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ON SUCCESSIVE DIFFERENTIATIONS OF THE ROTATION TENSOR: AN APPLICATION TO NONLINEAR BEAM ELEMENTS

TEODORO MERLINI AND MARCO MORANDINI

Successive differentiations of the rotation tensor are characterized by successive differential rotation vectors. Useful expressions of the differential rotation vectors for differentiations up to third order are derived. In the context of the exponential parameterization, explicit expressions for the differential maps (the maps providing the differential rotation vectors from the differentials of the parameters chosen) are obtained by resorting to an original infinite family of recursive subexponential maps. Useful properties of the mapping tensors are discussed.

The formulation is appropriate for nonlinear problems of computational solid mechanics, when spatial, incremental, and virtual variations of particle orientations must be dealt with together. As an application, the classical problem of modeling space-curved slender beams by finite elements is considered. The variational formulation and the nonlinear interpolation of the orientations, together with the relevant linearizations, consistently exploit the proposed differentiations and lead to an objective beam element. Two test cases are discussed.

1. Introduction

The motivation for a circumstantial study of the differentiations of the rotation tensor comes from specific demands in computational continuum mechanics by the finite element method and in the relevant variational formulations. As far as three-dimensional solids are concerned, the rotation field is manifestly an unknown variable in nonlinear mechanics of polar materials [Grekova and Zhilin 2001; Bauer et al. 2010], but is introduced as well as an unknown variable in some discrete representations with classical nonpolar materials [Simo et al. 1992; Atluri and Cazzani 1995; Merlini 1997]. Moreover, the rotation field is a primary unknown variable in Cosserat-type formulations of structured solid mechanics, namely beams, that behave as one-dimensional polar continua, and shells (refer to [Altenbach et al. 2010] and references therein), that feature a mixed polar/nonpolar constitutive behavior.

Multiple differentiations of the finite-rotation field are involved in nonlinear continuum mechanics problems. The particle orientations (that is, rotations from an absolute reference frame) within a body in any (deformed) configurations "differ" in general from each other, and the differential (hence the gradient) of the orientation field is used to define angular curvatures within the body (and hence angular strains by comparing curvatures in different configurations). Throughout the body deformation, the particle orientations undergo rotations; differential rotations are the unknowns of an incremental solution process in nonlinear boundary value problems. Following a variational approach with its discrete representation by the finite element method, one handles virtual functionals depending on virtual variable fields, so

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virtual (differential) rotations of the particle orientations must be accounted for as well. This outline of the computational strategy lets us see that we have to deal with (at least) three independent variations of the orientation field: a spatial variation within the material body (by itself a complicated task in three dimensions), an incremental variation along with the configuration evolution, and a virtual variation that enables us to cast a discrete equation set.

A consistent setting that allows one to handle successive derivatives of the rotation tensor in a systematic way is lacking in the literature. A number of papers on the derivatives of wider classes of tensor-valued functions of a tensor were published during the past decade. Often, these investigations were motivated in the field of continuum mechanics [Rosati 1999; Itskov 2002; Jog 2008] and in the modeling of nonlinear elastoplastic constitutive laws [de Souza Neto 2001]. Most papers concern the representation and the derivatives of isotropic tensor functions of either a symmetrical or a generally unsymmetrical tensor [Ortiz et al. 2001; de Souza Neto 2001; 2004; Itskov and Aksel 2002; Itskov 2003; Fung 2004; Lu 2004; Dui et al. 2006; Wang and Dui 2007; Jog 2008]. Some of these works deal explicitly with exponential functions, and in this context reference is due to the accurate and worthy paper [Najfeld and Havel 1995], published in a field far from the scope of engineering. Though some cues may come from these works, it seems difficult to bring them in the bed of orthogonal tensors, as the exponential functions of skew-symmetric tensors are.

In the case of orthogonal tensors, explicit formulae for the derivatives of the tangent map of the rotation tensor are available, for example, in [Borri et al. 1990; Ritto-Corrêa and Camotim 2002; Mäkinen 2008], but again a systematic setting of successive differentiations is missing. However, the specific properties of the special orthogonal group should help in obtaining closed-form expressions for any differentiations. This task is undertaken in this paper. Our approach is based on an original decomposition of the exponential map into an infinite family of recursive subexponential maps, whose lowest differentiations are affordable. This enables us to manage successive differentiations of the rotation tensor with analytical expressions that are safe to implement and exact within machine precision. The proposed methodology was implemented in a nonlinear finite element code and successfully tested with solids [Merlini and Morandini 2005] and shells [Merlini and Morandini 2011a]; in [Merlini and Morandini 2004b, Appendices] a brief description of the methodology was given.

The outline of the paper is as follows. In Section 2 the essential structure of successive differentiations of an orthogonal tensor is presented and appropriate differential rotation vectors are proposed, however no parameterization of the rotation tensor is introduced yet. In Section 3 the exponential map is assumed for the rotation tensor and a useful family of subexponential maps is conveniently set. The lowest differentiations of the subexponential maps are dealt with in Section 4 and are used in Section 5 in order to provide expressions for the differential maps of the rotation, that is, the maps from the differentials of the rotation vector to the differential rotation vectors themselves.

The remaining part of the paper deals with an application of the proposed formulation to a classical problem in computational mechanics, the finite-element modeling of space-curved slender beams. The essentials of Reissner–Simo beam variational mechanics are discussed in Section 6 and a beam element based on a consistent nonlinear interpolation is discussed in Section 7. The formulation is contrasted with the analogous paper [Ritto-Corrêa and Camotim 2002] to highlight the significance of adopting a nonlinear element. In Section 8, two popular numerical tests demonstrate the performance of both two-node and three-node beam elements. As further remarked in Section 9, the formulation is kept parameterization-free for as long as possible.

The style of the mathematical developments is kept as plain as possible. Abstract formalisms, typical of rotation math, are avoided as unnecessary to understanding matters that are meant for people with a mechanical background. When tensor-valued functions are differentiable — as rotations are, being exponential functions of the rotation vector — we by far prefer working with differentials than with directional derivatives. On the other hand, a heavy use of higher-order tensors is unavoidable in this context, and an index-free tensor notation is adopted throughout the paper to make it easier to follow complicated developments. Tensor notations and rules used in the paper are gathered in the Appendix.

