldsymbol{x})+W_{(\boldsymbol{\tau})}(\boldsymbol{U},\boldsymbol{U}')\big]\,d\boldsymbol{x}=-\int_S \boldsymbol{P}_{(\boldsymbol{\tau})}(\boldsymbol{D_z},\boldsymbol{n}(\boldsymbol{z}))\,\boldsymbol{U}(\boldsymbol{z})\,\boldsymbol{U}'(\boldsymbol{z})d_{\boldsymbol{z}}S,$$

where $\boldsymbol{\tau}=(\tau_1,\tau_2)$ *is an arbitrary vector.*

In the sequel we use the following two values $\boldsymbol{\tau}^{(1)}$ and $\boldsymbol{\tau}^{(2)}$ of the vector $\boldsymbol{\tau}$:

$$\boldsymbol{\tau}^{(1)}=(\mu,\,k_5),\qquad \boldsymbol{\tau}^{(2)}=(-\mu,\,-k_6). \tag{4-19}$$

By virtue of (3-2), (3-3), (4-1), (4-2), it follows from (4-14) and (4-18) that $\boldsymbol{P}_{(\boldsymbol{\tau}^{(1)})}=\boldsymbol{P}$ and $W_{(\boldsymbol{\tau}^{(1)})}=W$.

The operator $P_{(\boldsymbol{\tau})}(\boldsymbol{D_z},\boldsymbol{n})$ will be called *the generalized stress operator* in the linear theory of thermoelasticity with microtemperatures for microstretch solids.

5. Uniqueness theorems

In this section we prove the uniqueness theorems for the internal and external BVPs $(K)^+_{\boldsymbol{F},\boldsymbol{f}}$ and $(K)^-_{\boldsymbol{F},\boldsymbol{f}}$, where $K=I,II,\dots,XII$.

Theorem 5.1. *If the conditions*

$$\mu>0,\quad 3\lambda+2\mu>0,\quad \gamma>0,\quad (3\lambda+2\mu)\,\xi>3b^2, \tag{5-1}$$

$$k>0,\quad k_6+k_5>0\quad k_6-k_5>0,\quad 3k_4+k_5+k_6>0,\quad (k_1+T_0\,k_3)^2<4T_0\,k\,k_2 \tag{5-2}$$

are satisfied, the internal BVP $(K)^+_{\boldsymbol{F},\boldsymbol{f}}$ *admits at most one regular solution, where* $K=I,III,IV,V$.

Proof. Suppose that there are two regular solutions of the internal BVP $(K)^+_{\boldsymbol{F},\boldsymbol{f}}$. Then their difference $\boldsymbol{U}$ corresponds to zero data ($\boldsymbol{F}=\boldsymbol{f}=\boldsymbol{0}$), i.e., $\boldsymbol{U}$ is a regular solution of problem $(K)^+_{\boldsymbol{0},\boldsymbol{0}}$, where $K=I,III,IV,V$. If $\boldsymbol{U}=\boldsymbol{U}'$, we obtain from (4-5)–(4-8)

$$\int_{\Omega^+}\big[W^{(1)}(\boldsymbol{u},\boldsymbol{u})+(b\,\varphi-\beta\,\theta)\,\mathrm{div}\,\boldsymbol{u}\big]\,d\boldsymbol{x}=0, \tag{5-3}$$

$$\int_{\Omega^+}\big[W^{(2)}(\boldsymbol{w},\boldsymbol{w})+(k_2\boldsymbol{w}+k_3\,\mathrm{grad}\,\theta)\;\boldsymbol{w}\big]\,d\boldsymbol{x}=0, \tag{5-4}$$

$$\int_{\Omega^+}(k\,\mathrm{grad}\,\theta+k_1\boldsymbol{w})\,\mathrm{grad}\,\theta\,d\boldsymbol{x}=0, \tag{5-5}$$

$$\int_{\Omega^+}\big[(\gamma\,\mathrm{grad}\,\varphi-d\,\boldsymbol{w})\,\mathrm{grad}\,\varphi+(b\,\mathrm{div}\,\boldsymbol{u}-m\,\theta+\xi\,\varphi)\,\varphi\big]\,d\boldsymbol{x}=0. \tag{5-6}$$

Equations (5-4) and (5-5) imply

$$\int_{\Omega^+}\big[T_0\,W^{(2)}(\boldsymbol{w},\boldsymbol{w})+\big(T_0\,k_2|\boldsymbol{w}|^2+(k_1+T_0\,k_3)\boldsymbol{w}\,\mathrm{grad}\,\theta\;+k\,|\mathrm{grad}\,\theta|^2\big)\big]\,d\boldsymbol{x}=0. \tag{5-7}$$

