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INDENTATION AND RESIDUAL STRESS IN THE AXIALLY SYMMETRIC ELASTOPLASTIC CONTACT PROBLEM

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Most theoretical studies of mechanical indention, going back to Boussinesq, Hertz, and later Sneddon, address the relations between indenter pressure, indention size, and stress components. However, the relationship between indention and residual stress is also interesting. Here we use the Dorris and Nemat-Nasser method to derive a relation between the indention and the residual stress components for an axisymmetric load.

1. Introduction

The elastic contact problem plays a key role in interpreting experimental results of indention. This study was first considered by Boussinesq and Hertz in the late nineteenth century, and later Sneddon made major contributions. These authors derived general relationships among the load, the displacement, and the contact area for an axisymmetric indenter.

Also of interest is the relationship between the indention and the residual stress. This has been addressed for example in [Suresh and Giannakopoulos 1998], where it is stated that the residual stress cannot be determined using the loading theory of elasticity. In [Hao 2006] we made some progress in the study of the problem in the framework of the theory of finite elasticity. The paper continues that investigation, by considering the important unloading case. As in the previous paper, we derive the elastoplastic deformation is derived using the DNN method [Dorris and Nemat-Nasser 1980]. The elastic deformation is eliminated from the total deformation, leading to the residual plastic deformation. Thus we determine the relation between the residual stress and the residual plastic deformation.

2. Analysis of the axially symmetrical finite elastic-plastic case

Following Dorris and Nemat-Nasser we write, for the axially symmetrical case,

$$D_{ab} = 0.5(v_{a,b} + v_{b,a}), \qquad D_{\theta\theta} = v_2/x_2 \tag{1}$$

where x_1 and x_2 denote z and r, v_a is the increment displacement, $v_{a,b} = \partial v_a/\partial x_b$ and D_{ab}; $D_{\theta\theta}$ are the components of the rate tensor. Note that in [Dorris and Nemat-Nasser 1980] v_a is the velocity, but in this paper, it is the incremental displacement, whose dimension is length. Since we are dealing with small deformations superposed on initial stress body, the increment displacement v_a of the small deformation is a small part of the whole displacement.

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Still following Dorris and Nemat-Nasser, we use the current configuration as the reference one. We deal with the first Piola–Kirchhoff stress increment $\delta\sigma_{ab}$, where the index a denotes the direction of the stress and the index b denotes the normal to the surface subjected to $\delta\sigma_{ab}$ in the reference configuration (note that $\delta\sigma_{ab} \neq \delta\sigma_{ba}$). Only the incompressible case is considered; the compressible case can be derived from the results of the incompressible case and it will be studied in another paper.

We turn to the constitutive equations, still following [Dorris and Nemat-Nasser 1980]. Similarly to what is done in Appendix A — compare A — we can write

$$\delta\sigma_{ab} = \Phi_{abce} D_{ce} + P\delta_{ab} - D_{ac}T_{cb} - D_{bc}T_{ca} + v_{a,c}T_{cb}, \tag{2}$$

where Φ_{abce} will be discussed later, P is an unknown scalar function (hydrostatic pressure), and T_{ca} is the Cauchy stress. For the flow theory, the constitutive equation is

$$\Phi_{abce} = (2\mu\delta_{ac}\delta_{be} - 6A\mu^2 S T'_{ab} T'_{ce}/\overline{T}^2), \tag{3}$$

where μ is the elastic constant, S equals $(2h/3 + 2\mu)^{-1}$, T'_{ab} and D'_{ab} are the deviatoric parts of the Cauchy stress T_{ab} and rate D_{ab}, the scalar A is defined as 1 if $T'_{ab}D'_{ab} \geq 0$ and as 0 if $T'_{ab}D'_{ab} < 0$, and $\overline{T}^2 = 1.5T'_{ab}T'_{ab}$.

The value of h is given by

$$h = \left(\frac{1}{E_t} - \frac{1}{E}\right)^{-1},$$

where E is the initial Young's modulus and E_t is the instantaneous tangent modulus. E_t equals $d\sigma/d\epsilon = En(\sigma/\sigma_y)^{1-1/n}$ for $0 \leq n \leq 1$, σ is the true stress, σ_y is the yield stress and ϵ is the logarithmic strain (σ and ϵ are of simple tension or compression).

From total deformation theory, one has

$$\Phi_{abce} = 2\mu(\overline{\gamma}/\gamma_0)^{n-1}\delta_{ac}\delta_{be} - \frac{3h(1-n)}{n} T'_{ab}T'_{ce}/\overline{T}^2 \tag{4}$$

for some n satisfying $0 \leq n \leq 1$; here $\overline{\gamma}$ is the effective increment strain and γ_0 is a reference increment strain.

Because (3) and (4) are in similar form, from now on, only the flow theory case is discussed. When $A = 0$, i.e., $T'_{ab}D'_{ab} < 0$ or $\sigma < \sigma_y$, we are in the unloading case or the elastic case, and we can deal with this problem as in [Hao 2006]. The case $A = 1$, i.e., $T'_{ab}D'_{ab} \geq 0$ and $\sigma > \sigma_y$, is the loading case, to be considered in this paper. As the cone indention causes compressive stresses, in order to satisfy the requirement of the loading case $A = 1$, the residual stress must also be compressive.

Following [Dorris and Nemat-Nasser 1980], in view of the constitutive equations, one obtains

$$\delta\sigma_{22} = 2\mu v_{2,2} + 2\mu^2 S v_{1,1} + P, \tag{5}$$

$$\delta\sigma_{11} = (2\mu - 4\mu^2 S - T)v_{1,1} + P, \tag{6}$$

$$\delta\sigma_{21} = (2\mu - T)(v_{1,2} + v_{2,1})/2 + T v_{2,1}, \tag{7}$$

$$\delta\sigma_{12} = (2\mu - T)(v_{1,2} + v_{2,1})/2, \tag{8}$$

$$\delta\sigma_{\theta\theta} = 2\mu v_2/r + 2\mu^2 S v_{1,1} + P, \tag{9}$$

where T is the residual stress.

