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BUCKLING INSTABILITIES OF ELASTICALLY CONNECTED TIMOSHENKO BEAMS ON AN ELASTIC LAYER SUBJECTED TO AXIAL FORCES

VLADIMIR STOJANOVIĆ, PREDRAG KOZIĆ AND GORAN JANEVSKI

We study the buckling instability of a system of three simply supported elastic Timoshenko beams, joined together by Winkler elastic layers, and each subjected to the same compressive axial load. The model of the Timoshenko beam includes the effects of axial loading, shear deformation, and rotary inertia. Explicit analytical expressions are derived for the critical buckling load of single, double, and triple-beam systems. It can be observed from these expressions that the critical buckling load depends on the Winkler elastic layer stiffness modulus K, and that the instability of the system increases with an increase in the numbers of beams and elastic layers. These results are of considerable practical interest and have wide application in engineering practice.

1. Introduction

Vibration and buckling problems of beams and beam-columns on elastic layers occupy an important place in many fields of structural and layer engineering, occurring often in mechanical and civil engineering applications. Their solution demands modeling of the mechanical behavior of the beam, the mechanical behavior of the soil, and the form of the interaction between the beam and the soil.

As far as the beam is concerned, most engineering analyses are based on classical Bernoulli–Euler beam theory, in which straight lines or planes normal to the neutral beam axis remain straight and normal after deformation. This theory thus neglects the effect of transverse shear deformations, a condition that holds only in the case of slender beams. To confront this problem, the well-known Timoshenko beam model, in which the effect of transverse shear deflections is considered, can be used.

Matsunaga [1996] studied the buckling instabilities of a simply supported thick elastic beam subjected to axial stresses. Taking into account the effects of shear deformations and thickness changes, the buckling loads and buckling displacement modes of thick beams were obtained. Based on the power series expansion of displacement components, a set of fundamental equations of a one-dimensional higher-order beam theory was derived through the principle of virtual displacement. Several sets of truncated approximate theories were applied to solve the eigenvalue problems for a thick beam. The convergence properties of the buckling loads of a simply supported thick beam were examined in detail and comparison of the results with previously published ones was made.

On the basis of the Bernoulli–Euler beam theory, the properties of free transverse vibration and buckling of a double-beam system under compressive axial loading were investigated in [Zhang et al. 2008]. Explicit expressions were derived for the natural frequencies and the associated amplitude ratios of the two beams, and analytical solutions for the critical buckling load were obtained. The influence of the

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compressive axial loading on the response of the double-beam system was discussed. It was shown that the critical buckling load of the system was related to the axial compression ratio of the two beams and the Winkler elastic layer, and that the properties of free transverse vibration of the system greatly depended on the axial compressions.

Kelly and Srinivas [2009] investigated the problem concerning free vibration of a set of n axially loaded stretched Bernoulli–Euler beams connected by elastic layers and connected to a Winkler type layer. A normal-mode solution was applied to the governing partial differential equations to derive a set of coupled ordinary differential equations which were used to determine the natural frequencies and mode shapes. It was shown that the set of differential equations could be written in self-adjoint form with an appropriate inner product. An exact solution for the general case was obtained, but numerical procedures had to be used to determine the natural frequencies and mode shapes. The numerical procedure was difficult to apply, especially in determining higher frequencies. For the special case of identical beams, an exact expression for the natural frequencies was obtained in terms of the natural frequencies of a corresponding set of unstretched beams and the eigenvalues of the coupling matrix.

Stojanović et al. [2011] studied the influence of rotary inertia and shear on the free vibration and buckling of a double-beam system under axial loading. It was assumed that the system under consideration was composed of two parallel and homogeneous simply supported beams continuously joined by a Winkler elastic layer. Both beams had the same length. It was also supposed that the buckling could only occur in the plane where the double-beam system lay. Explicit expressions were derived for the natural frequencies and the associated amplitude ratio of the two beams, and the analytical solution of the critical buckling was obtained. The influence of the characteristics of the Winkler elastic layer on the natural frequencies and the critical buckling force was determined.

Li et al. [2008] analyzed an exact dynamic stiffness matrix which was established for an elastically connected three-beam system, composed of three parallel beams of uniform properties with uniformly distributed springs connecting them. The formulation included the effects of shear deformation and rotary inertia of the beams. The dynamic stiffness matrix was derived by rigorous use of the analytical solutions of the governing differential equations of motion of the three-beam system in free vibration. The use of the dynamic stiffness matrix to study the three vibration characteristics of the three-beam system was demonstrated by applying the Muller root-search algorithm.

De Rosa [1995] studied the free vibration frequencies of Timoshenko beams on a two-parameter elastic layer. Two variants of the equation of motion were deduced, in which the second-layer parameter was a function of the total rotation of the beam or a function of the rotation due to bending only.

Lazopoulos and Lazopoulos [2011], considering the influence of the microstructure, revisited the Timoshenko beam model, invoking Mindlin's strain gradient strain energy density function. The equations of motion were derived and the bending equilibrium equations were discussed. The solution of the static problem, for a simply supported beam loaded by a force at the middle of the beam, was defined and the first (least) eigenfrequency was found.

