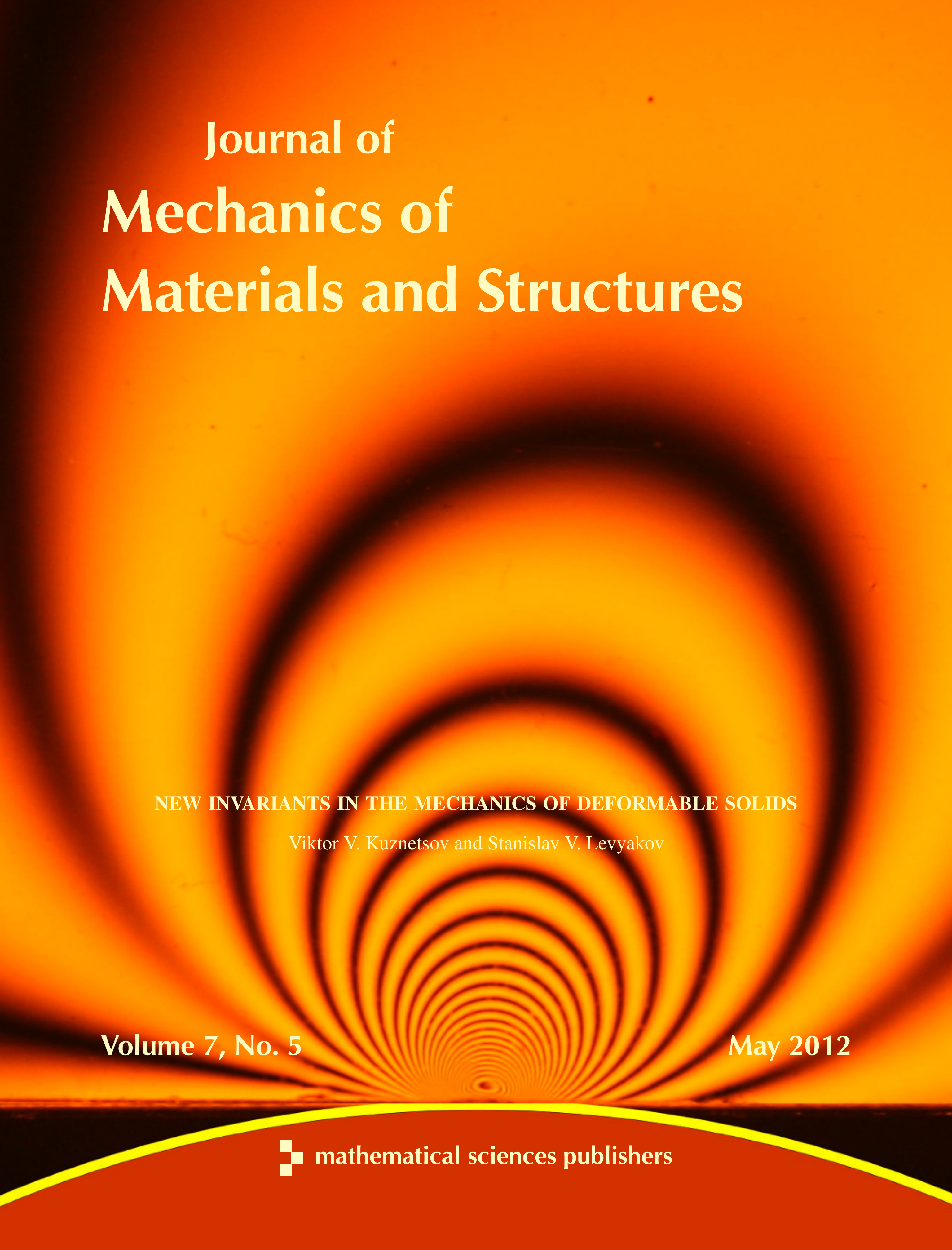

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Viktor V. Kuznetsov and Stanislav V. Levyakov

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NEW INVARIANTS IN THE MECHANICS OF DEFORMABLE SOLIDS

VIKTOR V. KUZNETSOV AND STANISLAV V. LEVYAKOV

A system of invariants of symmetric three-dimensional tensors of second order is proposed in a new form. The system contains three classical invariants of a tensor and three new invariants which depend on the components of two or three tensors. The system proposed allows one to express the strain energy density of a linear elastic anisotropic body and virtual work done by internal stresses in terms of the invariants for any constitutive law of the material. Application of the invariants to the derivation of the tetrahedron finite element of anisotropic solids is discussed.

