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THE IMPLICIT COROTATIONAL METHOD AND ITS USE IN THE DERIVATION OF NONLINEAR STRUCTURAL MODELS FOR BEAMS AND PLATES

GIOVANNI GARCEA, ANTONIO MADEO AND RAFFAELE CASCIARO

What we call the *implicit corotational method* is proposed as a tool to obtain geometrically exact nonlinear models for structural elements, such as beams or shells, undergoing finite rotations and small strains, starting from the basic solutions for the three-dimensional Cauchy continuum used in the corresponding linear modelings.

The idea is to use a local corotational description to decompose the deformation gradient in a stretch part followed by a finite rigid rotation. Referring to this corotational frame we can derive, from the linear stress tensor and the deformation gradient provided by linear elasticity, an accurate approximation for the nonlinear stress and strain tensors which implicitly assure frame invariance. The stress and strain fields recovered in this way as functions of generalized stress and strain resultants are then used within a mixed variational formulation allowing us to recover an objective nonlinear modeling directly suitable for FEM implementations through a black-box process which maintains the full details of the linear solutions, such as shear warping and other subtle effects.

The method is applied to the construction of three-dimensional beam and plate nonlinear models starting from the Saint-Venant rod and Kirchhoff and Mindlin–Reissner plate linear theories, respectively.

1. Introduction

Nonlinear analysis of slender elastic structures requires sound nonlinear modeling to achieve reliable and accurate solutions. The use of a frame-invariant (or objective) structural model, where stress and strain fields are unaffected by finite rigid motions of the body, is a necessary requirement for the analysis [Malvern 1969; Ibrahimbegovic and Taylor 2002]. While it is quite easy to satisfy frame-invariance for three-dimensional bodies using the Cauchy continuum, it can be difficult to obtain a coherent, simple enough modeling for slender structural elements, such as beams or shells, which are more conveniently described as one or two-dimensional fibred continua characterized by three-dimensional displacements and rotations.

In the last few years great efforts have been made to develop nonlinear models of beams, plates, and shells and their finite element implementations. The number of papers on this topic is formidable and this is due, in the authors' opinion, to the complexity of the nonlinear modeling process and to the sometimes unsatisfactory results obtained by FEM discretizations.

The great majority of beam and shell models are based on *geometric exact theories* such as those developed in [Cosserat and Cosserat 1909; Reissner 1972; Simo 1986; Wriggers and Gruttmann 1993; Antman 1995; Auricchio et al. 2008]. Models so generated are geometrically exact, that is, exactly frame-independent, but, being based on simplified assumptions in the constitutive laws relating strain and stress resultants, are generally unable to describe important details already present in the corresponding linear models. This is evident, for example, in the classical Antman–Simo nonlinear beam model where the assumed simplified constitutive law lacks the shear/torsional coupling manifested by the three-dimensional Saint-Venant linear solution [de Saint-Venant 1855] and more subtle nonlinear couplings associated with the section warping, such as the axial-torsional second-order coupling recognized in [Wagner 1936].

Models derived as Ritz–Galerkin approximations by introducing a three-dimensional displacement field in a variational principle allow for more detailed modeling, at least in principle. This approach was followed, for instance, in [Pai and Nayfeh 1994; Pai et al. 1998; Petrov and Géradin 1998a; 1998b; Nayfeh and Pai 2004; Pi et al. 2005; Kim et al. 2005; Chen et al. 2006]. However, extending to finite kinematics the three-dimensional displacement provided by the linear theory, like the nonlinear beam model of [Petrov and Géradin 1998a; 1998b], appears somewhat overcomplex and also requires ad hoc simplifications in order to eliminate spurious locking. Models obtained by the use of problem-dependent engineering nonlinear strain measures, like the beam and shell models of Nayfeh and Pai (see [Nayfeh and Pai 2004]), are only aimed at an essential simplified modeling.

On the other hand the availability of linear structural models for fibred continua derived, using a small displacements hypothesis, from three-dimensional Cauchy equations through appropriate assumptions on the statics and kinematics of the body is notable. Its use, as a basis to generate a corresponding nonlinear model, is then attractive due to the possibility of recovering all the effort spent in developing linear theories in a simplified context. The aim of this paper is to exploit this possibility through the use, in the continuum description, of the corotational approach initially proposed in a FEM context in [Wemper 1969; Belytschko and Glaum 1979; Rankin 1986; Nour-Omid and Rankin 1991] and used to construct a nonlinear finite element starting from a linear one.

In this paper we show that, by transferring this idea from the element to the continuum, we can derive a standard methodology to obtain a frame-indifferent nonlinear modeling which maintains all the details of the embedded linear solution. We call the proposed method the *implicit corotational method* (ICM).

The main idea is to associate a corotational frame (observer) to each point of the three-dimensional continuum so allowing the motion in the neighbor of the point to be split in a pure stretch followed by a pure rotation, according to the decomposition theorem [Malvern 1969; Bonet and Wood 1997]. It will be shown that, using the small strain hypothesis and rotation algebra, the linear stress and linear strain solution fields, when viewed in this corotational frame, can provide accurate approximations for the Biot nonlinear stress and strain tensor fields. Once the corotational rotation is appropriately defined, the local statics and kinematics of the model are recovered from the linear solution as a function of the stress/displacement resultants. Stress and strain fields are then introduced within a mixed variational principle in order to obtain the constitutive laws directly in terms of stress/strain resultants. This completes the ICM definition of the nonlinear model.

The thus-obtained nonlinear model retains all the details of the three-dimensional linear solution, including torsion/shear warping, while its objectivity is ensured implicitly. Furthermore, the use of the mixed formulation and the greater accuracy with which ICM recovers the stress field allow an accurate

description of the constitutive laws in terms of resultants. The proposed approach does not require any ad hoc assumptions about the structural model at hand, nor does it depend on any particular parametrization of the rotation tensor, but rather it actually behaves as a black-box tool able to translate known linear models into the corresponding nonlinear ones. Moreover, the direct use of a mixed (stress/strain) description provides an automatic and implicitly coherent methodology for generating models free of nonlinear locking effects [Garcea et al. 1998; 1999; 2002; 2005] in a format directly suitable for use in FEM implementations.

To better illustrate the features of ICM and to show its potential, the method will be implemented here in order to obtain a nonlinear three-dimensional beam model, derived from the general Saint-Venant linear rod theory, and two nonlinear plate models, derived from the Kirchhoff and Mindlin plate theories. We will also show that from these new structural models, through simplifying assumptions, models already available in the literature can be derived.

The paper is organized as follows: Section 2 introduces the variational framing used in our proposal. Section 3 gives a general overview of ICM and provides the main algebra which is involved in the proposed method. Section 4 presents and discusses some tutorial implementations to underline its main features and to show its effectiveness and reliability in recovering known noteworthy results from the literature. Sections 5 and 6 develop a nonlinear three-dimensional beam model and two nonlinear plate models, and also compare these with similar proposals available in the literature. Section 7 contains some comments and further insights. A summary of the results obtained and possible extensions are given in the concluding section.

The FEM implementation of the ICM models presented here, together with the derivation of nonlinear beam and plate finite elements and their validation through numerical testing, will be discussed in detail in our subsequent, related paper [Garcea et al. 2012].

2. Nonlinear analysis

Nonlinear analysis usually refers to a structure subjected to an assigned loading path $p[\lambda]$, λ being a load parameter controlling the loading process. The structural behavior is governed by the stationarity condition of its total potential energy $\Pi[u, \lambda]$, associated with the external load p and with the configuration u of the structure, with respect to all admissible changes in configuration δu. Using compact notation this condition is expressed by

$$\Pi'[u, \lambda] \delta u = 0, \quad u \in \mathcal{U}, \quad \delta u \in \mathcal{T}, \tag{1}$$

where $\mathcal{U}$ is the manifold of the admissible configurations, $\mathcal{T}$ its tangent space, and the prime stands for Frechèt differentiation with respect to u. Usually (and conveniently) the configuration is described making $\mathcal{U}$ a linear manifold, so $\mathcal{T}$ becomes independent from u. This choice, always possible with an opportune definition of the configuration variables, noticeably simplifies the variations algebra and also is well suited for FEM implementation, where it simplifies the interelement continuity conditions (see [Casciaro 2005]). Condition (1) states a relationship between λ and u describing a curve in $\{u, \lambda\}$ space. The goal of the analysis is to determine this curve, the *equilibrium path*, with particular accuracy in the evaluation of the maximum value λ_c of the load multiplier.

The potential energy can be split into two separate terms, the first expressing the internal strain energy and the second the external load work, that, for the sake of simplicity and without loss of generality, can be assumed as linear in u, as usual in FEM formulation:

$$\Pi[u, \lambda] := \Phi[u] - p[\lambda]u. \tag{2}$$

We also know (see [Garcea et al. 1998; 1999; 2002; 2005]) that, with an appropriate choice of configuration variables $u := \{\sigma, d\}$ and the corresponding choice $p := \{\bar{\varepsilon}, q\}$, these terms could be written in mixed form as

$$\Phi[u] := -\Psi[\sigma] + \sigma \varepsilon[d], \quad p[\lambda]u := q[\lambda]d + \bar{\varepsilon}[\lambda]\sigma, \tag{3}$$

σ and d being the stress and displacement fields, q and $\bar{\varepsilon}$ the external forces and distortions, $\varepsilon[d]$ the compatible strain field associated with d through kinematics, and $\Psi[\sigma]$ the *complementary strain energy* which is usually assumed as quadratic in σ:

$$\Psi[\sigma] := \tfrac{1}{2} C \sigma^2, \tag{4}$$

with C being the positive definite bilinear compliance operator. The $\varepsilon[d]$ function plays an important role in the analysis. Assuming the $\varepsilon[d]$ relationship is linear, we obtain a linear formulation. So we can translate from a linear to a geometrically nonlinear formulation simply by referring to a proper nonlinear expression for the geometrical relation $\varepsilon[d]$, which also implicitly defines the stress σ as work-associated with ε.

The invariance from superposed rigid body motions, that is, objectivity, is an essential prerequisite for the function $\varepsilon[d]$. This requirement is easily satisfied if referring to a three-dimensional Cauchy body, simply by taking d as the displacement field and ε as the Green–Lagrange strain tensor field (by this choice, σ will be the second Piola–Kirchhoff stress tensor field).

In many cases of practical relevance, such as for beams and plates, the three-dimensional behavior can be conveniently described using generalized quantities defined over a one-dimensional (the beam axis) or two-dimensional (the median plane of the plate) domain. In this way, the three-dimensional continuum is converted to a coherently derived one or two-dimensional model, which can more easily be treated. This modeling approach is really convenient and extensively used in linear analysis, but its extension to nonlinear analysis is not so obvious.

Equations (1)–(4) offer a convenient variational framing for the setup of a nonlinear model that only requires as ingredients suitable separate choices for the displacement and stress fields u and σ. When using a material description, stresses are unaffected by a change in observer, so the use of a separate evaluation is extremely convenient in nonlinear modeling. Furthermore we will show that, as usual, we have a better estimate of the stress than that of the displacements.

3. The implicit corotational method

The goal of the implicit corotational method (ICM) is to fully reuse, in a nonlinear context, the results of linear theories in order to obtain objective structural models in a format suitable for FEM implementations. It is shown that, at least in the presence of smooth small strains, an appropriate representation of the strains and stresses (making use of the Biot tensors) together with an appropriate choice of the local frame (through a corotational formulation) is the key to exploiting the linear solution in a nonlinear

setting. This can be achieved by standard methodology, without the need for ad hoc assumptions related to the particular problem at hand, as in [Nayfeh and Pai 2004].

3.1. *Kinematics preliminaries.* Material points $X \in \mathbb{E}^3$ of the reference configuration occupy position x in the current configuration. Rigid-body motions (see [Malvern 1969]) can be described in terms of *translations* and *rotations*. We denote by the vector v the translations and by R the rotations, where R belongs to the space of proper-orthogonal tensors, $R \in SO(3)$. A rigid-body motion characterized by a translation v and rotation R can also be viewed as a change of observer which suffers the inverse rotation $Q = R^T$ and the opposite translation $c = -v$ (T indicates the transpose).

If the positions viewed by the original observer and by the moved observer are denoted by $\bar{x}$ and x, respectively, the relationships between them are

$$\bar{x} = Q^T(x - c), \quad x = Q\bar{x} + c.$$

The displacements viewed by the two observers are denoted by $\bar{u}$ and u, respectively, and will be related by

$$\bar{u} = Q^T(X + u - c) - X, \tag{5a}$$

while the rotations are related by

$$\bar{R} = Q^T R. \tag{5b}$$

By assuming a fixed reference frame with origin O and basis vectors $\{e_1, e_2, e_3\}$ in $\mathbb{E}^3$, both translations and rotations can be defined in terms of three scalar parameters. While the translations are described conveniently by their (covariant) components, the rotations are expressed by the exponential mapping according to

$$R := I + \omega[w], \quad \omega[w] := \sum_{n=1}^{\infty} \frac{1}{n!} W^n, \tag{5c}$$

where W is the following skew-symmetric matrix, whose associated axial vector is $w = w_i e_i$:

$$W := \begin{bmatrix} \cdot & -w_3 & w_2 \\ w_3 & \cdot & -w_1 \\ -w_2 & w_1 & \cdot \end{bmatrix}. \tag{6}$$

3.2. *Kinematics of a three-dimensional Cauchy body.* Consider a Cauchy body (see [Malvern 1969]) referred to the fixed Cartesian frame $(O, \{e_1, e_2, e_3\})$ and let the reference configuration be stress-free. The deformation gradient is $F = I + \nabla u$ where I is the identity tensor and $\nabla(\cdot) := \partial(\cdot)/\partial X$ is the material gradient.