Keeping (5-2) in mind, (4-2) yields

$$W^{(2)}(\boldsymbol{w},\boldsymbol{w})\geq 0,\quad T_0\,k_2|\boldsymbol{w}|^2+(k_1+T_0\,k_3)\boldsymbol{w}\,\mathrm{grad}\,\theta\;+k\,|\mathrm{grad}\,\theta|^2\geq 0. \tag{5-8}$$

On the basis of (5-8) we obtain from (5-7) $\boldsymbol{w}(\boldsymbol{x})=\boldsymbol{0}$ and $\theta(\boldsymbol{x})=\text{const}$, for $\boldsymbol{x}\in\Omega^+$. In view of homogeneous boundary condition $\{\theta(z)\}^+=0$ it follows that

$$\boldsymbol{w}(\boldsymbol{x})=\boldsymbol{0}\quad\text{and}\quad\theta(\boldsymbol{x})=0\quad\text{for }\boldsymbol{x}\in\Omega^+. \tag{5-9}$$

By virtue of (5-9) from (5-3) and (5-6) we obtain

$$\int_{\Omega^+}\big[W^{(1)}(\boldsymbol{u},\boldsymbol{u})+2b\,\varphi\,\mathrm{div}\,\boldsymbol{u}+\xi\,|\varphi|^2+\gamma\,|\mathrm{grad}\,\varphi|^2\big]d\boldsymbol{x}=0, \tag{5-10}$$

Keeping (4-2) and (5-1) in mind, (5-10) gives

$$\varphi(\boldsymbol{x})=0, \tag{5-11}$$

$$W^{(1)}(\boldsymbol{u},\boldsymbol{u})=0\quad\text{for }\boldsymbol{x}\in\Omega^+. \tag{5-12}$$

Equations (5-1) and (5-12) show that $\boldsymbol{u}$ is the rigid displacement vector [Ieşan 2004], having the form

$$\boldsymbol{u}(\boldsymbol{x})=\boldsymbol{a}'+[\boldsymbol{a}''\times\boldsymbol{x}], \tag{5-13}$$

where $\boldsymbol{a}'$ and $\boldsymbol{a}''$ are arbitrary real constant three-component vectors and $[\boldsymbol{a}''\times\boldsymbol{x}]$ is the vector product of $\boldsymbol{a}''$ and $\boldsymbol{x}$. Keeping in mind the homogeneous boundary condition $\{\boldsymbol{u}(z)\}^+=\boldsymbol{0}$ from (5-13) we have $\boldsymbol{u}(\boldsymbol{x})=\boldsymbol{0}$ for $\boldsymbol{x}\in\Omega^+$. In view of (5-9) and (5-11) we get $\boldsymbol{U}(\boldsymbol{x})=\boldsymbol{0}$ for $\boldsymbol{x}\in\Omega^+$. Hence, the uniqueness of the solution of BVP $(K)^+_{\boldsymbol{F},\boldsymbol{f}}$ is proved, where $K=I, III, IV, V$. □

Theorem 5.1 leads to:

Theorem 5.2. *If the conditions (5-1) and (5-2) are satisfied, then any two regular solutions of the BVP* $(K)^+_{\boldsymbol{F},\boldsymbol{f}}$, $K=VI, VII, VIII, IX$, *differ only by an additive vector* $\boldsymbol{U}=(\boldsymbol{u},\boldsymbol{w},\theta,\varphi)$, *where*

$$\boldsymbol{u}(\boldsymbol{x})=\boldsymbol{a}'+[\boldsymbol{a}''\times\boldsymbol{x}],\quad \boldsymbol{w}(\boldsymbol{x})=\boldsymbol{0},\quad \theta(\boldsymbol{x})=\varphi(\boldsymbol{x})=0\qquad \textit{for } \boldsymbol{x}\in\Omega^+,$$

$\boldsymbol{a}'$ *and* $\boldsymbol{a}''$ *being arbitrary real constant three-component vectors.*

Theorem 5.3. *If the conditions (5-1) and (5-2) are satisfied, then any two regular solutions of the BVP* $(K)^+_{\boldsymbol{F},\boldsymbol{f}}$, $K=II, X$, *differ only by an additive vector* $\boldsymbol{U}=(\boldsymbol{u},\boldsymbol{w},\theta,\varphi)$, *where*

$$\boldsymbol{u}(\boldsymbol{x})=\boldsymbol{a}'+[\boldsymbol{a}''\times\boldsymbol{x}]+d_1\boldsymbol{x},\quad \boldsymbol{w}(\boldsymbol{x})=\boldsymbol{0},\quad \theta(\boldsymbol{x})=c_1,\quad \varphi(\boldsymbol{x})=d_2\qquad \textit{for } \boldsymbol{x}\in\Omega^+, \tag{5-14}$$