The homogeneous residual stress σ_R is equal to T. When the xy-plane is parallel to the surface, the residual stresses σ_x and σ_y are also equal to T. The first equilibrium equation is

$$\delta\sigma_{22,2} + (\delta\sigma_{22} - \delta\sigma_{\theta\theta})/x_2 + \delta\sigma_{21,1} = 0. \tag{10}$$

From the calculations in Appendix B, one obtains

$$2\mu v_{2,221} + 2\mu^2 S v_{1,121} + P_{,21} + (1/x_2)2\mu(v_{2,21} - v_{2,1}/x_2) + (2\mu - T)(v_{2,111} + v_{1,121})/2 + T v_{2,111} = 0 \tag{11}$$

The other equilibrium equation is

$$\delta\sigma_{11,1} + \delta\sigma_{12,2} + \delta\sigma_{12}/x_2 = 0. \tag{12}$$

Also in view of Appendix B, one has

$$(2\mu - 8\mu^2 S - T)v_{1,112}/2 + P_{,12} + (2\mu - T)(v_{1,222} + v_{1,22}/x_2 - v_{1,2}/x_2^2)/2 = 0, \tag{13}$$

Considering (11) and (13) and eliminating $P_{,12}$, one has

$$\begin{aligned} 2\mu v_{2,221} + 2\mu^2 S v_{1,121} + (1/x_2)2\mu(v_{2,21} - v_{2,1}/x_2) + (2\mu - T)(v_{2,111} + v_{1,211})/2 + T v_{2,111} \\ - (2\mu - 8\mu^2 S - T)v_{1,112}/2 - (2\mu - T)(v_{1,222} + v_{1,22}/x_2 - v_{1,2}/x_2^2)/2 = 0. \end{aligned} \tag{14}$$

Let $L(v_2) = v_{2,22} + v_{2,2}/x_2 - v_2/x_2^2$ and $L(v_{1,2}) = v_{1,222} + v_{1,22}/x_2 - v_{1,2}/x_2^2$. Substituting into (14), one obtains

$$2\mu L(v_2)_{,1} + 6\mu^2 S v_{1,112} + (2\mu + T)v_{2,111}/2 - (2\mu - T)L(v_{1,2})/2 = 0. \tag{15}$$

Let v_2 be F_{11}, where F_{11} is $\partial^2 F/\partial x_1^2$. In view of Appendix C, one has

$$(4\mu - 12\mu^2 S)L(F)_{11} + (2\mu + T)L^0 F_{1111} + (2\mu - T)L^2(F) = 0. \tag{16}$$

Let $G(s, x_1)$ be $\int_0^\infty x_2 F(x_1, x_2) J_1(s x_2)\, dx_2$ which is the Hankel transform of $F(x_1, x_2)$ with order 1 [Sneddon 1951]. Therefore, one has

$$\begin{aligned} 4\mu(1 - 3\mu S) \int_0^\infty x_2 L(F)_{11} J_1(s x_2)\, dx_2 \\ + (2\mu + T) \int_0^\infty x_2 (F)_{1111} J_1(s x_2)\, dx_2 + (2\mu - T) \int_0^\infty x_2 L^2(F) J_1(s x_2)\, dx_2 = 0. \end{aligned} \tag{17}$$

If $x_2 \to 0$ and ∞, $x_2 F \to 0$, we have

$$\begin{aligned} \int_0^\infty x_2 L(F) J_1(s x_2)\, dx_2 = -s^2 G(s, x_1), \\ \int_0^\infty x_2 L^2(F) J_1(s x_2)\, dx_2 = -s^2 \int_0^\infty x_2 L(F) J_1(s x_2)\, dx_2 = s^4 G(s, x_1). \end{aligned} \tag{18}$$

Substituting (18) into (17), one obtains

$$-4\mu s^2(1 - 3\mu S)G(s, x_1)_{,11} + (2\mu + T)G(s, x_1)_{,1111} + s^4(2\mu - T)G(s, x_1) = 0 \tag{19}$$

Let $G(s, x_1) = N(s)\exp(mx_1)$, where m is a function of s. We obtain $G(s, x_1)_{11} = N(s)\exp(mx_1)m^2$, $G(s, x_1)_{,1111} = N(s)\exp(mx_1)m^4$ and

$$\begin{aligned} s^4(1-Q) - 2s^2(1-3\mu S)m^2 + (1+Q)m^4 = 0, \\ m^2 = r^2 s^2 = s^2\{(1-3\mu S) \pm [Q^2 - 6\mu S - 9\mu^2 S^2]^{1/2}\}/(1+Q), \end{aligned} \tag{20}$$

where $r^2 = \{(1-3\mu S) \pm [Q^2 - 6\mu S - 9\mu^2 S^2]^{1/2}\}/(1+Q)$ and $Q = T/2\mu$. One can deal only with the case where there are two different real positive roots r_1^2 and r_2^2. It can be proved that the same results will be obtained in the case with two conjugate complex roots.

Letting $x_1 \to \infty$, $v_2 \to 0$, $F_1 \to 0$, $r_1, r_2 > 0$, similar to [Hao 2006], one obtains

$$\begin{aligned} G(s, x_1) = \int_0^\infty x_2 F(x_1, x_2) J_1(sx_2)\, dx_2 &= N_1(s)\exp(m_1 x_1) + N_2(s)\exp(m_2 x_1) \\ &= N_1(s)\exp(-r_1 s x_1) + N_2(s)\exp(-r_2 s x_1). \end{aligned} \tag{21}$$

Considering $x_1 = 0$, $\delta\sigma_{21} = 0$, in view of Appendix C, one has

$$G(s, x_1) = N_1(s)e^{-r_1 s x_1} + N_2(s)e^{-r_2 s x_1} = N_1(s)(e^{-r_1 s x_1} - Ue^{-r_2 s x_1}) \tag{22}$$

where

$$U = \frac{r_1^3 + r_1(2\mu - T)/(2\mu + T)}{r_2^3 + r_2(2\mu - T)/(2\mu + T)}.$$