By the polar decomposition theorem (see [Malvern 1969])

$$F = RU, \tag{7}$$

where $R[X]$ is the rotation tensor and $U[X]$ is the (symmetric, positive-definite) stretching tensor.

Recalling (5b), the rotation can be reduced to the identity ($\bar{R} = Q^T R = I$) simply by a change in the observer, making $Q = R$. An objective description requires that the strain measure be independent from R. This requirement is not satisfied, in general, by the so-called infinitesimal strain tensor, denoted

by $\boldsymbol{\varepsilon}_L$, which is obtained by expressing the deformation gradient $\nabla \boldsymbol{u}$ in its symmetric and skew-symmetric parts

$$\boldsymbol{\varepsilon}_L := \boldsymbol{E} = \tfrac{1}{2}(\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^T), \quad \boldsymbol{W} := \tfrac{1}{2}(\nabla \boldsymbol{u} - \nabla \boldsymbol{u}^T). \tag{8}$$

Possible objective strain measures are expressed in the form $\boldsymbol{\varepsilon}^{(n)} := (1/n)(\boldsymbol{U}^n - \boldsymbol{I})$ which, for $n = 1$ and $n = 2$ respectively, provides the *Biot strain tensor* $\boldsymbol{\varepsilon}_b$ and the *Green–Lagrange strain tensor* $\boldsymbol{\varepsilon}_g$:

$$\boldsymbol{\varepsilon}_b := \boldsymbol{R}^T \boldsymbol{F} - \boldsymbol{I}, \quad \boldsymbol{\varepsilon}_g := \tfrac{1}{2}(\boldsymbol{F}^T \boldsymbol{F} - \boldsymbol{I}). \tag{9}$$

The latter can be easily evaluated from $\nabla \boldsymbol{u}$ through a simple quadratic expression:

$$\boldsymbol{\varepsilon}_g = \boldsymbol{E} + \tfrac{1}{2} \nabla \boldsymbol{u}^T \nabla \boldsymbol{u}. \tag{10}$$

However, its identification in terms of linear strain, that is, $\boldsymbol{\varepsilon}_g \approx \boldsymbol{\varepsilon}_L$, requires $\|\nabla \boldsymbol{u}\| \ll 1$ which implies too restrictive hypotheses about the expected solutions to be used in practice.

The Biot strain $\boldsymbol{\varepsilon}_b$ seems to be a better candidate for that purpose. In fact, expressing $\boldsymbol{R}$ in the form (5c) we obtain the relation

$$\boldsymbol{\varepsilon}_b = \boldsymbol{E} + \tfrac{1}{2}(\boldsymbol{\omega}^T \nabla \boldsymbol{u} + \nabla \boldsymbol{u}^T \boldsymbol{\omega} - \boldsymbol{\omega}^T \boldsymbol{\omega}), \tag{11}$$

which implies that $\boldsymbol{\varepsilon}_b \approx \boldsymbol{\varepsilon}_L$ when $\boldsymbol{\omega} \approx \boldsymbol{0}$, obtained by an appropriate change of local observer. If we assume that $\boldsymbol{\varepsilon}_b$ and $\boldsymbol{\omega}$ are small enough (so that $\boldsymbol{\omega} \sim \boldsymbol{W}$), $\boldsymbol{\varepsilon}_b$ can be expressed by its second-order approximation

$$\boldsymbol{\varepsilon}_b \approx \tilde{\boldsymbol{\varepsilon}}_b := \boldsymbol{E} + \tfrac{1}{2}(\boldsymbol{E}\boldsymbol{W} - \boldsymbol{W}\boldsymbol{E} - \boldsymbol{W}^2). \tag{12}$$

Note that by (9), it is

$$\boldsymbol{\varepsilon}_b = \boldsymbol{\varepsilon}_g - \tfrac{1}{2}\boldsymbol{\varepsilon}_b^2 = \tilde{\boldsymbol{\varepsilon}}_b - \tfrac{1}{2}(\boldsymbol{\varepsilon}_b^2 - \boldsymbol{E}^2),$$

so the resulting error in using (12) can be evaluated as $(\boldsymbol{\varepsilon}_b^2 - \boldsymbol{E}^2)/2 \approx \mathbb{O}[\|\boldsymbol{\varepsilon}_b^2 \boldsymbol{\omega}\| + \|\boldsymbol{\varepsilon}_b \boldsymbol{\omega}^2\| + \|\boldsymbol{\omega}^4\|]$.

In our treatment, we deal with problems characterized by large displacements but small strains, so that we can assume $\|\boldsymbol{\varepsilon}_b\| \ll 1$. The key of our approach is to use a suitable change of observer so as to make at the same time the *rotation residual* as small as possible ($\boldsymbol{\omega} \approx \boldsymbol{0}$). That is, we set up the formulation so as to make $\boldsymbol{\omega}$ small enough to allow $\boldsymbol{\varepsilon}_b$ to be evaluated through (12).

From now on $\boldsymbol{R}$ will always denote the polar decomposition rotation in (7) and $\boldsymbol{Q}$ the change of observer rotation in (5a).

3.3. *Constitutive equation for a three-dimensional Cauchy body.*
Stress and strain are work conjugated, thus the selection of a strain measure among $\boldsymbol{\varepsilon}_L$, $\boldsymbol{\varepsilon}_g$, and $\boldsymbol{\varepsilon}_b$ implies the choice of the work-conjugate stress tensor, that is, $\boldsymbol{\sigma}_L$, $\boldsymbol{\sigma}_g$, and $\boldsymbol{\sigma}_b$. Here $\boldsymbol{\sigma}_L$ is the linear stress tensor, $\boldsymbol{\sigma}_g$ is the *second Piola–Kirchhoff stress tensor*, and the *Biot stress tensor* $\boldsymbol{\sigma}_b$ is defined by $\boldsymbol{\sigma}_b := \tfrac{1}{2}(\boldsymbol{U}\boldsymbol{\sigma}_g + \boldsymbol{\sigma}_g \boldsymbol{U})$.

As discussed above, by referring to an appropriate corotational frame such that $\boldsymbol{\omega} \approx \boldsymbol{0}$, we can recover $\boldsymbol{\varepsilon}_b$ using (12) from the displacement gradient $\nabla \boldsymbol{u}$ with great accuracy. To complete the recovery we also need to relate the nonlinear stress and strain tensors, namely $\boldsymbol{\sigma}_b$ and $\boldsymbol{\varepsilon}_b$, through constitutive equations.

Within the mixed formulation followed in Section 2, constitutive equations are obtained by zeroing the variation with respect to the stress field of the mixed form (3) of the strain energy. Denoting by $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$ any work conjugate strain and stress tensors, we can rewrite (3) as

$$\Phi := \int_{\mathcal{B}} (\boldsymbol{\sigma} \cdot \boldsymbol{\varepsilon} - \psi[\boldsymbol{\sigma}]) \, dV, \tag{13}$$

$\mathscr{B}$ being the reference body volume.

The linear solution is characterized by the complementary energy $\psi_L[\boldsymbol{\sigma}_L] := \frac{1}{2}\boldsymbol{\sigma}_L \cdot \boldsymbol{C}_L \boldsymbol{\sigma}_L$, $\boldsymbol{C}_L$ being the *elastic compliance* tensor. Taking the variation with respect to $\boldsymbol{\sigma}_L$, we have

$$\boldsymbol{\varepsilon}_L[\boldsymbol{\sigma}_L] := \boldsymbol{C}_L \boldsymbol{\sigma}_L. \tag{14}$$

So, assuming the same relationship between $\boldsymbol{\sigma}_b$ and $\boldsymbol{\varepsilon}_b$, that is,

$$\boldsymbol{\varepsilon}_b[\boldsymbol{\sigma}_b] := \boldsymbol{C}_b \boldsymbol{\sigma}_b, \quad \boldsymbol{C}_b \approx \boldsymbol{C}_L, \tag{15}$$

the identification $\boldsymbol{\sigma}_b \approx \boldsymbol{\sigma}_L$, with $\boldsymbol{\sigma}_L$ defined by (14) and $\boldsymbol{\varepsilon}_b$ by (12), will imply a local compatibility error

$$\boldsymbol{\varepsilon}_b - \boldsymbol{C}_b \boldsymbol{\sigma}_b \approx \frac{1}{2}(\boldsymbol{E}\boldsymbol{W} - \boldsymbol{W}\boldsymbol{E} - \boldsymbol{W}^2) = \mathbb{O}(\|\boldsymbol{\varepsilon}_b \boldsymbol{\omega}\| + \|\boldsymbol{\omega}^2\|), \tag{16}$$

which can be considered negligible for small $\boldsymbol{\omega}$.

This is a key result of our approach since it proves that the choice of the Biot tensors allows us to extend the constitutive laws of linear elasticity to the nonlinear range. On the contrary, if the Green–Lagrange strain tensor is adopted, the assumption

$$\boldsymbol{\varepsilon}_g[\boldsymbol{\sigma}_g] := \boldsymbol{C}_g \boldsymbol{\sigma}_g, \quad \boldsymbol{C}_g \approx \boldsymbol{C}_L,$$

implies

$$\boldsymbol{\varepsilon}_g - \boldsymbol{\varepsilon}_g[\boldsymbol{\sigma}_g] \approx \mathbb{O}(\|\boldsymbol{\varepsilon}_b^2\|),$$

even in the most favorable case of $\boldsymbol{\omega} \approx 0$, which can hardly be accepted.

3.4. *Use of linear solutions to set up a nonlinear model.* The previous considerations have shown that, when viewed by a local corotational observer rotating with the material neighborhood ($\boldsymbol{Q} \approx \boldsymbol{R}$), both the kinematic and constitutive equations of linear theory correspond to those of the nonlinear problem expressed in terms of the Biot (corotational) tensors, at least within our assumption of small strains.

We need to obtain sufficient information about the stress and strain fields in order to obtain an accurate Galerkin approximation of the nonlinear solution through the use of the variational formulation (1)–(4). So the possibility of relating the nonlinear solution, written in terms of corotational quantities, to the results of a linear theory is an important point in the setup of a general methodology for deriving nonlinear structural models from the available linear ones. This has great potential because the literature on linear theory is very vast and well consolidated and so the possibility of reusing linear results in a nonlinear context provides a real advantage.

Linear theory, during its long history, has yielded a rich variety of complex and elegant solutions for the Cauchy problem obtained with the assumption that displacements (rotations), besides their gradients, are small enough that the deformed configuration can be taken to coalesce into the undeformed configuration when enforcing balance laws. So their derivation implies the use of an appropriate frame (observer), suitable for filtering out rigid rotations in order to minimize this difference. Therefore, the corotational idea is in some way implicit in linear theories.

To better explain this concept, we consider, as an example, the Saint-Venant theory of elastic cylinders. The solution, expressed in terms of the stress field over the cross section domain and the out-of-plane warping of the section, is naturally referred to a local Cartesian system $\{x, y, z\}$, such that the x-axis

is orthogonal to the (average) section plane and the y and z-axes are aligned with its principal directions, that is, with an observer fixed to the cross section. Actually, besides the disturbance produced by the section warping, which is somewhat small being related only to the distortion of the longitudinal fibers due to torsion and shear, all quantities involved in the Saint-Venant solution can be identified as *corotational*.

The Saint-Venant stress solution σ_L can thus be quite naturally identified as a first-order approximation of the nonlinear solution in terms of Biot stresses $\sigma_b[y, z]$. In the same way, the displacement solution $\nabla u_L[y, z]$ can be identified as a first-order approximation of the corresponding nonlinear solution. The identification $\varepsilon_b[y, z] \approx \varepsilon_L[y, z]$ requires $\omega[y, z] \approx 0$ to obtain full geometrical consistency. Actually, this condition cannot be satisfied exactly, because of the pointwise differences in rotation due to the section warping, but we can generally assume a small enough ω to allow the Biot strains to be obtained from (12), by introducing the expressions for E and W directly provided by $\nabla u_L[y, z]$.

The previous discussion can easily be generalized for the nonlinear modeling of fibred continua such as beams or shells, whose description is obtained by splitting the original three-dimensional material reference X into a two or one-dimensional abscissa z lying on a *fiber* (the cross section, in the case of beams, or the transverse fiber, in the case of shells) and a one or two-dimensional abscissa s lying on its *support* (the line axis or the middle surface, respectively), depending on the model considered. The linear modeling defines both stress and displacement fields in terms of the generalized stress $t[s]$ and displacement (translation/rotation) $d[s]$ parameters:

$$\sigma_L[z, s] := \sigma_L[z, t], \quad \nabla u_L[z, s] := \nabla u_L[z, d]. \tag{17}$$

The nonlinear model is then generated, by identifying the linear solution as the Biot stress field, taking $\sigma_b[z, s] \approx \sigma_L[z, t]$, and by rewriting the second relation, through the use of an appropriate local corotational change from $d[s]$ to $\bar{d}[s]$ such that the Biot strains could also be recovered by introducing $\nabla \bar{u}_L = \nabla u_L[z, \bar{d}]$ into (12). We can introduce these fields into the expression of the potential energy (2)–(4), so completing the modeling.