1. Introduction

Invariants of the strain and stress tensors known as frame-indifferent (or objective) measures are considered in almost every book on the theory of elasticity (see, for example, [Sneddon and Berry 1958; Timoshenko and Goodier 1970; Antman 1995]). In contrast to the tensorial components whose values depend on the coordinate system chosen, the values of invariants remain unchanged regardless of the coordinate system used to evaluate them. In the theory of elasticity, invariants are introduced as coefficients of the characteristic polynomial equation of the standard eigenvalue problem to find the principal directions of the strain and stress tensors. Alternatively, the invariants can be obtained by decomposing the ratio of an elementary volume in a deformed state to that in the undeformed state into linear, quadratic, and cubic forms in the tensor components [Kuznetsov and Levyakov 2009].

In theoretical studies, invariants provide a convenient tool for a compact representation of constitution relations and strain energy density of a deformable solid. Since the strain energy is a scalar, it must be represented as a function of invariants of the tensors governing the stress-strain state of the solid. For an isotropic homogeneous material, this expression in terms of the stress or strain tensor invariants is known [Timoshenko and Goodier 1970]. But to the authors' best knowledge, no similar expression is available in the literature for anisotropic or even orthotropic materials.

In the present paper, the question of determining the strain energy density of an anisotropic solid in terms of invariants in the form different from [Timoshenko and Goodier 1970] is studied. Symmetric three-dimensional tensors of second order are considered and a system of invariants of the tensors is proposed. The system comprises three traditional invariants of the tensor and three new quantities called the combined invariants. It is shown that the strain energy density of a linear elastic anisotropic body can be represented as a function of these invariants.

The property of the invariants can be valuable in formulating computationally effective numerical algorithms for analysis of stresses and strain of deformable solids. In the finite element method, the stiffness equations are formulated in a local coordinate system introduced for each finite element of

Keywords: tensors, invariants, anisotropic material, strain, stress, tetrahedral finite element.

the structure. To construct the finite-element assemblage, these equations are transformed to global coordinate axes. The coordinate transformations can be eliminated using invariants, which can reduce computer time. The time reduction can be pronounced in the nonlinear analysis based on the Newton-Raphson type techniques where stiffness relations of each element are updated at each iteration. Below, application of the invariant based approach to the finite-element analysis of solids is briefly discussed.

2. Invariants of three-dimensional tensors

We consider a second-order three-dimensional tensor $\boldsymbol{S}$ with covariant components S_{ij} $(i, j = 1, 2, 3)$ determined in a convective coordinate system $\alpha^1, \alpha^2, \alpha^3$ characterized by metric tensor $\boldsymbol{A}$ with covariant components

$$A_{ij} = \frac{\partial \boldsymbol{R}}{\partial \alpha^i} \frac{\partial \boldsymbol{R}}{\partial \alpha^j},$$

where $\boldsymbol{R}$ is the position vector. The first, second, and third invariants of the tensor $\boldsymbol{S}$ in the metric $\boldsymbol{A}$ are given by

$$I_S = \frac{1}{2\Delta} \varepsilon_{ijk} \varepsilon_{pqr} S_{ip} A_{jq} A_{kr}, \tag{2-1}$$

$$I_{S^2} = \frac{1}{2\Delta} \varepsilon_{ijk} \varepsilon_{pqr} S_{ip} S_{jq} A_{kr}, \tag{2-2}$$

$$I_{S^3} = \frac{1}{6\Delta} \varepsilon_{ijk} \varepsilon_{pqr} S_{ip} S_{jq} S_{kr}, \tag{2-3}$$

where

$$\varepsilon_{ijk} = \tfrac{1}{2}(i - j)(j - k)(k - i) \tag{2-4}$$

is the Levi-Civita symbol and

$$\Delta = \det \|A_{ij}\| = \tfrac{1}{6} \varepsilon_{ijk} \varepsilon_{pqr} A_{ip} A_{jq} A_{kr} \tag{2-5}$$

is the discriminant of the metric tensor.