2. Structure of successive differentiations of an orthogonal tensor

From a merely geometric standpoint, a rotation is a tensor $\boldsymbol{\Phi}$ that transforms a frame of three vectors (say $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$) into another frame ($\boldsymbol{\Phi a}$, $\boldsymbol{\Phi b}$, and $\boldsymbol{\Phi c}$), while preserving the vector lengths and their mutual orientations, and hence the frame volume $\boldsymbol{a} \times \boldsymbol{b} \cdot \boldsymbol{c}$. To work so, a rotation tensor must obey the symmetrical tensor equation

$$\boldsymbol{\Phi}\boldsymbol{\Phi}^{\mathrm{T}} = \boldsymbol{I} \tag{1}$$

and the scalar condition $\det \boldsymbol{\Phi} = +1$. Equation (1) is called the *orthogonality condition* and represents six scalar constraints to fulfill in order to classify a tensor as a rotation; as a consequence, it makes a rotation tensor depend on just three scalar parameters.

2.1. ***Differential rotation vectors.*** It is well known that the differentiation of the rotation tensor is characterized by a skew-symmetric tensor, hence by the relevant axial vector [Pietraszkiewicz and Badur 1983; Cardona and Géradin 1988; Ibrahimbegović et al. 1995]. It has been found that any successive independent differentiation of the rotation tensor is in turn characterized by a further vector that depends on the differentiations of the preceding characteristic vectors themselves [Merlini and Morandini 2004a]. In this section we delve into this topic.

Let us start by differentiating the orthogonality condition (1). We denote successive independent variations with $\mathrm{d}_1, \mathrm{d}_2, \mathrm{d}_3, \dots$ and keep evaluating $\mathrm{d}_1(\boldsymbol{\Phi}\boldsymbol{\Phi}^{\mathrm{T}})$, $\mathrm{d}_2\mathrm{d}_1(\boldsymbol{\Phi}\boldsymbol{\Phi}^{\mathrm{T}})$, $\mathrm{d}_3\mathrm{d}_2\mathrm{d}_1(\boldsymbol{\Phi}\boldsymbol{\Phi}^{\mathrm{T}}), \dots$ in sequence. We use the symbolic notation $\mathrm{d}^n(\,)$ for a multiple differential $\mathrm{d}_n \dots \mathrm{d}_3\mathrm{d}_2\mathrm{d}_1(\,)$. Evaluation of the n-th differential $\mathrm{d}^n(\boldsymbol{\Phi}\boldsymbol{\Phi}^{\mathrm{T}})$ yields a sum of 2^n terms, specifically the extreme tensors, namely $\mathrm{d}^n\boldsymbol{\Phi}\,\boldsymbol{\Phi}^{\mathrm{T}}$ and its transpose $\boldsymbol{\Phi}\,\mathrm{d}^n\boldsymbol{\Phi}^{\mathrm{T}}$, and the bulk of the remaining tensors that contain lower-order differentials (up to $\mathrm{d}^{n-1}\boldsymbol{\Phi}$) of the rotation. Let us denote by $-2\boldsymbol{\Phi}^{\mathrm{S}}_{\mathrm{d}^n}$ the sum of this bulk of tensors, which is of course a symmetric tensor (we denote a symmetric tensor by a superscript $(\,)^{\mathrm{S}}$ and a skew-symmetric tensor by means of its axial vector $\boldsymbol{a}$ as $\boldsymbol{a}\times$, see (A.1)). Thus, a multiple differentiation of (1) ends up in a form like $\mathrm{d}^n(\boldsymbol{\Phi}\boldsymbol{\Phi}^{\mathrm{T}}) = \mathrm{d}^n\boldsymbol{\Phi}\,\boldsymbol{\Phi}^{\mathrm{T}} - 2\boldsymbol{\Phi}^{\mathrm{S}}_{\mathrm{d}^n} + \boldsymbol{\Phi}\,\mathrm{d}^n\boldsymbol{\Phi}^{\mathrm{T}} = \mathrm{d}^n\boldsymbol{\Phi}\,\boldsymbol{\Phi}^{\mathrm{T}} - \boldsymbol{\Phi}^{\mathrm{S}}_{\mathrm{d}^n} + (\mathrm{d}^n\boldsymbol{\Phi}\,\boldsymbol{\Phi}^{\mathrm{T}} - \boldsymbol{\Phi}^{\mathrm{S}}_{\mathrm{d}^n})^{\mathrm{T}} = \boldsymbol{0}$. This means that tensor $\mathrm{d}^n\boldsymbol{\Phi}\,\boldsymbol{\Phi}^{\mathrm{T}} - \boldsymbol{\Phi}^{\mathrm{S}}_{\mathrm{d}^n}$ is skew-symmetric, say $\boldsymbol{\varphi}_{\mathrm{d}^n}\times$, and finally we obtain the decomposition of tensor $\mathrm{d}^n\boldsymbol{\Phi}\,\boldsymbol{\Phi}^{\mathrm{T}}$ into its symmetric and skew-symmetric parts,

$$\mathrm{d}^n\boldsymbol{\Phi}\,\boldsymbol{\Phi}^{\mathrm{T}} = (\boldsymbol{\varphi}_{\mathrm{d}^n}\times) + \boldsymbol{\Phi}^{\mathrm{S}}_{\mathrm{d}^n}.$$

As it is well known, tensor $\mathrm{d}_1\boldsymbol{\Phi}\,\boldsymbol{\Phi}^{\mathrm{T}} = \boldsymbol{\varphi}_{\mathrm{d}_1}\times$ is skew-symmetric ($\boldsymbol{\Phi}^{\mathrm{S}}_{\mathrm{d}_1} = \boldsymbol{0}$) and $\boldsymbol{\varphi}_{\mathrm{d}_1}$ represents the characteristic vector of the first differential $\mathrm{d}_1\boldsymbol{\Phi}$ of the rotation tensor. The successive tensors $\mathrm{d}^n\boldsymbol{\Phi}\,\boldsymbol{\Phi}^{\mathrm{T}}$

with $n > 1$ have both symmetric and skew-symmetric parts, instead. The former ones ($\boldsymbol{\Phi}^{S}_{d^n}$) are algebraic functions of the characteristic vectors $\boldsymbol{\varphi}_{d^1}, \boldsymbol{\varphi}_{d^2}, \ldots, \boldsymbol{\varphi}_{d^{n-1}}$ of the preceding lower-order differentials, whereas the latter ones ($\boldsymbol{\varphi}_{d^n}\times$) contain the differentials of those characteristic vectors as well. Therefore, the axial vector $\boldsymbol{\varphi}_{d^n}$ can be assumed as a characteristic vector of the n-th differential $d^n\boldsymbol{\Phi}$ of the rotation tensor. Note that vectors $\boldsymbol{\varphi}_{d^n}$ with $n > 1$ also contain algebraic functions of the preceding characteristic vectors $\boldsymbol{\varphi}_{d^1}, \boldsymbol{\varphi}_{d^2}, \ldots, \boldsymbol{\varphi}_{d^{n-1}}$, thus there is some arbitrariness in the choice of the characteristic vector. Our choice is to assume the whole axial vector of $d^n\boldsymbol{\Phi}\boldsymbol{\Phi}^{T}$ as *the* characteristic vector of the differential $d^n\boldsymbol{\Phi}$.