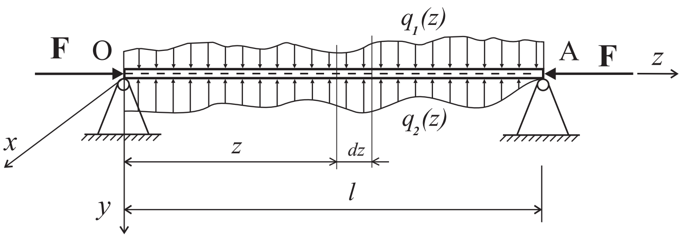
Miranda and Taghavi [2005] presented an approximate procedure to estimate floor acceleration demands in multistory buildings with the use of only a small number of parameters. Floor acceleration demands were computed using approximations of the first three modes of vibration of the building based on those of a continuum model consisting of a cantilever flexural beam connected laterally to a cantilever shear beam. The models had uniform stiffness along the height.

In this paper, the buckling instability of simply supported elastic Timoshenko beams, continuously joined by Winkler elastic layers, subjected to the same compressive axial load is studied. The beams have the same length l, and it is also supposed that the buckling can only occur in the plane where the system beams lie. The model of the Timoshenko beams includes the effects of axial loading, shear deformation, and rotary inertia. Explicit analytical expressions are determined for the critical buckling load of single, double, and triple-beam models. The critical buckling load for the triple-beam model is also determined using the trigonometric method. It can be observed from these expressions that the critical buckling load depends on the Winkler elastic layer stiffness modulus K, and that the instability of the system increases with an increase in the numbers of beams and elastic layers.

2. Formulation of the differential equations of the dynamic equilibrium and structural model

It can be seen that the literature on the dynamic analysis of elastically parallel-beam systems is concentrated primarily on the case of a double-beam system of two parallel simply supported beams continuously joined by a Winkler elastic layer. Very few research papers can be found that deal with the problem related to the elastically connected three-beam system. Those studies of this region are limited to the particular cases of identical beams with some prescribed boundary conditions. In most of these references, the simple Bernoulli–Euler beam theory has been used in deriving the necessary equation. Here, the basic differential equations of motion for the analysis will be deduced by considering a Timoshenko beam of length l (Figure 1a) subjected to an axial compressive force F, and to distributed lateral loads of intensity q_1 and q_2 which vary with the distance z along the beam. This will be applied on the basis of several assumptions:

- The behavior of the beam material is linear elastic.
- The cross-section is rigid and constant throughout the length of the beam and has one plane of symmetry.



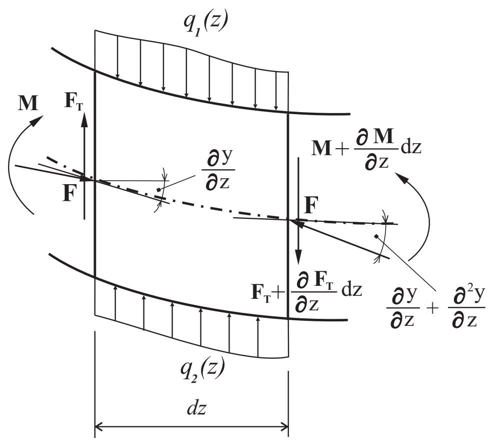


Figure 1. The physical model Timoshenko beam subjected to an axial compressive force F and to distributed lateral loads of intensity q_1 and q_2.

- Shear deformations of the cross-section of the beam are taken into account while elastic axial deformations are ignored.
- The equations are derived bearing in mind the geometric axial deformations.
- The axial forces F acting on the ends of the beam are not changed with time.

Consider the element of length dz between two cross-sections normal to the deflected axis of the beam (Figure 1b). Since the slope of the beam is small, the normal forces acting on the sides of the element can be taken as equal to the axial compressive force F. The shearing force F_T is related to the following relationship:

$$F_T = kGA\left(\frac{\partial w}{\partial z} - \psi\right), \tag{1}$$

where $w = w(z, t)$ is the displacement of a cross-section in the y-direction, $\partial w/\partial z$ is the global rotation of the cross-section, ψ is the bending rotation, G is the shear modulus, A is the area of the beam cross-section, and k is the shear factor. Analogously the relationship between bending moments M and bending angles $\psi = \psi(z, t)$ is given by

$$M = -EI_x \frac{\partial \psi}{\partial z}, \tag{2}$$

where E is the Young's modulus and I_x is the second moment of the area of the cross-section. Finally, the forces and moments of inertia are given by

$$f_I = -\rho A \frac{\partial^2 w}{\partial t^2}, \quad J_I = -\rho I_x \frac{\partial^2 \psi}{\partial t^2}, \tag{3}$$

respectively, where ρ is the mass density.

The forces acting on a differential layered-beam element are shown in Figure 1b. The dynamic-force equilibrium conditions of these forces are given by the following equations:

$$\rho A \frac{\partial^2 w}{\partial t^2} - kGA\left(\frac{\partial^2 w}{\partial z^2} - \frac{\partial \psi}{\partial z}\right) + F \frac{\partial^2 w}{\partial z^2} - q_1(z) + q_2(z) = 0, \tag{4a}$$

$$\rho I_x \frac{\partial^2 \psi}{\partial t^2} - EI_x \frac{\partial^2 \psi}{\partial z^2} - kGA\left(\frac{\partial w}{\partial z} - \psi\right) = 0. \tag{4b}$$

The development and solution of the differential equations of motion governing the free flexural vibrations of a system of three identical elastically connected beams, considering the effects of shear deformation and rotary inertia (Figure 2).