In these equations and below, summation is performed over repeated indices which run from 1 to 3. Formulas (2-1)–(2-3) are based on the results of [Kuznetsov and Levyakov 2009] and written here in a new unified manner to show the rule according to which the invariants are formed. Writing the invariants of the tensor $\boldsymbol{S}$ in terms of its principal values S_i, we obtain the well-known expressions [Timoshenko and Goodier 1970]

$$I_S = S_1 + S_2 + S_3, \quad I_{S^2} = S_1 S_2 + S_2 S_3 + S_3 S_1, \quad I_{S^3} = S_1 S_2 S_3. \tag{2-6}$$

Let the tensor $\boldsymbol{S}$ be decomposed into three symmetric tensors $\boldsymbol{U}$, $\boldsymbol{V}$, and $\boldsymbol{W}$ such that $\boldsymbol{S} = \boldsymbol{U} + \boldsymbol{V} + \boldsymbol{W}$. In this case, (2-1)–(2-3) become

$$I_S = I_U + I_V + I_W, \tag{2-7}$$

$$I_{S^2} = I_{U^2} + I_{V^2} + I_{W^2} + 2(I_{UV} + I_{UW} + I_{VW}), \tag{2-8}$$

$$I_{S^3} = I_{U^3} + I_{V^3} + I_{W^3} + 3(I_{UV^2} + I_{UW^2} + I_{VU^2} + I_{VW^2} + I_{WU^2} + I_{WV^2}) + 6I_{UVW}. \tag{2-9}$$

These expressions contain three known invariants determined by (2-1)–(2-3) and three new invariants given by

$$I_{UV} = \frac{1}{2\Delta}\varepsilon_{ijk}\varepsilon_{pqr}U_{ip}V_{jq}A_{kr}, \tag{2-10}$$

$$I_{UV^2} = \frac{1}{6\Delta}\varepsilon_{ijk}\varepsilon_{pqr}U_{ip}V_{jq}V_{kr}, \tag{2-11}$$

$$I_{UVW} = \frac{1}{6\Delta}\varepsilon_{ijk}\varepsilon_{pqr}U_{ip}V_{jq}W_{kr}. \tag{2-12}$$

We call these expressions, which depend on the components of two or three tensors, the combined invariants. One can easily show that decomposition of $\boldsymbol{S}$ into more than three symmetric tensors does not lead to any new invariants. Obviously, expressions (2-1)–(2-3) and (2-10)–(2-12) are valid for any symmetric three-dimensional tensors of second order. For this reason, we refer to them as templates.

The invariants determined by formulas (2-1)–(2-3) and (2-10)–(2-12) constitute a complete system that allows one to obtain invariant forms of the first, second, and third degrees containing components of any number of symmetric tensors. The invariant I_{UV} is the bilinear invariant of two tensors, I_{UV^2} is the linear-quadratic invariant of two tensors, and I_{UVW} is the trilinear invariant of three tensors. It should be noted that the invariant I_{UVW} is of general character: all the invariants mentioned above fall out as particular cases from (2-12), namely

$$I_S = 3I_{SAA}, \quad I_{S^2} = 3I_{SSA}, \quad I_{S^3} = I_{SSS}, \quad I_{UV} = 3I_{UVA}, \quad I_{UV^2} = I_{UVV}. \tag{2-13}$$

In Cartesian coordinates, where $A_{ij} = \delta_{ij}$ (the Kronecker symbol) and $\Delta = 1$, the invariants considered above become

$$I_U = \delta_{ij}U_{ij}, \qquad I_{U^2} = \tfrac{1}{2}\varepsilon_{ijk}\varepsilon_{iqr}U_{jq}U_{kr}, \qquad I_{U^3} = \tfrac{1}{6}\varepsilon_{ijk}\varepsilon_{pqr}U_{ip}U_{jq}U_{kr}, \tag{2-14}$$

$$I_{UV} = \tfrac{1}{2}\varepsilon_{ijk}\varepsilon_{iqr}U_{jq}V_{kr}, \quad I_{UV^2} = \tfrac{1}{6}\varepsilon_{ijk}\varepsilon_{pqr}U_{ip}V_{jq}V_{kr}, \quad I_{UVW} = \tfrac{1}{6}\varepsilon_{ijk}\varepsilon_{pqr}U_{ip}V_{jq}W_{kr}. \tag{2-15}$$