This process is described in greater detail in the following section. We only note here that, besides the use of the quadratic formula (12), all of the nonlinearity of the formulation reduces to the transformation rules relating the kinematic parameters $d[s]$ and $\bar{d}[s]$, which are governed by purely geometric laws (see Section 3.1). It is straightforward to treat the rotation $Q[s]$ of the observer as a finite rotation, without introducing any further approximations. In this way we can obtain a nonlinear modeling which is able to both exploit the full details of the corresponding three-dimensional linear solution and to satisfy the objectivity requirements with respect to the rigid motion of the fiber exactly.

3.5. *Basic elements of ICM.* The basic elements and assumptions of ICM are summarized schematically here:

(1) *Setup of the corotational observer.* By assuming that a linear solution is known in advance, in the form (17), we define for each point of the support an appropriate change in the observer $Q[s]$ suitable to make as small as possible the rotation residual $\omega[z, s]$ on the corresponding fiber $\mathcal{F}[s]$. This change is governed by a relationship between the two sets of kinematical parameters $d[s]$ and $\bar{d}[s]$, as viewed by the fixed and corotational observers. Usually $d[s]$ collects both displacement derivatives $u_{,s}$ and rotation

derivatives $\boldsymbol{R}_{,s}$, and these are transformed according to

$$\bar{\boldsymbol{u}}_{,s} := \boldsymbol{Q}^T(\boldsymbol{X}_{,s} + \boldsymbol{u}_{,s}) - \boldsymbol{X}_{,s}, \quad \bar{\boldsymbol{R}}_{,s} := \boldsymbol{Q}^T \boldsymbol{R}_{,s}. \tag{18a}$$

From now on we denote the derivative with a comma.

(2) *Recovering the stress and displacement fields by the linear solution.* When referring the linear solution to the corotational frame, we obtain an explicit expression for the linear tensors $\nabla \bar{\boldsymbol{u}}_L$ and $\boldsymbol{\sigma}_L$ in $\mathcal{S}[s]$:

$$\nabla \bar{\boldsymbol{u}}_L := \nabla \bar{\boldsymbol{u}}_L[\bar{\boldsymbol{d}}[s]], \quad \boldsymbol{\sigma}_L := \boldsymbol{\sigma}_L[t[s]]. \tag{18b}$$

The corotational (Biot) stress and strain tensors are then identified as

$$\boldsymbol{\sigma}_b := \boldsymbol{\sigma}_L, \quad \boldsymbol{\varepsilon}_b := \boldsymbol{E} + \tfrac{1}{2}(\boldsymbol{E}\boldsymbol{W} - \boldsymbol{W}\boldsymbol{E} - \boldsymbol{W}^2). \tag{18c}$$

The latter can reduce to $\boldsymbol{\varepsilon}_b := \boldsymbol{E}$, if we can assume $\boldsymbol{W}$ to be sufficiently small.

(3) *Constitutive equations.* The constitutive equations, in terms of the fiber generalized parameters $t[s]$ and $\bar{\boldsymbol{d}}[s]$, are recovered by entering the corotational tensors into the mixed form of the strain energy (3). When writing the strain work $\sigma \varepsilon$ and the complementary energy Ψ in the form

$$\sigma \varepsilon := \int \mathcal{W}[s]\,ds, \quad \Psi := \int \psi[s]\,ds, \tag{18d}$$

the local strain work $\mathcal{W}[s]$ and the local complementary energy $\psi[s]$ will be defined as

$$\mathcal{W}[s] := \int_{\mathcal{S}[s]} \boldsymbol{\sigma}_b[t] \cdot \boldsymbol{\varepsilon}_b[d]\,d\mathcal{S}, \tag{18e}$$

$$\psi[s] := \frac{1}{2} \int_{\mathcal{S}[s]} \boldsymbol{\sigma}_b[t] \cdot \boldsymbol{C}_b \boldsymbol{\sigma}_b[t]\,d\mathcal{S}, \tag{18f}$$

where $\boldsymbol{C}_b$ is the compliance tensor that ICM assumes as defined by linear theory as $\boldsymbol{C}_b \approx \boldsymbol{C}_L$. Performing the integration, the previous equations can be rewritten in terms of generalized parameters t and $\bar{\boldsymbol{d}}$ as:

$$\mathcal{W}[s] = t^T \boldsymbol{\varrho}[\bar{\boldsymbol{d}}], \quad \psi[s] = \tfrac{1}{2} t^T \boldsymbol{H} t, \tag{18g}$$

$\boldsymbol{\varrho}[\bar{\boldsymbol{d}}]$ being the generalized strain, work-conjugate with t, and $\boldsymbol{H}$ a generalized compliance operator. The local constitutive law is then obtained directly from (1) and (18g), by differentiating with respect to t:

$$\boldsymbol{\varrho}[\bar{\boldsymbol{d}}] = \boldsymbol{H} t. \tag{18h}$$

(4) *Change of observer.* To complete the nonlinear modeling we only need to express the generalized strain $\boldsymbol{\varrho}[s]$, obtained in Step (3) as a function of the corotational displacement parameters $\bar{\boldsymbol{d}}[s]$, in terms of $\boldsymbol{d}[s]$. As these are related by the change of observer relationships (18a), the frame independence is assured by definition.

The nonlinear model so obtained will satisfy the objectivity requirements with respect to rigid rotations of the fiber exactly, while exploiting the full details of the linear solution, including the nonlinear effects due to the fiber warping which are kept by (12) within second-order accuracy.

Note also that, being based on the mixed variational condition (1) through a separate description for the stress and displacement fields, the model provides in general a mixed approximate solution. Apart from

the simplified assumptions already contained in the linear theory, the error is, however, only produced by the pointwise differences (16) in the strain as evaluated from the displacement field or from the stress field, which is only related to the rotation residual $\boldsymbol{\omega}[s]$ due to warping. Moreover, its effect is further zeroed on average by the use of the integral formulation (18e)–(18h), so it can be considered as negligible.

Obviously, accepting less accuracy, a series of variants of the method are possible, such as the use of a linear approximation for $\boldsymbol{\varepsilon}_b$ ($\boldsymbol{\varepsilon}_b := \boldsymbol{E}$), by making $\boldsymbol{\omega} \approx \boldsymbol{0}$. Some of these variants will be discussed in the following section.

4. Some tutorial implementations of ICM

To better illustrate our proposal, it is convenient to consider the implementation of ICM in some simple tutorial contexts. This gives us the opportunity to clarify the details of the method, check its accuracy in practical cases, and also examine the effects of different approximations.

4.1. *Planar beam with rectangular cross section.*
Consider the planar beam in Figure 1, with a rectangular cross section of dimensions h and $b \ll h$. Assuming as reference its straight undeformed configuration, the beam is referred through a Cartesian material system $X := \{s, y\}$, s being aligned with the barycentric axis and y with the cross section.

Obtaining a linear solution. A linear solution is provided in this case by Saint-Venant–Jourasky theory [Timoshenko 1955]. By referring to an appropriate (corotational) spatial system $\bar{x} := \{\bar{x}_1, \bar{x}_2\}$, aligned with the current section $\mathscr{S} := \mathscr{S}[s]$ as shown in Figure 1, and denoting with $\sigma_L[s, y]$ and $\tau_L[s, y]$ the normal and tangential components of the stress field, these are locally (that is, on $\mathscr{S}$) given by:

$$\sigma_L := \frac{1}{A} N[s] - \frac{y}{J} M[s], \quad \tau_L := \frac{\zeta_{,y}[y]}{A} T[s], \tag{19a}$$

where, according to the usual notation, $A = bh$ and $J = bh^3/12$ are the area and inertia of the section, respectively, $\zeta[y]$ is the so-called stress function expressed in our case by

$$\zeta := \frac{3h^2 y - 4y^3}{2h^2}, \quad \int_S \zeta \, dA = 0, \quad \int_S \zeta_{,y} \, dA = A, \tag{19b}$$

and $N[s]$, $T[s]$, and $M[s]$ are the axial, shear and flexural strengths, respectively, defined, as usual, by

$$N := \int_S \sigma_L[s, y] \, dA, \quad T := \int_S \tau_L[s, y] \, dA, \quad M := \int_S y\sigma_L[s, y] \, dA. \tag{19c}$$

Furthermore, denoting with $\bar{u}_1[s, y]$ and $\bar{u}_2[s, y]$ the components of the displacement field in the spatial system $\{\bar{x}_1, \bar{x}_2\}$, these are locally (that is, in a neighborhood of $\mathscr{S}$) given by

$$\bar{u}_1[s + ds, y] = \bar{u}[s + ds] - \bar{\varphi}[s + ds]y + \gamma_L[s]w[y],$$
$$\bar{u}_2[s + ds, y] = \bar{v}[s + ds], \tag{20a}$$

where

$$\bar{u}[s + ds] := \varepsilon_L[s] \, ds \quad \text{and} \quad \bar{v}[s + ds] := (\theta[s] + \gamma_L[s]) \, ds + \mathbb{O}(ds^2), \tag{20b}$$

are the *average translations* of the section in the $\bar{x}_1$ and $\bar{x}_2$ directions,

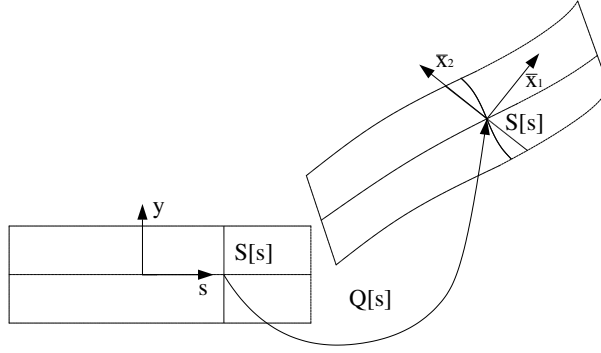$$\bar{\varphi}[s + ds] := \theta[s] + \chi_L[s] \, ds + \mathbb{O}(ds^2) \tag{20c}$$

Figure 1. Planar beam with thin rectangular cross section.

is its *average rotation*, and $w[y]$ is the so-called warping function defined by

$$w := \zeta/k - y, \quad k := \frac{1}{A}\int_S \zeta_{,y}^2\, dA = 6/5, \tag{20d}$$

k being the so-called shear factor and θ a rotation parameter which defines the orientation of the local frame chosen in order to maximize accuracy: for $\theta = 0$ (that is, $\bar{\varphi}[s] = 0$) the frame is locally aligned with the section and for $\theta = -\gamma_L$ (that is, $\bar{v}_{,s}[s] = 0$) it is aligned with the axis line.

The previous expressions allow the recovery of the displacement gradient field $\nabla u[y] = E[y] + W[y]$ on the current section $\mathscr{S}$. We obtain

$$E = \begin{bmatrix} \varepsilon_L - \chi_L y & \gamma_L \zeta_{,y}/2k \\ \text{sym.} & 0 \end{bmatrix}, \quad W = \begin{bmatrix} 0 & \gamma_L(\zeta_{,y}/2k - 1) - \theta \\ \text{skew} & 0 \end{bmatrix}. \tag{20e}$$

From the previous equations, we have

$$\varepsilon_L := \bar{u}_{,s}[s], \quad \gamma_L := \bar{v}_{,s}[s] - \bar{\varphi}[s], \quad \chi_L := \bar{\varphi}_{,s}[s], \tag{20f}$$

so, apart from the alignment parameter θ, the local kinematics $E[\bar{\mathscr{S}}]$ and $W[\bar{\mathscr{S}}]$ are completely defined by the local derivatives $\bar{u}_{,s}$, $\bar{v}_{,s}$, and $\bar{\varphi}_{,s}$ of the translations $\bar{u}[s]$ and $\bar{v}[s]$ and the rotation $\bar{\varphi}[s]$ associated with the section. It is also convenient to introduce the quantities

$$\bar{\alpha}[s+ds] := \bar{\varphi}[s+ds] - \theta[s]. \tag{20g}$$

Obviously we have $\bar{\alpha}[s] = 0$ and $\bar{\alpha}_{,s} = \bar{\varphi}_{,s}$. These conditions implicitly define the alignment of the local corotational observer associated with the section s. In fact assuming $\alpha[s]$ to be a finite rotation angle associated with the fiber $\mathscr{S}$, as viewed by a fixed observer, and $\bar{\alpha}$ its value viewed by a local observer, this implies a relative alignment angle $\bar{\alpha}[s] - \alpha[s] = -\alpha[s]$ between the two observers and defines the rotation matrix

$$Q[s] := \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}, \tag{20h}$$

relating the two observers.

Recovering nonlinear stresses, strains, and constitutive equations. As stated in the previous section, we assume that the Biot stress field $\boldsymbol{\sigma}_b$ is directly described by (19a), its relevant components being defined by $\sigma_{11} = \sigma_L$ and $\sigma_{12} = \tau_L$. By substitution into (18f), we obtain

$$\psi[s] := \frac{1}{2} \int_{\bar{\mathcal{G}}} \left(\frac{\sigma_{11}^2}{E} + \frac{\sigma_{12}^2}{G} \right) dA = \frac{1}{2} \left(\frac{N^2}{EA} + \frac{kT^2}{GA} + \frac{M^2}{EJ} \right), \tag{21a}$$

E and G being the normal and shear elastic moduli of the material. Conversely, the Biot strain field will be obtained by substituting (20e) into (12). The evaluation of internal work (18e) provides

$$\mathcal{W}[s] := \int_S (\sigma_{11} \varepsilon_{11} + 2\sigma_{12} \varepsilon_{12}) \, dA = N\varepsilon + T\gamma + M\chi, \tag{21b}$$

where the strain parameters ε, γ, and χ are defined by:

$$\varepsilon := \varepsilon_L + \tfrac{19}{48} \gamma_L^2 + \theta \gamma_L + \tfrac{1}{2} \theta^2, \quad \gamma := \gamma_L - \varepsilon_L (\theta + \tfrac{1}{2} \gamma_L), \quad \chi := \chi_L,$$

that is, recalling (20f) and (20g),

$$\varepsilon := \bar{u}_{,s} + \tfrac{19}{48} \bar{v}_{,s}^2 + \tfrac{10}{48} \theta \bar{v}_{,s} - \tfrac{5}{48} \theta^2, \quad \gamma := \bar{v}_{,s} - \theta - \tfrac{1}{2} \bar{u}_{,s} (\bar{v}_{,s} + \theta), \quad \chi := \bar{\alpha}_{,s}. \tag{21c}$$

By relating (21a) and (21b) through the Clapeyron equivalence $2\Phi_c = \mathcal{W}$, we also obtain

$$N = EA\varepsilon, \quad T = kGA\gamma, \quad M = EJ\chi, \tag{21d}$$

which provides the elastic laws for the nonlinear model.

