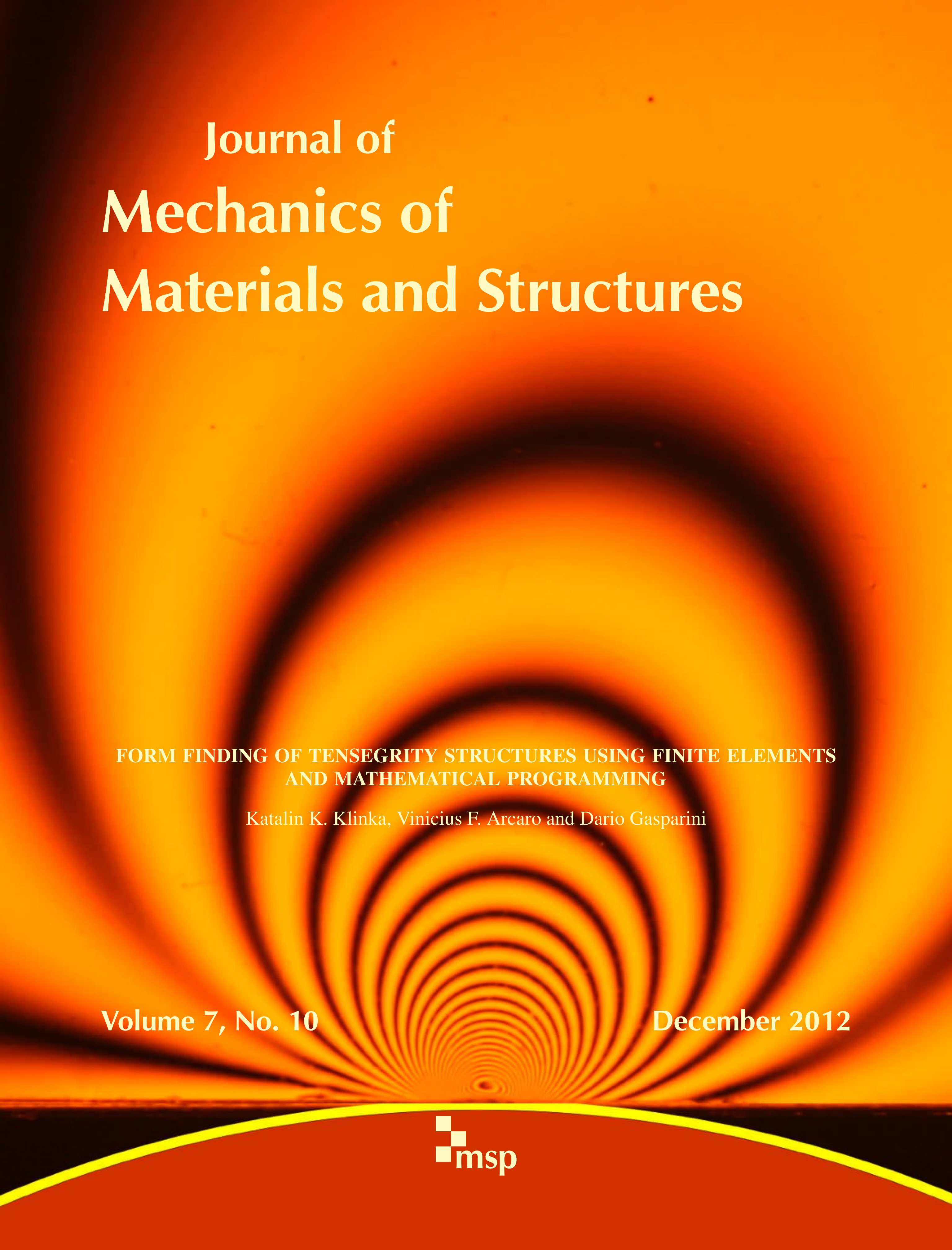


Journal of

Mechanics of Materials and Structures


# FORM FINDING OF TENSEGRITY STRUCTURES USING FINITE ELEMENTS AND MATHEMATICAL PROGRAMMING


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Volume 7, No. 10 December 2012

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JOURNAL OF MECHANICS OF MATERIALS AND STRUCTURES
Vol. 7, No. 10, 2012
dx.doi.org/10.2140/jomms.2012.7.899