Let α'^1, α'^2, α'^3 denote new convective curvilinear coordinates and assume that they are related to the coordinates α^1, α^2, α^3 by

$$\alpha^i = \alpha^i(\alpha'^1, \alpha'^2, \alpha'^3) \quad (i = 1, 2, 3). \tag{2-16}$$

Now we show that the expressions (2-1)–(2-3) and (2-10)–(2-12) retain their form under general coordinate transformations (2-16). To this end, it suffices to prove the invariance of the expression for the trilinear invariant of three tensors

$$I_{U'V'W'} = I_{UVW} \tag{2-17}$$

or

$$\frac{1}{6\Delta'}\varepsilon_{ijk}\varepsilon_{pqr}U'_{ip}V'_{jq}W'_{kr} = \frac{1}{6\Delta}\varepsilon_{ijk}\varepsilon_{pqr}U_{ip}V_{jq}W_{kr}. \tag{2-18}$$

Here the primed quantities are evaluated in the coordinates α'^i. The components of any tensor considered above are transformed according to the well-known rule

$$U'_{ij} = U_{pq}\frac{\partial\alpha^p}{\partial\alpha'^i}\frac{\partial\alpha^q}{\partial\alpha'^j}. \tag{2-19}$$

We write the useful relation

$$\varepsilon_{ijk}\frac{\partial\alpha^p}{\partial\alpha'^i}\frac{\partial\alpha^q}{\partial\alpha'^j}\frac{\partial\alpha^r}{\partial\alpha'^k}=\varepsilon_{pqr}\det\boldsymbol{J},\tag{2-20}$$

where $\boldsymbol{J}=\partial(\alpha^1,\alpha^2,\alpha^3)/\partial(\alpha'^1,\alpha'^2,\alpha'^3)$ is the Jacobian matrix of coordinate transformation.

Following (2-5), we write the discriminant of the metric tensor on the left side of (2-18) as

$$\Delta'=\tfrac{1}{6}\varepsilon_{ijk}\varepsilon_{pqr}A'_{ip}A'_{jq}A'_{kr}.\tag{2-21}$$

Expressing the components of the metric tensor A'_{ij} in terms of unprimed components according to the rule (2-19) and using relation (2-20), we obtain

$$\Delta'=(\det\boldsymbol{J})^2\Delta.\tag{2-22}$$

Performing similar manipulations for the nominator on the left side of (2-18), we find that

$$\varepsilon_{ijk}\varepsilon_{pqr}U'_{ip}V'_{jq}W'_{kr}=\varepsilon_{ijk}\varepsilon_{pqr}U_{ip}V_{jq}W_{kr}(\det\boldsymbol{J})^2.\tag{2-23}$$

Substituting (2-22) and (2-23) into the left side of (2-18), we prove the validity of (2-18).

3. Strain energy density

Now we use the results obtained above to express the stress energy density Π_V of a linear elastic anisotropic solid. In Cartesian coordinates referred to anisotropy axes of the material, we obtain

$$\Pi_V=\tfrac{1}{2}\sigma_{ij}S_{ij},\tag{3-1}$$

where σ_{ij} and S_{ij} are the stress and strain tensors, respectively, which in general are related through 21 elastic constants. We note that the strain tensor components can be related to the displacements by linear or geometrically nonlinear relations. Expression (3-1) can be written in the invariant form as

$$\Pi_V=\tfrac{1}{2}(I_\sigma I_S-2I_{\sigma S}),\tag{3-2}$$

where I_σ and I_S are the first invariants of the stress and strain tensors, respectively, and $I_{\sigma S}$ is the combined invariant of these two tensors. These quantities can be calculated using $(2\text{-}14)_1$ and $(2\text{-}14)_4$, in which $\boldsymbol{U}$ and $\boldsymbol{V}$ are replaced with $\boldsymbol{\sigma}$ and $\boldsymbol{S}$. To prove the equivalence of (3-1) and (3-2), it suffices to employ the relation

$$\varepsilon_{ijk}\varepsilon_{iqr}=\delta_{jq}\delta_{kr}-\delta_{jr}\delta_{kq}.\tag{3-3}$$

The invariants on the right side of (3-2) imply that the strain energy density retains its value for any coordinate system. We note that the invariants appearing in (3-2) can be calculated in arbitrary curvilinear coordinates α^i $(i=1,2,3)$ using templates (2-1) and (2-10).