Applying corotational kinematics. Equations (21a)–(21d) provide the explicit expressions for the local energy terms needed for the use of the variational condition (1)–(3). In order to complete the ICM nonlinear modeling, we need to relate the displacements $\bar{u}[s]$ and $\bar{v}[s]$ and the rotation $\bar{\varphi}[s]$ associated with the section $\mathcal{G}[s]$, defined by reference to the local corotational frame $\{\bar{x}_1, \bar{x}_2\}$, with the corresponding $u[s]$, $v[s]$, and $\varphi[s]$ referred to a global fixed frame x_1, x_2. The relationship is governed by (18a). The two systems are related by the rotation matrix (20h), so we have

$$\bar{u}_{,s}[s] = (u_{,s} + 1) \cos\alpha + v_{,s} \sin\alpha - 1, \quad \bar{v}_{,s}[s] = v_{,s} \cos\alpha - (u_{,s} + 1) \sin\alpha, \quad \bar{\alpha}_{,s}[s] = \alpha_{,s}. \tag{22}$$

This relationship, when coupled with (21c), provides the nonlinear relationships between the generalized strain $\boldsymbol{\varrho}[\boldsymbol{d}] = \{\varepsilon, \gamma, \chi\}$ and the generalized displacement $\boldsymbol{d} = \{u_{,s}, v_{,s}, \varphi_{,s}\}$, which completes the definition of the nonlinear model. Note that, in order to obtain an explicit expression for ε and γ, we need to set the alignment parameter θ in (21c). The ICM accuracy being related to the smallness of the rotation residual $\boldsymbol{\omega}$, an appropriate choice could be that of the zeroing tensor $\boldsymbol{W}$ in weighted average, that is,

$$\int_{\bar{S}} \zeta_{,y} (\theta - (1 - \psi_{,y}/2k)) \, dA = 0,$$

which provides $\theta = -\gamma_L/2$, and corresponds to an average alignment between the section and the axis line. By this choice we obtain

$$\varepsilon := \bar{u}_{,s} + \tfrac{1}{12} \bar{v}_{,s}^2, \quad \gamma := 2\bar{v}_{,s}, \quad \chi := \bar{\alpha}_{,s}. \tag{23}$$

However, other choices are also possible, for example, the choice $\theta = 0$ (aligned with the section) or $\theta = -\gamma_L$ (aligned with the line axis). Different choices only reflect how the alignment of the corotational frame treats the average rotation $\frac{1}{2}\gamma_L$ due to warping; it is accounted for exactly through (22) by the choice $\theta = -\gamma_L/2$ and only to second-order accuracy by the other choices, while the use of corotational algebra (22) always assures, by definition, frame independence with respect to a rigid rotation of the section. Different choices only produce a small $\mathbb{O}^3(\varepsilon_L, \gamma_L)$ difference in the final expression of $\varrho[d]$, so they can be considered as equivalent, within our assumption of small strains, and the actual choice could be suggested by practical convenience.

Further insights. Some further comments and remarks are useful.

(1) With regard to slender structures, we can expect $T[s]$ will be much smaller than $N[s]$ in practical contexts and then assume $\gamma \ll \varepsilon$. So, the quadratic term $\bar{v}_{,s}^2/12 = \gamma_L^2/48$ in (23) will be really very small and can be considered as irrelevant. If obliged we obtain, through (22),

$$\varepsilon = (1+u_{,s})\cos\alpha + v_{,s}\sin\alpha - 1, \quad \gamma = 2(v_{,s}\cos\alpha - (1+u_{,s})\sin\alpha), \quad \chi = \alpha_{,s}. \tag{24a}$$

Note that this approximation corresponds to making $\zeta_{,y}/k \approx 0$ in (20e), that is, because of (20d), assuming a linear warping $\bar{\gamma}w[y] \approx -\bar{\gamma}y$ in the evaluation $W[s]$.

(2) The effect of warping, being related to the shear strain $\gamma[s]$, can be considered so small as to be neglected in most practical cases. Making $w[y] \approx 0$, $W[s]$ will be zeroed by the choice $\theta = 0$ and consequently $\varepsilon_b \approx E$. We also have $\bar{\alpha}[s] = \bar{\varphi}[s]$, so the ICM kinematics reduce to the well known Antman beam kinematics [Antman 1995]:

$$\varepsilon = (1+u_{,s})\cos\varphi + v_{,s}\sin\varphi - 1, \quad \gamma = v_{,s}\cos\varphi - (1+u_{,s})\sin\varphi, \quad \chi = \varphi_{,s}. \tag{24b}$$

(3) By neglecting the shear strain everywhere, that is, assuming $\bar{\gamma} \approx 0$, we can use the condition $\gamma = 0$ in the second part of (24b) for relating φ to $u_{,s}$ and $v_{,s}$. With some algebra, (24b) becomes

$$\varepsilon = \sqrt{(1+u_{,s})^2 + v_{,s}^2} - 1, \quad \chi = \frac{v_{,ss} + v_{,ss}\,u_{,s} - v_{,s}\,u_{,ss}}{\sqrt{(1+u_{,s})^2 + v_{,s}^2}}, \tag{24c}$$

which coincide with the kinematic relationships developed in [Nayfeh and Pai 2004]. When assuming $\varepsilon \ll 1$, the previous equations can be simplified into

$$\varepsilon \approx \varepsilon + \tfrac{1}{2}\varepsilon^2 = u_{,s} + \tfrac{1}{2}(u_{,s}^2 + v_{,s}^2), \quad \chi \approx (1+\varepsilon)\chi = v_{,ss} + v_{,ss}\,u_{,s} - u_{,ss}\,v_{,s}. \tag{24d}$$

(4) All previous variants only differ in the treatment of shear and axial strains in the expression of $W[s]$. When, as generally happens for slender beams, $\gamma \ll 1$ and $\varepsilon \ll 1$, their differences become negligible. To give an example, we can refer to a simply supported Euler beam of length L, subjected to a compressive axial force $F[\lambda] := \lambda\pi^2 EJ/L^2$. The analytical solution for the postbuckling path coming from the use of (24b) can be found in [Salerno and Lanzo 1997] in the form

$$\lambda = \lambda_b + \tfrac{1}{2}\ddot{\lambda}_b\xi^2 + \cdots,$$

where ξ is the end-section rotation angle and, introducing stiffness ratios $\beta_s := EJ/GAL^2$, $\beta_a := EJ/EAL^2$, and $\beta := \beta_s - \beta_a$, the buckling load λ_b and postbuckling curvature $\ddot{\lambda}_b$ are defined by

$$\lambda_b + \lambda_b^2 \beta = 1, \quad \ddot{\lambda}_b = \frac{\lambda_b}{4} \left(\frac{1 + 4\lambda_b \beta}{1 + 2\lambda_b \beta} \right).$$

The same expressions can also be obtained from (24a), if $\beta := 2\beta_s - \beta_a$, and from (24c) and (24d), assuming $\beta_s \approx 0$ and $\beta_s \approx \beta_a$, respectively (the solutions can be recovered using algebraic manipulators, so their details are omitted here). We generally have $\beta_s \ll 1$ and $\beta_a \ll 1$, so all possible choices give essentially the same result ($\lambda_b \approx 1$, $\ddot{\lambda}_b \approx 1/4$) and can be considered as equivalent for practical purposes. We can also mention that usually $\beta_a < \beta_s$, so the simpler expressions (24d) could be considered better than (24c).

(5) Finally, for a better understanding of the role of (12) in our proposal, it is convenient to refer to the limit case where the local and fixed frames coincide, that is, $\bar{u},_s = u,_s$, $\bar{v},_s = v,_s$. Making $\gamma_L \approx 0$ and $\bar{\varepsilon} \ll 1$ into (21c), we obtain

$$\varepsilon = u,_s + \tfrac{1}{2} v,_s^2, \quad \chi = v,_{ss}, \tag{25}$$

which correspond to the standard second-order kinematics used in simplified (technical) modelings. Obviously, this equation satisfies objectivity only to a second-order accuracy, and then requires some external tool in FEM management to obtain frame invariance at the element level (see [Garcea et al. 2009]).

4.2. *Thin walled beam under axial force and torsion.* Consider a spatial beam with a thin-walled cross-shaped section and subjected to axial force and torsion, as shown in Figure 2, the section thickness t being small when compared with its size h. Assuming as reference its straight undeformed configuration the beam will be referred to through a Cartesian material system $X := \{s, y, z\}$, s being aligned with the barycentric axis and y and z lying in the cross section.

Obtaining a linear solution. A linear solution is provided in this case by the simplified Saint-Venant torsion theory for thin-walled beams [de Saint-Venant 1855]. By referring to an appropriate spatial
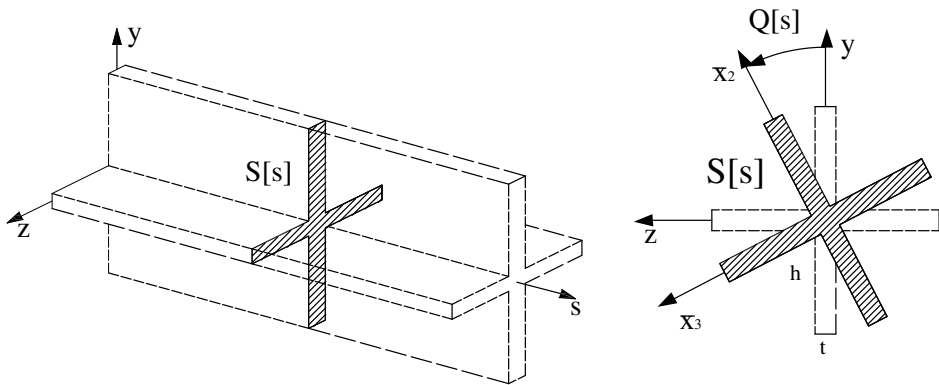

Figure 2. Cross-shaped beam under axial force and torsion.

system $\bar{x} := \{\bar{x}_1, \bar{x}_2, \bar{x}_3\}$, aligned with the current section $\mathscr{S} := \mathscr{S}[s]$ as shown in Figure 1, and denoting with $\sigma_{xx}[s, y, z]$, $\tau_{xy}[s, y, z]$, and $\tau_{xz}[s, y, z]$ the normal and tangential components of the stress field, these are locally (that is, on $\mathscr{S}$) given by:

$$\sigma_{xx} = N[s]/A, \quad \tau_{xy} = (w[y, z]_{,y} - z) M_t[s]/J_t, \quad \tau_{xz} = (w[y, z]_{,z} + y) M_t[s]/J_t, \quad (26a)$$

where $w[y, z]$ is an auxiliary warping which, in the case considered, can be evaluated as

$$w[y, z] \approx \begin{cases} yz & \text{if } -t/2 \le y \le t/2, \\ -yz & \text{if } -t/2 \le z \le t/2. \end{cases} \quad (26b)$$

$A \approx 2bt$ is the area of the section, J_t is its torsional inertia, defined by

$$J_t := \int_{\bar{S}} \{(w[y, z]_{,y} - z)^2 + (w[y, z]_{,z} + y)^2\} \, dA \approx \tfrac{2}{3} t^3 h, \quad (26c)$$

and N and M are the strength resultants

$$N := \int_{\bar{S}} \sigma_{xx} \, dA, \quad M := \int_{\bar{S}} (\tau_{xz} y - \tau_{xy} z) \, dA. \quad (26d)$$

Furthermore, denoting with $\bar{u}_1[s, y, z]$, $\bar{u}_2[s, y, z]$, and $\bar{u}_3[s, y, z]$ the components of the displacement field in the corotational frame, these are locally given by:

$$\begin{aligned} \bar{u}_1[s+ds, y, z] &= \bar{u}[s+ds] + \chi_L[s] w[y, z], \\ \bar{u}_2[s+ds, y, z] &= -z\bar{\varphi}[s+ds], \\ \bar{u}_3[s+ds, y, z] &= y\bar{\varphi}[s+ds], \end{aligned} \quad (27a)$$

where $\bar{u}$ and $\bar{\varphi}$ are the *average axial displacement* and the *torsional rotation* of the section:

$$\bar{u}[s+ds] := \varepsilon_L[s] \, ds, \quad \bar{\varphi}[s+ds] := \chi_L[s] \, ds, \quad (27b)$$

The previous expressions allow the recovery of the displacement gradient $\nabla u[y, z] = E[y, z] + W[y, z]$ field on the current section $S[s]$. We obtain

$$E = \begin{bmatrix} \varepsilon_L & \tfrac{1}{2}(w_{,y} - z)\chi_L & \tfrac{1}{2}(w_{,z} + y)\chi_L \\ \tfrac{1}{2}(w_{,y} - z)\chi_L & 0 & 0 \\ \tfrac{1}{2}(w_{,z} - y)\chi_L & 0 & 0 \end{bmatrix},$$

$$W = \begin{bmatrix} 0 & \tfrac{1}{2}(w_{,y} + z)\chi_L & \tfrac{1}{2}(w_{,z} - y)\chi_L \\ -\tfrac{1}{2}(w_{,y} + z)\chi_L & 0 & 0 \\ -\tfrac{1}{2}(w_{,z} - y)\chi_L & 0 & 0 \end{bmatrix}.$$

From previous equations, we have

$$\varepsilon_L[s] = \bar{u}_{,s}[s], \quad \chi_L[s] := \bar{\varphi}_{,s}[s], \quad (27c)$$

so the local kinematic $E[y, z] + W[y, z]$ is completely defined by the local derivatives of the displacement $\bar{u}[s]$ and the rotation $\bar{\varphi}[s]$ associated with the section.