Each beam is made of material with a Young's modulus E and mass density ρ, and has a uniform cross-section of area A and moment of inertia $I = I_x$. Each beam is subjected to the same compressive axial loading. The first beam is connected to a Winkler layer of stiffness modulus K, and the second and third beams are also connected by a continuous linear elastic layer of Winkler type of the same stiffness modulus K. The transverse displacement of the beams is $w_i = w_i(z, t)$, $i = 1, 2, 3$, and $\psi_i = \psi_i(z, t)$, $i = 1, 2, 3$, are the bending rotations. If we apply the abovementioned procedure to a differential element

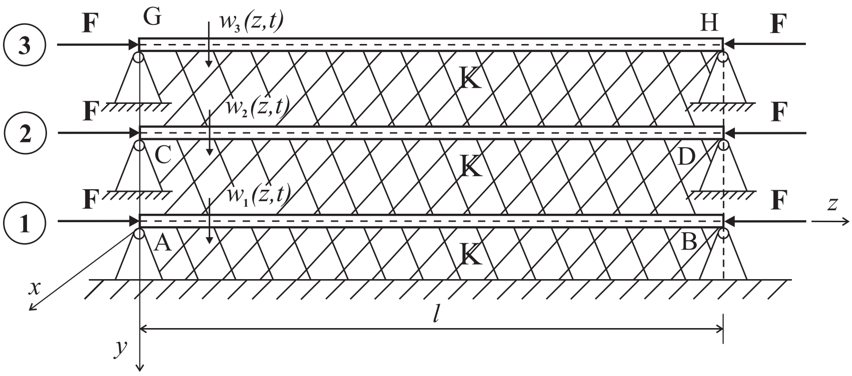


Figure 2. Three identical Timoshenko beams supported on a Winkler elastic layer.

of each beam, the following set of coupled differential equations will be obtained:

$$\begin{aligned}\rho A\frac{\partial^2 w_1}{\partial t^2}-kGA\left(\frac{\partial^2 w_1}{\partial z^2}-\frac{\partial \psi_1}{\partial z}\right)+F\frac{\partial^2 w_1}{\partial z^2}+2Kw_1-Kw_2=0,\\ \rho I\frac{\partial^2 \psi_1}{\partial t^2}-EI\frac{\partial^2 \psi_1}{\partial z^2}-kGA\left(\frac{\partial w_1}{\partial z}-\psi_1\right)=0,\end{aligned}\tag{5}$$

$$\begin{aligned}\rho A\frac{\partial^2 w_2}{\partial t^2}-kGA\left(\frac{\partial^2 w_2}{\partial z^2}-\frac{\partial \psi_2}{\partial z}\right)+F\frac{\partial^2 w_2}{\partial z^2}-Kw_1+2Kw_2-Kw_3=0,\\ \rho I\frac{\partial^2 \psi_2}{\partial t^2}-EI\frac{\partial^2 \psi_2}{\partial z^2}-kGA\left(\frac{\partial w_2}{\partial z}-\psi_2\right)=0,\end{aligned}\tag{6}$$

$$\begin{aligned}\rho A\frac{\partial^2 w_3}{\partial t^2}-kGA\left(\frac{\partial^2 w_3}{\partial z^2}-\frac{\partial \psi_3}{\partial z}\right)+F\frac{\partial^2 w_3}{\partial z^2}-Kw_2+Kw_3=0,\\ \rho I\frac{\partial^2 \psi_3}{\partial t^2}-EI\frac{\partial^2 \psi_3}{\partial z^2}-kGA\left(\frac{\partial w_3}{\partial z}-\psi_3\right)=0.\end{aligned}\tag{7}$$

3. The axial buckling load of the elastically connected identical three Timoshenko beams

The stability behavior of simply supported Timoshenko-beam systems on a Winkler elastic layer is of great interest to both practicing engineers and researchers. The usual approach to formulating this problem is to include the layer reaction in the corresponding differential equation of the beam. The buckling of an elastically connected simply supported Timoshenko beam under some static compressive axial load is investigated. The analytical solution for the critical buckling load of the system is derived. The second-order partial differential equations (5), (6), and (7) can be further reduced, by eliminating ψ_1, ψ_2, and ψ_3, respectively, to the following system of fourth-order partial differential equations:

$$\begin{aligned}EI\left(1-\frac{F}{kAG}\right)\frac{\partial^4 w_1}{\partial z^4}+\left(\rho A+2\frac{K\rho I}{kAG}\right)\frac{\partial^2 w_1}{\partial t^2}-\frac{K\rho I}{kAG}\frac{\partial^2 w_2}{\partial t^2}+\left(F-2\frac{KEI}{kAG}\right)\frac{\partial^2 w_1}{\partial z^2}\\ +\frac{KEI}{kAG}\frac{\partial^2 w_2}{\partial z^2}-\left(\rho I+\frac{\rho EI}{kG}-\frac{F\rho I}{kAG}\right)\frac{\partial^4 w_1}{\partial z^2\partial t^2}+\frac{\rho^2 I}{kG}\frac{\partial^4 w_1}{\partial t^4}+2Kw_1-Kw_2=0,\end{aligned}\tag{8a}$$