Now, the stress component $\delta\sigma_{11}$ and v_1 are discussed. According to Appendix C, one has

$$\delta\sigma_{11} = (2\mu - T)[U(r_2^2 + 1) - (r_1^2 + 1)] \int_0^\infty s J_0(sx_2) s^3 N_1(s)\, ds/2, \tag{23}$$

$$v_1 = (r_1 - Ur_2) \int_0^\infty s J_0(sx_2) s^2 N_1(s)\, ds. \tag{24}$$

The boundary conditions are

$$\begin{aligned} (r_1 - Ur_2)a^4 \int_0^\infty s^3 J_0(sx_2) N_1(s)\, ds &= a^4[v_1(x_2)]_{x_1=0} && \text{for } x_1 = 0,\ a \ge x_2 \ge 0, \\ \int_0^\infty s^4 J_0(sx_2) N_1(s)\, ds &= 0 && \text{for } x_1 = 0,\ x_2 > a, \end{aligned} \tag{25}$$

where a is the radius of contact area, which will be discussed in detail later. Finally,

$$p^3 N_1(s) = p^3 N_1(p/a) = f(p), \quad s = p/a. \tag{26}$$

3. The circular cone and the residual stress

We now turn to a circular cone on a half-space and consider the residual stress. Let α be the angle of the circular cone (the angle between the axis of symmetry and the generatrix). Then

$$[v_1(x_2)]_{x_1=0} = b + a\cot\alpha(1 - x_2/a) \quad \text{for } a \ge x_2 \ge 0 \tag{27}$$

and

$$a^4[v_1(x_2)]_{x_1=0} = a^4[b + a\cot\alpha(1 - x_2/a)] = (r_1 - Ur_2)(A_0 + A_1 x_2/a) \quad \text{for } a \geq x_2 \geq 0 \tag{28}$$

On the foundation of that the stress component $\delta\sigma_{11}$ is finite at the edge of the punch, the relation between b and a can be obtained. Similar to [Hao 2006], we obtain

$$f(P) = \frac{1}{\sqrt{\pi}}\left\{A_0\left(\cos p + p\int_0^1 u\sin(pu)\,du\right)\frac{\Gamma(1)}{\Gamma(3/2)} + A_1\left(\cos p + p\int_0^1 u^2\sin(pu)\,du\right)\frac{\Gamma(3/2)}{\Gamma(2)}\right\}. \tag{29}$$

According to [Hao 2006] and Appendix D, for the compressive force R on the cone, one obtains

$$R = \pi(2\mu - T)a^2\cot\alpha\,\frac{r_2r_1(r_1r_2 - \beta) + (r_2^2 + r_1r_2 + r_1^2) + \beta}{2r_2r_1(r_2^2 + 2r_1r_2 + r_1^2)^{1/2}}, \tag{30}$$

where $\beta = \dfrac{2\mu - T}{2\mu + T}$ and r_1, r_2 can be calculated from the equalities $r_2^2r_1^2 = \beta,\ \ r_2^2 + r_1^2 = \dfrac{2\mu - 6\mu^2 S}{2\mu + T}$.

The contact radius a is

$$a = \left(\frac{R[r_2r_1(r_2^2 + 2r_1r_2 + r_1^2)^{1/2}]}{\pi(2\mu - T)[r_2r_1(r_1r_2 - \beta) + (r_2^2 + r_1r_2 + r_1^2) + \beta]\cot\alpha}\right)^{1/2}, \tag{31}$$

from which the contact area πa^2 is easily calculated. The penetration depth is

$$v_1(x_1, x_2)_{x_1=0, x_2=0} = \tfrac{1}{2}\pi a\cot\alpha = \frac{1}{2}\left(\frac{\pi R\cot\alpha[r_2r_1(r_2^2 + 2r_1r_2 + r_1^2)^{1/2}]}{(2\mu - T)[r_2r_1(r_1r_2 - \beta) + (r_2^2 + r_1r_2 + r_1^2) + \beta]}\right)^{1/2}. \tag{32}$$

4. The unloading of the indenter

Lastly, as the residual stress is determined by the indention, the unloading of the indenter is now discussed. For the unloading case, let A be 0 in (3), i.e., $S = 0$ in (20). One obtains two roots r_1, r_2. Then, all other related values (penetration depth, contact area) of the purely elastic case are obtained.

For an example, the penetration depth in the unloading case is studied.

$$h = 0.5\{\pi R\cot\alpha[r_2r_1(r_2 + r_1)]/(2\mu - T)[r_2r_1(r_1r_2 - r_2^2r_1^2) + (r_2^2 + r_1r_2 + r_1^2) + r_2^2r_1^2]\}^{1/2} \tag{33}$$

where h is the penetration depth of the purely elastic case and the values of r_1, r_2 can be obtained where $S = 0$ in (20). Subtracting it from the elastoplastic case, one obtains the penetration depth for the residual plastic deformation case.

5. Results and explanation

As an illustration, we take the specific example considered in [Hao 2006]. The parameters are $\mu = 10$–100 GPa, $\sigma_y =$ yield stress $= 200$ MPa, $R = 0.46$ kg and $\alpha = \pi/12$. The results are in Figures 1–3.