Recovering nonlinear stresses, strains, and constitutive equations. Assuming that the Biot stress field $\boldsymbol{\sigma}_b$ is defined by the linear solution (26a), its relevant components being defined by $\sigma_{11} := \sigma_{xx}$, $\sigma_{12} := \tau_{xy}$, and $\sigma_{13} := \tau_{xz}$, the complementary energy (18f) and the strain work (18e) become

$$\psi[s] := \frac{1}{2} \int_A \left\{ \frac{\sigma_{11}^2}{E} + \frac{\sigma_{12}^2 + \sigma_{13}^2}{G} \right\} dA = \frac{1}{2} \left(\frac{N}{EA} + \frac{M_t}{GJ_t} \right), \tag{28a}$$

$$\mathcal{W} = \int_A \left\{ \sigma_{11}\varepsilon_{11} + 2\tau_{12}\varepsilon_{12} + 2\tau_{13}\varepsilon_{13} \right\} dA = N\varepsilon[s] + M\chi[s], \tag{28b}$$

where, introducing the *polar inertia*

$$J_1 := \int_{\mathscr{S}} (y^2 + z^2) \, dA \approx \frac{1}{6} tb^3, \tag{28c}$$

the strain generalized parameters ε and χ are expressed by

$$\varepsilon[s] = \bar{u}_{,s} + \frac{(4J_1 - J_t)}{8A} \bar{\varphi}_{,s}^2, \quad \chi[s] = \bar{\varphi}_{,s} - \frac{1}{2}\bar{u}_{,s}\,\bar{\varphi}_{,s}, \tag{28d}$$

where use is made of (27c).

By relating (28a) and (28b) through Clapeyron's equivalence, we also obtain the elastic laws

$$N = EA\varepsilon, \quad M = GJ\chi, \tag{28e}$$

so completing the modeling, in the corotational reference frame.

Applying corotational kinematics. To complete the ICM modeling, we need to relate the generalized displacement derivatives $\bar{u}_s$ and $\bar{\varphi}_{,s}$, referring the corotational frame to the corresponding one referring to the fixed frame. As the relative rotation $\boldsymbol{Q}$ between the two systems is, in our case, described by a rotation vector aligned with the x_1 axis, we simply have $\bar{u}_{,s} = u_{,s}$, $\bar{\varphi}_{,s} = \varphi_{,s}$, and so we finally obtain

$$\varepsilon[s] = u_{,s} + \frac{(4J_1 - J_t)}{8A} \varphi_{,s}^2, \quad \chi[s] = \left(1 - \frac{1}{2}u_{,s}\right)\varphi_{,s}, \tag{29a}$$

which defines the nonlinear beam kinematics and, together with (28a) and (28b) completes the definition of the ICM modeling.

Note that terms J_1 and J_t appearing in the first section of (28d) take into account the rotations due to the helicoid distortion of longitudinal fibers and those due to shear strains, respectively. So we have $J_t \ll J_1$ in general (actually we have $J_t = \frac{2}{3}bt^3$ and $J_1 = \frac{1}{6}tb^3$). So, neglecting the small term $\frac{1}{2}u_{,s}$, (29a) can be simplified into

$$\varepsilon[s] := \bar{u}_{,s} + \frac{1}{2}\frac{J_1}{A}\varphi_{,s}^2, \quad \chi[s] := \varphi_{,s}, \tag{30}$$

which coincides with that derived by Wagner [1936] in his study of beams in torsion.

It is also worth mentioning that the term $\frac{1}{2}\varphi_{,s}^2 J_1/A$ in the expression of ε implies a nonlinear coupling between axial elongation and torsional curvature which comes from warping and is generally neglected in approaches based on rigid motion kinematics of the section. This coupling, which we call the Wagner effect, can however play an important role in activating flexural/torsional buckling of slender thin-walled beams, as we will show in our subsequent, related paper [Garcea et al. 2012], so generally it cannot be ignored.

5. Nonlinear beam model based on Saint-Venant general rod theory

ICM is now applied to recover a nonlinear three-dimensional beam model based on the Saint-Venant (SV) general rod theory [de Saint-Venant 1855]. For full details of the SV problem we refer readers to classic works such as [Timoshenko 1955; Fichera 1977] while a numerical solution of the problem can be found in [Petrolo and Casciaro 2004]. Interesting, also for comparison with the model presented here, is [Petrov and Géradin 1998a; 1998b].

We use the same notation introduced in the tutorial section and also assume the beam is straight in its reference configuration and is referred to by a Cartesian material system $X := \{s, y, z\}$, s being aligned with the barycentric axis and y and z with the principal direction of the cross section. We will denote with A the area and with

$$J_1 := \int_{\mathscr{S}} (y^2 + z^2)\, dA, \quad J_2 := \int_{\mathscr{S}} z^2\, dA, \quad J_3 := \int_{\mathscr{S}} y^2\, dA,$$

the polar and flexural inertia of the section. The linear solution is referred to a corotational spatial system $\bar{x} := \{\bar{x}_1, \bar{x}_2, \bar{x}_3\}$, aligned with the current section $\mathscr{S} := \mathscr{S}[s]$, as shown in Figure 3, $Q[s]$ being the rotation matrix relating the two systems.

5.1. *Obtaining the SV linear solution: statics.* ICM assumes that the Biot stress tensor components in the material frame are coincident with the stress components recovered by the linear SV solution in the corotational frame. From now on, we express the Biot stress directly as obtained by the Saint-Venant solution obtaining

$$\boldsymbol{\sigma}_b = \begin{bmatrix} \sigma & \boldsymbol{\tau}^T \\ \boldsymbol{\tau} & \mathbf{0} \end{bmatrix},$$

where $\sigma := \sigma_{11}$ is the normal stress and $\boldsymbol{\tau} := [\sigma_{12}, \sigma_{13}]^T$ collects in a single vector the tangential stresses.

Introducing the force and moment resultants $N = \{N_1, N_2, N_3\}$ and $M = \{M_1, M_2, M_3\}$ over the section, as usual defined by

$$\begin{aligned} N_1 &= \int_{\mathscr{S}} \sigma_{11}\, dA, & N_2 &= \int_{\mathscr{S}} \sigma_{12}\, dA, & N_3 &= \int_{\mathscr{S}} \sigma_{13}\, dA, \\ M_1 &= \int_{\mathscr{S}} (y\sigma_{13} - z\sigma_{12})\, dA, & M_2 &= \int_{\mathscr{S}} z\sigma_{11}\, dA, & M_3 &= -\int_{\mathscr{S}} y\sigma_{11}\, dA, \end{aligned} \tag{31a}$$
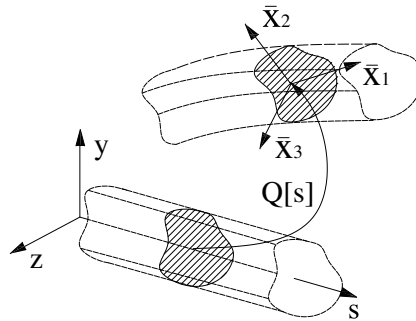

Figure 3. SV three-dimensional beam, material, and corotational frames.

and collecting them into the strength vectors

$$
t_\sigma := \begin{bmatrix} N_1 \\ M_2 \\ M_3 \end{bmatrix}, \quad t_\tau := \begin{bmatrix} M_1 \\ N_2 \\ N_3 \end{bmatrix}, \tag{31b}
$$

the stress solution can be expressed as:

$$
\sigma = D_\sigma[y, z]t_\sigma, \quad \tau = D_\tau[y, z]t_\tau, \tag{31c}
$$

where

$$
D_\sigma[y, z] := \left[\frac{1}{A}, \frac{z}{J_2}, -\frac{y}{J_3} \right], \quad D_\tau[y, z] := \begin{bmatrix} d_2^T[y, z] \\ d_3^T[y, z] \end{bmatrix}, \tag{31d}
$$

and vectorial stress functions $d_2[y, z]$ and $d_3[y, z]$ are defined by a Laplace/Neumann differential problem whose numerical solution can be easily obtained through a finite element discretization of the section as, for instance, performed in [Petrolo and Casciaro 2004]. Also as in that reference, the interpolation (31c) allows us to obtain the expression of the complementary energy $\psi[s]$ associated with the current section $\mathcal{S}[s]$ in terms of the strength vectors t_σ and t_τ:

$$
\psi[s] := \frac{1}{2} \left(\frac{1}{E} t_\sigma^T H_\sigma t_\sigma + \frac{1}{G} t_\tau^T H_\tau t_\tau \right), \tag{32a}
$$

where

$$
H_\sigma := \int_{\mathcal{S}} D_\sigma^T D_\sigma \, dA, \quad H_\tau := \int_{\mathcal{S}} D_\tau^T D_\tau \, dA. \tag{32b}
$$

Matrix H_σ, gives the contribution of normal stresses over the section and, being the corotational local frame aligned with the principal frame of the section, it is simply obtained as a diagonal matrix

$$
H_\sigma = \mathrm{diag} \left[\frac{1}{A} \ \ \frac{1}{J_2} \ \ \frac{1}{J_3} \right].
$$

Matrix H_τ gives the contribution of tangential stresses and generally results in a full symmetric matrix which can however be obtained as a by-product of the FEM solution process which provides the stress functions $d_2[y, z]$ and $d_3[y, z]$ (see [Petrolo and Casciaro 2004] for details). It is convenient to introduce the vector quantities

$$
t = \begin{bmatrix} N \\ M \end{bmatrix} \tag{32c}
$$

in order to arrange (32a) in the more compact form

$$
\psi[s] := \tfrac{1}{2} t^T H t. \tag{32d}
$$

5.2. *Obtaining the SV linear solution: kinematics.* As the in-plane deformation of the section must be inessential coming from the Poisson effect, the SV displacement solution, in a neighbor of $\mathcal{S}[s]$, can be expressed as

$$
\begin{aligned}
\bar{u}_1[s + ds, y, z] &= ds(\varepsilon_{L1} - y\chi_{L3} + z\chi_{L2}) + w[y, z], \\
\bar{u}_2[s + ds, y, z] &= ds(\varepsilon_{L2} - z\chi_{L1}) + \mathbb{O}(ds^2), \\
\bar{u}_3[s + ds, y, z] &= ds(\varepsilon_{L3} + y\chi_{L1}) + \mathbb{O}(ds^2),
\end{aligned} \tag{33}
$$

where the out-of-plane warping function $w[y, z]$, which can be assumed without loss of generality to satisfy the conditions

$$\int_A w[y, z] \, dA = \int_A y w[y, z] \, dA = \int_A z w[y, z] \, dA = 0, \tag{34}$$

is related to stress functions $d_2[y, z]$ and $d_3[y, z]$, as we will show in the sequel, and ε_{Lk} and χ_{Lk} are strain parameters to be related to the stress parameters N_k and M_k through the section constitutive law. Therefore, the displacement gradient $\nabla \bar{\boldsymbol{u}}$ assumes in $\mathscr{S}$ the following expression:

$$\nabla \bar{\boldsymbol{u}} = \begin{bmatrix} \varepsilon_L & \nabla w^T \\ \boldsymbol{\theta} & \boldsymbol{0} \end{bmatrix}, \tag{35a}$$

where

$$\varepsilon_L := \varepsilon_{L1} - y \chi_{L3} + z \chi_{L2}, \quad \boldsymbol{\theta} := \begin{bmatrix} \varepsilon_{L2} - z \chi_{L1} \\ \varepsilon_{L3} + y \chi_{L1} \end{bmatrix}, \quad \nabla \boldsymbol{w} := \begin{bmatrix} w_{,y} \\ w_{,z} \end{bmatrix}, \tag{35b}$$

and its symmetric and skew-symmetric parts become

$$\bar{\boldsymbol{E}} := \begin{bmatrix} \varepsilon_L & \frac{1}{2}(\boldsymbol{\theta} + \nabla w)^T \\ \frac{1}{2}(\boldsymbol{\theta} + \nabla w) & \boldsymbol{0} \end{bmatrix}, \quad \bar{\boldsymbol{W}} := \begin{bmatrix} 0 & -\frac{1}{2}(\boldsymbol{\theta} - \nabla w)^T \\ \frac{1}{2}(\boldsymbol{\theta} - \nabla w) & \boldsymbol{0} \end{bmatrix}. \tag{35c}$$