$$EI\left(1-\frac{F}{kAG}\right)\frac{\partial^4 w_2}{\partial z^4}-\frac{K\rho I}{kAG}\frac{\partial^2 w_1}{\partial t^2}+\left(\rho A+2\frac{K\rho I}{kAG}\right)\frac{\partial^2 w_2}{\partial t^2}-\frac{K\rho I}{kAG}\frac{\partial^2 w_3}{\partial t^2}+\frac{KEI}{kAG}\frac{\partial^2 w_1}{\partial z^2}+\left(F-\frac{KEI}{kAG}\right)\frac{\partial^2 w_2}{\partial z^2}$$
$$+\frac{KEI}{kAG}\frac{\partial^2 w_3}{\partial z^2}-\left(\rho I+\frac{\rho EI}{kG}-\frac{F\rho I}{kAG}\right)\frac{\partial^4 w_2}{\partial z^2\partial t^2}+\frac{\rho^2 I}{kG}\frac{\partial^4 w_2}{\partial t^4}-Kw_1+2Kw_2-Kw_3=0, \quad (8b)$$

$$EI\left(1-\frac{F}{kAG}\right)\frac{\partial^4 w_3}{\partial z^4}-\frac{K\rho I}{kAG}\frac{\partial^2 w_2}{\partial t^2}+\left(\rho A+\frac{K\rho I}{kAG}\right)\frac{\partial^2 w_3}{\partial t^2}+\frac{KEI}{kAG}\frac{\partial^2 w_2}{\partial z^2}+\left(F-\frac{KEI}{kAG}\right)\frac{\partial^2 w_3}{\partial z^2}$$
$$-\left(\rho I+\frac{\rho EI}{kG}-\frac{F\rho I}{kAG}\right)\frac{\partial^4 w_3}{\partial z^2\partial t^2}+\frac{\rho^2 I}{kG}\frac{\partial^4 w_3}{\partial t^4}-Kw_2+Kw_3=0. \quad (8c)$$

The initial conditions in general form and the boundary conditions for simply supported beams of the same length l are assumed as follows:

$$w_i(z,0)=w_{i0}(z), \quad \dot{w}_i(z,0)=v_{i0}(z), \quad \psi_i(z,0)=\psi_{i0}(z), \quad \dot{\psi}_i(z,0)=\omega_{i0}(z), \quad (9)$$
$$w_i(z,0)=w_i''(0,t)=w_i(l,0)=w_i''(l,t)=0, \quad i=1,2,3. \quad (10)$$

Assuming time-harmonic motion and using the separation of variables and the solutions of (8), the governing boundary conditions (10) can be written in the form

$$w_i(z,t)=\sum_{n=1}^{\infty}X_n(z)T_{in}(t), \quad i=1,2,3, \quad (11)$$

where $T_{in}(t)$, $i=1,2,3$, is the unknown time function and $X_n(z)$ is the known mode shape function for a simply supported single beam, which is defined as

$$X_n(x)=\sin(k_n z), \quad k_n=n\pi/l, \quad n=1,2,3,\ldots. \quad (12)$$

Introducing the general solutions (11) into (8) one gets the system of ordinary differential equations

$$\sum_{n=1}^{\infty}\left\{\frac{1}{C_s^2}\frac{d^4T_{1n}}{dt^4}+\left[1+C_r^2k_n^2\left(1+\frac{C_b^2}{C_s^2C_r^4}\right)+\frac{1}{C_s^2}(2H-F\eta)\right]\frac{d^2T_{1n}}{dt^2}\right.$$
$$\left.-\frac{H}{C_s^2}\frac{d^2T_{2n}}{dt^2}+\left[C_b^2k_n^4+(2H-F\eta)\left(1+\frac{C_b^2}{C_s^2C_r^2}k_n^2\right)\right]T_{1n}-H\left(1+\frac{C_b^2}{C_s^2C_r^2}k_n^2\right)T_{2n}\right\}=0, \quad (13a)$$

$$\sum_{n=1}^{\infty}\left\{\frac{1}{C_s^2}\frac{d^4T_{2n}}{dt^4}-\frac{H}{C_s^2}\frac{d^2T_{1n}}{dt^2}+\left[1+C_r^2k_n^2\left(1+\frac{C_b^2}{C_s^2C_r^4}\right)+\frac{1}{C_s^2}(2H-F\eta)\right]\frac{d^2T_{2n}}{dt^2}-\frac{H}{C_s^2}\frac{d^2T_{3n}}{dt^2}\right.$$
$$\left.-H\left(1+\frac{C_b^2}{C_s^2C_r^2}k_n^2\right)T_{1n}+\left[C_b^2k_n^4+(2H-F\eta)\left(1+\frac{C_b^2}{C_s^2C_r^2}k_n^2\right)\right]T_{2n}-H\left(1+\frac{C_b^2}{C_s^2C_r^2}k_n^2\right)T_{3n}\right\}=0, \quad (13b)$$