# FORM FINDING OF TENSEGRITY STRUCTURES USING FINITE ELEMENTS AND MATHEMATICAL PROGRAMMING

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We show that the minimization of total potential energy is the general principle behind the well-known rule of maximizing some lengths of a truss mechanism to define a tensegrity. Moreover, the latter rule is a special case, due to the usual high values of the modulus of elasticity. An innovative mathematical model is presented for finding the form of tensegrity structures, based on the finite element method and on mathematical programming. A special line element that shows constant stress for any displacement of its nodes is used to define a prestressed equilibrium configuration. Form finding is formulated as an unconstrained nonlinear programming problem, where the objective function is the total potential energy and the displacements of the nodal points are the unknowns. A connection is made with the geometric shape minimization problem, defined by a constrained nonlinear programming problem. A quasi-Newton method is used, which avoids the evaluation of the tangent stiffness matrix.

## 1. Introduction

Maxwell [1864] wrote: "In those cases in which stiffness can be produced with a smaller number of lines, certain conditions must be fulfilled, rendering the case one of a maximum or minimum value of one or more of its lines. The stiffness of such frames is of an inferior order, as a small disturbing force may produce a displacement infinite in comparison with itself". In [Calladine 1978], the author who made the connection between tensegrity structures and the exceptions to Maxwell's rule writes that presumably Maxwell intended to refer to a maximum or minimum value of the length of one or more of its lines. An explanation for Maxwell's obscure remark about maximum or minimum values based on the principle of minimum total potential energy is presented. A review of the important literature related to form finding methods for tensegrity structures is given in [Tibert and Pellegrino 2003] and more recently in [Hernández Juan and Mirats Tur 2008]. These methods can be classified into kinematical and statical methods. This text concentrates on the total potential energy minimization method for form finding. A special line element that shows constant stress for any displacement of its nodes is used to define a prestressed equilibrium configuration. The form finding is formulated as an unconstrained nonlinear programming problem, where the objective function is the total potential energy and the displacements of the nodal points are the unknowns. Another approach, which minimizes the total potential energy by modifying the lengths of selected elements, is described in [Pagitz and Miratz Tur 2009]. A quasi-Newton method is used, which avoids the evaluation of the tangent stiffness matrix. An interesting connection is made between minimizing the total potential energy, which is defined by an unconstrained nonlinear programming problem, and the geometric shape minimization problem, which is defined by a constrained nonlinear programming problem. The strain energy for a line element can be interpreted as

*Keywords:* cable, element, line, minimization, nonlinear, optimization, tensegrity.

a penalty function, as it imposes resistance for changing the length of the element. The total potential energy minimization method for the analysis of cable structures was first described in [Pietrzak 1978]. The following conventions apply unless otherwise specified or made clear by the context. A Greek letter expresses a scalar. A lower case letter represents a column vector.

## 2. Line element definition

Figure 1 shows the geometry of the element. The nodes are labeled 1 and 2; a superscript associates a variable with its corresponding node. The nodal displacements, given by vectors $u$, transform the element from its initial configuration to its final configuration. The strain is assumed constant along the element.

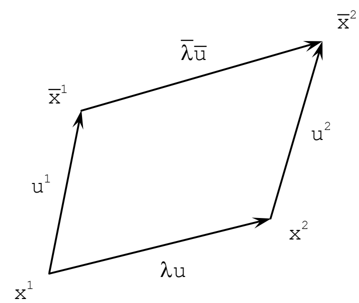



**Figure 1.** Line element.

## 3. Engineering strain

The vector $u$ is a unity vector. Note that $\lambda$ represents the undeformed length of the element. The nodal displacements vectors are numbered according to its node numbers. The deformed length can be written as follows:

$$z = \frac{u^2 - u^1}{\lambda} \qquad \Rightarrow \qquad \bar{\lambda}\bar{u} = \lambda u + \lambda z, \tag{1}$$

$$\delta = 2u^T z + z^T z \qquad \Rightarrow \qquad \bar{\lambda} = \lambda\sqrt{1+\delta}. \tag{2}$$