By collecting the six strain constants ε_{L1}, ε_{L2}, ε_{L3}, χ_{L1}, χ_{L2}, and χ_{L3} into the vectors

$$\bar{\boldsymbol{d}}_\sigma := \begin{bmatrix} \varepsilon_{L1} \\ \chi_{L2} \\ \chi_{L3} \end{bmatrix}, \quad \bar{\boldsymbol{d}}_\tau := \begin{bmatrix} \chi_{L1} \\ \varepsilon_{L2} \\ \varepsilon_{L3} \end{bmatrix}, \tag{35d}$$

we can express all the strain quantities in (35b) using the following interpolation defined as a solution of the linear SV problem:

$$\varepsilon_L := \boldsymbol{D}_\varepsilon[y, z] \bar{\boldsymbol{d}}_\sigma, \quad \boldsymbol{\theta} := \boldsymbol{D}_\theta[y, z] \bar{\boldsymbol{d}}_\tau, \quad \nabla w := \boldsymbol{D}_w[y, z] \bar{\boldsymbol{d}}_\tau, \tag{35e}$$

$\boldsymbol{D}_\epsilon$ and $\boldsymbol{D}_\theta$ being the interpolation matrices directly defined by

$$\boldsymbol{D}_\varepsilon := \begin{bmatrix} 1 & z & -y \end{bmatrix}, \quad \boldsymbol{D}_\theta := \begin{bmatrix} -z & 1 & 0 \\ y & 0 & 1 \end{bmatrix}, \tag{35f}$$

while $\boldsymbol{D}_w[y, z]$ is related to the stress interpolation (31c) through the elastic laws $\sigma_{12} = G\varepsilon_{12}$ and $\sigma_{13} = G\varepsilon_{13}$. Using (31c) and (32b), we obtain

$$\boldsymbol{\gamma}_L := \boldsymbol{\theta} + \nabla w = \boldsymbol{D}_\gamma[y, z] d_\tau, \quad \boldsymbol{D}_\gamma := \boldsymbol{D}_\tau \boldsymbol{H}_\tau^{-1}, \tag{35g}$$

and therefore

$$\boldsymbol{D}_w := \boldsymbol{D}_\gamma[y, z] - \boldsymbol{D}_\theta. \tag{35h}$$

Linear evaluation for Biot strains. We have already shown that in some cases (for example, for compact sections), the effects of warping can be ignored. That is, we can assume $\overline{W} \approx \mathbf{0}$ in the Biot strain expression (12), without introducing appreciable errors. From this assumption we obtain $\boldsymbol{\varepsilon}_b = \overline{E}$ and the internal work becomes the same as in the linear elastic case

$$\mathcal{W} := \boldsymbol{t}_\sigma^T \boldsymbol{\varrho}_\sigma + \boldsymbol{t}_\tau^T \boldsymbol{\varrho}_\tau, \quad \boldsymbol{\varrho}_\sigma := \bar{\boldsymbol{d}}_\sigma, \quad \boldsymbol{\varrho}_\tau := \bar{\boldsymbol{d}}_\tau. \tag{36a}$$

So, by rearranging the strain parameters in a single vector

$$\boldsymbol{\varrho} := \boldsymbol{d}_L = \begin{bmatrix} \boldsymbol{\varepsilon}_L \\ \boldsymbol{\chi}_L \end{bmatrix}, \quad \boldsymbol{\varepsilon}_L := \begin{bmatrix} \epsilon_{L1} \\ \epsilon_{L2} \\ \epsilon_{L3} \end{bmatrix}, \quad \boldsymbol{\chi}_L := \begin{bmatrix} \chi_{L1} \\ \chi_{L2} \\ \chi_{L3} \end{bmatrix}, \tag{36b}$$

and recalling (32d), the stationarity of the section strain energy $\psi[\boldsymbol{t}] - \mathcal{W}[\boldsymbol{t}, \boldsymbol{\omega}_L]$ with respect to stress vector $\boldsymbol{t}$ directly provides the constitutive law

$$\boldsymbol{t} = \boldsymbol{H}^{-1} \boldsymbol{\varrho}. \tag{36c}$$

Quadratic evaluation for Biot strains. More generally, the effect of warping cannot be ignored, even if we can assume $\overline{W}[y, z]$ is small enough to allow $\boldsymbol{\varepsilon}_b$ to be evaluated by the quadratic formula (12). Using (35c) we obtain $\boldsymbol{\varepsilon}_b[y, z]$ in the form

$$\boldsymbol{\varepsilon}_b = \begin{bmatrix} \varepsilon & \boldsymbol{\gamma}^T/2 \\ \boldsymbol{\gamma}/2 & \boldsymbol{\Gamma} \end{bmatrix}, \tag{37a}$$

where its relevant components $\varepsilon[y, z]$ and $\boldsymbol{\gamma}[y, z]$ are defined by

$$\varepsilon = \varepsilon_L + \tfrac{1}{2} \boldsymbol{\theta}^T \boldsymbol{\theta} - \tfrac{1}{8} \boldsymbol{\gamma}_L^T \boldsymbol{\gamma}_L, \quad \boldsymbol{\gamma} = \boldsymbol{\gamma}_L + \tfrac{1}{2} \varepsilon_L \boldsymbol{\gamma}_L - \varepsilon_L \boldsymbol{\theta}, \tag{37b}$$

and 2×2 matrix $\boldsymbol{\Gamma}$ is inessential in our treatment, being multiplied by zero stresses when evaluating the strain work $\mathcal{W}$.

Using (31c) and (37b) and introducing the strain parameter vectors

$$\boldsymbol{\varrho}_\sigma := \begin{bmatrix} \varepsilon_{b1} \\ \chi_{b2} \\ \chi_{b3} \end{bmatrix}, \quad \boldsymbol{\varrho}_\tau := \begin{bmatrix} \chi_{b1} \\ \varepsilon_{b2} \\ \varepsilon_{b3} \end{bmatrix}, \tag{37c}$$

we have

$$\mathcal{W} := \int_{\mathcal{S}} (\sigma \varepsilon + \boldsymbol{\tau}^T \boldsymbol{\gamma}) \, dA = \boldsymbol{t}_\sigma^T \boldsymbol{\varrho}_\sigma + \boldsymbol{t}_\tau^T \boldsymbol{\varrho}_\tau, \tag{37d}$$

where

$$\boldsymbol{\varrho}_\sigma = \begin{bmatrix} \varepsilon_{L1} + \dfrac{1}{2}\left(\dfrac{J_1}{A}\chi_{L1}^2 + \epsilon_{L2}^2 + \epsilon_{L3}^2\right) - \dfrac{1}{8}\bar{\boldsymbol{d}}_\tau^T \boldsymbol{H}_\tau^{-1} \bar{\boldsymbol{d}}_\tau \\[2mm] \chi_{L2} - \chi_{L1}\epsilon_{L2} + \dfrac{1}{2J_2}\chi_{L1}^2 J_{2r} - \dfrac{1}{8J_2}\bar{\boldsymbol{d}}_\tau^T \boldsymbol{A}_3 \bar{\boldsymbol{d}}_\tau \\[2mm] \chi_{L3} - \chi_{L1}\epsilon_{L3} - \dfrac{1}{2J_3}\chi_{L1}^2 J_{3r} + \dfrac{1}{8J_3}\bar{\boldsymbol{d}}_\tau^T \boldsymbol{A}_2 \bar{\boldsymbol{d}}_\tau \end{bmatrix}, \tag{37e}$$

$$\boldsymbol{\varrho}_\tau = \left(1 - \tfrac{1}{2}\varepsilon_{L1}\right)\bar{\boldsymbol{d}}_\tau + \tfrac{1}{2}(\chi_{L2}\boldsymbol{B}_3 - \chi_{L3}\boldsymbol{B}_2)\bar{\boldsymbol{d}}_\tau, \tag{37f}$$

hte matrices A_2, A_3, B_2, and B_3 being defined by

$$A_2 := \int_A y D_\gamma^T D_\gamma \, dA, \qquad\qquad A_3 := \int_A z D_\gamma^T D_\gamma \, dA$$

$$B_2 := \int_A y D_\tau^T (D_\gamma - 2D_\theta) \, dA, \qquad B_3 := \int_A z D_\tau^T (D_\gamma - 2D_\theta) \, dA, \tag{37g}$$

and

$$J_{2r} := \int_{\mathscr{S}} z(y^2 + z^2) \, dA, \quad J_{3r} := \int_{\mathscr{S}} y(y^2 + z^2) \, dA. \tag{37h}$$

Rearranging the strain parameters in a single vector

$$\varrho[\bar{d}_L] := \begin{bmatrix} \boldsymbol{\varepsilon} \\ \boldsymbol{\chi} \end{bmatrix}, \quad \boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon_{b1} \\ \varepsilon_{b2} \\ \varepsilon_{b3} \end{bmatrix}, \quad \boldsymbol{\chi} = \begin{bmatrix} \chi_{b1} \\ \chi_{b2} \\ \chi_{b3} \end{bmatrix}, \tag{38}$$

and combining (37c) and (32c) we finally obtain the constitutive law in the form

$$t = H^{-1} \varrho[\bar{d}_L]. \tag{39}$$

Simplified quadratic evaluation for Biot strains. The previous equations simplify if the contribution of shear strain $\nabla w + \boldsymbol{\theta}$ is ignored in the expression of W. As shown in the tutorial section its effect is generally very small and can be neglected without introducing appreciable errors. With this assumption we obtain

$$\varepsilon_b[y, z] \approx \varepsilon_L + \tfrac{1}{2} \boldsymbol{\theta}^T \boldsymbol{\theta}, \quad \boldsymbol{\gamma}_b[y, z] \approx \boldsymbol{\gamma}_L - \varepsilon_L \boldsymbol{\theta}, \tag{40a}$$

which provides

$$\varrho_\sigma \approx \begin{bmatrix} \varepsilon_{L1} \\ \chi_{L2} \\ \chi_{L3} \end{bmatrix} + \frac{1}{2} \begin{bmatrix} \dfrac{J_1}{A} \chi_{L1}^2 + \epsilon_{L2}^2 + \epsilon_{L3}^2 \\[2mm] -2\chi_{L1}\varepsilon_{L2} + \dfrac{1}{J_2} \chi_{L1}^2 J_{2r} \\[2mm] -2\chi_{L1}\varepsilon_{L3} - \dfrac{1}{J_3} \chi_{L1}^2 J_{3r} \end{bmatrix}, \tag{40b}$$

$$\varrho_\tau \approx (1 - \epsilon_{L1})\bar{d}_\tau - (\chi_{L2} C_3 - \chi_{L3} C_2)\bar{d}_\tau, \tag{40c}$$

where

$$C_2 := \int_A y D_\tau^T D_\theta \, dA, \quad C_3 := \int_A z D_\tau^T D_\theta \, dA, \tag{40d}$$

These expressions can be further simplified if neglecting nonlinear terms related to the shear strain in the case of a symmetric section. We obtain

$$\varrho_\sigma \approx \begin{bmatrix} \varepsilon_{L1} + \dfrac{1}{2} \dfrac{J_1}{A} \chi_{L1}^2 \\[2mm] \chi_{L2} \\[1mm] \chi_{L3} \end{bmatrix}, \quad \varrho_\tau \approx \begin{bmatrix} \chi_{L1} \\ \varepsilon_{L2} \\ \varepsilon_{L3} \end{bmatrix}. \tag{41}$$

Note that the remaining nonlinear term $\frac{1}{2}\chi_{L1}^2 J_1/A$ takes into account the so-called Wagner axial/torsional coupling.

5.3. *Applying corotational kinematics.* Note that the strain parameters ε_k in (33) correspond to the derivatives, with respect to s, of the components $\bar{u}_k[s, 0, 0]$ of the average displacement $\bar{\boldsymbol{u}}[s]$ of the section. Analogously, χ_k correspond to the derivatives, with respect to s, of the components $\bar{\varphi}_k$ of the average rotation vector $\bar{\boldsymbol{\varphi}}[s]$ associated with the section $\mathcal{S}$. So the SV kinematical solution is completely defined by the generalized strain parameters

$$\bar{\boldsymbol{u}},_s[s] = \boldsymbol{\varepsilon}_L, \quad \bar{\boldsymbol{\varphi}},_s[s] = \boldsymbol{\chi}_L, \tag{42a}$$

the first $\bar{\boldsymbol{u}},_s$ being a derivative of a displacement and the second a derivative of a rotation vector. Recalling that $\bar{\boldsymbol{\varphi}}[s] = \boldsymbol{0}$, the latter can be related to a derivative of a rotation matrix $\bar{\boldsymbol{R}}[s]$ through (6) and (5c):

$$\bar{\boldsymbol{R}},_s[s] := \mathrm{spin}(\bar{\boldsymbol{\varphi}},_s) = \begin{bmatrix} \cdot & -\bar{\varphi}_3,_s & \bar{\varphi}_2,_s \\ \bar{\varphi}_3,_s & \cdot & -\bar{\varphi}_1,_s \\ -\bar{\varphi}_2,_s & \bar{\varphi}_1,_s & \cdot \end{bmatrix}. \tag{42b}$$

Displacements $\bar{\boldsymbol{u}}[s]$ and rotations $\bar{\boldsymbol{R}}[s]$, defined by reference to the local corotational frame $\{\bar{x}_1, \bar{x}_2, \bar{x}_3\}$, are related to the corresponding ones $\boldsymbol{u}[s]$ and $\boldsymbol{R}[s]$, referring to a global fixed frame through a change in the observer characterized by a relative rotation $\boldsymbol{Q}[s] = \boldsymbol{R}[s]\bar{\boldsymbol{R}}[s]^T$. As $\bar{\boldsymbol{R}} = \boldsymbol{I}$, we have $\boldsymbol{Q} = \boldsymbol{R}$, and therefore from (18a) we obtain

$$\bar{\boldsymbol{u}},_s[s] = \boldsymbol{R}[s]^T(\boldsymbol{u},_s + \boldsymbol{e}_1) - \boldsymbol{e}_1, \quad \bar{\boldsymbol{R}},_s[s] = \boldsymbol{R}^T \boldsymbol{R},_s. \tag{42c}$$

Note that, if using a linear evaluation for the Biot strain, that is, if referring to (36), we recover the Antman–Simo nonlinear beam model kinematics (see [Simo 1986; Antman 1995]). We also recover appropriate constitutive laws directly derived from the Saint-Venant theory without the need for ad hoc assumptions. With the use of the complete quadratic evaluation (37) for Biot strains, we also recover the full subtle effects due to the section warping. When using the simplified quadratic evaluation (40) we obtain a quite simple expression which still however takes into account the nonlinear Wagner coupling due to torsional distortion. A FEM implementation of this model is given in our subsequent, related paper [Garcea et al. 2012], and its results also show the strong influence of Wagner coupling in cases of flexural/torsional buckling.