$\boldsymbol{a}'$ *and* $\boldsymbol{a}''$ *being arbitrary real constant three-component vectors*, c_1 *an arbitrary real constant*,

$$d_1=\frac{\beta\,\xi-b\,m}{(3\lambda+2\mu)\xi-3b^2}\,c_1,\quad \textit{and}\quad d_2=\frac{(3\lambda+2\mu)m-3b\beta}{(3\lambda+2\mu)\xi-3b^2}\,c_1.$$

Proof. The difference $\boldsymbol{U}$ between two regular solutions of the BVP $(K)^+_{\boldsymbol{F},\boldsymbol{f}}$ is a regular solution of the homogeneous BVP $(K)^+_{\boldsymbol{0},\boldsymbol{0}}$, where $K=II, X$. It may be shown similarly that

$$\boldsymbol{w}(\boldsymbol{x})=\boldsymbol{0},\quad \theta(\boldsymbol{x})=c_1,\quad \text{for }\boldsymbol{x}\in\Omega^+, \tag{5-15}$$

where c_1 is an arbitrary real constant. On the basis of (5-16) the vector $(\boldsymbol{u}, \varphi)$ is a regular solution in Ω^+ of the nonhomogeneous system

$$\begin{aligned} &\mu\,\Delta\boldsymbol{u} + (\lambda+\mu)\,\mathrm{grad}\,\mathrm{div}\,\boldsymbol{u} + b\,\mathrm{grad}\,\varphi = \boldsymbol{0},\\ &(\gamma\Delta - \xi)\,\varphi - b\,\mathrm{div}\,\boldsymbol{u} = -m\,c_1, \end{aligned} \tag{5-16}$$

satisfying the nonhomogeneous boundary condition

$$\{\boldsymbol{P}^{(1)}(\boldsymbol{D}_z, \boldsymbol{n})\boldsymbol{u}(z) + b\varphi\,\boldsymbol{n}\}^+ = c_1\beta\,\boldsymbol{n}(z), \quad \left\{\frac{\partial\varphi(z)}{\partial\boldsymbol{n}(z)}\right\}^+ = 0 \qquad \text{for } z \in S. \tag{5-17}$$

We introduce the notation

$$\tilde{\boldsymbol{u}}(\boldsymbol{x}) = \boldsymbol{u}(\boldsymbol{x}) - d_1\,\boldsymbol{x}, \quad \tilde{\varphi}(\boldsymbol{x}) = \varphi(\boldsymbol{x}) - d_2. \tag{5-18}$$

Thanks to (5-17), (5-18), and the equalities $3b\,d_1 + \xi\,d_2 = mc_1$, $(3\lambda+2\mu)\,d_1 + b\,d_2 = \beta\,c_1$, the vector $(\tilde{\boldsymbol{u}}, \tilde{\varphi})$ is the regular solution of the homogeneous BVP

$$\begin{aligned} &\mu\,\Delta\tilde{\boldsymbol{u}}(\boldsymbol{x}) + (\lambda+\mu)\,\mathrm{grad}\,\mathrm{div}\,\tilde{\boldsymbol{u}}(\boldsymbol{x}) + b\,\mathrm{grad}\,\tilde{\varphi}(\boldsymbol{x}) = \boldsymbol{0},\\ &(\gamma\Delta - \xi)\,\tilde{\varphi}(\boldsymbol{x}) - b\,\mathrm{div}\,\tilde{\boldsymbol{u}}(\boldsymbol{x}) = 0,\\ &\{\boldsymbol{P}^{(1)}(\boldsymbol{D}_z, \boldsymbol{n})\tilde{\boldsymbol{u}}(z) + b\,\tilde{\varphi}\,\boldsymbol{n}\}^+ = \boldsymbol{0}, \quad \left\{\frac{\partial\tilde{\varphi}(z)}{\partial\boldsymbol{n}(z)}\right\}^+ = 0 \end{aligned} \tag{5-19}$$

for $\boldsymbol{x} \in \Omega^+$ and $z \in S$. It is easily to see that the Green's formulae for $\tilde{\boldsymbol{u}}$ and $\tilde{\varphi}$ have the form