The above characterization of successive differentials of the rotation tensor is now made explicit for differentiations up to third order. We denote three successive independent variations by δ, ∂, and d. In computational finite elasticity, such symbols are conveniently associated respectively with virtual variations, incremental variations, and spatial variations — the latter being derivatives along either a one-coordinate domain (beams), or surface gradients on a two-dimensional domain (shells), or even gradients on a three-dimensional solid domain. After evaluation of $\delta(\boldsymbol{\Phi}\boldsymbol{\Phi}^{T}) = \mathbf{0}$, $\partial\delta(\boldsymbol{\Phi}\boldsymbol{\Phi}^{T}) = \mathbf{0}$, and $d\partial\delta(\boldsymbol{\Phi}\boldsymbol{\Phi}^{T}) = \mathbf{0}$, we easily obtain

$$\begin{aligned}
\delta\boldsymbol{\Phi}\boldsymbol{\Phi}^{T} &= \boldsymbol{\varphi}_{\delta}\times, \\
\partial\delta\boldsymbol{\Phi}\boldsymbol{\Phi}^{T} &= \boldsymbol{\varphi}_{\partial\delta}\times + \tfrac{1}{2}(\boldsymbol{\varphi}_{\partial}\times\boldsymbol{\varphi}_{\delta}\times + \boldsymbol{\varphi}_{\delta}\times\boldsymbol{\varphi}_{\partial}\times), \\
d\partial\delta\boldsymbol{\Phi}\boldsymbol{\Phi}^{T} &= \boldsymbol{\varphi}_{d\partial\delta}\times + \tfrac{1}{2}(\boldsymbol{\varphi}_{d\partial}\times\boldsymbol{\varphi}_{\delta}\times + \boldsymbol{\varphi}_{\partial\delta}\times\boldsymbol{\varphi}_{d}\times + \boldsymbol{\varphi}_{\delta d}\times\boldsymbol{\varphi}_{\partial}\times + \boldsymbol{\varphi}_{d}\times\boldsymbol{\varphi}_{\partial\delta}\times + \boldsymbol{\varphi}_{\partial}\times\boldsymbol{\varphi}_{\delta d}\times + \boldsymbol{\varphi}_{\delta}\times\boldsymbol{\varphi}_{d\partial}\times),
\end{aligned} \tag{2}$$

where $\boldsymbol{\varphi}_{\delta}$, $\boldsymbol{\varphi}_{\partial\delta}$, and $\boldsymbol{\varphi}_{d\partial\delta}$ are proposed here as the characteristic vectors introduced at each successive variation.

It is worth noting that no parameterization of the rotation tensor is implied in the foregoing characterization. In spite of that, the first characteristic vector $\boldsymbol{\varphi}_{\delta}$ has been properly referred to as the virtual rotation vector [Borri et al. 1990]. A simple reason for such terminology comes from considering the rotation built with the infinitesimal rotation vector $\boldsymbol{\varphi}_{\delta}$, that is, $\exp(\boldsymbol{\varphi}_{\delta}\times) \cong \boldsymbol{I} + \boldsymbol{\varphi}_{\delta}\times$, and appending it to a rotation $\boldsymbol{\Phi}$; the composed rotation becomes $\exp(\boldsymbol{\varphi}_{\delta}\times)\boldsymbol{\Phi} \cong (\boldsymbol{I} + \boldsymbol{\varphi}_{\delta}\times)\boldsymbol{\Phi} = \boldsymbol{\Phi} + \delta\boldsymbol{\Phi}$ and matches the expression of what is often understood — perhaps improperly — as a varied rotation. However, it is worth stressing that $\boldsymbol{\varphi}_{\delta}$ is an (independent) infinitesimal differential rotation vector and by no means has to coincide (in general) with the differential $\delta\boldsymbol{\varphi}$ of the rotation vector $\boldsymbol{\varphi} = \operatorname{ax}\ln\boldsymbol{\Phi}$ [Borri et al. 1990]. Note that vector $\boldsymbol{\varphi}_{\delta}$ is also related to the concept of spin and is often given a perhaps misleading notation using the differential symbol δ applied at a different variable, for example, $\delta\boldsymbol{\omega}$ in [Ritto-Corrêa and Camotim 2002]. We by far prefer the notation with an appended subscript δ to the rotation vector itself, a notation that stresses unequivocally that $\boldsymbol{\varphi}_{\delta}$ is not the variation of a vector-valued function.

We extend the same notation convention to vectors $\boldsymbol{\varphi}_{\partial\delta}$ and $\boldsymbol{\varphi}_{d\partial\delta}$ and adopt a general terminology for the characteristic vectors of any differentials of the rotation tensor by calling them *differential rotation vectors*. These vectors account for the variations of the differential vectors of lower order, as it is shown next.