6. Nonlinear plate model based on Mindlin–Reissner plate theory

ICM is further applied in this section for recovering nonlinear plate models based on the Mindlin–Reissner and Kirchhoff plate theories. The aim is to obtain objective models suitable for the nonlinear analysis of thin walled structures, such as those already analyzed in [Lanzo et al. 1995; Lanzo and Garcea 1996; Casciaro et al. 1998; Garcea 2001].

We assume the plate to be planar in its reference configuration and refer to a material system $\boldsymbol{X} = \{x_1, x_2, x_3\}$ where x_1 and x_2 lie in the middle plane, as shown in Figure 4, and $-h/2 \le x_3 \le h/2$, h being the plate thickness, moves along the transverse fiber $\mathcal{S} = \mathcal{S}(\boldsymbol{s})$, $\boldsymbol{s} := \{x_1, x_2\}$ being a two-dimensional abscissa moving in the middle plane. The linear solution applies to a neighbor of $\mathcal{S}$ and will be referred
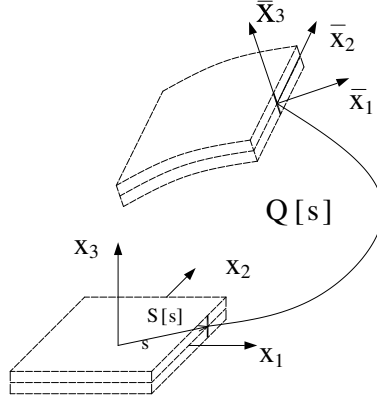
Figure 4. Plate model: kinematics.

to a corotational spatial frame $\bar{x} = \{\bar{x}_1, \bar{x}_2, \bar{x}_3\}$ aligned with the fiber, $Q[s]$ being the rotation matrix relating the fixed and corotational frames.

6.1. ***Obtaining a linear solution: Statics.*** The stress solution in linear Mindlin–Reissner plate theory corresponding to fiber $\mathcal{S}[x_1, x_2]$, identified as Biot strains, can be described by

$$\sigma := \frac{1}{h} \begin{bmatrix} N_{11} & N_{12} & \zeta_{,3}\, T_1 \\ \cdot & N_{22} & \zeta_{,3}\, T_2 \\ \text{sym.} & \cdot & 0 \end{bmatrix} + \frac{z}{J} \begin{bmatrix} M_{11} & M_{12} & 0 \\ \cdot & M_{22} & 0 \\ \text{sym.} & \cdot & 0 \end{bmatrix}, \tag{43a}$$

where $J := h^3/12$ is the flexural inertia of the plate, $\zeta[x_3]$ is the stress function defined by (see also (19b))

$$\zeta := \frac{3h^2 x_3 - 4x_3^3}{2h^2}, \quad \int_{-h/2}^{h/2} \zeta\, dx_3 = 0, \quad \int_{-h/2}^{h/2} \zeta_{,3}\, dx_3 = h, \tag{43b}$$

and, making $i, j = 1, 2$,

$$N_{ij}[s] := \int_{-h/2}^{h/2} \sigma_{ij}\, dx_3, \quad T_i[s] := \int_{-h/2}^{h/2} \sigma_{i3}\, dx_3, \quad M_{ij}[s] := \int_{-h/2}^{h/2} z\sigma_{ij}\, dx_3, \tag{43c}$$

are the plate strengths we collect into the in-plane strength $N := \{N_{11}, N_{12}, N_{22}\}$, the shear strength $T := \{T_1, T_2\}$, and the bending moment $M := [M_{11}, M_{22}, M_{12}]$.

Obtaining a linear solution: kinematics. The displacement field is provided by Mindlin–Reissner theory and can be expressed, in a neighbor of $\mathcal{S}[s]$, in the form

$$\begin{aligned} \bar{u}_1[s+ds, x_3] &:= \bar{u}_{01}[s+ds] + x_3\bar{\varphi}_2[s+ds] + w[x_3]\bar{u}_{03,1}[s], \\ \bar{u}_2[s+ds, x_3] &:= \bar{u}_{02}[s+ds] - x_3\bar{\varphi}_1[s+ds] + w[x_3]\bar{u}_{03,2}[s], \\ \bar{u}_3[s+ds, x_3] &:= \bar{u}_{03}[s+ds], \end{aligned} \tag{44a}$$

where

$$\begin{aligned} \bar{u}_{0i}[s+ds] &:= \bar{u}_{0i,1}[s]\,dx_1 + \bar{u}_{0i,2}[s]\,dx_2, & i &= 1, 2, 3, \\ \bar{\varphi}_i[s+ds] &:= \bar{\varphi}_{i,1}[s]\,dx_1 + \bar{\varphi}_{i,2}[s]\,dx_2, & i &= 1, 2, \end{aligned} \tag{44b}$$

are the components of mean displacement and rotation vectors of the fiber $\mathscr{S}[s]$, and $w[x_3]$ the warping function defined by

$$w := \zeta/k - x_3, \quad k := \frac{1}{A} \int_S \zeta_{,3}^2 \, dA = 6/5. \tag{44c}$$

Note that assumption (44b) provides $\bar{\varphi}_1[s] = \bar{\varphi}_2[s] = 0$, where the alignment of the $\bar{x}_1$ and $\bar{x}_2$ axes is implicitly set to be orthogonal to the fiber. To complete the alignment we need to fix their in-plane drilling orientation by correspondence to $\bar{\varphi}_3[s] = 0$. We assume that it is set such that

$$\bar{u}_{01,2} = \bar{u}_{02,1}. \tag{44d}$$

The expression for the displacement gradient consequently becomes

$$\nabla \bar{u} := \begin{bmatrix} \bar{u}_{01,1} + \bar{\varphi}_{2,1} \, x_3 & \bar{u}_{02,1} + \bar{\varphi}_{2,2} \, x_3 & \bar{u}_{03,1} \, w_{,3} \\ \bar{u}_{02,1} - \bar{\varphi}_{1,1} \, x_3 & \bar{u}_{02,2} - \bar{\varphi}_{1,2} \, x_3 & \bar{u}_{03,2} \, w_{,3} \\ \bar{u}_{03,1} & \bar{u}_{03,2} & 0 \end{bmatrix}. \tag{44e}$$

6.2. *Recovering nonlinear strains and constitutive equations.* Starting from the stress interpolation (43a), the complementary energy is obtained in the standard form

$$\psi[s] := \tfrac{1}{2} \{ N^T C_n N + T^T C_t T + M^T C_m M \}, \tag{45a}$$

where C_n, C_t, and C_m are the standard matrices of elastic moduli

$$C_n := \frac{hE}{1-v^2} \begin{bmatrix} 1 & v & 0 \\ v & 1 & 0 \\ 0 & 0 & (1-v)/2 \end{bmatrix}, \quad C_t := \frac{5Gh}{6} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad C_m := \frac{J}{h} C_n. \tag{45b}$$

Collecting the stress parameters in a single vector $t := \{N, T, M\}$, (45a) can be written in compact form as

$$\psi[s] = \tfrac{1}{2} t^T H t. \tag{45c}$$

The generalized strain parameters can be obtained by comparing the previous equation with the expression of strain work

$$\mathscr{W}[s] = \int_{x_3=-h/2}^{h/2} \sigma_b \cdot \varepsilon_b \, dx_3,$$

where the Biot stress σ_b is directly identified with the linear solution (43a) and the corresponding strain ε_b with the displacement solution (44e) given by the quadratic approximation (12). Performing integration, we obtain

$$\mathscr{W}[s] = t^T \varrho \quad \Rightarrow \quad \varrho = H t, \tag{46}$$

the generalized strain parameter vector ϱ being defined by

$$\varrho := \begin{bmatrix} \varepsilon \\ \gamma \\ \chi \end{bmatrix} = \varrho_L + \varrho_Q, \quad \varrho_L = \begin{bmatrix} \varepsilon_L \\ \gamma_L \\ \chi_L \end{bmatrix}, \quad \varrho_Q = \begin{bmatrix} \varepsilon_Q \\ \gamma_Q \\ \chi_Q \end{bmatrix}, \tag{47a}$$

where

$$\boldsymbol{\varepsilon}_L := \begin{bmatrix} \bar{u}_{01,1} \\ \bar{u}_{02,2} \\ \bar{u}_{01,2} + \bar{u}_{02,1} \end{bmatrix}, \quad \boldsymbol{\gamma}_L := \begin{bmatrix} \bar{u}_{03,1} \\ \bar{u}_{03,2} \end{bmatrix}, \quad \boldsymbol{\chi}_L := \begin{bmatrix} \bar{\varphi}_{2,1} \\ -\bar{\varphi}_{1,2} \\ \bar{\varphi}_{2,2} - \bar{\varphi}_{1,1} \end{bmatrix}, \tag{47b}$$

and

$$\boldsymbol{\varepsilon}_Q := \begin{bmatrix} \dfrac{19}{48}\bar{u}_{03,1}^2 + \dfrac{J}{8h}(2\bar{\varphi}_{2,2}\,\bar{\varphi}_{1,1} - \bar{\varphi}_{2,2}^2 + 3\bar{\varphi}_{1,1}^2) \\ \dfrac{19}{48}\bar{u}_{03,2}^2 + \dfrac{J}{8h}(2\bar{\varphi}_{2,2}\,\bar{\varphi}_{1,1} - \bar{\varphi}_{1,1}^2 + 3\bar{\varphi}_{2,2}^2) \\ \dfrac{19}{24}\bar{u}_{03,1}\,\bar{u}_{03,2} + \dfrac{J}{2h}(\bar{\varphi}_{1,1} + \bar{\varphi}_{2,2})(\bar{\varphi}_{1,2} + \bar{\varphi}_{2,1}) \end{bmatrix},$$

$$\boldsymbol{\gamma}_Q := -\frac{1}{2}\begin{bmatrix} \bar{u}_{03,2}\,\bar{u}_{02,1} + \bar{u}_{03,1}\,\bar{u}_{01,1} \\ \bar{u}_{03,1}\,\bar{u}_{02,1} + \bar{u}_{03,2}\,\bar{u}_{02,2} \end{bmatrix},$$

$$\boldsymbol{\chi}_Q := \frac{1}{2}\begin{bmatrix} -\bar{u}_{02,1}\,\bar{\varphi}_{1,1} - \bar{u}_{02,1}\,\bar{\varphi}_{2,2} \\ \bar{u}_{02,1}\,\bar{\varphi}_{1,1} + \bar{u}_{02,1}\,\bar{\varphi}_{2,2} \\ (\bar{u}_{01,1} + \bar{u}_{02,2})\bar{\varphi}_{2,2} - (\bar{u}_{02,2} - \bar{u}_{01,1})\bar{\varphi}_{1,1} \end{bmatrix}. \tag{47c}$$

6.3. *Applying corotational kinematics.* The plate kinematics are completely defined by the generalized displacements $\bar{u}_{0,1}[s]$, $\bar{u}_{0,2}[s]$, $\bar{\varphi}_{,1}[s]$, and $\bar{\varphi}_{,2}[s]$, which correspond to the derivatives of the average displacement and rotation vectors associated with the fiber $\mathcal{S}[s]$ and referred to the corotational system $\boldsymbol{x}$. We have $\bar{\varphi}[s] = \boldsymbol{0}$, so the latter can be related to a derivative of a rotation matrix $\bar{\boldsymbol{R}}[s]$ through (6) and (5c). We obtain

$$\bar{\boldsymbol{R}}[s] = \boldsymbol{I}, \quad \bar{\boldsymbol{R}}_{,1}[s] = \text{spin}(\bar{\boldsymbol{\varphi}}_{,1}[s]), \quad \bar{\boldsymbol{R}}_{,2}[s] = \text{spin}(\bar{\boldsymbol{\varphi}}_{,2}[s]). \tag{48a}$$

In order to complete the nonlinear modeling we only need to relate this local description, in the corotational frame, to the corresponding description in a fixed global frame through changing the observer algebra (18a). Condition $\boldsymbol{Q}[s]^T\boldsymbol{R}[s] = \boldsymbol{I}$ furnishes $\boldsymbol{Q}[s] = \boldsymbol{R}[s]$ and so, from (18a), with $i = 1, 2$,

$$\bar{\boldsymbol{u}}_{0,i}[s] = \boldsymbol{R}^T(\boldsymbol{u}_{0,i} + \boldsymbol{e}_i) - \boldsymbol{e}_i, \quad \bar{\boldsymbol{R}}_{,i}[s] = \boldsymbol{R}^T\boldsymbol{R}_{,i}. \tag{48b}$$