$$\begin{aligned} &\int_{\Omega^+}\big[(\boldsymbol{A}^{(1)}\,\tilde{\boldsymbol{u}} + b\,\mathrm{grad}\,\tilde{\varphi})\tilde{\boldsymbol{u}} + W^{(1)}(\tilde{\boldsymbol{u}}, \tilde{\boldsymbol{u}}) + b\,\tilde{\varphi}\,\mathrm{div}\,\tilde{\boldsymbol{u}}\big]\,d\boldsymbol{x} = \int_S\big[\boldsymbol{P}^{(1)}(\boldsymbol{D}_z, \boldsymbol{n}) + b\tilde{\varphi}\,\boldsymbol{n}\big]\tilde{\boldsymbol{u}}\,d_zS,\\ &\int_{\Omega^+}\big[\big((\gamma\Delta - \xi)\,\tilde{\varphi} - b\,\mathrm{div}\,\tilde{\boldsymbol{u}}\big)\tilde{\varphi} + \big(\gamma|\mathrm{grad}\,\tilde{\varphi}|^2 + \xi\,|\tilde{\varphi}|^2 + b\,\tilde{\varphi}\,\mathrm{div}\,\tilde{\boldsymbol{u}}\big)\big]\,d\boldsymbol{x} = \gamma\int_S \frac{\partial\tilde{\varphi}}{\partial\boldsymbol{n}}\,\tilde{\varphi}\,d_zS. \end{aligned} \tag{5-20}$$

Keeping in mind (5-19) from (5-20) we obtain

$$\int_{\Omega^+}\big[W^{(1)}(\tilde{\boldsymbol{u}}, \tilde{\boldsymbol{u}}) + 2\,b\,\tilde{\varphi}\,\mathrm{div}\,\tilde{\boldsymbol{u}} + \xi\,|\tilde{\varphi}|^2 + \gamma\,|\mathrm{grad}\,\tilde{\varphi}|^2\big]\,d\boldsymbol{x} = 0. \tag{5-21}$$

On account of (5-1) from (5-21) it follows that

$$\tilde{\boldsymbol{u}}(\boldsymbol{x}) = \boldsymbol{a}' + [\boldsymbol{a}'' \times \boldsymbol{x}] \quad \text{and} \quad \tilde{\varphi}(\boldsymbol{x}) = 0 \quad \text{for } \boldsymbol{x} \in \Omega^+, \tag{5-22}$$

where $\boldsymbol{a}'$ and $\boldsymbol{a}''$ are arbitrary real constant three-component vectors. Using (5-15), (5-18) and (5-22) we get (5-14). Hence, the theorem is proved. □

Theorem 5.3 leads to:

Theorem 5.4. *If the conditions (5-1) and (5-2) are satisfied, any two regular solutions of the BVP $(K)^+_{\boldsymbol{F},f}$, $K = XI, XII$, differ only by additive vector $\boldsymbol{U} = (\boldsymbol{u}, \boldsymbol{w}, \theta, \varphi)$, where*

$$\boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{w}(\boldsymbol{x}) = \boldsymbol{0}, \quad \theta(\boldsymbol{x}) = c_1, \quad \varphi(\boldsymbol{x}) = d_3 \qquad \text{for } \boldsymbol{x} \in \Omega^+, \tag{5-23}$$

c_1 is an arbitrary real constant, and $d_3 = m\,c_1/\xi$.

Now let us establish the uniqueness of regular solutions of the external BVPs.

Theorem 5.5. *If conditions (5-1) and (5-2) are satisfied, then the external BVP* $(K)^{-}_{\boldsymbol{F},\boldsymbol{f}}$ *admits at most one regular solution, where* $K = I, II, \dots, XII$.

Proof. Suppose that there are two regular solutions of the external BVP $(K)^{-}_{\boldsymbol{F},\boldsymbol{f}}$, where $K = I, II, \dots, XII$. Then their difference $\boldsymbol{U}$ corresponds to zero data ($\boldsymbol{F} = \boldsymbol{f} = \boldsymbol{0}$), i.e., $\boldsymbol{U}$ is a regular solution of problem $(K)^{-}_{\boldsymbol{0},\boldsymbol{0}}$.