In a similar manner, one can easily show that the virtual work δW done by the internal stresses is invariant for any constitutive law:

$$\delta W=I_\sigma I_{\delta S}-2I_{\sigma(\delta S)},\tag{3-4}$$

where invariants on the right side can be evaluated using templates (2-14) and (2-15) in Cartesian coordinates or (2-1) and (2-10) in curvilinear coordinates.

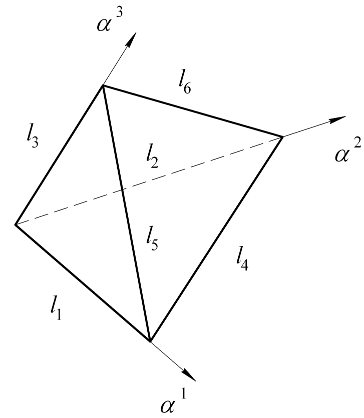

Figure 1. Tetrahedral finite element and notation.

4. Finite-element application

Let us determine the strain and stress invariants and the strain energy density for a tetrahedron finite element of a three-dimensional solid (Figure 1). The components of the metric tensor are expressed in terms of the squared lengths l_i of the elemental edges:

$$\begin{gathered} A_{11}=l_1^2, \quad A_{22}=l_2^2, \quad A_{33}=l_3^2, \quad A_{12}=A_{21}=\tfrac{1}{2}(l_1^2+l_2^2-l_4^2), \\ A_{13}=A_{31}=\tfrac{1}{2}(l_1^2+l_3^2-l_5^2), \quad A_{23}=A_{32}=\tfrac{1}{2}(l_2^2+l_3^2-l_6^2). \end{gathered} \tag{4-1}$$

As the physical strains and stresses, we use the normal strains S_i and stresses σ_i $(i=1,\dots,6)$ in the direction of the elemental edges. Covariant components of the strain tensor are related to the physical strains by the expressions

$$\begin{gathered} S_{11}=l_1^2S_1, \quad S_{22}=l_2^2S_2, \quad S_{33}=l_3^2S_3, \quad S_{12}=S_{21}=\tfrac{1}{2}(l_1^2S_1+l_2^2S_2-l_4^2S_4), \\ S_{13}=S_{31}=\tfrac{1}{2}(l_1^2S_1+l_3^2S_3-l_5^2S_5), \quad S_{23}=S_{32}=\tfrac{1}{2}(l_2^2S_2+l_3^2S_3-l_6^2S_6). \end{gathered} \tag{4-2}$$

Similar relations can be written for the stress components.

Thus, the stress and strain invariants and the strain energy density can be determined by the normal components of the tensors. According to the terminology of [Argyris et al. 1979], these components are referred to as natural components which are in harmony with the tetrahedron geometry. The strain energy of the element can be evaluated by approximating the natural strains in terms of the nodal parameters and integrating over the element volume $V=\sqrt{\Delta}/6$.

Once the equilibrium state has been determined and the natural strains have been obtained, one can compute the principal values of the strain tensor as roots of the cubic equation

$$S^3-I_SS^2+I_{SS}S-I_{SSS}=0. \tag{4-3}$$

The principal stresses can be obtained in a similar manner.

A complete system of invariants of two-dimensional tensors defined on a plane or a surface is given in the Appendix.

5. Concluding remarks

A system of invariants of symmetric three-dimensional tensors of second order has been proposed. The invariants have been determined by ratios of certain third-order determinants depending on the covariant components of the tensors. The system comprises the well-known three invariants of one tensor and three new combined invariants which depend on the components of two or three tensors. The first new invariant is a bilinear invariant of two tensors, the second is a linear-quadratic invariant of two tensors, and the third is a trilinear invariant of three tensors. These invariants have been expressed in a unified manner to show the rule according to which they are formed. The trilinear invariant of three tensors is of general structure in the sense that the other invariants considered in the paper fall out as particular cases. This implies that the system of invariants is complete. It can be used to form any invariant forms of the first, second, and third degrees which depend on the components of any number of tensors and also to prove the invariance of other expressions. It has been shown that the strain-energy density and virtual work of an anisotropic solid can be expressed in terms of invariants where the bilinear invariant of the strain and stress tensors plays a key role.