6.4. *Relation with nonlinear plate modeling by Simo.* By denoting with $\{\boldsymbol{i}_1, \boldsymbol{i}_2, \boldsymbol{i}_3\}$ and $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ the versors of the local frame $\bar{\boldsymbol{x}}$ and the global frame $\boldsymbol{x}$, we can express $\boldsymbol{R}[s]$ in the form

$$\boldsymbol{R} = [\boldsymbol{i}_1, \boldsymbol{i}_2, \boldsymbol{i}_3]. \tag{49a}$$

Moreover, by introducing the *plate directors* $\boldsymbol{a}_i$ defined by

$$\boldsymbol{a}_i := \boldsymbol{e}_i + \boldsymbol{u}_{0,i}, \quad i = 1, 2, \tag{49b}$$

(48b) can be rewritten as

$$\bar{\boldsymbol{u}}_{0,i} = \begin{bmatrix} \boldsymbol{i}_1 \cdot \boldsymbol{a}_i \\ \boldsymbol{i}_2 \cdot \boldsymbol{a}_i \\ \boldsymbol{i}_3 \cdot \boldsymbol{a}_i \end{bmatrix} - \boldsymbol{e}_i, \quad \bar{\boldsymbol{\varphi}}_{,i} = \begin{bmatrix} \boldsymbol{i}_3 \cdot \boldsymbol{i}_{2,i} \\ -\boldsymbol{i}_3 \cdot \boldsymbol{i}_{1,i} \\ -\boldsymbol{i}_1 \cdot \boldsymbol{i}_{2,i} \end{bmatrix}, \tag{49c}$$

and, consequently, obliging the quadratic contribution ϱ_Q in (47), that is, accepting a linear approximation for the strain parameters, we obtain

$$\boldsymbol{\varepsilon}_L \approx \begin{bmatrix} \boldsymbol{i}_1 \cdot \boldsymbol{a}_1 - 1 \\ \boldsymbol{i}_2 \cdot \boldsymbol{a}_2 - 1 \\ \boldsymbol{i}_1 \cdot \boldsymbol{a}_2 + \boldsymbol{i}_2 \cdot \boldsymbol{a}_1 \end{bmatrix}, \quad \boldsymbol{\gamma}_L \approx \begin{bmatrix} \boldsymbol{i}_3 \cdot \boldsymbol{a}_1 \\ \boldsymbol{i}_3 \cdot \boldsymbol{a}_2 \end{bmatrix}, \quad \boldsymbol{\chi}_L \approx \begin{bmatrix} \boldsymbol{i}_3 \cdot \boldsymbol{i}_{1,1} \\ \boldsymbol{i}_3 \cdot \boldsymbol{i}_{2,2} \\ \boldsymbol{i}_3 \cdot (\boldsymbol{i}_{1,2} + \boldsymbol{i}_{2,1}) \end{bmatrix}, \tag{49d}$$

which coincides with what is derived in [Simo et al. 1990].

6.5. *Rotation-free modeling for Kirchhoff thin plate.*
In some cases, for very thin plates, the effect of out-of-plane shear strain will become irrelevant, so we can assume the Kirchhoff hypotheses $\boldsymbol{\gamma}_L \approx \boldsymbol{0}$, $\boldsymbol{T}^T \boldsymbol{C}_t \boldsymbol{T} \approx 0$, that is

$$\begin{matrix} \boldsymbol{i}_3 \cdot \boldsymbol{a}_1 = 0 \\ \boldsymbol{i}_3 \cdot \boldsymbol{a}_2 = 0 \end{matrix} \quad \Rightarrow \quad \boldsymbol{i}_3 = \frac{\boldsymbol{a}_1 \wedge \boldsymbol{a}_2}{|\boldsymbol{a}_1 \wedge \boldsymbol{a}_2|}. \tag{50a}$$

We can also assume the in-plane strain $\boldsymbol{\varepsilon}$ is sufficiently small to be neglected in comparison with unity. Under this assumption, we obtain

$$\boldsymbol{a}_i \cdot \boldsymbol{a}_j \approx \boldsymbol{i}_i \cdot \boldsymbol{a}_j + \boldsymbol{i}_j \cdot \boldsymbol{a}_i, \quad (\boldsymbol{a}_1 \wedge \boldsymbol{a}_2) \cdot \boldsymbol{a}_{i,j} \approx \boldsymbol{i}_3 \cdot \boldsymbol{i}_{i,j}, \quad i,j = 1,2, \tag{50b}$$

so (49c) simplifies into

$$\boldsymbol{\varepsilon}_L = \frac{1}{2} \begin{bmatrix} \boldsymbol{a}_1 \cdot \boldsymbol{a}_1 - 1 \\ \boldsymbol{a}_2 \cdot \boldsymbol{a}_2 - 1 \\ 2\boldsymbol{a}_1 \cdot \boldsymbol{a}_2 \end{bmatrix}, \quad \boldsymbol{\chi}_L = \begin{bmatrix} (\boldsymbol{a}_1 \wedge \boldsymbol{a}_2) \cdot \boldsymbol{a}_{1,1} \\ (\boldsymbol{a}_1 \wedge \boldsymbol{a}_2) \cdot \boldsymbol{a}_{2,2} \\ (\boldsymbol{a}_1 \wedge \boldsymbol{a}_2) \cdot (\boldsymbol{a}_{1,2} + \boldsymbol{a}_{2,1}) \end{bmatrix}. \tag{50c}$$

This formulation does not make explicit reference to rotation matrices and could be convenient in FEM discretizations which do not make use of nodal rotations, such as those based on spline interpolations (for example, see [Garcea 2001]). A FEM implementation of this modeling will be discussed in our subsequent, related paper [Garcea et al. 2012].

7. Further comments and remarks

Some further comments are useful for a better understanding of the proposed method and of its possible extensions.

7.1. *Linear or quadratic recovery of Biot strains.*
The quadratic terms in the local expression of the strain (12), due to warping, are usually very small and, at least some of them, can be ignored without introducing appreciable errors. In many cases a direct identification of the linear strain, such as the Biot strain, could be sufficient for practical purposes.

This is actually true in the two implementation examples previously discussed. Warping has a negligible effect on the plate modeling discussed in Section 6, generally allowing the linearized formula (47) for the strain evaluation. In the absence of torsional distortions, the warping could also be neglected in the beam model discussed in Section 5, the only relevant contribution of warping being the Wagner coupling. Note however that warping can have a noticeable effect in other possible implementations such as, for instance, cases of laminated plates and beams.

7.2. *Errors inherited from linear theory.* In the implementation examples described before, the parent linear solutions have been derived from standard theories for beams and plates. The nonlinear ICM models obtained will obviously inherit all the approximations contained in these theories. More refined models can be obtained, by the same procedure, using more sophisticated theories, such as the thin-walled beam theory of Vlasov, the so-called generalized beam theories [Goncalves et al. 2009], and the anisotropic theories for plates [Nayfeh and Pai 2004].

7.3. *Linear assumptions in nonlinear analysis.* The initial assumption in ICM is that the nonlinear solution could be locally described by the linear one, at least qualitatively. However we must consider that, in some cases, a nonlinear solution can differ noticeably from the linear one, because of the activation of local buckling phenomena characterized by deformation patterns which play a negligible role in the linear solution and so are taken out in the linear modeling. Thin-walled beams or stiffened panels, for example, can be modeled as one-dimensional fibred continua or as equivalent orthotropic plates through homogenization criteria; this can be sufficient in linear analysis but is unable to deal with complex coupled buckling behaviors which can be better described by more detailed modeling like plates assemblage (for example, see [Lanzo and Garcea 1996; Casciaro et al. 1998]). A certain care is then always needed in selecting the appropriate parent linear theory in order to obtain a reliable and accurate nonlinear modeling.

7.4. *Mixed versus compatible formulation.* The mixed format, used for recovering constitutive laws in terms of generalized stress and strain parameters, plays an important role in the ICM approach. It corresponds to a better use of the first-order information achieved from linear theory. In fact, it allows the use of information on both the displacement field $\boldsymbol{u}_L$ and the stress field $\boldsymbol{\sigma}_L$, which define the linear solution.

We can derive the constitutive laws in an alternative way, that is, by using a compatible formulation where the strain energy is defined only in terms of $\boldsymbol{\varepsilon}_b[\boldsymbol{u}]$:

$$\Phi_c := \frac{1}{2} \int \boldsymbol{\varepsilon}_b \cdot \boldsymbol{C}^{-1} \boldsymbol{\varepsilon}_b \, dV. \tag{51a}$$

This expression can also be rewritten in the form

$$\Phi_c := \int \left(\boldsymbol{\varepsilon}_b \cdot \boldsymbol{\sigma} - \tfrac{1}{2} \boldsymbol{\sigma} \cdot \boldsymbol{C} \boldsymbol{\sigma} \right) dV, \quad \boldsymbol{\sigma} := \boldsymbol{C}^{-1} \boldsymbol{\varepsilon}_b, \tag{51b}$$

which looks very similar to the mixed expression

$$\Phi_m := \int \left(\boldsymbol{\varepsilon}_b \cdot \boldsymbol{\sigma}_b - \tfrac{1}{2} \boldsymbol{\sigma}_b \cdot \boldsymbol{C} \boldsymbol{\sigma}_b \right) dV. \tag{51c}$$

Note however that the two expressions differ in the definition of the stress field which is obtained from $\boldsymbol{\varepsilon}_b$, through the constitutive relation $\boldsymbol{\sigma} = \boldsymbol{C}^{-1} \boldsymbol{\varepsilon}_b$ in (51b), or directly recovered as $\boldsymbol{\sigma}_b \approx \boldsymbol{\sigma}_L$ from the linear solution in (51c). These two evaluations do not coincide as already discussed in Section 3.3.

The stresses are an important part of the linear solution, whose main goal is their accurate recovery. To assume that $\boldsymbol{\sigma}_b$ is directly recovered as $\boldsymbol{\sigma}_L$ can then be generally considered a more reliable evaluation than that obtained from the displacements in a more elaborate way, which also includes differentiations. Actually, compatible formulations tend to generate spurious third and fourth-order terms in (51a) which can produce some locking in the resulting nonlinear modeling. Generally they need some ad hoc treatment to avoid this problem.

7.5. *Obtaining more than quadratic local accuracy.* Note that the possibility of obtaining an accurate recovery of nonlinear kinematics is strictly related to the occurrence of two circumstances: the fact that the strain can be considered small enough that the local motion is governed by its rigid part, and the fact that a rigid rotation R can be completely defined, through expansion (5c), by its first-order estimate W. We actually made use of these properties when deriving the second-order recovery relation (12).

We can obtain more accurate expressions by setting further terms in the expansion (5c) of the rotation. This could allow, at least in principle, full kinematical coherence in the recovery of rotations. However great care has to be taken in the appropriate definition of the kinematical quantities involved and a more precise evaluation of the stretch part of the motion could be required, such as, for instance, that obtained by extending the linear solution by perturbation procedures [Nayfeh 1981]. Even if the use of (12) is generally sufficient for practical applications, research in this direction could be interesting.

8. Conclusion

In the current state of the art in structural mechanics there is a lack of a suitable nonlinear models for use in nonlinear FEM analysis, in comparison with the amounts of those available for linear analysis. The purpose of this paper was reducing this gap by developing a general tool able to exploit information from existing linear models for fibered continua, such as beams or plate, to set up corresponding nonlinear models.

We proposed a general procedure, called the implicit corotational method (ICM), which consists of two logical steps. In the first, we exploit the stress and displacement fields provided by the linear theory to derive appropriate expressions for the nonlinear fields as viewed by a moving local corotational observer; in the second we transfer this description to a fixed global frame directly exploiting the change in observer algebra, so completing the nonlinear modeling and assuring frame invariance, by definition.

The former step only needs a linear solution be available. The recovery of the nonlinear fields is straightforward through a standard procedure: the stress field is directly taken from the linear field and the strain field is obtained from the linear displacements by the quadratic formula (12). The latter step introduces the geometrical nonlinearities due to the finite motion of the fiber in the modeling, simply by exploiting the objectivity requirements in a constructive way. Only the simple standard algebra defined by (18a) is involved in this step and its implementation is straightforward and does not require any ad hoc adaptation to fit the particular problem at hand.

ICM was implemented in two special, but still technically relevant, contexts: that of three-dimensional beams, according to Saint-Venant general rod theory, and plates, according to Mindlin–Reissner and Kirchhoff plate theories, a homogeneous isotropic material being assumed in both cases. We discussed these cases in detail and have shown that the resulting models can actually recover all the richness of the underlying linear solutions. We also indicated the effect of different simplification choices and have shown that, by appropriate simplifying assumptions, we can recover existing nonlinear models (for example, the beam and plate models of [Simo 1986; Antman 1995]). The method could however easily be generalized to anisotropic materials and also applied in different contexts, such as, for instance, Vlasov thin-walled beam theory, generalized beam theories [Goncalves et al. 2009], or laminated plate theory [Nayfeh and Pai 2004]. In all cases the advantage is that it provides fully objective nonlinear models by a black-box procedure which only needs the corresponding linear model to be already available.

In the our subsequent, related paper [Garcea et al. 2012] we also show that nonlinear models derived through ICM are directly suitable for numerical implementations through the use of a standard FEM technology. Numerical results will also show their accuracy and robustness in different technically relevant contexts.

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