2.2. ***Evaluation of successive differential rotation vectors.*** Expressions for evaluating the second and third differential rotation vectors follow from their definitions as axial vectors, that is, $\boldsymbol{\varphi}_{\partial\delta} = \operatorname{ax}(\partial\delta\boldsymbol{\Phi}\boldsymbol{\Phi}^{T})$ and $\boldsymbol{\varphi}_{d\partial\delta} = \operatorname{ax}(d\partial\delta\boldsymbol{\Phi}\boldsymbol{\Phi}^{T})$. From

$$\boldsymbol{\varphi}_{\partial\delta}\times = \tfrac{1}{2}(\partial\delta\boldsymbol{\Phi}\boldsymbol{\Phi}^{T} - (\partial\delta\boldsymbol{\Phi}\boldsymbol{\Phi}^{T})^{T}) \quad \text{and} \quad \boldsymbol{\varphi}_{d\partial\delta}\times = \tfrac{1}{2}(d\partial\delta\boldsymbol{\Phi}\boldsymbol{\Phi}^{T} - (d\partial\delta\boldsymbol{\Phi}\boldsymbol{\Phi}^{T})^{T})$$

we obtain, after some algebraic manipulations involving (2), the expressions

$$\begin{aligned}\boldsymbol{\varphi}_{\partial\delta} &= \partial\boldsymbol{\varphi}_{\delta} - \tfrac{1}{2}\boldsymbol{\varphi}_{\partial} \times \boldsymbol{\varphi}_{\delta},\\ \boldsymbol{\varphi}_{\mathrm{d}\partial\delta} &= \mathrm{d}\boldsymbol{\varphi}_{\partial\delta} - \tfrac{1}{2}\boldsymbol{\varphi}_{\mathrm{d}} \times \boldsymbol{\varphi}_{\partial\delta} - \tfrac{1}{2}(\boldsymbol{\varphi}_{\mathrm{d}} \otimes \boldsymbol{\varphi}_{\partial} \cdot \boldsymbol{\varphi}_{\delta} + (\boldsymbol{\varphi}_{\partial} \otimes \boldsymbol{\varphi}_{\delta})^{\mathrm{S}} \cdot \boldsymbol{\varphi}_{\mathrm{d}})\\ &= \mathrm{d}\partial\boldsymbol{\varphi}_{\delta} - \tfrac{1}{2}\boldsymbol{\varphi}_{\mathrm{d}\partial} \times \boldsymbol{\varphi}_{\delta} - \tfrac{1}{2}(\boldsymbol{\varphi}_{\mathrm{d}} \times \boldsymbol{\varphi}_{\partial\delta} + \boldsymbol{\varphi}_{\partial} \times \boldsymbol{\varphi}_{\delta\mathrm{d}}) - (\boldsymbol{\varphi}_{\mathrm{d}} \otimes \boldsymbol{\varphi}_{\partial})^{\mathrm{S}} \cdot \boldsymbol{\varphi}_{\delta},\end{aligned} \tag{3}$$

where the symbols d, ∂, and δ may exchange cyclically. Other expressions, symmetrical with respect to the variations d, ∂, and δ, easily follow from (3):

$$\begin{aligned}\boldsymbol{\varphi}_{\partial\delta} &= \tfrac{1}{2}(\partial\boldsymbol{\varphi}_{\delta} + \delta\boldsymbol{\varphi}_{\partial}),\\ \boldsymbol{\varphi}_{\mathrm{d}\partial\delta} &= \tfrac{1}{3}(\mathrm{d}\boldsymbol{\varphi}_{\partial\delta} + \partial\boldsymbol{\varphi}_{\delta\mathrm{d}} + \delta\boldsymbol{\varphi}_{\mathrm{d}\partial}) - \tfrac{1}{6}(\boldsymbol{\varphi}_{\mathrm{d}} \times \boldsymbol{\varphi}_{\partial\delta} + \boldsymbol{\varphi}_{\partial} \times \boldsymbol{\varphi}_{\delta\mathrm{d}} + \boldsymbol{\varphi}_{\delta} \times \boldsymbol{\varphi}_{\mathrm{d}\partial})\\ &\qquad - \tfrac{1}{3}(\boldsymbol{\varphi}_{\mathrm{d}} \otimes \boldsymbol{\varphi}_{\partial} \cdot \boldsymbol{\varphi}_{\delta} + \boldsymbol{\varphi}_{\partial} \otimes \boldsymbol{\varphi}_{\delta} \cdot \boldsymbol{\varphi}_{\mathrm{d}} + \boldsymbol{\varphi}_{\delta} \otimes \boldsymbol{\varphi}_{\mathrm{d}} \cdot \boldsymbol{\varphi}_{\partial})\\ &= \tfrac{1}{3}(\mathrm{d}\partial\boldsymbol{\varphi}_{\delta} + \partial\delta\boldsymbol{\varphi}_{\mathrm{d}} + \delta\mathrm{d}\boldsymbol{\varphi}_{\partial}) - \tfrac{1}{6}(\boldsymbol{\varphi}_{\mathrm{d}} \times \boldsymbol{\varphi}_{\partial\delta} + \boldsymbol{\varphi}_{\partial} \times \boldsymbol{\varphi}_{\delta\mathrm{d}} + \boldsymbol{\varphi}_{\delta} \times \boldsymbol{\varphi}_{\mathrm{d}\partial})\\ &\qquad - \tfrac{1}{3}(\boldsymbol{\varphi}_{\mathrm{d}} \otimes \boldsymbol{\varphi}_{\partial} \cdot \boldsymbol{\varphi}_{\delta} + \boldsymbol{\varphi}_{\partial} \otimes \boldsymbol{\varphi}_{\delta} \cdot \boldsymbol{\varphi}_{\mathrm{d}} + \boldsymbol{\varphi}_{\delta} \otimes \boldsymbol{\varphi}_{\mathrm{d}} \cdot \boldsymbol{\varphi}_{\partial}),\end{aligned} \tag{4}$$

together with the identity

$$\mathrm{d}\boldsymbol{\varphi}_{\partial\delta} + \partial\boldsymbol{\varphi}_{\delta\mathrm{d}} + \delta\boldsymbol{\varphi}_{\mathrm{d}\partial} = \mathrm{d}\partial\boldsymbol{\varphi}_{\delta} + \partial\delta\boldsymbol{\varphi}_{\mathrm{d}} + \delta\mathrm{d}\boldsymbol{\varphi}_{\partial}.$$

In practical applications, a parameterization of the rotation tensor must be resorted to, and the differential rotation vectors are solved for the differentials of the parameters chosen. Evaluation of the first differential vector follows from its definition $\boldsymbol{\varphi}_{\delta} = \mathrm{ax}(\delta\boldsymbol{\Phi}\boldsymbol{\Phi}^{\mathrm{T}})$; then, the next differential vectors $\boldsymbol{\varphi}_{\partial\delta}$ and $\boldsymbol{\varphi}_{\mathrm{d}\partial\delta}$ are evaluated, via (3) or (4), by differentiating the lower-order differential vectors themselves—which in turn is a parameterization-dependent operation. Such customized evaluations of the differential vectors are addressed in Section 5, with focus on our preferred natural parameterization.