In a similar way as in the proof of Theorem 4.4 we obtain

$$\int_{\Omega^-} \big[W^{(1)}(\boldsymbol{u},\boldsymbol{u}) + (b\,\varphi - \beta\,\theta)\operatorname{div}\boldsymbol{u}\big]\,d\boldsymbol{x} = 0, \tag{5-24}$$

$$\int_{\Omega^-} \big[W^{(2)}(\boldsymbol{w},\boldsymbol{w}) + (k_2\boldsymbol{w} + k_3\operatorname{grad}\theta)\;\boldsymbol{w}\big]\,d\boldsymbol{x} = 0, \tag{5-25}$$

$$\int_{\Omega^-} (k\operatorname{grad}\theta + k_1\boldsymbol{w})\operatorname{grad}\theta\,d\boldsymbol{x} = 0, \tag{5-26}$$

$$\int_{\Omega^-} \big[(\gamma\operatorname{grad}\varphi - d\,\boldsymbol{w})\operatorname{grad}\varphi + (b\operatorname{div}\boldsymbol{u} - m\,\theta + \xi\,\varphi)\,\varphi\big]\,d\boldsymbol{x} = 0. \tag{5-27}$$

Equations (5-25) and (5-26) imply $\boldsymbol{w}(\boldsymbol{x}) = \boldsymbol{0}$ and $\theta(\boldsymbol{x}) = \text{const}$, for $\boldsymbol{x} \in \Omega^-$. In view of condition (3-1) we get $\theta(\boldsymbol{x}) = 0$ for $\boldsymbol{x} \in \Omega^-$; hence,

$$\boldsymbol{w}(\boldsymbol{x}) = \boldsymbol{0} \quad \text{and} \quad \theta(\boldsymbol{x}) = 0 \qquad \text{for } \boldsymbol{x} \in \Omega^-. \tag{5-28}$$

Taking (5-28) into account, (5-24) and (5-27) yield

$$\int_{\Omega^-} \big[W^{(1)}(\boldsymbol{u},\boldsymbol{u}) + 2b\,\varphi\operatorname{div}\boldsymbol{u} + \xi\,|\varphi|^2 + \gamma\,|\operatorname{grad}\varphi|^2\big]\,d\boldsymbol{x} = 0. \tag{5-29}$$

Keeping in mind (3-1) and (5-1) from Eq. (5-29) we have

$$\boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{0} \quad \text{and} \quad \varphi(\boldsymbol{x}) = 0 \qquad \text{for } \boldsymbol{x} \in \Omega^-, \tag{5-30}$$

and in view of (5-28) we get $\boldsymbol{U}(\boldsymbol{x}) = \boldsymbol{0}$ for $\boldsymbol{x} \in \Omega^-$, as desired. □

6. Uniqueness theorems under weak conditions

In this section we use Theorems 4.3 and 4.4 to prove the uniqueness of regular solutions of the problems $(I)^{+}_{\boldsymbol{F},\boldsymbol{f}}$ and $(I)^{-}_{\boldsymbol{F},\boldsymbol{f}}$ under weaker conditions that (5-1) and (5-2).

Theorem 6.1. *If the conditions*

$$\mu > 0, \quad \lambda + 2\mu > 0, \quad \gamma > 0, \quad (\lambda + 2\mu)\,\xi > b^2, \tag{6-1}$$

$$k > 0, \quad k_6 > 0, \quad k_7 > 0, \quad (k_1 + T_0\,k_3)^2 < 4T_0\,k\,k_2 \tag{6-2}$$

are satisfied, the internal BVP $(K)^{+}_{\boldsymbol{F},\boldsymbol{f}}$ *admits at most one regular solution, where* $k_7 = k_4 + k_5 + k_6$ *and* $K = I, IV$.

Proof. Suppose that there are two regular solutions of problem $(K)^{+}_{\boldsymbol{F},\boldsymbol{f}}$. Their difference $\boldsymbol{U}$ is a regular solution of problem $(K)^{+}_{\boldsymbol{0},\boldsymbol{0}}$, where $K = I, IV$. If $\boldsymbol{U} = \boldsymbol{U}'$ and $\boldsymbol{\tau} = \boldsymbol{\tau}^{(2)}$ (see (4-19)), it follows from (4-7), (4-8), (4-15), and (4-16) that

$$\int_{\Omega^+}\big[W^{(1)}_{(-\mu)}(\boldsymbol{u},\boldsymbol{u})+(b\,\varphi-\beta\,\theta)\operatorname{div}\boldsymbol{u}\big]\,d\boldsymbol{x}=0, \tag{6-3}$$

$$\int_{\Omega^+}\big[W^{(2)}_{(-k_6)}(\boldsymbol{w},\boldsymbol{w})+(k_2\boldsymbol{w}+k_3\operatorname{grad}\theta)\;\boldsymbol{w}\big]\,d\boldsymbol{x}=0, \tag{6-4}$$