In the figure, N is a function of the elastic shear modulus μ and the plastic constant $k = E_t/E$ (recall that E is the initial Young's modulus and E_t is the instantaneous modulus). This parameter N equals $\{(\mu^{1/2}/3\mu_0)^k k^{(1-k)}\}$, where $\mu_0 = 7$ Gpa. When $E_t = 0$ i.e. $k = 0$ or $N = 0$, the material is soft. When $k = 1$ or $N = (\mu^{1/2}/3\mu_0)$, the material is tough. Therefore, the parameter N is a coefficient to determine

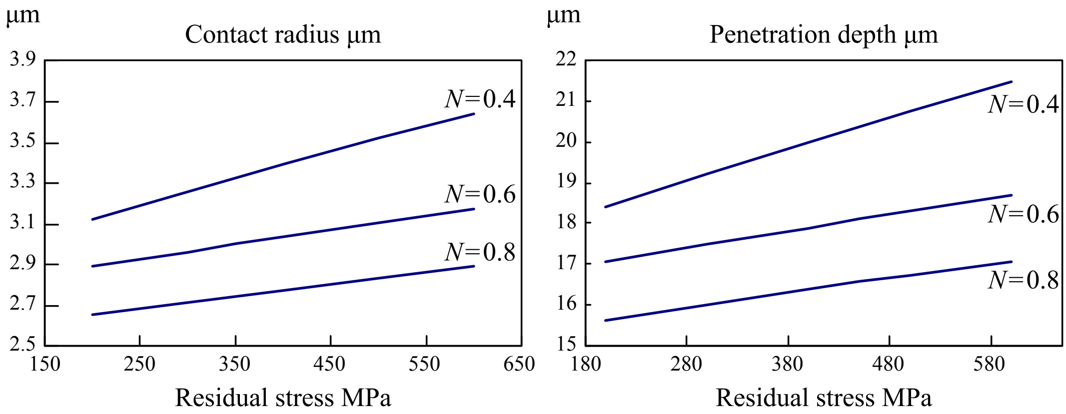

Figure 1. Elastoplastic case: contact radius (left) and penetration depth (right) versus residual stress.

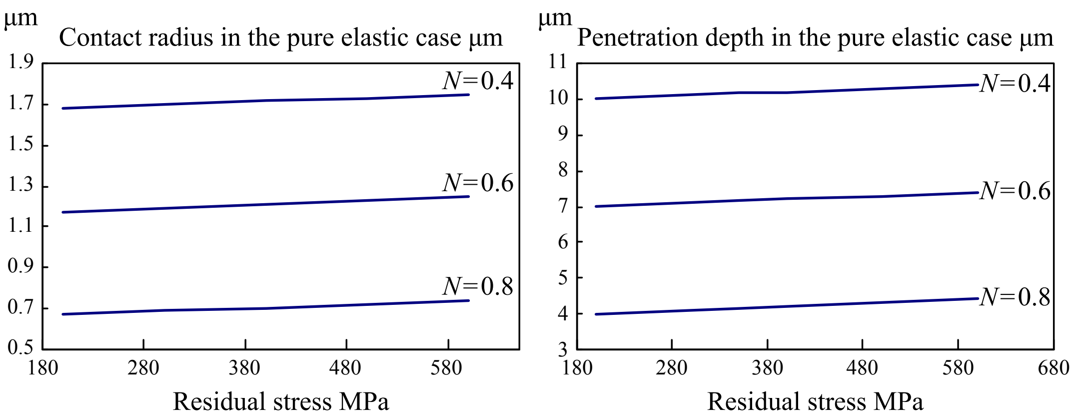

Figure 2. Purely elastic case: contact radius (left) and penetration depth (right) versus residual stress.

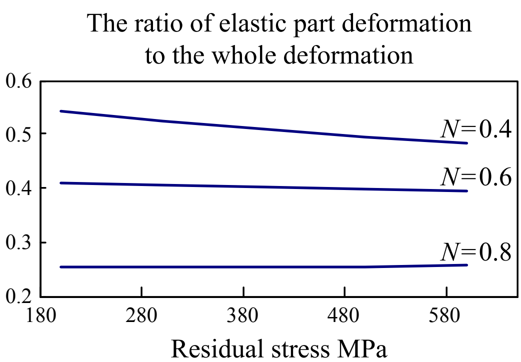

Figure 3. Ratio of elastic part deformation to the whole deformation, plotted versus residual stress.

the softness or toughness of materials. Figure 1 shows that in the elastoplastic case, the contact radius and the penetration depth vary with the residual stress. However, in the purely elastic case ($k = 1$), when N is constant, we see in Figure 2 that the contact radius and penetration depth are nearly constant. These figures show that the shear modulus is almost an identified factor to determine the contact area, the contact radius and the penetration depth. These figures also show that the effect of Jaumann rate is not notable for this case. Therefore, in the purely elastic case, when the Jaumann rate is even considered, we hardly determine the residual stress according to the indention. This means that in order to determine the residual stress the plastic deformation must be considered. Figure 3 shows that when N is a constant, the larger the residual stress is, the smaller the ratio of elastic part deformation to whole deformation becomes. Naturally, the larger the residual stress becomes, the larger the plastic part deformation also becomes.

6. Conclusions

In this paper, the axially symmetrical elastoplastic contact problem and its application are studied. The relation among the penetration depth, the contact radius and the residual stress has been determined. Besides, the unloading case is considered. The essence of this method is to deviate from the linear theory to consider Jaumann rate. When deviating from the linear theory a little, this difficult problem can be studied easily. For an example, when studying the effects of air inside crack in the piezoelectric materials, we must deal with the opening crack after deformation because before deformation, the crack is closed and no air can be in it. Therefore, in this case, the replacement of the boundary after deformation by that before deformation leads to great deviation. In order to avoid it, we use the approximate boundary after deformation. The approximate boundary after deformation is the boundary before deformation plus the displacement. Naturally, the displacement is found for the body before deformation as in the theory of elasticity. On the basis of this consideration, the semipermeable boundary condition was suggested in [Hao and Shen 1994].

Appendix A. Jaumann rate increment, reference and current configurations

According to [Dorris and Nemat-Nasser 1980], for an incompressible body, the Jaumann rate is

$$\dot{\sigma}_{ab} = DT_{ab}/Dt + (u_{a,c} - D'_{ac})T_{cb} - T_{ac}D'_{cb}, \qquad D'_{ab} = 0.5(u_{a,b} + u_{b,a})$$

where u_a is the velocity component, σ_{ab} is the Piola stress component and T_{ab} is the Cauchy stress component. For convenience, only the hypo-elastic solid is dealt with

$$\dot{\sigma}_{ba} = (2\mu D'_{ab} + \dot{P}\delta_{ab}) + (u_{a,c} - D'_{ac})T_{cb} - T_{ac}D'_{cb}$$

Using the variables $\delta\sigma_{ba}$ and D_{ab} from (1) to replace $\dot{\sigma}_{ba}$ and D'_{ab} from A, we obtain

$$\delta\sigma_{ba} = (2\mu D_{ab} + P\delta_{ab}) + (v_{a,c} - D_{ac})T_{cb} - T_{ac}D_{cb}$$

where $D_{ab} = 0.5(v_{a,b} + v_{b,a})$, v_a is the increment displacement component and $D_{\theta\theta} = v_2/x_2$. There are two configurations in our work. The configuration of the body after deformation is the current configuration. That before deformation can be the reference configuration.