$$\sum_{n=1}^{\infty}\left\{\frac{1}{C_s^2}\frac{d^4T_{3n}}{dt^4}-\frac{H}{C_s^2}\frac{d^2T_{2n}}{dt^2}+\left[1+C_r^2k_n^2\left(1+\frac{C_b^2}{C_s^2C_r^4}\right)+\frac{1}{C_s^2}(H-F\eta)\right]\frac{d^2T_{3n}}{dt^2}\right.$$
$$\left.-H\left(1+\frac{C_b^2}{C_s^2C_r^2}k_n^2\right)T_{2n}+\left[C_b^2k_n^4+(H-F\eta)\left(1+\frac{C_b^2}{C_s^2C_r^2}k_n^2\right)\right]T_{3n}\right\}=0, \quad (13c)$$

where

$$H = \frac{K}{\rho A}, \qquad \eta = \frac{k_n^2}{\rho A}.$$

The coefficients

$$C_b = \sqrt{\frac{EI}{\rho A}}, \qquad C_s = \sqrt{\frac{GAk}{\rho I}}, \qquad C_r = \sqrt{\frac{I}{A}},$$

related to the bending stiffness, shear stiffness, and rotational effects, respectively, are now introduced. The shear beam model, the Rayleigh beam model, and the simple Euler beam model can be obtained from the Timoshenko beam model by setting C_r to zero (that is, ignoring the rotational effect), C_s to infinity (ignoring the shear effect), and setting both C_r to zero and C_s to infinity, respectively.

The solutions of (13a), (13b), and (13c) can be assumed to have the following forms:

$$T_{1n} = A_{1n} e^{j\omega_n t}, \qquad T_{2n} = A_{2n} e^{j\omega_n t}, \qquad T_{3n} = A_{3n} e^{j\omega_n t}, \qquad j = \sqrt{-1}, \tag{14}$$

where ω_n denotes the natural frequency of the system. Substituting (14) into (13) results in the following system of homogeneous algebraic equations for the unknown constants A_{1n}, A_{2n}, and A_{3n}:

$$\left\{ \frac{\omega_n^4}{C_s^2} - \left[1 + C_r^2 k_n^2 \left(1 + \frac{C_b^2}{C_s^2 C_r^4} \right) + \frac{1}{C_s^2}(2H - F\eta) \right] \omega_n^2 + \left[C_b^2 k_n^4 + (2H - F\eta) \left(1 + \frac{C_b^2}{C_s^2 C_r^2} k_n^2 \right) \right] \right\} A_{1n} + H \left[\frac{\omega_n^2}{C_s^2} - \left(1 + \frac{C_b^2}{C_s^2 C_r^2} k_n^2 \right) \right] A_{2n} = 0, \tag{15a}$$

$$H \left[\frac{\omega_n^2}{C_s^2} - \left(1 + \frac{C_b^2}{C_s^2 C_r^2} k_n^2 \right) \right] A_{1n} + \left\{ \frac{\omega_n^4}{C_s^2} - \left[1 + C_r^2 k_n^2 \left(1 + \frac{C_b^2}{C_s^2 C_r^4} \right) + \frac{1}{C_s^2}(2H - F\eta) \right] \omega_n^2 + \left[C_b^2 k_n^4 + (2H - F\eta) \left(1 + \frac{C_b^2}{C_s^2 C_r^2} k_n^2 \right) \right] \right\} A_{2n} + H \left[\frac{\omega_n^2}{C_s^2} - \left(1 + \frac{C_b^2}{C_s^2 C_r^2} k_n^2 \right) \right] A_{3n} = 0, \tag{15b}$$

$$H \left[\frac{\omega_n^2}{C_s^2} - \left(1 + \frac{C_b^2}{C_s^2 C_r^2} k_n^2 \right) \right] A_{2n} + \left\{ \frac{\omega_n^4}{C_s^2} - \left[1 + C_r^2 k_n^2 \left(1 + \frac{C_b^2}{C_s^2 C_r^4} \right) + \frac{1}{C_s^2}(H - F\eta) \right] \omega_n^2 + \left[C_b^2 k_n^4 + (H - F\eta) \left(1 + \frac{C_b^2}{C_s^2 C_r^2} k_n^2 \right) \right] \right\} A_{3n} = 0. \tag{15c}$$

Equations (15) have nontrivial solutions when the determinant of the system matrix coefficients of A_{1n}, A_{2n}, and A_{3n} is zero. This yields the following frequency (characteristic) equation, which is a twelfth-order polynomial in ω_n. When the natural frequency of the system vanishes under the axial loading, the system begins to buckle. By introducing $\omega_n = 0$ into (15) expressed in matrix form one gets

$$\begin{bmatrix} x + 2RH & -HR & 0 \\ -HR & x + 2RH & -HR \\ 0 & -HR & x + RH \end{bmatrix} \begin{Bmatrix} A_{1n} \\ A_{2n} \\ A_{3n} \end{Bmatrix} = 0, \tag{16}$$

where

$$R = 1 + \frac{C_b^2}{C_s^2 C_r^2} k_n^2, \qquad x = C_b^2 k_n^4 - RF\eta. \tag{17}$$

