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WAVE VELOCITY FORMULAS TO EVALUATE ELASTIC CONSTANTS OF SOFT BIOLOGICAL TISSUES

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We use the equations governing infinitesimal motions superimposed on a finite deformation in order to establish formulas for the velocity of (plane homogeneous) shear bulk waves and surface Rayleigh waves propagating in soft biological tissues subject to uniaxial tension or compression. Soft biological tissues are characterized as transversely isotropic incompressible nonlinearly elastic solids. The constitutive model is given as an strain-energy density expanded up to fourth order in terms of the Green strain tensor. The velocity formulas are written as $\rho v^2 = a_0 + a_1 e + a_2 e^2$ where ρ is the mass density, v is the wave velocity, a_k are functions in terms of the elastic constants and e is the elongation in the loading direction. These formulas can be used to evaluate the elastic constants since they determine the exact behavior of the elastic constants of second, third, and fourth orders in the incompressible limit.

1. Introduction

Soft biological tissues were generally considered incompressible and isotropic under the early days of their analysis. In more recent years they have been recognized as highly anisotropic due to the presence of collagen fibers [Holzapfel et al. 2000]. Determination of the acoustoelastic coefficients in incompressible solids and the limiting values of the coefficients of nonlinearity for elastic wave propagation, among other studies, has very recently attracted a lot of attention since these analyses give an opportunity to capture the mechanical properties of these materials (see, for instance, Destrade et al. [Destrade et al. 2010b] and references therein). For other applications dealing with linearized dynamics we refer to [Bigoni et al. 2007; 2008] and the references therein.

Hamilton et al. [2004] analyzed a strain-energy density suitable for incompressible isotropic elastic solids such as gels and phantoms, namely

$$W = \mu I_2 + (A/3) I_3 + D I_2^2, \tag{1}$$

where

$$I_2 = \mathrm{tr}(\boldsymbol{E}^2),\ \ I_3 = \mathrm{tr}(\boldsymbol{E}^3), \tag{2}$$

$\boldsymbol{E}$ is the Green strain tensor and μ, A, and D are second-, third-, and fourth-order elastic constants, respectively (the order given by the exponent of $\boldsymbol{E}$). A very similar expansion to the one given in (1) was originally derived in [Ogden 1974].

Indeed, several investigations have been carried out to determine the elastic constants μ, A and D using shear bulk nonlinear waves [Gennisson et al. 2007; Renier et al. 2007; 2008] and small-amplitude

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waves propagating in incompressible solids subject to homogeneous deformations [Destrade et al. 2010b] (linearized waves). It should be noted that if the analysis of a material only includes the small deformation regime then it is enough to consider (1) up to fourth order in strains.

In contrast to gels and phantoms, soft biological tissues are anisotropic solids due to the presence of oriented collagen fiber bundles [Holzapfel et al. 2000; Destrade et al. 2010a]. It is thus required a model other than (1) to account for the anisotropic behavior of these solids. A transversely isotropic model has been proposed in [Destrade et al. 2010a], in which the strain-energy density of third order (actually, it is the most general third order expansion) is given by

$$W = \mu\, I_2 + \tfrac{1}{3} A\, I_3 + \alpha_1\, I_4^2 + \alpha_2\, I_5 + \alpha_3\, I_2\, I_4 + \alpha_4\, I_4^3 + \alpha_5\, I_4\, I_5, \tag{3}$$

where I_2 and I_3 are given in (2) and

$$I_4 = \boldsymbol{M} \cdot (\boldsymbol{E}\boldsymbol{M}), \qquad I_5 = \boldsymbol{M} \cdot (\boldsymbol{E}^2\boldsymbol{M}), \tag{4}$$

are anisotropic invariants where $\boldsymbol{M}$ is the unit vector that gives the undeformed fiber direction. It follows that μ, α_1, α_2 and A, α_3, α_4, α_5 are second- and third-order elastic constants, respectively. To evaluate the elastic constants μ, A, α_k ($k = \overline{1, 5}$, where the overline means $k = 1, \dots, 5$) the authors established a formula for the velocity of shear bulk waves. This formula is a first-order polynomial in the elongation e, defined by $\lambda = 1 + e$, where λ is the principal stretch in the direction of the fibers and the uniaxial tension. The speeds of infinitesimal waves expressed in terms of third- and fourth-order constants does provide a basis for the acousto-elastic evaluation of the material constants [Destrade and Ogden 2010].

To make the model more accurate and representative of soft biological tissue we consider a fourth-order strain-energy function (actually, the most general fourth order expansion), namely (see also [Destrade et al. 2010a])

$$W = \mu\, I_2 + \tfrac{1}{3} A\, I_3 + \alpha_1\, I_4^2 + \alpha_2\, I_5 + \alpha_3\, I_2\, I_4 + \alpha_4\, I_4^3 + \alpha_5\, I_4\, I_5 \\ + \alpha_6\, I_2^2 + \alpha_7\, I_2\, I_4^2 + \alpha_8\, I_2\, I_5 + \alpha_9\, I_4^4 + \alpha_{10}\, I_5^2 + \alpha_{11}\, I_3\, I_4, \tag{5}$$

where $\alpha_6, \dots, \alpha_{11}$ are fourth-order elastic constants. In order to determine the elastic constants μ, A, and α_k ($k = \overline{1, 11}$), we develop formulas for the velocity of (homogeneous plane) shear bulk waves and surface Rayleigh waves which are second-order polynomials of the elongation e. When $\alpha_k = 0$, $k = \overline{1, 11}$, $k \neq 6$ and α_6 is denoted D, these formulas coincide with the corresponding approximate formulas obtained in [Destrade et al. 2010b]. The results show that linear corrections to the acoustoelastic wave speed formulas involve second- and third-order constants, and that quadratic corrections involve second-, third-, and fourth-order constants, in agreement with [Hoger 1999].

The layout of the paper is as follows. In Section 2, we introduce briefly the main governing equations while Sections 3 and 4 are devoted to the acousto-elastic analysis of (5). In Section 5 some conclusions are outlined.