Appendix B. Derivation of (11) and (13)

$$\delta\sigma_{22,2} + (\delta\sigma_{22} - \delta\sigma_{\theta\theta})/x_2 + \delta\sigma_{21,1} = 0$$

$$2\mu v_{2,22} + 2\mu^2 S v_{1,12} + P_{,2} + (1/x_2)2\mu(v_{2,2} - v_2/x_2) + (2\mu - T)(v_{2,11} + v_{1,12})/2 + T v_{2,11} = 0$$

$$2\mu v_{2,221} + 2\mu^2 S v_{1,121} + P_{,21} + (1/x_2)2\mu(v_{2,21} - v_{2,1}/x_2) + (2\mu - T)(v_{2,111} + v_{1,121})/2 + T v_{2,111} = 0$$

$$\delta\sigma_{11,1} + \delta\sigma_{12,2} + \delta\sigma_{12}/x_2 = 0$$

$$(2\mu - 4\mu^2 S - T)v_{1,11} + P_{,1} + (2\mu - T)(v_{2,12} + v_{1,22})/2 + (2\mu - T)(v_{2,1} + v_{1,2})/2x_2 = 0$$

In order to consider the incompressible equation $v_{1,1} + v_{2,2} + v_2/x_2 = 0$, which can become $v_{1,11} + v_{2,12} + v_{2,1}/x_2 = 0$, we have

$$(2\mu - 8\mu^2 S - T)v_{1,11}/2 + P_{,1} + (2\mu - T)[(v_{2,12} + v_{1,11} + v_{1,22})/2 + (v_{2,1} + v_{1,2})/2x_2] = 0 \quad (34)$$

$$(2\mu - 8\mu^2 S - T)v_{1,11}/2 + P_{,1} + (2\mu - T)(v_{1,22} + v_{1,2}/x_2)/2 = 0 \quad (35)$$

$$(2\mu - 8\mu^2 S - T)v_{1,112}/2 + P_{,12} + (2\mu - T)(v_{1,222} + v_{1,22}/x_2 - v_{1,2}/x_2^2)/2 = 0 \quad (36)$$

Appendix C. Derivation of (16), $\delta\sigma_{11}$ and v_1

Letting $v_2 = F_{11}$, from the incompressible equation $v_{1,1} + v_{2,2} + v_2/x_2 = 0$, we have

$$\begin{aligned} v_{1,1} &= (-v_{2,2} - v_2/x_2) = (-F_2 - F/x_2)_{,11}, \quad v_1 = -(F_2 + F/x_2)_{,1} \\ v_{1,2} &= (-F_2 - F/x_2)_{,12} = -(F_{22} + F_2/x_2 - F/x_2^2)_{,1} = -L(F)_{,1}, \quad v_{1,12} = -L(F)_{,11}, \\ v_{1,112} &= -L(v_2)_1 = -L(F)_{111} \end{aligned} \quad (37)$$

We have

$$2\mu L(F)_{,111} - 6\mu^2 S L(F)_{111} + (2\mu + T)F_{,11111}/2 + (2\mu - T)L^2(F)_{,1}/2 = 0$$

$$(4\mu - 12\mu^2 S)L(F)_{11} + (2\mu + T)F_{1111} + (2\mu - T)L^2(F) = 0$$

$$\begin{aligned} \delta\sigma_{21} &= (2\mu - T)(v_{1,2} + v_{2,1})/2 + T v_{2,1} = (2\mu - T)[-(F_{,12} + F_{,1}/x_2)_{,2} + F_{,111}]/2 + T F_{,111} \\ &= (2\mu - T)[F_{,111} - L(F)_1]/2 + T F_{,111} = [(2\mu + T)F_{,111} - (2\mu - T)L(F)_1]/2 \end{aligned}$$

Considering $G(s, x_1) = N_1(s)e^{-r_1 s x_1} + N_2(s)e^{-r_2 s x_1}$ and $\int_0^\infty x_2 L(F) J_1(s x_2)\, dx_2 = -s^2 G(s, x_1)$, one obtains

$$\int_0^\infty x_2 \delta\sigma_{21} J_1(s x_2)\, dx_2 = (2\mu + T)G(s, x_1)_{,111}/2 + (2\mu - T)s^2 G(s, x_1)_{,1}/2$$

$$\delta\sigma_{21} = [(2\mu + T)/2] \int_0^\infty s\{G(s, x_1)_{111} + [(2\mu - T)/(2\mu + T)]s^2 G(s, x_1)_1\} J_1(s x_2)\, ds$$

Considering $G(s, x_1) = N_1(s)e^{-r_1 s x_1} + N_2(s)e^{-r_2 s x_1}$, $G(s, x_1)_1 = -[N_1(s) r_1 s e^{-r_1 s x_1} + N_2(s) r_2 s e^{-r_2 s x_1}]$, $G(s, x_1)_{11} = N_1(s) r_1^2 s^2 e^{-r_1 s x_1} + N_2(s) r_2^2 s^2 e^{-r_2 s x_1}$, $G(s, x_1)_{111} = -[N_1(s) r_1^3 s^3 e^{-r_1 s x_1} + N_2(s) r_2^3 s^3 e^{-r_2 s x_1}]$, one has

$$\delta\sigma_{21} = -[(2\mu+T)/2]\int_0^\infty s\{N_1(s)r_1^3 s^3 e^{-r_1 s x_1} + N_2(s)r_2^3 s^3 e^{-r_2 s x_1} + [(2\mu-T)/(2\mu+T)] s^2[N_1(s)r_1 s e^{-r_1 s x_1} + N_2(s)r_2 s e^{-r_2 s x_1}]\}J_1(s x_2)\,ds \quad (38)$$