The unit vector parallel to the element in its final configuration can be written as

$$\bar{u} = \frac{u+z}{\sqrt{1+\delta}}. \tag{3}$$

The engineering strain can be written as

$$\epsilon = \frac{\bar{\lambda} - \lambda}{\lambda}. \tag{4}$$

Inaccuracy often results from severe cancellation that occurs when nearly equal values are subtracted [Goldberg 1991]. In order to avoid it, the previous expression should be evaluated as

$$\epsilon = \frac{\delta}{\sqrt{1+\delta}+1}. \tag{5}$$

## 4. Variable stress element

Considering $\sigma$ as the conjugate stress to the engineering strain $\epsilon$ and $\alpha$ as the undeformed area of the element, the potential strain energy and its gradient with respect to the nodal displacements can be written as

$$\phi = \alpha\lambda \int_0^{\epsilon} \sigma(\xi)\, d\xi, \tag{6}$$

$$\frac{\partial\phi}{\partial u_i^1} = -\alpha\sigma(\epsilon)\bar{u}_i, \quad \frac{\partial\phi}{\partial u_i^2} = +\alpha\sigma(\epsilon)\bar{u}_i. \tag{7}$$

The gradient can be interpreted as internal forces acting on nodes of the element.

***Stress and strain.*** Consider stress as a linear function of strain with $E$ as the modulus of elasticity. The potential strain energy can be written as

$$\phi = \frac{\alpha E(\bar{\lambda}-\lambda)^2}{2\lambda}. \tag{8}$$

The strain energy can be interpreted as a penalty function with the modulus of elasticity as the penalty parameter. The modulus of elasticity, which is usually large, imposes resistance to changing the length of the elements.

## 5. Constant stress element

A constant stress element can be defined by imposing a constant stress $\sigma$. The potential strain energy can be written as

$$\phi = \alpha\lambda \int_0^{\epsilon} \sigma\, d\xi = \alpha\sigma(\bar{\lambda}-\lambda). \tag{9}$$

The potential strain energy is equal to the force multiplied by the relative displacement between the nodes. In the expression for the strain energy, the undeformed length can be eliminated because it does not depend on the nodal displacements. Its permanence in the expression would only add constants, one for each element, to the total potential strain energy function. To minimize a function plus a constant is equivalent to minimize the function only. Therefore, the potential strain energy can be replaced by

$$\phi = \alpha\sigma\bar{\lambda}. \tag{10}$$

The strain energy is simply the final length of the element multiplied by the imposed constant force. The gradient with respect to the nodal displacements can be written as

$$\frac{\partial\phi}{\partial u_i^1} = -\alpha\sigma\bar{u}_i, \quad \frac{\partial\phi}{\partial u_i^2} = +\alpha\sigma\bar{u}_i. \tag{11}$$

Note that the components of the gradient are given by a scalar (stress multiplied by area) multiplying the unit vector parallel to the element in its final configuration. The gradient can be interpreted as internal forces with constant modulus acting on nodes of the element. The element shows constant stress for any displacement of its nodes. A similar element was described in [Meek 1971]. The element was called variable initial length element.

## 6. Form finding

The initial configuration of a tensegrity structure is defined as the configuration of zero nodal displacements for all its nodes. A form finding strategy can be defined as: Starting with an initial configuration, select some elements as constant stress elements by specifying a stress value. Find the prestressed equilibrium configuration by minimizing the total potential strain energy.

## 7. Equilibrium configuration

Considering $C$ as the set of constant stress elements, $V$ as the set of variable stress elements and $u$ as the vector of unknown displacements, the total potential strain energy function and its gradient can be written as

$$\pi(u) = \sum_{e \in C} \phi_e + \sum_{e \in V} \phi_e, \tag{12}$$

$$\nabla\pi(u) = \sum_{e \in C} \nabla\phi_e + \sum_{e \in V} \nabla\phi_e. \tag{13}$$

The stable equilibrium configurations correspond to local minimum points of the total potential energy function, which in the absence of external forces reduces to the total potential strain energy function. In order to find the local minimum points of a nonlinear multivariate function, the general strategy that can be used is: Choose a starting point and move in a given direction such that the function decreases. Find the minimum point in this direction and use it as a new starting point. Continue this way until a local minimum point is reached. The problem of finding the minimum points of a nonlinear multivariate function is replaced by a sequence of sub problems, each one consisting of finding the minimum of a univariate nonlinear function. In the quasi-Newton methods, starting with the unit matrix, a positive definite approximation to the inverse of the Hessian matrix is updated at each iteration. This update is made using only values of the gradient vector. A direction such that the function decreases is calculated as minus the product of this approximation of the inverse of the Hessian matrix and the gradient vector calculated at the starting point of each iteration. Consequently, it is not necessary to solve any system of equations. Moreover, the analytical derivation of an expression for the Hessian matrix is not necessary. Note that by minimizing the total potential energy function it is almost impossible to find an unstable equilibrium configuration, which corresponds to a local maximum point. The only exception is that it is possible to find a saddle point, that is, the point is a local minimum and also a local maximum. However, even in this improbable situation, a direction of negative curvature to continue toward a local minimum point can be found as described in [Gill and Murray 1974]. It is important to emphasize that minimizing total potential energy to find equilibrium configurations does not require support constraints to prevent rigid body motion. The computer code uses the limited memory BFGS to tackle large-scale problems as described in [Nocedal and Wright 2006]. It also employs a line search procedure through cubic interpolation [loc. cit.].