2. Expressions of components of the fourth-order elasticity tensor

We consider an incompressible transversely isotropic elastic body $\mathcal{B}$, which possesses a natural unstrained state $\mathcal{B}_0$ and a finitely deformed (pre-stressed) equilibrium state $\mathcal{B}_e$. A small time-dependent motion is superimposed upon this pre-stressed equilibrium configuration to reach a final material state $\mathcal{B}_t$, called current configuration. The position vectors of a representative particle are denoted by X_A, $x_i(\boldsymbol{X})$, $\tilde{x}_i(\boldsymbol{X}, t)$ in $\mathcal{B}_0$, $\mathcal{B}_e$ and $\mathcal{B}_t$, respectively. The deformation gradient tensor associated with the deformations $\mathcal{B}_0 \to \mathcal{B}_t$ and $\mathcal{B}_0 \to \mathcal{B}_e$ are denoted by $\boldsymbol{F}$ and $\bar{\boldsymbol{F}}$ and given in component form by

$$F_{iA} = \frac{\partial \tilde{x}_i}{\partial X_A}, \quad \bar{F}_{iA} = \frac{\partial x_i}{\partial X_A}. \tag{6}$$

It is clear from (6) that

$$F_{iA} = (\delta_{ij} + u_{i,j})\bar{F}_{jA}, \tag{7}$$

where δ_{ij} is the Kronecker operator, $u_i(\boldsymbol{X}, t)$ denotes the small time-dependent displacement associated with the deformation $\mathcal{B}_e \to \mathcal{B}_t$ and a comma indicates differentiation with respect to the indicated spatial coordinate in $\mathcal{B}_e$.

Suppose that the body is a soft tissue with one preferred direction associated with a family of parallel fibers of collagen. We denote by $\boldsymbol{M}$ the unit vector in that direction when the solid is unloaded and at rest. Then, the strain-energy function W of the body, per unit volume at $\mathcal{B}_0$, may be expressed by (5) (see [Destrade et al. 2010a]). It is well-known that $\boldsymbol{E} = (\boldsymbol{C} - \boldsymbol{I})/2$, where $\boldsymbol{C} = \boldsymbol{F}^T\boldsymbol{F}$ is the right Cauchy–Green strain tensor and $\boldsymbol{I}$ is the identity tensor. In the absence of body forces, the equations of motion may be expressed in the following form (see [Prikazchikov and Rogerson 2003]):

$$\frac{\partial S_{Ai}}{\partial X_A} = \rho \ddot{u}_i \;\text{ or }\; \frac{\partial}{\partial x_m}(\bar{F}_{mA} S_{Ai}) = \rho \ddot{u}_i, \quad S_{Ai} = \frac{\partial W^*}{\partial F_{iA}}, \quad W^* = W - p(J-1), \quad J = \det \boldsymbol{F}, \tag{8}$$

where a superposed dot indicates differentiation with respect to the time t, $\bar{\boldsymbol{F}}$ is a constant tensor, S_{Ai} are the components of the nominal stress tensor and p plays the role of a Lagrange multiplier and may be understood as a pressure (in $\mathcal{B}_t$) associated with the incompressibility constraint. Since the quantities associated with the deformation $\mathcal{B}_e \to \mathcal{B}_t$ are small in comparison with the corresponding quantities associated with the deformations $\mathcal{B}_0 \to \mathcal{B}_e$ we have

$$S_{Ai} \approx S_{Ai}(\bar{\boldsymbol{F}}, \bar{p}) + \frac{\partial S_{Ai}}{\partial F_{kB}}(\bar{\boldsymbol{F}}, \bar{p}) u_{k,m} \bar{F}_{mB} + p^* \frac{\partial S_{Ai}}{\partial p}(\bar{\boldsymbol{F}}, \bar{p}), \tag{9}$$

where $\bar{p} = p(\bar{\boldsymbol{F}})$ and $p^* = p - \bar{p}$ is the time-dependent pressure increment. On use of the linear approximation (9) into $(8)_2$, the linearized equations of motion are obtained and can be written as

$$A_{jilk} u_{k,lj} - p^*_{,i} = \rho \ddot{u}_i, \tag{10}$$

where

$$A_{ijkl} = \bar{F}_{iA} \bar{F}_{kB} \frac{\partial^2 W}{\partial F_{jA} \partial F_{lB}} \bigg|_{\boldsymbol{F}=\bar{\boldsymbol{F}}}, \tag{11}$$

are the components of the so-called fourth-order elasticity tensor. It is not difficult to verify that

$$A_{piqj} = \bar{F}_{p\alpha} \bar{F}_{q\beta} \biggl[\frac{1}{2} \delta_{ij} \biggl(\frac{\partial W}{\partial E_{\alpha\beta}} + \frac{\partial W}{\partial E_{\beta\alpha}} \biggr) + \frac{1}{4} \biggl(\bar{F}_{in} \bar{F}_{jy} \frac{\partial^2 W}{\partial E_{\alpha n} \partial E_{\beta y}} + \bar{F}_{in} \bar{F}_{jx} \frac{\partial^2 W}{\partial E_{\alpha n} \partial E_{x\beta}} + \bar{F}_{im} \bar{F}_{jy} \frac{\partial^2 W}{\partial E_{m\alpha} \partial E_{\beta y}} + \bar{F}_{im} \bar{F}_{jx} \frac{\partial^2 W}{\partial E_{m\alpha} \partial E_{x\beta}} \biggr) \biggr] \bigg|_{\boldsymbol{F}=\bar{\boldsymbol{F}}}, \tag{12}$$

where

$$\frac{\partial W}{\partial E_{mn}} = \sum_{k=2}^{5} \frac{\partial W}{\partial I_k} \frac{\partial I_k}{\partial E_{mn}}, \tag{13}$$

$$\frac{\partial^2 W}{\partial E_{mn}\partial E_{xy}} = \sum_{k=2}^{5} \frac{\partial W}{\partial I_k} \frac{\partial^2 I_k}{\partial E_{mn}\partial E_{xy}} + \sum_{k=2}^{5}\sum_{l=2}^{5} \frac{\partial^2 W}{\partial I_k \partial I_l} \frac{\partial I_k}{\partial E_{mn}} \frac{\partial I_k}{\partial E_{xy}}, \tag{14}$$

and

$$\begin{aligned}
&\frac{\partial I_2}{\partial E_{mn}} = 2E_{nm}, && \frac{\partial^2 I_2}{\partial E_{mn}\partial E_{xy}} = 2\delta_{nx}\delta_{my},\\
&\frac{\partial I_3}{\partial E_{mn}} = 3E_{nk}E_{km}, && \frac{\partial^2 I_3}{\partial E_{mn}\partial E_{xy}} = 3(\delta_{nx}\delta_{ky}E_{km} + \delta_{kx}\delta_{my}E_{nk}),\\
&\frac{\partial I_4}{\partial E_{mn}} = M_m M_n, && \frac{\partial^2 I_4}{\partial E_{mn}\partial E_{xy}} = 0,\\
&\frac{\partial I_5}{\partial E_{mn}} = M_m E_{nj} M_j + M_i E_{im} M_n, && \frac{\partial^2 I_5}{\partial E_{mn}\partial E_{xy}} = M_m M_y \delta_{nx} + M_x M_n \delta_{my}.
\end{aligned} \tag{15}$$