As an example of application of the invariants, relations have been given to evaluate the strain energy of a tetrahedron finite element of an anisotropic solid in terms of natural physical components of the strains and stresses in the direction of the element edges.

Appendix. Invariants of two-dimensional tensors

Let a symmetric two-dimensional tensor $\boldsymbol{E}$ with covariant components E_{ij} $(i, j = 1, 2)$ be determined in a convective curvilinear coordinate system α^1, α^2 on a plane or surface characterized by the metric tensor a with covariant components a_{ij}. The first and second invariants of the tensor $\boldsymbol{E}$ in metrics $\boldsymbol{a}$ are given by

$$I_E = \frac{1}{\Delta} e_{mp} e_{nq} a_{pq} E_{mn}, \quad I_{E^2} = \frac{1}{2\Delta} e_{mp} e_{nq} E_{pq} E_{mn}, \quad \Delta = \frac{1}{2} e_{mp} e_{nq} a_{pq} a_{mn}, \tag{A-1}$$

where e_{mp} is the permutation tensor whose components are given by $e_{11} = e_{22} = 0$ and $e_{12} = -e_{21} = 1$ and Δ is the discriminant of the metric tensor. In (A-1) and below, summation is performed over repeated indices running from 1 to 2.

We assume that the tensor $\boldsymbol{E}$ is a sum of two tensors $\boldsymbol{U}$ and $\boldsymbol{V}$. In this case, the invariants in (A-1) become

$$I_E = I_U + I_V, \quad I_{E^2} = I_{U^2} + I_{V^2} + 2I_{UV}. \tag{A-2}$$

Here I_{UV} is the combined invariant of two tensors written as

$$I_{UV} = \frac{1}{2\Delta} e_{mp} e_{nq} U_{pq} V_{mn}. \tag{A-3}$$

This invariant is of general nature since the first and second invariants can be obtained from (A-3) as particular cases:

$$I_E = 2I_{Ea}, \quad I_{E^2} = I_{EE}. \tag{A-4}$$

In Cartesian coordinates, where $a_{ij} = \delta_{ij}$ and $\Delta = 1$, the complete system of two-dimensional invariants becomes

$$I_E = \delta_{mn} E_{mn}, \quad I_{E^2} = \tfrac{1}{2} e_{mp} e_{nq} E_{pq} E_{mn}, \quad I_{UV} = \tfrac{1}{2} e_{mp} e_{nq} U_{pq} V_{mn}, \tag{A-5}$$

where $\boldsymbol{U}$ and $\boldsymbol{V}$ are symmetric two-dimensional tensors.

Now we consider an anisotropic linear elastic body under plane-stress conditions and denote the covariant components of the strain and stress tensors by S_{ij} and σ_{ij}, respectively. In this case, the strain energy density Π_V and the virtual work of stresses δW for any constitutive law of the material is written in Cartesian coordinates as

$$\Pi_V = \tfrac{1}{2} \sigma_{ij} S_{ij}, \quad \delta W = \sigma_{ij} \delta S_{ij}. \tag{A-6}$$

Using (A-4), one can easily show that (A-6) is equivalent to

$$\Pi_V = \tfrac{1}{2}(I_\sigma I_S - 2 I_{\sigma S}), \quad \delta W = I_\sigma I_{(\delta S)} - 2 I_{\sigma(\delta S)}. \tag{A-7}$$

As can be seen, relations (A-7) are invariants and retain their form in any coordinates. In curvilinear coordinates, the invariants in (A-7) can be determined by (A-1) and (A-3).

It is worth noting that in the theory of shells, the bending strain tensor $\boldsymbol{\kappa}$ and transverse shear strain tensor $\boldsymbol{\Gamma}$ are introduced. As a result, one obtains combined invariants like $I_{S\kappa}$, $I_{S\Gamma}$, $I_{\kappa\Gamma}$, and so on, which can be determined by the template relation (A-3). Invariants of two-dimensional tensors for triangular shell finite elements are discussed in [Kuznetsov and Levyakov 2009].

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VIKTOR V. KUZNETSOV: vku1952@yandex.ru
Department of Engineering Mathematics, Novosibirsk State Technical University, Novosibirsk, 630092, Russia

STANISLAV V. LEVYAKOV: stan-levyakov@yandex.ru
Dapartment of Engineering Mathematics, Novosibirsk State Technical University, Novosibirsk, 630092, Russia

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