**7.1.** ***Geometrical shape minimization.*** Due to the fact that the modulus of elasticity is usually a big number, the problem of minimizing the total potential strain energy can be interpreted as an equality constrained nonlinear programming problem converted to an unconstrained nonlinear programming problem by the quadratic penalty method. This interpretation leads to an extension of the mathematical model for geometric shape minimization described in [Arcaro and Klinka 2009].

Special case 1: A structure with the same modulus of elasticity for all elements, area equal to 1 for all elements and stress equal to 1 (tension) for all constant stress elements. Minimizing the total potential strain energy can be interpreted as minimizing the sum of the lengths of the constant stress elements while keeping the lengths of the variable stress elements.

$$\operatorname{Min} \pi(u) = +\sum_{e\in C} \bar{\lambda} + \frac{E}{2} \sum_{e\in V} \frac{(\bar{\lambda}-\lambda)^2}{\lambda} \tag{14}$$

Special case 2: A structure with the same modulus of elasticity for all elements, area equal to 1 for all elements and stress equal to $-1$ (compression) for all constant stress elements. Minimizing the total potential strain energy can be interpreted as maximizing the sum of the lengths of the constant stress elements while keeping the lengths of the variable stress elements.

$$\operatorname{Min} \pi(u) = -\sum_{e\in C} \bar{\lambda} + \frac{E}{2} \sum_{e\in V} \frac{(\bar{\lambda}-\lambda)^2}{\lambda} \tag{15}$$

## 8. Examples

Elements shown in red are in compression. Elements shown in blue are in tension. Constant stress elements are shown in green in the initial configuration.

**Example 1.** A straight prismoid with height = 3. The bottom and top regular triangles are inscribed in a circle of radius = 1. It is composed by 3 constant stress elements and 9 variable stress elements.

Special case 1: Figure 2 shows the initial shape on the left, the final shape with $E = 1000$ on the center and the final shape with $E = 10$ on the right. The constant stress elements are shown in blue in the final configuration. The top triangle rotates 150 degrees clockwise relative to the bottom triangle.

Special case 2: Figure 3 shows the initial shape on the left, the final shape with $E = 1000$ on the center and the final shape with $E = 10$ on the right. The constant stress elements are shown in red in the final configuration. The top triangle rotates 30 degrees counterclockwise relative to the bottom triangle.

Table 1 shows the lengths of the constant stress elements in the initial and final configurations.

**Example 2.** Figure 4 shows the geometry of a sculpture called the stella octangula, which was proposed by Hungarian architect, sculptor and author David Georges Emmerich. An extensive description of his works is given in [Chassagnoux 2006]. An analysis of this structure, using the dynamic relaxation method, is described in [Motro 2011]. Recently, a modified dynamic relaxation algorithm for the analysis of tensegrity structures was proposed in [Ali et al. 2011].

| | Initial | $E = 1000$ | $E = 10$ |
|---|---|---|---|
| $\sigma = +1$ | 3.4641 | 2.3473 | 1.5184 |
| $\sigma = -1$ | 3.4641 | 3.5329 | 3.7782 |

**Table 1.** Initial and prestressed configurations for Example 1.

The geometry consists of 18 elements with length equal to $s$ and 6 diagonal elements with length equal to $s\sqrt{3}$. The connectivity of the diagonal elements is also shown in Figure 4. The coordinates of the vertices appear in Table 2, where we have set

$$r = \frac{s}{\sqrt{3}}, \quad h = \frac{s}{\sqrt{6}}. \tag{16}$$

A stella octangula with parameter $s = 1$, $E = 1000$ and all elements with area = 1. There are support constraints on nodes 1, 2 and 3 to prevent rigid body motion. According the definition given in [Zhang et al. 2006], a regular tensegrity is characterized by equal length for the elements in tension and by equal length for the elements in compression. A nonregular tensegrity can be generated by imposing different