Alternative expressions, based on corotational instead of direct differentiations of lower-order differential vectors, are available for $\boldsymbol{\varphi}_{\partial\delta}$ and $\boldsymbol{\varphi}_{\mathrm{d}\partial\delta}$. Introducing the corotational differentiations

$$\begin{aligned}\boldsymbol{\Phi}\,\partial(\boldsymbol{\Phi}^{\mathrm{T}}\boldsymbol{\varphi}_{\delta}) &= \partial\boldsymbol{\varphi}_{\delta} - \boldsymbol{\varphi}_{\partial} \times \boldsymbol{\varphi}_{\delta},\\ \boldsymbol{\Phi}\,\mathrm{d}(\boldsymbol{\Phi}^{\mathrm{T}}\boldsymbol{\varphi}_{\partial\delta}) &= \mathrm{d}\boldsymbol{\varphi}_{\partial\delta} - \boldsymbol{\varphi}_{\mathrm{d}} \times \boldsymbol{\varphi}_{\partial\delta},\\ \boldsymbol{\Phi}\,\mathrm{d}\partial(\boldsymbol{\Phi}^{\mathrm{T}}\boldsymbol{\varphi}_{\delta}) &= \mathrm{d}\partial\boldsymbol{\varphi}_{\delta} - \boldsymbol{\varphi}_{\mathrm{d}\partial} \times \boldsymbol{\varphi}_{\delta} - (\boldsymbol{\varphi}_{\mathrm{d}} \times \boldsymbol{\varphi}_{\partial\delta} + \boldsymbol{\varphi}_{\partial} \times \boldsymbol{\varphi}_{\delta\mathrm{d}}),\end{aligned} \tag{5}$$

and the related formulae

$$\begin{aligned}\boldsymbol{\Phi}\,\mathrm{d}(\boldsymbol{\Phi}^{\mathrm{T}}\boldsymbol{\varphi}_{\partial\delta}) + \boldsymbol{\Phi}\,\partial(\boldsymbol{\Phi}^{\mathrm{T}}\boldsymbol{\varphi}_{\delta\mathrm{d}}) + \boldsymbol{\Phi}\,\delta(\boldsymbol{\Phi}^{\mathrm{T}}\boldsymbol{\varphi}_{\mathrm{d}\partial}) &= \boldsymbol{\Phi}\,\mathrm{d}\partial(\boldsymbol{\Phi}^{\mathrm{T}}\boldsymbol{\varphi}_{\delta}) + \boldsymbol{\Phi}\,\partial\delta(\boldsymbol{\Phi}^{\mathrm{T}}\boldsymbol{\varphi}_{\mathrm{d}}) + \boldsymbol{\Phi}\,\delta\mathrm{d}(\boldsymbol{\Phi}^{\mathrm{T}}\boldsymbol{\varphi}_{\partial})\\ &= \mathrm{d}\boldsymbol{\varphi}_{\partial\delta} + \partial\boldsymbol{\varphi}_{\delta\mathrm{d}} + \delta\boldsymbol{\varphi}_{\mathrm{d}\partial} - (\boldsymbol{\varphi}_{\mathrm{d}} \times \boldsymbol{\varphi}_{\partial\delta} + \boldsymbol{\varphi}_{\partial} \times \boldsymbol{\varphi}_{\delta\mathrm{d}} + \boldsymbol{\varphi}_{\delta} \times \boldsymbol{\varphi}_{\mathrm{d}\partial})\\ &= \mathrm{d}\partial\boldsymbol{\varphi}_{\delta} + \partial\delta\boldsymbol{\varphi}_{\mathrm{d}} + \delta\mathrm{d}\boldsymbol{\varphi}_{\partial} - (\boldsymbol{\varphi}_{\mathrm{d}} \times \boldsymbol{\varphi}_{\partial\delta} + \boldsymbol{\varphi}_{\partial} \times \boldsymbol{\varphi}_{\delta\mathrm{d}} + \boldsymbol{\varphi}_{\delta} \times \boldsymbol{\varphi}_{\mathrm{d}\partial}),\end{aligned}$$

equation (3) can be written

$$\begin{aligned}\boldsymbol{\varphi}_{\partial\delta} &= \boldsymbol{\Phi}\,\partial(\boldsymbol{\Phi}^{\mathrm{T}}\boldsymbol{\varphi}_{\delta}) + \tfrac{1}{2}\boldsymbol{\varphi}_{\partial} \times \boldsymbol{\varphi}_{\delta},\\ \boldsymbol{\varphi}_{\mathrm{d}\partial\delta} &= \boldsymbol{\Phi}\,\mathrm{d}(\boldsymbol{\Phi}^{\mathrm{T}}\boldsymbol{\varphi}_{\partial\delta}) + \tfrac{1}{2}\boldsymbol{\varphi}_{\mathrm{d}} \times \boldsymbol{\varphi}_{\partial\delta} - \tfrac{1}{2}(\boldsymbol{\varphi}_{\mathrm{d}} \otimes \boldsymbol{\varphi}_{\partial} \cdot \boldsymbol{\varphi}_{\delta} + (\boldsymbol{\varphi}_{\partial} \otimes \boldsymbol{\varphi}_{\delta})^{\mathrm{S}} \cdot \boldsymbol{\varphi}_{\mathrm{d}})\\ &= \boldsymbol{\Phi}\,\mathrm{d}\partial(\boldsymbol{\Phi}^{\mathrm{T}}\boldsymbol{\varphi}_{\delta}) + \tfrac{1}{2}\boldsymbol{\varphi}_{\mathrm{d}\partial} \times \boldsymbol{\varphi}_{\delta} + \tfrac{1}{2}(\boldsymbol{\varphi}_{\mathrm{d}} \times \boldsymbol{\varphi}_{\partial\delta} + \boldsymbol{\varphi}_{\partial} \times \boldsymbol{\varphi}_{\delta\mathrm{d}}) - (\boldsymbol{\varphi}_{\mathrm{d}} \otimes \boldsymbol{\varphi}_{\partial})^{\mathrm{S}} \cdot \boldsymbol{\varphi}_{\delta},\end{aligned} \tag{6}$$