$$\int_{\Omega^+}\big[k\,|\operatorname{grad}\theta|^2+k_1\boldsymbol{w}\operatorname{grad}\theta\big]\,d\boldsymbol{x}=0, \tag{6-5}$$

$$\int_{\Omega^+}\big[\gamma\,|\operatorname{grad}\varphi|^2-d\;\boldsymbol{w}\operatorname{grad}\varphi+b\,\varphi\operatorname{div}\boldsymbol{u}-m\,\theta\,\varphi+\xi\,|\varphi|^2\big]\,d\boldsymbol{x}=0, \tag{6-6}$$

where

$$W^{(1)}_{(-\mu)}(\boldsymbol{u},\boldsymbol{u})=(\lambda+2\mu)\,|\operatorname{div}\boldsymbol{u}|^2+\mu\,|\operatorname{curl}\boldsymbol{u}|^2,\qquad W^{(2)}_{(-k_6)}(\boldsymbol{w},\boldsymbol{w})=k_7\,|\operatorname{div}\boldsymbol{w}|^2+k_6\,|\operatorname{curl}\boldsymbol{w}|^2.$$

From (6-4) and (6-5) it follows that

$$\int_{\Omega^+}\big[T_0\,W^{(2)}_{(-k_6)}(\boldsymbol{w},\boldsymbol{w})+\big(T_0\,k_2|\boldsymbol{w}|^2+(k_1+T_0\,k_3)\boldsymbol{w}\operatorname{grad}\theta\;+k\,|\operatorname{grad}\theta|^2\big)\big]\,d\boldsymbol{x}=0. \tag{6-7}$$

Keeping in mind (6-2), we have from (6-7)

$$\boldsymbol{w}(\boldsymbol{x})=\mathbf{0},\qquad \theta(\boldsymbol{x})=\text{const}\quad \text{for } \boldsymbol{x}\in\Omega^+.$$

By the homogeneous boundary condition we get $\theta(\boldsymbol{x})=0$ for $\boldsymbol{x}\in\Omega^+$, and from (6-3) and (6-6) we get

$$\int_{\Omega^+}\big[\mu\,|\operatorname{curl}\boldsymbol{u}|^2+(\lambda+2\mu)\,|\operatorname{div}\boldsymbol{u}|^2+2b\,\varphi\operatorname{div}\boldsymbol{u}+\xi\,|\varphi|^2+\gamma\,|\operatorname{grad}\varphi|^2\big]\,d\boldsymbol{x}=0. \tag{6-8}$$

By (6-1) from (6-8) we obtain $\varphi(\boldsymbol{x})=0,\ \operatorname{div}\boldsymbol{u}(\boldsymbol{x})=\mathbf{0},\ \operatorname{curl}\boldsymbol{u}(\boldsymbol{x})=\mathbf{0}$ for $\boldsymbol{x}\in\Omega^+$. Hence, $\boldsymbol{u}$ is a regular solution of the BVP

$$\Delta\,\boldsymbol{u}(\boldsymbol{x})=\mathbf{0},\qquad \{\boldsymbol{u}(\boldsymbol{z})\}^+=\mathbf{0}\quad \text{for } \boldsymbol{x}\in\Omega^+,\ \boldsymbol{z}\in S. \tag{6-9}$$

This implies $\boldsymbol{u}(\boldsymbol{x})=\mathbf{0}$ for $\boldsymbol{x}\in\Omega^+$, as needed. □

Theorem 6.2. *If the conditions (6-1) and (6-2) are satisfied, any two regular solutions of the BVP $(XI)^+_{\boldsymbol{F},\boldsymbol{f}}$ may differ only by an additive vector $\boldsymbol{U}=(\boldsymbol{u},\boldsymbol{w},\theta,\varphi)$, where $\boldsymbol{U}$ is given by (5-23), c_1 is an arbitrary real constant and d_3 is defined in Theorem 5.4.*