It is clear from (12) that $A_{ijkl} = A_{klij}$. The incremental condition of incompressibility follows and is of the form

$$u_{i,i} = 0. \tag{16}$$

3. Formulas for the velocity of shear bulk waves

We now describe the special loading and geometry case that will be used in the sections that follow. Consider a rectangular block of a soft transversely isotropic incompressible elastic solid whose faces in the unstressed state $\mathcal{B}_0$ are parallel to the (X_1, X_2)-, (X_2, X_3)-, (X_3, X_1)-planes and with the fiber direction $\boldsymbol{M}$ parallel to the X_1-direction (i. e. the fibers are parallel to OX_1). Suppose that the sample is under uniaxial tension or compression with the direction of tension parallel to the X_1-axis. It is easy to see that the sample is subject to a equi-biaxial deformation, namely

$$x_1 = \lambda_1 X_1, \;\; x_2 = \lambda_2 X_2, \;\; x_3 = \lambda_3 X_3, \tag{17}$$

in which

$$\lambda_1 = \lambda, \;\; \lambda_2 = \lambda_3 = \lambda^{-1/2}, \;\; \lambda > 0, \tag{18}$$

where λ_k are the principal stretches of deformation. Note that the faces of the deformed block are parallel to the (x_1, x_2)-, (x_2, x_3)-, (x_3, x_1)-planes. In the case under consideration we have

$$\boldsymbol{F} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}, \;\; \boldsymbol{E} = \frac{1}{2}\begin{pmatrix} \lambda_1^2 - 1 & 0 & 0 \\ 0 & \lambda_2^2 - 1 & 0 \\ 0 & 0 & \lambda_3^2 - 1 \end{pmatrix}, \tag{19}$$

and

$$I_2 = E_{11}^2 + E_{22}^2 + E_{33}^2, \;\; I_3 = E_{11}^3 + E_{22}^3 + E_{33}^3, \;\; I_4 = E_{11}, \;\; I_5 = E_{11}^2, \tag{20}$$

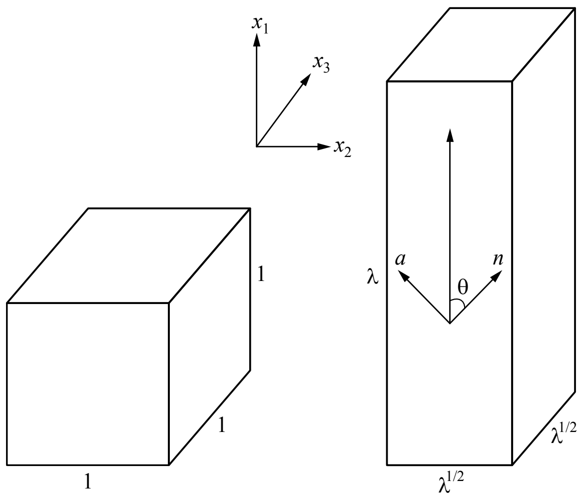

Figure 1. Geometry of cases addressed in Sections 3 and 4. The rectangular block on the left gives the undeformed configuration while the one on the right gives the deformed configuration under the conditions at hand indicating the principal stretches of deformation. Waves travel in the (x_1, x_2)-plane. In that plane, we denote $\boldsymbol{n}$ as the unit vector in the direction of propagation and we denote $\boldsymbol{a}$ as the unit vector orthogonal to $\boldsymbol{n}$.

where $E_{kk} = (\lambda_k^2 - 1)/2$. With the focus on (5) we apply the equation of motion and the incompressibility condition to the analysis of homogeneous plane waves.

Remark 1. Using (12)–(15) together with (19) and (20) it is easy to find that there are only 15 nonzero components of the fourth-order elasticity tensor, namely A_{iijj}, A_{ijij} $(i, j = 1, 2, 3,\ i \neq j)$ and A_{ijji} $(i, j = 1, 2, 3,\ \ i \neq j)$.

Consider waves traveling in the (x_1, x_2)-plane. In that plane denote $\boldsymbol{n}$ as the unit vector in the direction of propagation and $\boldsymbol{a}$ as the unit vector orthogonal to $\boldsymbol{n}$ (see Figure 1). From [Ogden 2007], for example, it is known that there exist two shear bulk waves, one of which is polarized along $\boldsymbol{a}$ and travels with velocity v_{1a}, and the other is polarized along $\boldsymbol{b} = \boldsymbol{a} \times \boldsymbol{n}$ and travels with velocity v_{1b}. These velocities are determined by (see also [Destrade et al. 2010b])

$$\rho v_{1a}^2 = (\gamma_{12} + \gamma_{21} - 2\beta_{12})c_\theta^4 + 2(\beta_{12} - \gamma_{21})c_\theta^2 + \gamma_{21}, \qquad \rho v_{1b}^2 = \gamma_{13}c_\theta^2 + \gamma_{23}s_\theta^2, \tag{21}$$

where θ is the angle between $\boldsymbol{n}$, the direction of propagation, and the x_1-direction, $c_\theta^n := \cos^n\theta$, $s_\theta^n := \sin^n\theta$, and γ_{ij} and β_{ij} $(i, j = 1, 2, 3,\ i \neq j)$ are given by

$$\gamma_{ij} = A_{ijij}, \qquad 2\beta_{ij} = A_{iiii} + A_{jjjj} - 2(A_{iijj} + A_{ijji}), \tag{22}$$

with no sum on repeated indices in formulas (22). Note that while $\beta_{ij} = \beta_{ji}$ (due to $A_{ijkl} = A_{klij}$), it is easy to see that $\gamma_{ij} \neq \gamma_{ji}$ in general. The velocities in (21) are written as polynomials in terms of c_θ^n and s_θ^n.