and the symmetrical forms (4) become

$$
\begin{aligned}
\boldsymbol{\varphi}_{\partial\delta} &= \tfrac{1}{2}\,(\boldsymbol{\Phi}\,\partial(\boldsymbol{\Phi}^{\mathrm{T}}\boldsymbol{\varphi}_{\delta}) + \boldsymbol{\Phi}\,\delta(\boldsymbol{\Phi}^{\mathrm{T}}\boldsymbol{\varphi}_{\partial})),\\
\boldsymbol{\varphi}_{\mathrm{d}\partial\delta} &= \tfrac{1}{3}\,(\boldsymbol{\Phi}\,\mathrm{d}(\boldsymbol{\Phi}^{\mathrm{T}}\boldsymbol{\varphi}_{\partial\delta}) + \boldsymbol{\Phi}\,\partial(\boldsymbol{\Phi}^{\mathrm{T}}\boldsymbol{\varphi}_{\delta\mathrm{d}}) + \boldsymbol{\Phi}\,\delta(\boldsymbol{\Phi}^{\mathrm{T}}\boldsymbol{\varphi}_{\mathrm{d}\partial})) + \tfrac{1}{6}\,(\boldsymbol{\varphi}_{\mathrm{d}}\times\boldsymbol{\varphi}_{\partial\delta} + \boldsymbol{\varphi}_{\partial}\times\boldsymbol{\varphi}_{\delta\mathrm{d}} + \boldsymbol{\varphi}_{\delta}\times\boldsymbol{\varphi}_{\mathrm{d}\partial})\\
&\qquad - \tfrac{1}{3}\,(\boldsymbol{\varphi}_{\mathrm{d}}\otimes\boldsymbol{\varphi}_{\partial}\cdot\boldsymbol{\varphi}_{\delta} + \boldsymbol{\varphi}_{\partial}\otimes\boldsymbol{\varphi}_{\delta}\cdot\boldsymbol{\varphi}_{\mathrm{d}} + \boldsymbol{\varphi}_{\delta}\otimes\boldsymbol{\varphi}_{\mathrm{d}}\cdot\boldsymbol{\varphi}_{\partial})\\
&= \tfrac{1}{3}\,(\boldsymbol{\Phi}\,\mathrm{d}\partial(\boldsymbol{\Phi}^{\mathrm{T}}\boldsymbol{\varphi}_{\delta}) + \boldsymbol{\Phi}\,\partial\delta(\boldsymbol{\Phi}^{\mathrm{T}}\boldsymbol{\varphi}_{\mathrm{d}}) + \boldsymbol{\Phi}\,\delta\mathrm{d}(\boldsymbol{\Phi}^{\mathrm{T}}\boldsymbol{\varphi}_{\partial})) + \tfrac{1}{6}\,(\boldsymbol{\varphi}_{\mathrm{d}}\times\boldsymbol{\varphi}_{\partial\delta} + \boldsymbol{\varphi}_{\partial}\times\boldsymbol{\varphi}_{\delta\mathrm{d}} + \boldsymbol{\varphi}_{\delta}\times\boldsymbol{\varphi}_{\mathrm{d}\partial})\\
&\qquad - \tfrac{1}{3}\,(\boldsymbol{\varphi}_{\mathrm{d}}\otimes\boldsymbol{\varphi}_{\partial}\cdot\boldsymbol{\varphi}_{\delta} + \boldsymbol{\varphi}_{\partial}\otimes\boldsymbol{\varphi}_{\delta}\cdot\boldsymbol{\varphi}_{\mathrm{d}} + \boldsymbol{\varphi}_{\delta}\otimes\boldsymbol{\varphi}_{\mathrm{d}}\cdot\boldsymbol{\varphi}_{\partial}).
\end{aligned}
$$

Combining (3), (5), and (6), some remarkable relations follow:

$$
\begin{aligned}
\boldsymbol{\Phi}\,\partial(\boldsymbol{\Phi}^{\mathrm{T}}\boldsymbol{\varphi}_{\delta}) &= \partial\boldsymbol{\varphi}_{\delta} - \boldsymbol{\varphi}_{\partial}\times\boldsymbol{\varphi}_{\delta}\\
&= \delta\boldsymbol{\varphi}_{\partial},
\end{aligned}
\tag{7}
$$

$$
\begin{aligned}
\boldsymbol{\Phi}\,\mathrm{d}(\boldsymbol{\Phi}^{\mathrm{T}}\boldsymbol{\varphi}_{\partial\delta}) &= \mathrm{d}\boldsymbol{\varphi}_{\partial\delta} - \boldsymbol{\varphi}_{\mathrm{d}}\times\boldsymbol{\varphi}_{\partial\delta}\\
&= \partial\delta\boldsymbol{\varphi}_{\mathrm{d}} - \tfrac{1}{2}\,(\boldsymbol{\varphi}_{\partial}\times\delta\boldsymbol{\varphi}_{\mathrm{d}} + \boldsymbol{\varphi}_{\delta}\times\partial\boldsymbol{\varphi}_{\mathrm{d}}),
\end{aligned}
\tag{8}
$$

$$
\begin{aligned}
\boldsymbol{\Phi}\,\mathrm{d}\partial(\boldsymbol{\Phi}^{\mathrm{T}}\boldsymbol{\varphi}_{\delta}) &= \mathrm{d}\partial\boldsymbol{\varphi}_{\delta} - \boldsymbol{\varphi}_{\mathrm{d}\partial}\times\boldsymbol{\varphi}_{\delta} - (\boldsymbol{\varphi}_{\mathrm{d}}\times\boldsymbol{\varphi}_{\partial\delta} + \boldsymbol{\varphi}_{\partial}\times\boldsymbol{\varphi}_{\delta\mathrm{d}})\\
&= \delta\boldsymbol{\varphi}_{\mathrm{d}\partial} - \tfrac{1}{2}\,(\boldsymbol{\varphi}_{\mathrm{d}}\times\delta\boldsymbol{\varphi}_{\partial} + \boldsymbol{\varphi}_{\partial}\times\delta\boldsymbol{\varphi}_{\mathrm{d}}).
\end{aligned}
\tag{9}
$$

They will be exploited in Section 5.

3. The exponential and subexponential maps

The structure of the differentiations of the rotation tensor discussed above has general validity, independently of any particular parameterization. From now on, a specific parameterization is assumed instead. We adopt the natural vectorial parameterization and resort to the so-called *exponential map*,

$$
\boldsymbol{\Phi} = \exp(\boldsymbol{\varphi}\times) = \sum_{n=0}^{\infty}\frac{1}{n!}\boldsymbol{\varphi}\times^{n},
\tag{10}
$$

where $\boldsymbol{\varphi}\times$ is the skew-symmetric tensor built on the rotation vector $\boldsymbol{\varphi}$ (refer, for example, to [Argyris 1982; Ritto-Corrêa and Camotim 2002; Bauchau and Trainelli 2003; Mäkinen 2008]).