Proof. The difference $\boldsymbol{U}$ between two regular solutions of the BVP $(XI)^+_{\boldsymbol{F},\boldsymbol{f}}$ is a regular solution of the homogeneous BVP $(XI)^+_{\mathbf{0},\mathbf{0}}$. Using Green's formula (4-16) and (6-2), we can show as above that

$$\boldsymbol{w}(\boldsymbol{x})=\mathbf{0}\quad\text{and}\quad \theta(\boldsymbol{x})=c_1\qquad \text{for } \boldsymbol{x}\in\Omega^+,$$

and the vector function $\boldsymbol{u}$ and function φ from a regular solution in Ω^+ of the nonhomogeneous system

$$\mu\,\Delta\boldsymbol{u}+(\lambda+\mu)\operatorname{grad}\operatorname{div}\boldsymbol{u}+b\operatorname{grad}\varphi=\mathbf{0},$$
$$(\gamma\,\Delta-\xi)\,\varphi-b\operatorname{div}\boldsymbol{u}=-m\,c_1,$$

satisfying the homogeneous boundary condition

$$\{\boldsymbol{u}(\boldsymbol{z})\}^+=0,\qquad \left\{\frac{\partial\,\varphi(\boldsymbol{z})}{\partial\boldsymbol{n}(\boldsymbol{z})}\right\}^+=0\qquad \text{for } \boldsymbol{z}\in S,$$

where c_1 is an arbitrary real constant. If we introduce

$$\tilde{\varphi}(\boldsymbol{x}) = \varphi(\boldsymbol{x}) - d_3, \tag{6-10}$$

the vector $(\boldsymbol{u}, \tilde{\varphi})$ is then a regular solution of the homogeneous BVP

$$\begin{aligned} &\mu\, \Delta \boldsymbol{u}(\boldsymbol{x}) + (\lambda+\mu)\,\mathrm{grad}\,\mathrm{div}\,\boldsymbol{u}(\boldsymbol{x}) + b\,\mathrm{grad}\,\tilde{\varphi}(\boldsymbol{x}) = \boldsymbol{0}, \\ &(\gamma\,\Delta - \xi)\,\tilde{\varphi}(\boldsymbol{x}) - b\,\mathrm{div}\,\boldsymbol{u}(\boldsymbol{x}) = 0, \quad \{\boldsymbol{u}(\boldsymbol{z})\}^{+} = \boldsymbol{0}, \quad \left\{\frac{\partial \tilde{\varphi}(\boldsymbol{z})}{\partial \boldsymbol{n}(\boldsymbol{z})}\right\}^{+} = 0 \end{aligned} \tag{6-11}$$

for $\boldsymbol{x} \in \Omega^+$ and $\boldsymbol{z} \in S$. It is easily to see that by virtue of (6-11) the Green's formulas (4-8) and (4-16) for $\boldsymbol{u}$ and $\tilde{\varphi}$ take on the form

$$\int_{\Omega^+} \big[W^{(1)}_{(-\mu)}(\boldsymbol{u},\boldsymbol{u}) + b\,\tilde{\varphi}\,\mathrm{div}\,\boldsymbol{u}\big]\,d\boldsymbol{x} = 0, \quad \int_{\Omega^+} \big[\big(\gamma\,|\mathrm{grad}\,\tilde{\varphi}|^2 + \xi\,|\tilde{\varphi}|^2 + b\,\tilde{\varphi}\,\mathrm{div}\,\boldsymbol{u}\big)\big]\,d\boldsymbol{x} = 0, \tag{6-12}$$

and on the basis of (6-7) we obtain from (6-12)

$$\int_{\Omega^+} \big[(\lambda+2\mu)\,|\mathrm{div}\,\boldsymbol{u}|^2 + 2\,b\,\tilde{\varphi}\,\mathrm{div}\,\boldsymbol{u} + \xi\,|\tilde{\varphi}|^2 + \mu\,|\mathrm{curl}\,\boldsymbol{u}|^2 + \gamma\,|\mathrm{grad}\,\tilde{\varphi}|^2\big]\,d\boldsymbol{x} = 0. \tag{6-13}$$

Taking (6-1) into account, (6-13) implies that

$$\boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{a}' + [\boldsymbol{a}'' \times \boldsymbol{x}] \quad \text{and} \quad \tilde{\varphi}(\boldsymbol{x}) = 0 \qquad \text{for } \boldsymbol{x} \in \Omega^+, \tag{6-14}$$

where $\boldsymbol{a}'$ and $\boldsymbol{a}''$ are arbitrary real constant three-component vectors. Using the homogeneous boundary condition (6-11), we obtain from (6-14) that $\boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{0}$ for $\boldsymbol{x} \in \Omega^+$, and using (6-14) we get from (6-10) that $\varphi(\boldsymbol{x}) = d_3$ for $\boldsymbol{x} \in \Omega^+$, as needed. □

Theorem 6.3. *If conditions (6-1) and (6-2) are satisfied, the external BVP $(K)^-_{\boldsymbol{F},\boldsymbol{f}}$ admit at most one regular solution, where $K = I, II, XI$.*