If $x_1 = 0$, we get from (38)

$$\delta\sigma_{21}^{=} - [(2\mu+T)/2] \times \int_0^\infty s\{N_1(s)r_1^3 s^3 + N_2(s)r_2^3 s^3 + [(2\mu-T)/(2\mu+T)]s^2[N_1(s)r_1 s + N_2(s)r_2 s]\}J_1(s x_2)\,ds$$

Considering $x_1 = 0$, $\delta\sigma_{21} = 0$, one has

$$N_1(s)r_1^3 + N_2(s)r_2^3 + [(2\mu-T)/(2\mu+T)][N_1(s)r_1 + N_2(s)r_2]\} = 0$$

$$N_1(s)r_1^3 + [(2\mu-T)/(2\mu+T)]N_1(s)r_1 + N_2(s)r_2^3 + [(2\mu-T)/(2\mu+T)]N_2(s)r_2 = 0$$

$$N_1(s)[r_1^3 + (2\mu-T)/(2\mu+T)r_1] + N_2(s)[r_2^3 + (2\mu-T)/(2\mu+T)r_2] = 0$$

$$N_2(s) = -N_1(s)[r_1^3 + (2\mu-T)/(2\mu+T)r_1]/[r_2^3 + (2\mu-T)/(2\mu+T)r_2]$$

$$G(s,x_1) = N_1(s)\{e^{-r_1 s x_1} - e^{-r_2 s x_1}[r_1^3 + (2\mu-T)/(2\mu+T)r_1]/[r_2^3 + (2\mu-T)/(2\mu+T)r_2]\}$$

$$G(s,x_1) = N_1(s)(e^{-r_1 s x_1} - U e^{-r_2 s x_1})$$

where

$$U = \frac{r_1^3 + (2\mu-T)/(2\mu+T)r_1}{r_2^3 + (2\mu-T)/(2\mu+T)r_2} = (r_1/r_2)\frac{r_1^2 + (2\mu-T)/(2\mu+T)}{r_2^2 + (2\mu-T)/(2\mu+T)} = (r_1/r_2)\frac{r_1^2+\beta}{r_2^2+\beta} \quad (39)$$

and $\beta = (2\mu-T)/(2\mu+T) = r_2^2 r_1^2$.

In view of (35), one has $(2\mu - 8\mu^2 S - T)v_{1,11}/2 + P_{,1} + (2\mu-T)(v_{1,22} + v_{1,2}/r)/2 = 0$. Considering $v_{1=} - (F_{,2} + F/x_2)_{,1}$, one obtains

$$-(2\mu - 8\mu^2 S - T)(F_{,2} + F/x_2)_{,111}/2 + P_{,1} - (2\mu-T)[(F_{,2}+F/x_2)_{,221} + (F_{,2}+F/x_2)_{,21}/x_2]/2 = 0$$

$$P = (2\mu - 8\mu^2 S - T)(F_{,2} + F/x_2)_{,11}/2 + (2\mu-T)[(F_{,2}+F/x_2)_{,22} + (F_{,2}+F/x_2)_{,2}/x_2]/2$$

$$P = (2\mu - 8\mu^2 S - T)(F_{,2} + F/x_2)_{,11}/2 + (2\mu-T)[L(F)_{,2} + L(F)/x_2]/2$$

In view of the (6), one has

$$\begin{aligned}
\delta\sigma_{11} &= (2\mu - 4\mu^2 S - T)_{v_{1,1}} + P \\
&= -(2\mu - 4\mu^2 S - T)(F_{,2} + F/x_2)_{,11} + (2\mu - 8\mu^2 S - T)(F_{,2}+F/x_2)_{,11}/2 \\
&\qquad + (2\mu-T)[L(F)_{,2} + L(F)/r]/2 \\
&= -(2\mu-T)(F_{,2}+F/x_2)_{,11}/2 + (2\mu-T)[L(F)_{,2} + L(F)/r]/2 \\
&= (2\mu-T)[-(F_{,2}+F/x_2)_{,11} + L(F)_{,2} + L(F)/x_2]/2 \\
\delta\sigma_{11,2} &= (2\mu-T)[-(F_{,2}+F/x_2)_{,211} + L(F)_{,22} + L(F)_{,2}/x_2 - L(F)/x_2^2]/2 \\
&= (2\mu-T)[-L(F)_{,11}/2 + L^2(F)]/2
\end{aligned}$$

Now, we discuss the order-one Hankel transform of the preceding quantity.

$$\begin{aligned}\int_0^\infty x_2 J_1(sx_2)\delta\sigma_{11,2}\,dx_2\\ &= (2\mu - T)\int_0^\infty x_2 J_1(sx_2)[-L(F)_{,11}/2 + L^2(F)]/2\,dx_2\\ &= (2\mu - T)\,\frac{1}{2}\left[-\int_0^\infty x_2 J_1(sx_2)L(F)_{,11}\,dx_2 + \int_0^\infty x_2 J_1(sx_2)L^2(F)\,dx_2\right]\\ &= (2\mu - T)[s^2 G_{11} + s^4 G]/2 = (2\mu - T)s^4 N_1(s)[(r_1^2+1)e^{-r_1 s x_1} - U(r_2^2+1)e^{-r_2 s x_1}]\end{aligned}$$

Its Hankel retransform is

$$\delta\sigma_{11,2} = (2\mu - T)\int_0^\infty sJ_1(sx_2)s^4 N_1(s)[(r_1^2+1)e^{-r_1 s x_1} - U(r_2^2+1)e^{-r_2 s x_1}]\,ds/2 \tag{40}$$

For $x_1 = 0$, from (40), one has

$$\delta\sigma_{11,2} = (2\mu - T)[(r_1^2+1) - U(r_2^2+1)]\int_0^\infty sJ_1(sx_2)s^4 N_1(s)\,ds/2$$

$$\delta\sigma_{11} = (2\mu - T)[(r_1^2+1) - U(r_2^2+1)]\int_0^\infty s\int J_1(sx_2)\,dx_2 s^4 N_1(s)\,ds/2$$