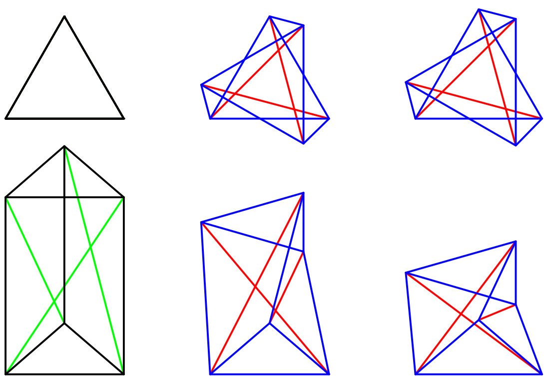


**Figure 2.** Initial and prestressed configurations for Example 1, special case 1.

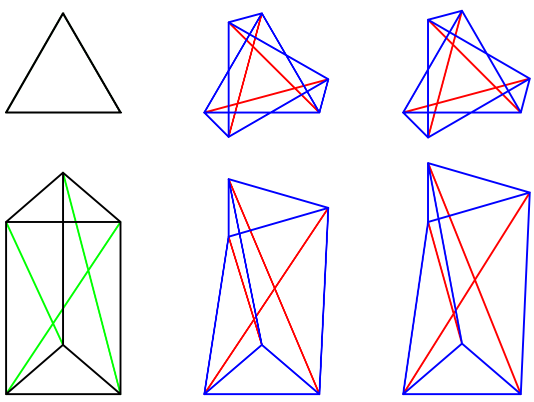


**Figure 3.** Initial and prestressed configurations for Example 1, special case 2.

| Node | Coord-$X$ | Coord-$Y$ | Coord-$Z$ |
|---|---|---|---|
| 1 | $-s/2$ | $-r/2$ | $h$ |
| 2 | $s/2$ | $-r/2$ | $h$ |
| 3 | 0 | $r$ | $h$ |
| 4 | 0 | $-2r$ | $h$ |
| 5 | $s$ | $r$ | $h$ |
| 6 | $-s$ | $r$ | $h$ |
| 7 | $s$ | $-r$ | $-h$ |
| 8 | $-s$ | $-r$ | $-h$ |
| 9 | 0 | $2r$ | $-h$ |
| 10 | 0 | $-r$ | $-h$ |
| 11 | $-s/2$ | $r/2$ | $-h$ |
| 12 | $s/2$ | $r/2$ | $-h$ |

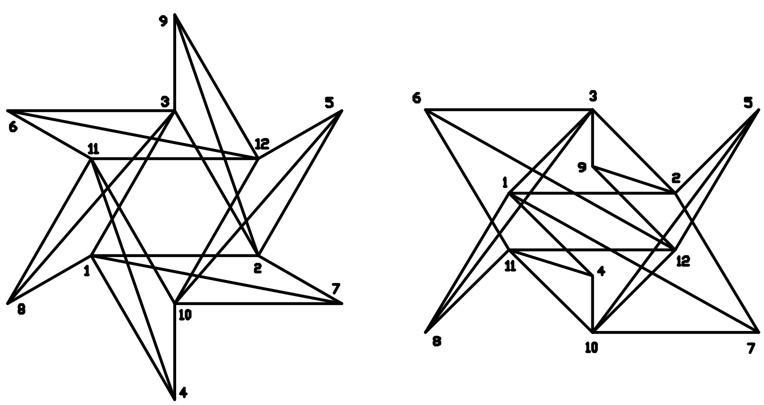
**Table 2.** Coordinates of the vertices.

stress values for selected elements of a regular tensegrity. The regular tensegrity can be recovered by imposing equal stress values for the same selected elements on the previously generated nonregular tensegrity. Another approach to generate a nonregular tensegrity, which is based on the dynamic relaxation method, is presented in [Tibert and Pellegrino 2003].

Generating the nonregular tensegrity: The stress values for the diagonal elements of the regular stella octangula and the lengths for the diagonal elements of its prestressed configuration are shown in the left half of Table 3.

Figure 5 shows the initial configuration (regular stella octangula) in the first column and its prestressed configuration (nonregular stella octangula) in the second column.