In this section we propose a helpful representation of the rotation tensor by means of a family of recursive subexponential maps; such representation will be profitably exploited in the parameterized differentiations in Sections 4 and 5. However, let us introduce this family in the realm of ordinary scalar functions first.

3.1. ***The family of subexponential functions.*** Consider the exponential function and the relevant series expansion $X(x) = \exp(x) = \sum_{n=0}^{\infty}(1/n!)x^{n}$. Collect all terms after the first (unity) and take the second term itself as the common factor of a subsequent series $X_1(x)$. Operate in the same way on $X_1(x)$ to define a subsequent series $X_2(x)$, and so on. By proceeding recursively, an infinite family of functions

$X_0(x)$, $X_1(x)$, $X_2(x)$, ... is defined:

$$\begin{aligned}
X(x) &= \underbrace{1+x+\frac{x^2}{2}+\frac{x^3}{6}+\cdots}_{X_0(x)} && = X_0(x) \\
&= 1+x\underbrace{\Big(1+\frac{x}{2}+\frac{x^2}{6}+\frac{x^3}{24}+\cdots\Big)}_{X_1(x)} && = 1+xX_1(x) \\
&= 1+x\Big(1+\frac{x}{2}\underbrace{\Big(1+\frac{x}{3}+\frac{x^2}{12}+\frac{x^3}{60}+\cdots\Big)}_{X_2(x)}\Big) && = 1+x\Big(1+\frac{x}{2}X_2(x)\Big) \\
&= 1+x\bigg(1+\frac{x}{2}\Big(1+\frac{x}{3}\underbrace{\Big(1+\frac{x}{4}+\frac{x^2}{20}+\frac{x^3}{120}+\cdots\Big)}_{X_3(x)}\Big)\bigg) && = 1+x\Big(1+\frac{x}{2}\Big(1+\frac{x}{3}X_3(x)\Big)\Big) \\
&= \cdots && = \cdots .
\end{aligned} \tag{11}$$

In (11), functions $X_m(x)$ ($m = 0, 1, 2, \dots$) are nested together inside each other as matryoshkas. Alternatively, the exponential function can be given one of the following forms of finite series:

$$\begin{aligned}
X(x) &= X_0(x) \\
&= 1+xX_1(x) \\
&= 1+x+\tfrac{1}{2}x^2X_2(x) \\
&= \cdots \\
&= \sum_{n=0}^{m-1}\frac{1}{n!}x^n+\frac{1}{m!}x^mX_m(x).
\end{aligned} \tag{12}$$

The meaning of (12) is clear: truncating the series expansion of $X(x)$ at the m-th term is exact, provided the last retained term is multiplied by the m-th element of the family $X_0(x)$, $X_1(x)$, $X_2(x)$, ... Thus, the elements of this family behave as plugs to properly truncate the series expansion of $X(x)$.

Functions $X_m(x)$ are equipped with the following series expansions:

$$X_m(x) = \exp_m(x) = \sum_{n=0}^{\infty}\frac{m!}{(n+m)!}x^n \qquad (\forall\, m \geq 0), \tag{13}$$

where we resort to the symbolic notation $\exp_m(x)$ owing to the evident similarity with the exponential expansion. We refer to the set of functions $\exp_m(x)$ as the family of *subexponential* functions (a plot of the first few is illustrated in Figure 1). The base function $\exp_0(x)$ is the exponential function itself, whereas at very high integer m the subexponential function approaches unity, $\exp_{m\to\infty}(x) \to 1$.

It is clear from the nesting representation (11) that each subexponential function can in turn be expressed via some higher subexponential functions. This recursive character allows us to extend the representation (12) to the whole family, in the form of the relation

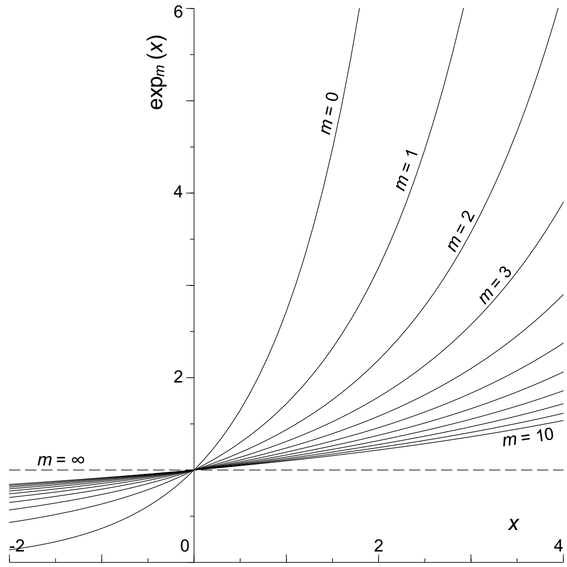

Figure 1. The family of subexponential functions.

$$X_m(x) = \sum_{n=0}^{l-1} \frac{m!}{(n+m)!} x^n + \frac{m!}{(l+m)!} x^l X_{l+m}(x) \qquad (\forall\, m \geq 0, \forall\, l \geq 1), \tag{14}$$

which can also be obtained directly from (13).

3.2. ***The family of subexponential maps.*** The family of *subexponential maps* $\boldsymbol{\Phi}_0$, $\boldsymbol{\Phi}_1$, $\boldsymbol{\Phi}_2, \ldots$ of the rotation tensor is introduced exactly the same way as above. Tensors $\boldsymbol{\Phi}_m$ $(m = 0, 1, 2, \ldots)$ take the series expansion form

$$\boldsymbol{\Phi}_m = \exp_m(\boldsymbol{\varphi}\times) = \sum_{n=0}^{\infty} \frac{m!}{(n+m)!} \boldsymbol{\varphi}\times^n \qquad (\forall\, m \geq 0). \tag{15}$$

The first element $\exp_0(\boldsymbol{\varphi}\times)$ of the family coincides with the exponential map (10) and the asymptotic element is the unit tensor, $\exp_{m\to\infty}(\boldsymbol{\varphi}\times) \to \boldsymbol{I}$. The subexponential maps are nested together just like the subexponential functions are in (11) and allow us to properly truncate the exponential map as in (12). This truncation property applies recursively to each element of the family, so, as in (14), we can state

$$\boldsymbol{\Phi}_m = \sum_{n=0}^{l-1} \frac{m!}{(n+m)!} \boldsymbol{\varphi}\times^n + \frac{m!}{(l+m)!} \boldsymbol{\varphi}\times^l \boldsymbol{\Phi}_{l+m} \qquad (\forall\, m \geq 0, \forall\, l \geq 1).$$

In particular, the strongest truncation ($l = 1$),

$$\boldsymbol{\Phi}_m = \boldsymbol{I} + \frac{1}{m+1} \boldsymbol{\varphi}\times\boldsymbol{\Phi}_{m+1} \qquad (\forall\, m \geq 0), \tag{16}$$

lets us envisage a nested representation of the rotation tensor employing all the subexponential maps in cascade.