Now, consider a sufficiently small elongation e defined by $\lambda_1 = 1 + e$. Expanding γ_{ij} and β_{ij} into Maclaurin series up to second order in e by means of (22) and using (12)–(15), under the (uniaxial) conditions at hand, we obtain for the coefficients of the polynomial in (21)$_1$, after a long computation,

$$2\beta_{12}-\gamma_{12}-\gamma_{21}=2\alpha_1+2(4\alpha_1+3\alpha_2+3\alpha_3+3\alpha_4+2\alpha_5)e$$
$$+3\big(6\mu+3A+4\alpha_1+4\alpha_2+6\alpha_3+9\alpha_4+8\alpha_5+6\alpha_6+5\alpha_7+4\alpha_8+4\alpha_9+\tfrac{8}{3}\alpha_{10}+\tfrac{3}{2}\alpha_{11}\big)e^2, \quad (23)$$
$$2(\beta_{12}-\gamma_{21})=2\alpha_1+(3\mu+10\alpha_1+8\alpha_2+6\alpha_3+6\alpha_4+4\alpha_5)e$$
$$+\big(21\mu+\tfrac{39}{4}A+17\alpha_1+17\alpha_2+\tfrac{45}{2}\alpha_3+30\alpha_4+27\alpha_5+18\alpha_6+15\alpha_7+12\alpha_8+12\alpha_9+8\alpha_{10}+\tfrac{9}{2}\alpha_{11}\big)e^2, \quad (24)$$
$$4\gamma_{21}=4\mu+2\alpha_2+(A+2\alpha_2+4\alpha_3+2\alpha_5)e+(8\mu+4A+2\alpha_3+3\alpha_5+12\alpha_6+4\alpha_7+7\alpha_8+4\alpha_{10}+3\alpha_{11})e^2. \quad (25)$$

Note that by taking $\alpha_k=0$, $k=\overline{6,11}$, in the expressions (23)–(25) we obtain the expansions given in [Destrade et al. 2010a, (19)].

Introducing (23)–(25) into (21)$_1$ yields

$$\begin{aligned}\rho v_{1a}^2=&\tfrac{1}{2}\alpha_1 s_{2\theta}^2+\mu+\tfrac{1}{2}\alpha_2+\big[3\mu c_\theta^2+\tfrac{1}{4}A+\big(10c_\theta^2-8c_\theta^4\big)\alpha_1+\big(8c_\theta^2-6c_\theta^4+\tfrac{1}{2}\big)\alpha_2\\&+\big(\tfrac{3}{2}s_{2\theta}^2+1\big)\alpha_3+\tfrac{3}{2}s_{2\theta}^2\alpha_4+\big(s_{2\theta}^2+\tfrac{1}{2}\big)\alpha_5\big]e+\big[\big(21c_\theta^2-18c_\theta^4+2\big)\mu\\&+\big(\tfrac{39}{4}c_\theta^2-9c_\theta^4+1\big)A+\big(17c_\theta^2-12c_\theta^4\big)(\alpha_1+\alpha_2)+\big(\tfrac{45}{2}c_\theta^2-18c_\theta^4+\tfrac{1}{2}\big)\alpha_3\\&+\big(30c_\theta^2-27c_\theta^4\big)\alpha_4+\big(27c_\theta^2-24c_\theta^4+\tfrac{3}{4}\big)\alpha_5+\big(\tfrac{9}{2}s_{2\theta}^2+1\big)\alpha_6+\big(\tfrac{15}{4}s_{2\theta}^2+1\big)\alpha_7\\&+\big(3s_{2\theta}^2+\tfrac{7}{4}\big)\alpha_8+3s_{2\theta}^2\alpha_9+\big(2s_{2\theta}^2+1\big)\alpha_{10}+\big(\tfrac{9}{8}s_{2\theta}^2+\tfrac{3}{4}\big)\alpha_{11}\big]e^2.\end{aligned} \quad (26)$$

In a parallel way, we use (12)–(15) to calculate γ_{13} $(=\gamma_{12})$ and γ_{23}, which are the polynomial-term coefficients in (21)$_2$. Their approximations up second order in e are derived expanding them into Maclaurin series and disregarding all terms equal to and higher than e^3 in the expansions. The values are

$$\begin{aligned}\gamma_{13}=&\mu+\tfrac{1}{2}\alpha_2+\big(3\mu+\tfrac{1}{4}A+2\alpha_1+\tfrac{5}{2}\alpha_2+\alpha_3+\tfrac{1}{2}\alpha_5\big)e\\&+\big(5\mu+\tfrac{7}{4}A+5\alpha_1+5\alpha_2+5\alpha_3+3\alpha_4+\tfrac{15}{4}\alpha_5+3\alpha_6+\alpha_7+\tfrac{7}{4}\alpha_8+\alpha_{10}+\tfrac{3}{4}\alpha_{11}\big)e^2,\\ \gamma_{23}=&\mu+\big(-3\mu-\tfrac{1}{2}A+\alpha_3\big)e+\big(5\mu+\tfrac{7}{4}A-\tfrac{5}{2}\alpha_3+3\alpha_6+\alpha_7+\alpha_8-\tfrac{3}{2}\alpha_{11}\big)e^2.\end{aligned} \quad (27)$$

Introducing (27) into (21)$_2$ one gets the approximation of ρv_{1b}^2 in terms of e, which is

$$\begin{aligned}\rho v_{1b}^2=&\mu+\tfrac{1}{2}c_\theta^2\alpha_2+\big[3(\mu+\tfrac{1}{8}A)c_{2\theta}-\tfrac{1}{8}A+2c_\theta^2\alpha_1+\tfrac{5}{2}c_\theta^2\alpha_2+\alpha_3+\tfrac{1}{2}\alpha_5\big]e\\&+\big[5\mu+\tfrac{7}{4}A+5c_\theta^2\alpha_1+5c_\theta^2\alpha_2+\tfrac{5}{2}(2c_\theta^2-s_\theta^2)\alpha_3+3c_\theta^2\alpha_4+\tfrac{15}{4}c_\theta^2\alpha_5\\&+3\alpha_6+\alpha_7+\tfrac{7}{4}(c_\theta^2+4s_\theta^2)\alpha_8+c_\theta^2\alpha_{10}+\tfrac{3}{4}(c_\theta^2-2s_\theta^2)\alpha_{11}\big]e^2.\end{aligned} \quad (28)$$

As noticed before, if $\alpha_k=0$, $k=\overline{1,11}$, $k\neq 6$ and α_6 is denoted D one gets the expressions in [Destrade et al. 2010b, (11)]. Note that when $\theta=0$ the two shear velocities coincide and can be written as

$$\begin{aligned}\rho v_{1a}^2=\rho v_{1b}^2=&\mu+\tfrac{1}{2}\alpha_2+\big(3\mu+\tfrac{1}{4}A+2\alpha_1+\tfrac{5}{2}\alpha_2+\alpha_3+\tfrac{1}{2}\alpha_5\big)e\\&+\big(5\mu+\tfrac{7}{4}A+5\alpha_1+5\alpha_2+5\alpha_3+3\alpha_4+\tfrac{15}{4}\alpha_5+3\alpha_6+\alpha_7+\tfrac{7}{4}\alpha_8+\alpha_{10}+\tfrac{3}{4}\alpha_{11}\big)e^2.\end{aligned} \quad (29)$$

The result in [Destrade et al. 2010b, (12)] is a special case of the approximation (29) when $\alpha_k=0$, $k=\overline{1,11}$, $k\neq 6$.