Recovering the regular tensegrity: We can also apply the procedure using as input the nonregular stella octangula obtained in the example immediately above. The stress values for the diagonal elements of the nonregular stella octangula and the lengths for the diagonal elements of its prestressed configuration are shown in the right half of Table 3.




| Elem | Node | Node |
|---|---|---|
| 3 | 4 | 11 |
| 6 | 5 | 10 |
| 9 | 6 | 12 |
| 12 | 7 | 1 |
| 15 | 8 | 3 |
| 18 | 9 | 2 |

**Figure 4.** The stella octangula: geometry and connectivity of diagonal elements.

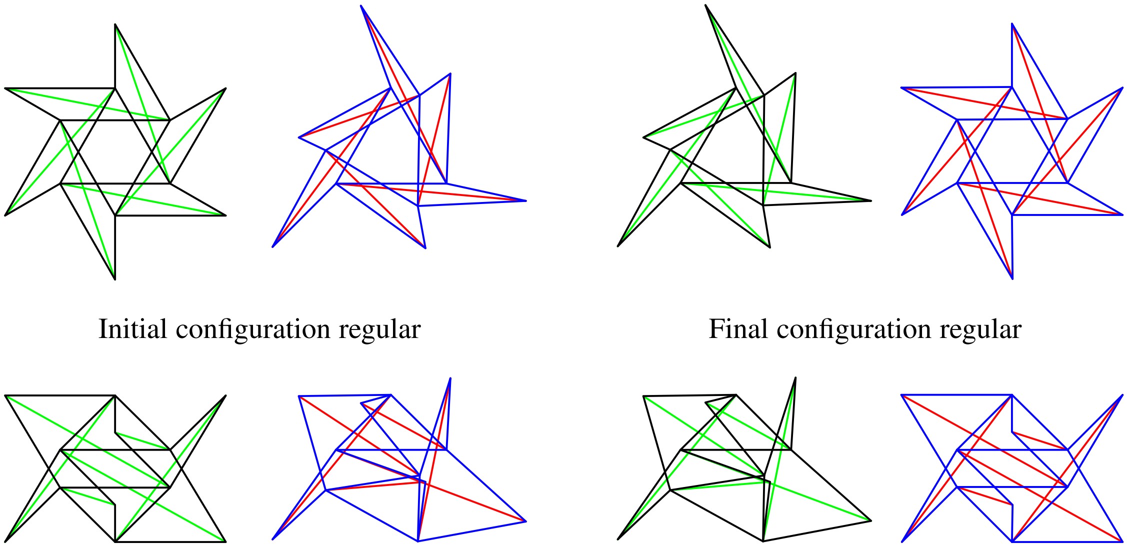



**Figure 5.** Stella octangula: Initial and prestressed configurations when the initial configuration is regular (left block), and when the final configuration is regular (right block).

Figure 5 shows the initial configuration (nonregular stella octangula) in the third column its prestressed configuration (regular stella octangula) in the last column.

## Conclusions

The principle of minimum total potential energy is a fundamental one in physics, and lies at the basis of the mathematical model presented here for form finding of tensegrity structures, using the finite element method and mathematical programming. The proposed approach can generate a nonregular tensegrity starting from a regular tensegrity. Additionally, the regular tensegrity can be recovered by imposing equal stresses on the previously selected constant stress elements of the nonregular tensegrity. The use of a quasi-Newton method to minimize the total potential energy function has several advantages over solving the equilibrium equations in nonlinear mechanics: It allows the analysis of under constrained structures even without support constraints to prevent rigid body motion. It is not necessary to derive the

| Elem | Stress | Length |
|---|---|---|
| 3 | $-1.25$ | 1.4573 |
| 6 | $-1.50$ | 1.5664 |
| 9 | $-1.75$ | 1.6312 |
| 12 | $-2.00$ | 1.8578 |
| 15 | $-2.25$ | 1.8899 |
| 18 | $-2.50$ | 1.8914 |

| Elem | Stress | Length |
|---|---|---|
| 3 | $-1.00$ | 1.7343 |
| 6 | $-1.00$ | 1.7345 |
| 9 | $-1.00$ | 1.7348 |
| 12 | $-1.00$ | 1.7351 |
| 15 | $-1.00$ | 1.7353 |
| 18 | $-1.00$ | 1.7357 |

**Table 3.** Nonregular (left) and regular (right) stella octangula.

tangent stiffness matrix. It is not necessary to solve any system of equations. It can handle large-scale problems with efficiency.

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Received 3 Nov 2011. Revised 26 Oct 2012. Accepted 30 Oct 2012.

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Journal of Mechanics of Materials and Structures

Volume 7, No. 10 December 2012



1559-3959(2012)7:10;1-#