The existence of nontrivial solutions for A_{1n}, A_{2n}, and A_{3n} requires that the determinant of the coefficient matrix vanish. This gives the cubic characteristic equation $x^3+5RHx^2+6(RH)^2x+(RH)^3=0$, or

$$\left(\frac{x}{RH}\right)^3+5\left(\frac{x}{RH}\right)^2+6\left(\frac{x}{RH}\right)+1=0. \tag{18}$$

Solution of the characteristic equation. We solve (18) using a standard method. Denote the coefficients by $a_0=1,\ a_1=5,\ a_2=6$, and $a_3=1$, and set

$$p=\frac{a_2}{a_0}-\frac{a_1^2}{3a_0^2} \quad\text{and}\quad q=\frac{a_3}{a_0}-\frac{a_1a_2}{3a_0^2}+\frac{2a_1^3}{27a_0^3}. \tag{19}$$

The discriminant $D=\frac{1}{4}q^2+\frac{1}{27}p^3$ is negative, so there are three roots for $x/(RH)$, given by

$$-\frac{a_1}{3a_0}+2\sqrt{-\frac{p}{3}}\cos\frac{\theta+2k\pi}{3}, \qquad \text{with } \theta=\cos^{-1}\left[-\frac{q}{2}\left(-\frac{p}{3}\right)^{-\frac{3}{2}}\right] \text{ and } k=0,\,1,\,2.$$

Thus the roots are

$$\frac{x_1}{RH}=-0.19806, \qquad \frac{x_2}{RH}=-1.55496, \qquad \frac{x_3}{RH}=-3.24698. \tag{20}$$

Substituting into (17), we obtain the buckling loads for different vibration modes n:

$$F_b^{\mathrm{I}}=\frac{0.198062H}{\eta}+\frac{C_b^2k_n^4}{R\eta}, \qquad F_b^{\mathrm{II}}=\frac{1.554962H}{\eta}+\frac{C_b^2k_n^4}{R\eta}, \qquad F_b^{\mathrm{III}}=\frac{3.24698H}{\eta}+\frac{C_b^2k_n^4}{R\eta}. \tag{21}$$

As can be seen, the values of the buckling loads F_b^{I}, F_b^{II}, and F_b^{III} are positive and $F_b^{\mathrm{I}}<F_b^{\mathrm{II}}<F_b^{\mathrm{III}}$. Thus F_b^{I} is the critical buckling load:

$$F_b^{\mathrm{cr}}=\frac{0.198062Kl^2}{\pi^2n^2}+\frac{EI\pi^2n^2}{l^2\left(1+\dfrac{EI}{GAk}\dfrac{\pi^2n^2}{l^2}\right)}. \tag{22}$$

For $K=0$ from (22) we obtain

$$P_n=\frac{EI\pi^2n^2}{l^2\left(1+\dfrac{EI}{GAk}\dfrac{\pi^2n^2}{l^2}\right)},$$

which is the critical buckling load corresponding to the number n of the Timoshenko beams as shown in [Timoshenko and Gere 1964, p. 134]. Setting $n=1$ in the preceding equation we obtain

$$P=\frac{EI\pi^2}{l^2\left(1+\dfrac{EI}{GAk}\dfrac{\pi^2}{l^2}\right)}.$$

This is the smallest load at which the beam ceases to be in stable equilibrium.

Remark. An alternative, but equivalent, method of solution is given in [Rašković 1965, pp. 157–166].

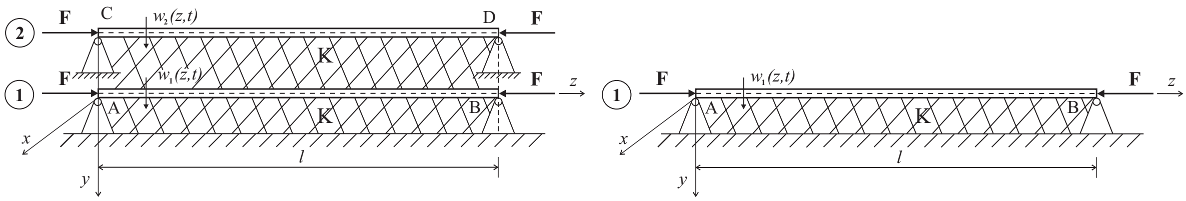


Figure 3. Left: system with two identical Timoshenko beams supported on a Winkler elastic layer. Right: analogous system with a single Timoshenko beam.