Let us turn our attention to consider shear waves that travel in the (x_2,x_3)-plane. Now, by θ we denote the angle between the direction of propagation of the plane wave and the x_2-axis. Then, it is clear that

$\boldsymbol{n} = [0, \cos\theta, \sin\theta]^T$, $\boldsymbol{a} = [0, \sin\theta, -\cos\theta]^T$. The speed v_{2a} of the shear bulk wave polarized along $\boldsymbol{a}$ is given by $(21)_1$ with the indices 12 and 21 replaced by 23 and 32, respectively. In this case, it follows easily using (12)–(15) that $\gamma_{23} = \gamma_{32} = \beta_{23}$. This makes the dependence of the shear wave on θ to vanish and one finally writes the speed v_{2a} in terms of e as

$$\rho v_{2a}^2 = \gamma_{32} = \mu + \left(-3\mu - \tfrac{1}{2}A + \alpha_3\right)e + \left(5\mu + \tfrac{7}{4}A - \tfrac{5}{2}\alpha_3 + 3\alpha_6 + \alpha_7 + \alpha_8 - \tfrac{3}{2}\alpha_{11}\right)e^2. \tag{30}$$

The approximation [Destrade et al. 2010b, (13)] is obtained from (30) by making $\alpha_k = 0$ for $k = \overline{1, 11}$, $k \neq 6$, and replacing α_6 by D.

Lastly, consider waves that travel in the (x_1, x_3)-plane. In this case, θ is the angle between the direction of propagation of the plane wave and the x_1-axis. Using (12)–(15), it follows that $\gamma_{12} = \gamma_{13}$, $\gamma_{21} = \gamma_{31}$ and $\beta_{12} = \beta_{13}$. From these facts it is obvious that the secular equations in this case are exactly the ones obtained for waves propagating in the (x_1, x_2)-plane. This is consistent with the transversely isotropic character of the strain-induced anisotropy.

4. Formulas for the velocity of Rayleigh waves

We turn our attention to the analysis of Rayleigh surface waves. In what follows, by RWkm ($k, m = 1, 2, 3$, $k \neq m$) we denote, for simplicity, a Rayleigh wave propagating along the x_k-direction, and attenuating in the x_m-direction, i.e., we consider a half space occupying the region $x_m < 0$ in the reference configuration with boundary $x_m = 0$ and surfaces waves propagating in the direction x_k.

4A. *Secular equations.*

Remark 2. According to Remark 1, the equations of motion (10) for the incremental displacements u_i, the incremental equation (16) of incompressibility, and the expressions of the incremental traction components are for Rayleigh surface waves the same as those for pre-stressed incompressible isotropic elastic materials (see [Vinh 2010] and references therein). Moreover, using (12)–(15), (19) and (20) one can see that the relations

$$A_{ijji} = A_{jiij} = A_{ijij} - \lambda_i \frac{\partial W}{\partial \lambda_i}, \tag{31}$$

still hold for the (uniaxial) cases under consideration. Therefore, the secular equations of Rayleigh waves for transversely isotropic materials under the conditions considered here are the same as the ones obtained for pre-stressed incompressible isotropic elastic materials.

Let us consider first the RW12 that travels with velocity v. Following Remark 2 and according to [Dowaikh and Ogden 1990], the secular equation of the Rayleigh wave RW12 is (see also [Vinh 2010; Prikazchikov and Rogerson 2004; Vinh and Giang 2010])

$$\gamma_{21}(\gamma_{12} - \rho v^2) + (2\beta_{12} + 2\gamma_{21}^* - \rho v^2)\left[\gamma_{21}(\gamma_{12} - \rho v^2)\right]^{1/2} = (\gamma_{21}^*)^2, \quad 0 < \rho v^2 < \gamma_{12}, \tag{32}$$

where γ_{12}, γ_{21}, and β_{12} are defined by (22), $\gamma_{mk}^* = \gamma_{mk} - \sigma_m$ ($m, k = 1, 2, 3,\ m \neq k$) and the σ_i are the principal stresses of the Cauchy stress tensor, which are given by [Ogden 1984]

$$\sigma_i = \lambda_i \frac{\partial W}{\partial \lambda_i} - \bar{p} \quad (i = 1, 2, 3). \tag{33}$$

Similarly, the secular equation of RWkm can be written as

$$\gamma_{mk}(\gamma_{km}-\rho v^2)+(2\beta_{km}+2\gamma^*_{mk}-\rho v^2)\big[\gamma_{mk}(\gamma_{km}-\rho v^2)\big]^{1/2}=(\gamma^*_{mk})^2, \quad 0<\rho v^2<\gamma_{km}, \tag{34}$$

where γ_{mk} and β_{mk} are given by (22). Under the conditions at hand, it follows that $\sigma_2=\sigma_3=0$, and, furthermore, $\gamma^*_{2k}=\gamma_{2k}$ $(k=1,3)$, $\gamma^*_{3k}=\gamma_{3k}$ $(k=1,2)$. The strong-ellipticity condition (see [Ogden and Singh 2011], for instance) requires that $\gamma_{km}>0$ $(k,m=1,2,3,k\neq m)$. The following results show that it seems natural to consider expansions of strain energy functions in terms of the invariants of $\boldsymbol{E}$. Formulas for the Rayleigh surface waves obtained as polynomials of e depend only on some of the terms in which the strain-energy function W maybe expanded. More precisely, it is shown that linear polynomials of e depend on the coefficients included up to the third-order terms of the strain-energy function W. On the other hand, second-order polynomials in e depend also on the coefficients included up to the fourth-order terms of the strain-energy function W. We focus first on the first-order approximation for the velocity to clarify the analysis.

4B. ***First-order approximations for the velocity.*** In this section we obtain formulas for the velocity of the RWkm given as first-order polynomials in e, i.e., we obtain

$$\rho v^2_{km}=a_{km}+b_{km}e, \tag{35}$$

where v_{km} is the velocity of RWkm. It follows that these equations include μ, A, a_k, $k=\overline{1,5}$ and can be used to determine the elastic coefficients associated with the third-order strain-energy function (3).