Considering $dJ_0(u)/du = -J_1(u)$, one has

$$\delta\sigma_{11} = (2\mu - T)[U(r_2^2+1) - (r_1^2+1)]\int_0^\infty sJ_0(sx_2)s^3 N_1(s)\,ds/2$$

According to (37), one has

$$v_{1,2} = -(F_{,2} + F/x_2)_{,12} = -L(F)_{,1} \tag{41}$$

$$\int_0^\infty x_2 J_1(\xi x_2)v_{1,2}\,dx_2 = -\int_0^\infty x_2 J_1(\xi x_2)L(F)_{,1}\,dx_2 = s^2\Big[\int_0^\infty x_2 J_1(\xi x_2)F\,dx_2\Big]_{,1} = s^2 G_{,1} \tag{42}$$

$$v_{1,2} = -\int_0^\infty sJ_1(sx_2)s^3 N_1(s)(r_1 e^{-r_1 s x_1} - Ur_2 e^{-r_2 s x_1})\,ds \tag{43}$$

$$v_{1,2} = -\int_0^\infty sJ_1(sx_2)s^3 N_1(s)(r_1 - Ur_2)\,ds \quad v_1 = -\int_0^\infty s\int J_1(sx_2)\,dx_2 s^3 N_1(s)(r_1 - Ur_2)\,ds \tag{44}$$

$$v_1 = (r_1 - Ur_2)\int_0^\infty sJ_0(sx_2)s^2 N_1(s)\,ds \tag{45}$$

Appendix D. The circular cone

Letting $a^4[v_1(x_2)]_{x_1=0} = a^4[b + a\cot\alpha(1-\rho)] = (r_1 - Ur_2)g(\rho) = (r_1 - Ur_2)(A_0 + A_1\rho)$, one has

$$A_0 = (b + a\cot\alpha)a^4/(r_1 - Ur_2), \qquad A_1 = -a^5\cot\alpha/(r_1 - Ur_2) \qquad \rho = x_2/a$$

where α is the angle of the circular cone (the angle between the axis of symmetry and the generatrix).

Considering (29), one obtains from [Gradshteyn and Ryzhik 1965]

$$f(p) = 2(A_0/\pi + A_1/2)\sin p/p + A_1(\cos p - 1)/p^2$$

As $p^3 N_1 = f(p)$ and $(sa) = p$, one has

$$p^3 N_1 = 2(A_0/\pi + A_1/2)\sin p/p + A_1(\cos p - 1)/p^2 \tag{46}$$

$$\delta\sigma_{11} = (2\mu - T)[U(r_2^2+1) - (r_1^2+1)]\int_0^\infty s J_0(sx_2)s^3 N_1(s)\,ds/2 \tag{47}$$

$$\delta\sigma_{11} = (2\mu - T)[U(r_2^2+1) - (r_1^2+1)]\int_0^\infty s J_0(sx_2)\left[(2A_0/\pi + A_1)\frac{\sin(sa)}{sa} + A_1\frac{\cos(sa)-1}{(sa)^2}\right]ds/2 \tag{48}$$

As the integral $\int_0^\infty J_0(p)\sin p\,dp$ is divergence, for the finiteness of stress component $\delta\sigma_{11}$ at the edge of the punch, we have $(2A_0/\pi + A_1) = 0$, that is, $b = a\cot\alpha(\pi/2 - 1)$. Hence

$$v(x_2)_{x_1=x_2=0} = b + a\cot\alpha = 0.5\pi a\cot\alpha$$

Because $\int_0^\infty J_0(pp)(\cos p - 1)/p\,dp = -\cosh^{-1}(1/\rho)$, one has

$$\delta\sigma_{11} = (2\mu - T)[U(r_2^2+1) - (r_1^2+1)]A_1 a^{-5}\int_0^\infty sa J_0(sax_2/a)[(\cos(sa)-1)/(sa)^2]\,dsa/2 \tag{49}$$

$$\delta\sigma_{11} = -(2\mu - T)[U(r_2^2+1) - (r_1^2+1)]a^{-5}A_1\cosh^{-1}(a/x_2)/2 \tag{50}$$

As $\int_0^a[\cosh^{-1}(a/x_2)x_2]\,dx_2 = 0.5a^2$ and $A_1 = -a^5\cot\alpha/(r_1 - Ur_2)$, one obtains for the compressive force R on the cone

$$\begin{aligned} R &= -2\pi\int_0^a[\delta\sigma_{11}]_{x_1=o}x_2\,dx_2 = 2\pi(2\mu - T)[U(r_2^2+1) - (r_1^2+1)]a^{-5}A_1\int_0^a[\cosh^{-1}(a/x_2)x_2]\,dx_2/2 \\ &= \pi(2\mu - T)[U(r_2^2+1) - (r_1^2+1)]a^{-3}A_1/2 = \pi(2\mu - T)[(r_1^2+1) - U(r_2^2+1)]\,a^2\frac{\cot\alpha}{2r_1 - 2Ur_2} \end{aligned}$$