Proof. Suppose that there are two regular solutions of problem $(K)^-_{\boldsymbol{F},\boldsymbol{f}}$. Their difference $\boldsymbol{U}$ is a regular solution of problem $(K)^-_{\boldsymbol{0},\boldsymbol{0}}$, where $K = I, II, XI$. If $\boldsymbol{U} = \boldsymbol{U}'$ and $\boldsymbol{\tau} = \boldsymbol{\tau}^{(2)}$, we have from (4-17) and (4-19)

$$\begin{aligned} &\int_{\Omega^-} \big[W^{(1)}_{(-\mu)}(\boldsymbol{u},\boldsymbol{u}) + (b\,\varphi - \beta\,\theta)\,\mathrm{div}\,\boldsymbol{u}\big]\,d\boldsymbol{x} = 0, \\ &\int_{\Omega^-} \big[T_0\,W^{(2)}_{(-k_6)}(\boldsymbol{w},\boldsymbol{w}) + \big(T_0\,k_2|\boldsymbol{w}|^2 + (k_1 + T_0\,k_3)\boldsymbol{w}\,\mathrm{grad}\,\theta\; + k\,|\mathrm{grad}\,\theta|^2\big)\,\big]\,d\boldsymbol{x} = 0, \\ &\int_{\Omega^-} \big[\gamma\,|\mathrm{grad}\,\varphi|^2 - d\,\boldsymbol{w}\,\mathrm{grad}\,\varphi + b\,\varphi\,\mathrm{div}\,\boldsymbol{u} - m\,\theta\,\varphi + \xi\,|\varphi|^2\big]\,d\boldsymbol{x} = 0. \end{aligned} \tag{6-15}$$

Similarly, taking (3-1), (6-1), and (6-2) into account, we obtain from (6-15) that $\boldsymbol{w}(\boldsymbol{x}) = \boldsymbol{0}$, $\theta(\boldsymbol{x}) = \varphi(\boldsymbol{x}) = 0$, $\mathrm{div}\,\boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{0}$, and $\mathrm{curl}\,\boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{0}$ for $\boldsymbol{x} \in \Omega^-$. Hence, $\boldsymbol{u}$ is regular solution of the BVP

$$\Delta\,\boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{0}, \quad \{\boldsymbol{u}(\boldsymbol{z})\}^- = \boldsymbol{0} \qquad \text{for } \boldsymbol{x} \in \Omega^-,\ \boldsymbol{z} \in S. \tag{6-16}$$

Therefore (6-16) shows that $\boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{0}$ for $\boldsymbol{x} \in \Omega^-$. □

Remark 6.4. From (5-1) and (5-2) we have (6-1) and (6-2), respectively. Indeed, (5-1) and (5-2) imply

$$\lambda + 2\mu = \tfrac{1}{3}\big((3\lambda+2\mu)+4\mu\big) > 0, \quad k_6 = \tfrac{1}{2}\big((k_6+k_5)+(k_6-k_5)\big) > 0, \quad k_7 = \tfrac{1}{3}\big((3k_4+k_5+k_6)+2(k_6+k_5)\big) > 0.$$

7. Concluding remarks

(1) In [Knops and Payne 1971], the uniqueness theorems of the first BVP (on the boundary given the displacement vector) and the second BVP (on the boundary given the stress vector) in the classical theory of elasticity are proved under the conditions $\mu > 0,\ \lambda + 2\mu > 0$ and $\mu > 0,\ 3\lambda + 2\mu > 0$, respectively.

(2) Using the uniqueness Theorems 5.1–5.4 and 6.1–6.3 it is possible to prove the existence theorems in the equilibrium theory of thermoelasticity with microtemperatures for microstretch solids by means of the potential method and the theory of singular integral equations.

(3) The conditions (5-1), (5-2) and (6-1), (6-2) are sufficient for the uniqueness of solutions of BVPs in the theory of equilibrium thermoelasticity with microtemperatures for microstretch solids occupying arbitrary 3D domains with a smooth surface. Establishing necessary conditions for the uniqueness of solutions is an open problem in the classical theory of thermoelasticity [Kupradze et al. 1979], the theory of thermoelasticity with microtemperatures [Ieşan and Quintanilla 2000], the micropolar theory of thermoelasticity, theories of micromorphic elasticity and thermomicrostretch elastic solid [Eringen 1999], and in the theory of thermoelasticity with microtemperatures for microstretch solids [Ieşan 2001]. The necessary condition for uniqueness of solutions have been established only in the classical theory of elasticity (see [Knops and Payne 1971; Fosdick et al. 2007], for details).

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