Expression of v_{12} *associated with* RW12. It is readily verified that $(32)_1$ in terms of $\eta=\sqrt{(\gamma_{12}-\rho v^2)/\gamma_{21}}$ is of the form (see also [Destrade et al. 2010b; Dowaikh and Ogden 1990])

$$\eta^3+\eta^2+g(e)\eta-1=0, \tag{36}$$

where $g(e):=(2\beta_{12}+2\gamma_{21}-\gamma_{12})/\gamma_{21}$. For our purposes it is sufficient to expand $g(e)$ up to first order in e. It is not difficult to obtain that $g(e)=g_0+g_1e+\mathrm{O}(e^2)$ where

$$\begin{aligned} g_0&=\frac{6\mu+4\alpha_1+\alpha_2}{2\mu+\alpha_2},\\ g_1&=\frac{3A/2+16\alpha_1+15\alpha_2+18\alpha_3+12\alpha_4+11\alpha_5}{2\mu+\alpha_2}-\frac{(6\mu+4\alpha_1+3\alpha_2)(A/2+\alpha_2+2\alpha_3+\alpha_5)}{(2\mu+\alpha_2)^2}.\end{aligned} \tag{37}$$

Equation (36) can be rewritten as

$$F[\eta,e]\equiv\eta^3+\eta^2+g(e)\eta-1=0. \tag{38}$$

To obtain the the first-order approximation in e of ρv^2 it is sufficient to expand η as

$$\eta=\eta_0+\eta_1\,e,\qquad \eta_0:=\eta(0),\qquad \eta_1=\eta'(0), \tag{39}$$

where η_0 is a solution of the equation

$$\eta^3+\eta^2+g_0\eta-1=0. \tag{40}$$

The value η_0 corresponds to the Rayleigh wave propagating in the incompressible transversely isotropic elastic solids (without pre-stresses) and, according to [Ogden and Vinh 2004], η_0 is given by

$$\frac{1}{3}\Big[-1+\sqrt[3]{\tfrac{1}{2}[9\Delta+16+3\sqrt{3}\sqrt{\Delta(4\Delta^2-13\Delta+32)}]}-\sqrt[3]{\tfrac{1}{2}[9\Delta+16-3\sqrt{3}\sqrt{\Delta(4\Delta^2-13\Delta+32)}]}\Big], \tag{41}$$

where $\Delta=g_0+1=(8\mu+4\alpha_1+2\alpha_2)/(2\mu+\alpha_2)$. Note that η_0 depends only on the second-order elastic constants μ, α_1, α_2 and $\eta_0=0.2956$ when $\alpha_1=\alpha_2=0$ (for which $g_0=3,\ \Delta=4$).

Since $\phi(e)=F[\eta(e),e]\equiv 0$, it is easy to get that $\phi'(e)=0$, $\phi''(e)=0$ as well as the remaining derivatives. Using (38) and $\phi'(e)=0$ it follows that

$$\eta'(e)=-\frac{\partial F(\eta(e),e)/\partial e}{\partial F(\eta(e),e)/\partial\eta}=-\frac{g'(e)\eta}{3\eta^2+2\eta+g(e)}, \tag{42}$$

and therefore

$$\eta_1=\eta'(0)=-\frac{\partial F(\eta_0,0)/\partial e}{\partial F(\eta_0,0)/\partial\eta}=-\frac{g_1\eta_0}{3\eta_0^2+2\eta_0+g_0}. \tag{43}$$

Now, introducing γ_{12} and γ_{21}, which are given by (22), into the relation $\rho v^2=\gamma_{12}-\gamma_{21}(\eta_0+\eta_1 e)^2$ and expanding the resulting expression up to first order in e, we obtained

$$\rho v_{12}^2=s_0+s_1 e, \tag{44}$$

where

$$\begin{aligned} s_0&=(1-\eta_0^2)(\mu+\tfrac{1}{2}\alpha_2),\\ s_1&=(3-2\eta_0\eta_1)\mu+\tfrac{1}{4}(1-\eta_0^2)A+2\alpha_1+\tfrac{1}{2}(5-2\eta_0\eta_1-\eta_0^2)\alpha_2+(1-\eta_0^2)(\alpha_3+\alpha_5). \end{aligned} \tag{45}$$

The values η_0 and η_1 are obtained using (41) and (43), respectively, by means of (37). It is clear that ρv_{12}^2 is a function of μ, A, a_k $(k=\overline{1,5})$ and e.

Expression of v_{23} *associated with* RW23. According to (34) and noting that $\sigma_3=0$, the secular equation of the RW23 $(k=2,\ m=3)$ takes the form

$$\gamma_{32}(\gamma_{23}-\rho v^2)+(2\beta_{23}+2\gamma_{32}-\rho v^2)\big[\gamma_{32}(\gamma_{23}-\rho v^2)\big]^{1/2}=(\gamma_{32})^2,\ \ 0<\rho v^2<\gamma_{21}. \tag{46}$$

In terms of the variable $\eta=\sqrt{(\gamma_{23}-\rho v^2)/\gamma_{32}}$, (46) can be rewritten as

$$\eta^3+\eta^2+g^{(23)}(e)\eta-1=0, \tag{47}$$

where $g^{(23)}(e)=(2\beta_{23}+2\gamma_{32}-\gamma_{23})/\gamma_{32}$. Since $\gamma_{23}=\gamma_{32}=\beta_{23}$, as mentioned just before Equation (30), it follows that $g^{(23)}(e)=3$ and, therefore, that $\eta=\eta_0$ where η_0 is given by (41). Taking into account $(27)_2$, the first-order approximation of $\rho v_{23}^2=\gamma_{23}(1-\eta_0^2)$ is finally

$$\rho v_{23}^2=(1-\eta_0^2)[\mu+(-3\mu-A/2)e+\alpha_3]. \tag{48}$$

Expression of v_{21} *associated with* RW21. According to (34) the secular equation of the RW21 is

$$\gamma_{12}(\gamma_{21}-\rho v^2)+(2\beta_{21}+2\gamma_{12}^*-\rho v^2)[\gamma_{12}(\gamma_{21}-\rho v^2)]^{1/2}=(\gamma_{12}^*)^2,\ \ 0<\rho v^2<\gamma_{21}. \tag{49}$$

Using (12)–(15) one can see that $\lambda_1\,\partial W/\partial\lambda_1 - \lambda_2\,\partial W/\partial\lambda_2 = \gamma_{12} - \gamma_{21}$. From this fact and the relation $\sigma_1 - \sigma_2 = \lambda_1\,\partial W/\partial\lambda_1 - \lambda_2\,\partial W/\partial\lambda_2$ (obtained from (33)) and $\sigma_2 = 0$ it follows that $\sigma_1 = \gamma_{12} - \gamma_{21}$. Thus $\gamma^*_{12} = \gamma_{21}$, and (49) now becomes

$$\gamma_{12}(\gamma_{21} - \rho v^2) + (2\beta_{21} + 2\gamma_{21} - \rho v^2)\big[\gamma_{12}(\gamma_{21} - \rho v^2)\big]^{1/2} = (\gamma_{21})^2, \quad 0 < \rho v^2 < \gamma_{21}. \tag{50}$$