From (39), we know that $U = (r_1/r_2)\dfrac{r_1^2+\beta}{r_2^2+\beta}$, where $\beta = (2\mu - T)/(2\mu + T) = r_2^2 r_1^2$. Then, one has

$$(r_1 - Ur_2) = r_1\{1 - [r_1^2 + r_2^2 r_1^2]/[r_2^2 + r_2^2 r_1^2]\} = r_1\{r_2^2 - r_1^2\}/[r_2^2 + r_2^2 r_1^2] \tag{51}$$

and

$$\begin{aligned} (r_1^2 + 1 - Ur_2^2 - U) &= \{r_1^2 + 1 - (r_1 r_2)[r_1^2 + r_2^2 r_1^2]/[r_2^2 + r_2^2 r_1^2] - (r_1/r_2)[r_1^2 + r_2^2 r_1^2]/[r_2^2 + r_2^2 r_1^2]\} \\ &= \frac{(r_1^2+1)[r_2^2 + r_2^2 r_1^2] - (r_1 r_2)[r_1^2 + r_2^2 r_1^2] - (r_1/r_2)[r_1^2 + r_2^2 r_1^2]}{r_2^2 + r_2^2 r_1^2} \\ &= \frac{r_1^2 r_2^2 + r_2^2 r_1^2 - (r_1 r_2)[r_1^2 + r_2^2 r_1^2]}{r_2^2 + r_2^2 r_1^2} + \frac{[r_2^2 + r_2^2 r_1^2] - (r_1/r_2)[r_1^2 + r_2^2 r_1^2]}{r_2^2 + r_2^2 r_1^2} \\ &= \frac{r_1[r_1 r_2(r_2 - r_1) + (r_1 - r_2)r_2^2 r_1^2] + [r_2^2 - (r_1/r_2)r_1^2] + r_2^2 r_1^2 - (r_1/r_2)r_2^2 r_1^2}{r_2^2 + r_2^2 r_1^2} \\ &= (r_2 - r_1)\{r_1[r_1 r_2 - r_2^2 r_1^2] + [r_2^2 + r_1 r_2 + r_1^2]/r_2 + r_2^2 r_1^2/r_2\}/[r_2^2 + r_2^2 r_1^2] \\ &= (r_2 - r_1)r_2 r_1 r_1 r_2 - r_2^2 r_1^2 + [r_2^2 + r_1 r_2 + r_1^2] + r_2^2 r_1^2\}/[(r_2^2 + r_2^2 r_1^2)r_2] \end{aligned} \tag{52}$$

Considering the equations (51) and (52), one knows

$$\frac{(r_1^2+1)-U(r_2^2+1)}{r_1-Ur_2}=\frac{(r_2-r_1)[r_2r_1(r_1r_2-r_2^2r_1^2)+(r_2^2+r_1r_2+r_1^2)+r_2^2r_1^2]/[(r_2^2+r_2^2r_1^2)r_2]}{r_1\{r_2^2-r_1^2\}/(r_2^2+r_2^2r_1^2)}$$
$$=\{-r_2^3r_1^3+(r_2^2+r_1r_2+r_1^2)+2r_2^2r_1^2\}/r_2r_1(r_2+r_1)$$
$$=\{-r_2^3r_1^3+[r_2^2+r_1r_2+r_1^2]+2r_2^2r_1^2\}/[r_2r_1(r_2^2+2r_1r_2+r_1^2)^{1/2}]$$

Considering r_1 and r_2 are the two positive roots of the equation $p_1+p_2m^2/s^2+p_3m^4/s^4=0$, where $p_3=(1+Q)$, $p_2=-(2-6\mu S)$, $p_1=(1-Q)$ and $Q=T/2\mu$, one has

$$r_2r_1=(p_1/p_3)^{1/2}=(1-Q)^{1/2}/(1+Q)^{1/2}=(2\mu-T)^{1/2}/(2\mu+T)^{1/2},$$
$$r_2^2+r_1^2=-p_2/p_3=(2-6\mu S)/(1+Q)=(2\mu-6\mu^2S)/(2\mu+T)$$

where S can be found from (3). Then, one obtains (30). The contact radius a is then given by (31) and the penetration depth by (32). Finally, from the expression for a and the values $(r_2^2+r_1^2)=-p_2/p_3$, $r_1r_2=(p_1/p_3)^{1/2}$, $\beta=(p_1/p_3)$ and $T/2\mu=Q$, one obtains

$$\pi a^2=(R/2\mu)\{(p_1/p_3)^{1/2}[2(p_1/p_3)^{1/2}-p_2/p_3]^{1/2}\}$$
$$/\{(1-Q)[(p_1/p_3-p_1^{3/2}/p_3^{3/2})+(p_1/p_3)^{1/2}-p_2/p_3+p_1/p_3]\cot\alpha\} \quad (53)$$

Substituting $-p_2/p_3=+2(1-3\mu S)/(1+Q)$, $(p_1/p_3)=(1-Q)/(1+Q)$, $(p_1/p_3)^{1/2}=(1-Q)^{1/2}/(1+Q)^{1/2}$ into (53), one has

$$\pi a^2=(R/2\mu)tg\alpha\{[1/(1-Q^2)^{1/2}][2(1-Q)^{1/2}/(1+Q)^{1/2}+2(1-3\mu S)/(1+Q)]^{1/2}\}$$
$$/\{(1-Q)^{1/2}/(1+Q)^{1/2}[(1-Q)^{1/2}/(1+Q)^{1/2}-(1-Q)/(1+Q)]$$
$$+[(1-Q)^{1/2}/(1+Q)^{1/2}+2(1-3\mu S)/(1+Q)]+(1-Q)/(1+Q)\} \quad (54)$$

References

[Dorris and Nemat-Nasser 1980] J. F. Dorris and S. Nemat-Nasser, "Instability of a layer on a half space", *J. Appl. Mech. (ASME)* **47** (1980), 304–312.

[Gradshteyn and Ryzhik 1965] I. S. Gradshteyn and I. M. Ryzhik, *Table of integrals, series and products*, Scripta Technica, Academic Press, New York, 1965.

[Hao 2006] T. H. Hao, "Axially symmetrical contact problem of finite elasticity and its application to estimating residual stresses by cone indentation", *J. Mech. Mater. Struct.* **1**:8 (2006), 1367–1384.

[Hao and Shen 1994] T. H. Hao and Z. Y. Shen, "A new electric boundary condition of electric fracture mechanics and its application", *Eng. Fract. Mech.* **47**:6 (1994), 793–802.

[Sneddon 1951] I. N. Sneddon, *Fourier transforms*, McGraw-Hill, New York, 1951.

[Suresh and Giannakopoulos 1998] S. Suresh and A. E. Giannakopoulos, "A new method for estimating residual stresses by instrumented sharp indentation", *Acta Mater.* **46** (1998), 5755–5767.

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