Critical buckling load for system with fewer Timoshenko beams. The preceding analysis was also applied to a system with two beams instead of three (Figure 3, left) and a system with a single beam resting on a Winkler elastic layer (Figure 3, right). The computation is easier in these cases, in that the characteristic equation is quadratic or linear, respectively. For the case of two beams we get

$$F_b^{\mathrm{I}} = \frac{0.382H}{\eta} + \frac{C_b^2 k_n^4}{R\eta}, \qquad F_b^{\mathrm{II}} = \frac{2.618H}{\eta} + \frac{C_b^2 k_n^4}{R\eta};$$

thus F_b^{I} is the critical buckling load corresponding to vibration mode n for this system:

$$F_b^{\mathrm{cr}} = \frac{0.382Kl^2}{\pi^2 n^2} + \frac{EI\pi^2 n^2}{l^2\left(1 + \dfrac{EI}{GAk}\dfrac{\pi^2 n^2}{l^2}\right)}. \tag{23}$$

For the case of a single beam we have

$$F_b^{\mathrm{cr}} = \frac{Kl^2}{\pi^2 n^2} + \frac{EI\pi^2 n^2}{l^2\left(1 + \dfrac{EI}{GAk}\dfrac{\pi^2 n^2}{l^2}\right)}. \tag{23a}$$

4. Numerical results

We ran numerical calculations for the system with parameters

$$\begin{gathered} E = 1\times 10^{10}\,\mathrm{Nm}^{-2}, \quad G = 0.417\times 10^{10}\,\mathrm{Nm}^{-2}, \quad k = 5/6, \quad K_0 = 2\times 10^5\,\mathrm{Nm}^{-2}, \\ \rho = 2\times 10^3\,\mathrm{kgm}^{-3}, \quad l = 10\,\mathrm{m}, \quad A = 5\times 10^{-2}\,\mathrm{m}^2, \quad I = 4\times 10^{-4}\,\mathrm{m}^4, \end{gathered} \tag{24}$$

as in [Zhang et al. 2008]. If we introduce a nondimensional value $\xi = h/l$, the ratio of the cross-sectional height h to the beam length l, we can write the surface and moment of inertia of the cross-section of the beam as a function of the nondimensional value ξ as

$$A = h^2 = (\xi l)^2, \qquad I = \frac{h^4}{12} = \frac{(\xi l)^4}{12}. \tag{25}$$

The change in the critical buckling load in the function of the nondimensional value ξ is given in Figures 4 and 5. These diagrams represent the variation of the critical buckling load for systems with triple, double, and single Timoshenko beams obtained by analytical expressions (22), (23), and (23a) for the different parameters of the system (24). Figure 4 shows the diagrams obtained for different values of the stiffness

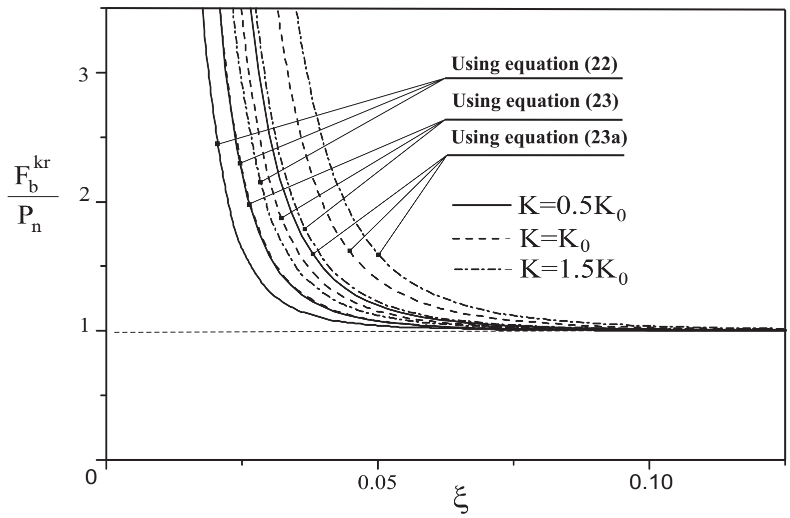


Figure 4. Effect of the nondimensional value ξ on the critical buckling load F_b^{cr} for different values of K and $n = 1$.

modulus $K = 0.5K_0$, K_0, and $1.5K_0$ and for vibration mode $n = 1$. It can be seen that the critical buckling load increases with an increase in the stiffness modulus K. Figure 5 shows diagrams of the critical buckling load for different values of the vibration mode $n = 1, 2, 3$, and for stiffness modulus $K = K_0$. It can be seen that the critical buckling load decreases with an increase in the vibration mode n.

In Figure 6, the static stability regions for the first vibration mode $n = 1$ are represented for systems with triple, double, and single Timoshenko beams supported on a Winkler elastic layer. It can be seen that the static stability region is largest in the case of a single beam. For the system with two beams, the static stability region is reduced, and even more so for three beams.

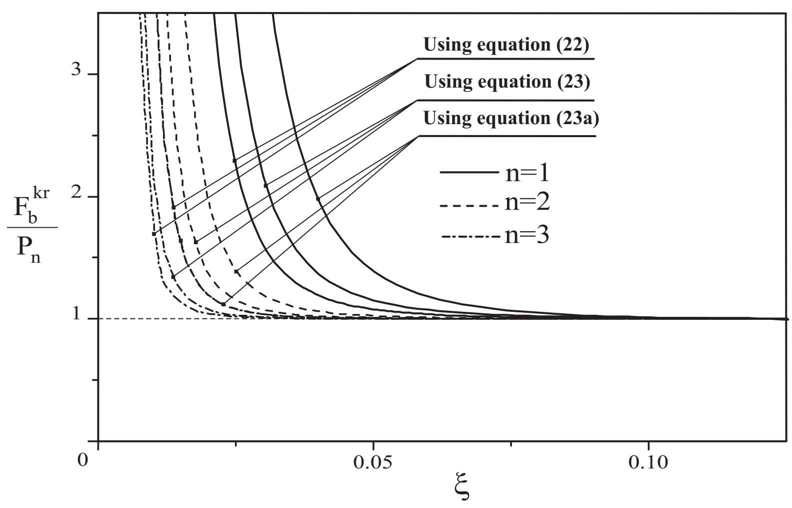


Figure 5. Effect of the nondimensional value ξ on the critical buckling load F_b^{cr} for different values of n and $K = K_0$.