In terms of the variable $\eta = \sqrt{(\gamma_{21} - \rho v^2)/\gamma_{12}}$, Equation (50) can be written as

$$\eta^3 + \eta^2 + g^{(21)}(e)\eta - h(e) = 0, \tag{51}$$

where $g^{(21)}(e) := (2\beta_{21} + \gamma_{21})/\gamma_{12}$, $h(e) := \gamma_{21}^2/\gamma_{12}^2$. Up to first order, the expansions of $g^{(21)}(e)$ and $h(e)$ are $g^{(21)}(e) = g_0^{(21)} + g_1^{(21)}e + \mathrm{O}(e^2)$ and $h(e) = 1 - h_1 e + \mathrm{O}(e^2)$, where

$$\begin{aligned} g_0^{(21)} &= \frac{6\mu+4\alpha_1+3\alpha_2}{2\mu+\alpha_2}, \\ g_1^{(21)} &= \frac{6\mu+\frac{3}{2}A+20\alpha_1+19\alpha_2+18\alpha_3+12\alpha_4+11\alpha_5}{2\mu+\alpha_2} - \frac{6\mu+4\alpha_1+3\alpha_2}{(2\mu+\alpha_2)^2}(6\mu+\tfrac{1}{2}A+4\alpha_1+5\alpha_2+2\alpha_3+\alpha_5), \\ h_1 &= \frac{4(3\mu+2\alpha_1+2\alpha_2)}{2\mu+\alpha_2}. \end{aligned} \tag{52}$$

Following the same procedure used to get the first-order approximation of ρv_{12}^2, now, we have

$$\rho v_{21}^2 = s_0^{(21)} + s_1^{(21)}\,e, \tag{53}$$

where

$$\begin{aligned} s_0^{(21)} &= (1-\eta_0^2)\big(\mu + \tfrac{1}{2}\alpha_2\big), \\ s_1^{(21)} &= -(2\eta_0\eta_1 + 3\eta_0^2)\mu + \tfrac{1}{4}(1-\eta_0^2)A - 2\eta_0^2\alpha_1 + \tfrac{1}{2}(1 - 2\eta_0\eta_1 - 5\eta_0^2)\alpha_2 + \tfrac{1}{2}(1-\eta_0^2)(2\alpha_3+\alpha_5), \end{aligned} \tag{54}$$

in which η_0 is calculated by (41) and

$$\eta_1 = -\frac{g_1^{(21)}\eta_0 + h_1}{3\eta_0^2 + 2\eta_0 + g_0^{(21)}}. \tag{55}$$

4C. ***Second-order approximations for the velocity.*** We now extend the above analysis to include fourth-order terms in the strain-energy function. For that reason, it is necessary to obtain formulas for the velocity of the RWkm given as second-order polynomials in e. We follow closely the notation used in the different cases analyzed in Section 4B.

Expression of v_{12} *associated with* RW12. In order to create second-order approximations for the velocity of RW12 we need to expand $g(e)$ into a Maclaurin series up to second order in e. One can write

$$g(e) = g_0 + g_1 e + g_2 e^2 + \mathrm{O}(e^3),$$

where g_0, g_1 are given by (37) and

$$\begin{aligned} g_2 = {} & \frac{2}{2\mu+\alpha_2}\big(24\mu+12A+12\alpha_1+12\alpha_2+\tfrac{39}{2}\alpha_3+27\alpha_4+\tfrac{105}{4}\alpha_5+27\alpha_6+18\alpha_7+\tfrac{69}{4}\alpha_8+12\alpha_9+11\alpha_{10}+\tfrac{27}{4}\alpha_{11}\big) \\ & -\frac{2(6\mu+4\alpha_1+3\alpha_2)}{(2\mu+\alpha_2)^2}\big(2\mu+A+\tfrac{1}{2}\alpha_3+\tfrac{3}{4}\alpha_5+3\alpha_6+\alpha_7+\tfrac{7}{4}\alpha_8+\alpha_{10}+\tfrac{3}{4}\alpha_{11}\big) \\ & -\frac{A+2\alpha_2+4\alpha_3+2\alpha_5}{(2\mu+\alpha_3)^3}\big(12\mu\alpha_3+6\alpha_2\alpha_3+8\mu\alpha_5+4\alpha_2\alpha_5+12\mu\alpha_2+6\alpha_2^2 \\ & \qquad +12\mu\alpha_4+6\alpha_2\alpha_4+16\mu\alpha_1+6\alpha_1\alpha_2-4\alpha_1\alpha_3-A\alpha_1-2\alpha_1\alpha_5\big). \end{aligned} \tag{56}$$

Up to second order in e the expansion of $\eta(e)$ is $\eta=\eta_0+\eta_1 e+\eta_2 e^2$, where η_0 and η_1 are given by (41) and (43), respectively, and η_2 is to be determined. Using (38) and $\phi''(e)=0$, it is obtained that

$$\eta''(e) = -\frac{\dfrac{\partial^2 F}{\partial \eta^2}\eta'^2+2\dfrac{\partial^2 F}{\partial\eta\partial e}\eta'+\dfrac{\partial^2 F}{\partial e^2}}{\dfrac{\partial F}{\partial \eta}}\Bigg|_{(\eta(e),e)}, \tag{57}$$

and, therefore, that

$$\eta_2 = \tfrac{1}{2}\eta''(0) = -\frac{(3\eta_0+1)\eta_1^2+g_1\eta_1+g_2\eta_0}{3\eta_0^2+2\eta_0+g_0}. \tag{58}$$

Expanding $\rho v^2=\gamma_{12}-\gamma_{21}(\eta_0+\eta_1\, e+\eta_2\, e^2)^2$ up to second order in e yields

$$\rho v_{12}^2 = s_0+s_1\, e+s_2\, e^2, \tag{59}$$

where s_0 and s_1 are given by (45) and

$$\begin{aligned} s_2 = {} & (5-2\eta_0^2-2\eta_0\eta_2-\eta_1^2)\mu+(7-2\eta_0\eta_1-4\eta_0^2)A/4+5\alpha_1+[5-\eta_0(\eta_1+\eta_2)-\eta_1^2/2]\alpha_2 \\ & +(5-2\eta_0\eta_1-\eta_0^2/2)\alpha_3+3\alpha_4+(15-4\eta_0\eta_1-3\eta_0^2)\alpha_5/4 \\ & +3(1-\eta_0^2)\alpha_6+(1-\eta_0^2)\alpha_7+7(1-\eta_0^2)\alpha_8/4+(1-\eta_0^2)\alpha_{10}+3(1-\eta_0^2)\alpha_{11}/4. \end{aligned} \tag{60}$$