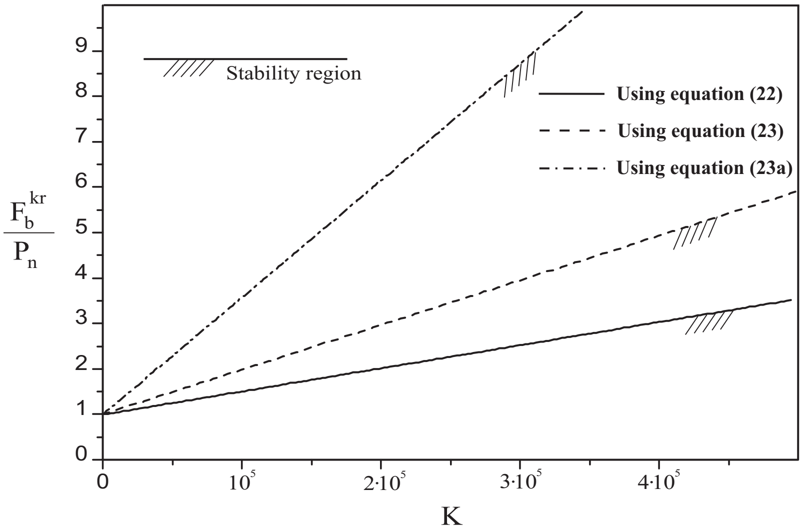


Figure 6. Influence of the stiffness modulus K on the static stability region for $n = 1$.

Conclusions

In the present paper, the equations of dynamic equilibrium and the equations of natural vibration of a triple Timoshenko beam elastically connected to a Winkler elastic layer are formulated. In order to derive these equations, the influence of constant axial forces at the ends of the same beams (second-order theory), as well as the influence of the elastic layer on the beams, are taken into account. Using the classical Bernoulli–Fourier method, the solutions of the differential equations of motion for the system are formulated. The explicit expressions for the critical buckling loads of the systems with triple, double, and single Timoshenko beams are obtained. The critical buckling load for the triple-beam model is also determined using the trigonometric method. The thus determined values for the critical buckling load are only slightly different from the values determined by the numerical solution of the characteristic equation. It is observed from the numerical results that the static stability region is influenced by the Winkler layer of stiffness modulus K and the number of Timoshenko beams. The static stability region of the triple and double-Timoshenko-beam systems is always smaller than that of the single-beam system. It can be concluded that an increase in the number of elastically connected Timoshenko beams leads to a reduction of the static stability region for the same system parameters.

References

[De Rosa 1995] M. A. De Rosa, "Free vibrations of Timoshenko-beams on two-parameter elastic foundation", *Comput. Struct.* **57** (1995), 151–156.

[Kelly and Srinivas 2009] G. S. Kelly and S. Srinivas, "Free vibrations of elastically connected stretched beams", *J. Sound Vib.* **326** (2009), 883–893.

[Lazopoulos and Lazopoulos 2011] K. A. Lazopoulos and K. A. Lazopoulos, "On a strain gradient elastic Timoshenko beam model", *Z. Angew. Math. Mech.* **91**:11 (2011), 875–882.

[Li et al. 2008] J. Li, Y. Chen, and H. Hua, "Exact dynamic stiffness matrix of a Timoshenko three-beam system", *Int. J. Mech. Sci.* **50** (2008), 1023–1034.

[Matsunaga 1996] H. Matsunaga, "Buckling instabilities of thick elastic beams subjected to axial stresses", *Comput. Struct.* **59** (1996), 859–868.

[Miranda and Taghavi 2005] E. Miranda and S. Taghavi, "Approximate floor acceleration demands in multistory buildings, I: Formulation", *J. Struct. Eng.* (*ASCE*) **131** (2005), 203–211.

[Rašković 1965] D. Rašković, *Teorija oscilacija*, Građevinska knjiga, Beograd, 1965. A scan can be found at http://tinyurl.com/RaskovicTeorijaOscilacija.

[Stojanović et al. 2011] V. Stojanović, P. Kozić, R. Pavlović, and G. Janevski, "Effect of rotary inertia and shear on vibration and buckling of a double beam system under compressive axial loading", *Arch. Appl. Mech.* **81**:12 (2011), 1993–2005.

[Timoshenko and Gere 1964] S. P. Timoshenko and J. M. Gere, *Theory of elastic stability*, 2nd ed., McGraw-Hill, 1964.

[Zhang et al. 2008] Q. Y. Zhang, Y. Lu, L. S. Wang, and X. Liu, "Vibration and buckling of a double-beam system under compressive axial loading", *J. Sound Vib.* **318** (2008), 341–352.

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