Relation (59), where s_0 and s_1 are given by (45) and s_2 is given by (60), is the second-order approximation for the velocity. Now, consider that $\alpha_k=0$, $k=\overline{1,11}$, $k\neq 6$. Then, using (37) and (56) one obtains that $g_0=3$, $g_1=0$, $g_2=18+9(A/\mu)+18(\alpha_6/\mu)$. Similarly, using (41), (43) and (58) one obtains that $\eta_0=0.2956$, $\eta_1=0$ and $\eta_2=-(1.3806+0.6903(A/\mu)+1.3806(\alpha_6/\mu))$. Introducing these results into (45) and (60) it is easy to obtain that

$$s_0=0.9126\mu,\;\; s_1=3\mu+0.9126A/4,\;\; s_2=5.642\mu+2.071A+3.554\alpha_6, \tag{61}$$

which coincide with the coefficients of the approximation in [Destrade et al. 2010b, (19)], where the coefficient D is simply α_6.

Expression of v_{23} associated with RW23. Introducing the expansion $(27)_2$ of γ_{23} into the relation $\rho v_{23}^2=\gamma_{23}(1-\eta_0^2)$ one obtains the second-order approximation as

$$\rho v_{23}^2=(1-\eta_0^2)[\mu+(-3\mu-A/2+\alpha_3)e+\big(5\mu+\tfrac{7}{4}A-\tfrac{5}{2}\alpha_3+3\alpha_6+\alpha_7+\alpha_8-\tfrac{3}{2}\alpha_{11}\big)e^2]. \tag{62}$$

Expression of v_{21} associated with RW21. Following the same procedure used to obtain the second-order expansion for ρv_{12}^2, one can write in this case that

$$\rho v_{21}^2 = s_0^{(21)} + s_1^{(21)}\, e + s_2^{(21)}\, e^2, \tag{63}$$

where $s_0^{(21)}$ and $s_1^{(21)}$ are determined using (54) and $s_2^{(21)}$ is given by

$$\begin{aligned} s_2^{(21)} \;=\; & (2 - 2\eta_0\eta_2 - \eta_1^2 - 6\eta_0\eta_1 - 5\eta_0^2)\mu + \big(1 - \tfrac{1}{2}\eta_0\eta_1 - \tfrac{7}{4}\eta_0^2\big)A - (4\eta_1 + 5\eta_0)\eta_0 a_1 \\ & - \big(\eta_0\eta_2 + \tfrac{1}{2}\eta_1^2 + 5\eta_0\eta_1 + 5\eta_0^2\big)a_2 + \big(\tfrac{1}{2} - 2\eta_0\eta_1 - 5\eta_0^2\big)a_3 - 3\eta_0^2 a_4 \\ & + \big(\tfrac{3}{4} - \eta_0\eta_1 - \tfrac{15}{4}\eta_0^2\big)a_5 + (1 - \eta_0^2)\big(3a_6 + a_7 + \tfrac{7}{4}a_8 + a_{10} + \tfrac{3}{4}a_{11}\big), \end{aligned} \tag{64}$$

where η_0 and η_1 are determined using (41) and (55), respectively, and η_2 is

$$\eta_2 = -\frac{(3\eta_0 + 1)\eta_1^2 + g_1^{(21)}\eta_1 + g_2^{(21)}\eta_0 - h_2}{3\eta_0^2 + 2\eta_0 + g_0^{(21)}}. \tag{65}$$

In (65), $g_0^{(21)}$ and $g_1^{(21)}$ are determined using (52) and the remaining symbols are given by

$$\begin{aligned} g_2^{(21)} = & \frac{2}{2\mu+a_2}\big(27\mu+\tfrac{51}{4}A+17a_1+17a_2+24a_3+30a_4+\tfrac{117}{4}a_5+27a_6+18a_7+\tfrac{69}{4}a_8+12a_9+11a_{10}+\tfrac{27}{4}a_{11}\big) \\ & -2\,\frac{6\mu + 4a_1 + 3a_2}{(2\mu + a_2)^2}\big(5\mu + \tfrac{7}{4}A + 5a_1 + 5a_2 + 5a_3 + 3a_4 + \tfrac{15}{4}a_5 + 3a_6 + a_7 + \tfrac{7}{4}a_8 + a_{10} + \tfrac{3}{4}a_{11}\big) \\ & +4\,\frac{3\mu+\frac{A}{4}+2a_1+\frac{5}{2}a_2+a_3+\frac{1}{2}a_5}{(2\mu+a_2)^3}\big[(12\mu+4a_1+2a_2-12a_3-12a_4-8a_5)\mu \\ & \qquad +(A+8a_1+6a_2+4a_3+2a_5)a_1-(2a_2+6a_3+6a_4+4a_5)a_2\big] \end{aligned} \tag{66}$$

and

$$\begin{aligned} h_2 = \frac{1}{(2\mu + a_2)^2}\big[& (84\mu + 104a_1 + 104a_2 - 12a_3 - 24a_4 - 12a_5)\mu \\ & +(4A + 48a_1 + 16a_3 + 8a_5)a_1 + (A + 8a_1 + 36a_2 - 2a_3 - 12a_4 - 4a_5)a_2\big]. \end{aligned} \tag{67}$$

5. Conclusions

The purpose of this analysis is to evaluate the mechanical properties of transversely isotropic incompressible nonlinear elastic materials such as certain soft biological tissues. We have considered an expanded strain energy function in terms of the Green strain tensor. More in particular we have focused on an energy function with elastic constants of second, third, and fourth orders in the Green strain tensor (see (5)). Homogeneous plane waves and Rayleigh surface waves have been examined in conjunction with the strain energy function (5). The speeds of shear waves and Rayleigh waves in the incompressible model (5) have been obtained. The formulas developed can be used to determine the elastic coefficients included in (5), although, it is not an easy task. The equations obtained in [Destrade et al. 2010b] are recovered from their corresponding formulas obtained in this paper. It has been noted that formulas for the speeds of Rayleigh waves that are linear in e depend on the coefficients included up to third-order terms in the strain-energy function (5). On the other hand, the speeds of Rayleigh waves given as second-order polynomials in e depend also on the coefficients included up to fourth-order terms in the strain-energy function (5). This is particularly important since even though physical acousticians are interested in third order constants for anisotropic solids, workers in nonlinear elasticity, and furthermore, in soft biological tissues, work with finite extensions involving fourth order constants.

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