It can be noted that all the subexponential maps $\exp_m(\boldsymbol{\varphi}\times)$ share the same eigenvector $\boldsymbol{\varphi}$; they are transparent to their argument, $\boldsymbol{\varphi}\times\boldsymbol{\Phi}_m=\boldsymbol{\Phi}_m\boldsymbol{\varphi}\times$, and commute with each other, $\boldsymbol{\Phi}_m\boldsymbol{\Phi}_n=\boldsymbol{\Phi}_n\boldsymbol{\Phi}_m$. Other useful identities involving subsequent tensors $\boldsymbol{\Phi}_m$ (with $m\geq 1$) are easily proven:

$$\begin{aligned}\boldsymbol{\varphi}\times\boldsymbol{\Phi}_m &= m(\boldsymbol{\Phi}_{m-1}-\boldsymbol{I}),\\ \boldsymbol{\Phi}_{m-1}-\boldsymbol{\Phi}_m &= \frac{1}{m}\sum_{n=1}^{\infty}\frac{m!}{(n+m)!}\boldsymbol{\varphi}\times^n\boldsymbol{\Phi}_{n+m}=\Big(\frac{1}{m}\boldsymbol{\Phi}_m-\frac{1}{m+1}\boldsymbol{\Phi}_{m+1}\Big)\boldsymbol{\varphi}\times,\\ (\boldsymbol{\Phi}_m-\boldsymbol{\Phi}_{m+1})\boldsymbol{\varphi}\times &= \boldsymbol{I}+m\boldsymbol{\Phi}_{m-1}-(m+1)\boldsymbol{\Phi}_m,\\ \boldsymbol{\Phi}_m\boldsymbol{\Phi}_0^{\mathrm{T}} &= \sum_{n=1}^{m}\frac{(-1)^{n-1}m!}{n!(m-n)!}\boldsymbol{\Phi}_n^{\mathrm{T}}.\end{aligned}\tag{17}$$

Finally, the following property is quoted:

$$(\boldsymbol{\Phi}_m-\boldsymbol{\Phi}_n)\boldsymbol{I}^{\times}\boldsymbol{\varphi}\times=\boldsymbol{\varphi}\times\boldsymbol{I}^{\times}(\boldsymbol{\Phi}_m-\boldsymbol{\Phi}_n)\qquad(\forall\, m,n),$$

which can be proved using the expansion (15) and the tensor identities (A.7), (A.13), and (A.14). Tensor $\boldsymbol{I}^{\times}$ is the unitary third-order tensor of skew-symmetrical nature, namely Ricci's tensor (A.6).

3.3. *Compact form.* Series expansions $\boldsymbol{\Phi}_m$ can be brought to forms made of a finite number of terms and more suited for computations. This is accomplished by recursively using the formula $\boldsymbol{\varphi}\times^3+\varphi^2\boldsymbol{\varphi}\times=\boldsymbol{0}$, where φ is the magnitude of vector $\boldsymbol{\varphi}$, namely the rotation angle. This formula follows from the Cayley–Hamilton theorem applied to skew-symmetric tensor $\boldsymbol{\varphi}\times$, or it can be drawn directly from tensor identity (A.9); it can be generalized to any power $\boldsymbol{\varphi}\times^n$ ($\forall\, n\geq 1$) in the forms

$$\begin{aligned}\boldsymbol{\varphi}\times^{2n-1}&=(-1)^{n-1}\varphi^{2(n-1)}\boldsymbol{\varphi}\times,\\ \boldsymbol{\varphi}\times^{2n}&=(-1)^{n-1}\varphi^{2(n-1)}\boldsymbol{\varphi}\times^2.\end{aligned}$$

Using these formulae and the trigonometric series expansions in (15) yields

$$\begin{aligned}\boldsymbol{\Phi}_0&=\boldsymbol{I}+a\boldsymbol{\varphi}\times+b_0\boldsymbol{\varphi}\times^2,\\ \boldsymbol{\Phi}_m&=\boldsymbol{I}+mb_{m-1}\boldsymbol{\varphi}\times+b_m\boldsymbol{\varphi}\times^2\qquad(\forall\, m\geq 1).\end{aligned}\tag{18}$$

Equation (18) is referred to as the *compact form* of the family of tensors $\boldsymbol{\Phi}_m$; they stand on three minimal tensorial bases, the identity $\boldsymbol{I}$ and tensors $\boldsymbol{\varphi}\times$ and $\boldsymbol{\varphi}\times^2$. The coefficients a and b_m are recursive functions of the rotation angle,

$$\begin{aligned}a&=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\varphi^{2n}=\frac{1}{\varphi}\sin\varphi,\\ b_0&=0!\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+2)!}\varphi^{2n}=\frac{1}{\varphi^2}(1-\cos\varphi),\\ b_1&=1!\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+3)!}\varphi^{2n}=\frac{1}{\varphi^2}(1-a),\\ b_m&=m!\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+2+m)!}\varphi^{2n}=\frac{1}{\varphi^2}(1-m(m-1)b_{m-2})\qquad(\forall\, m\geq 2).\end{aligned}\tag{19}$$

Details about the computational implementation of these coefficient functions are discussed in [Merlini and Morandini 2004b, Appendix A].

It is seen that $(18)_1$ is the well-known Euler–Rodrigues formula of the rotation tensor [Cheng and Gupta 1989]. The expression for $\boldsymbol{\Phi}_1$ in $(18)_2$ corresponds to another well-known tensor in the finite-rotation literature, sometimes referred to as the associated differential tensor [Bauchau and Trainelli 2003]. This tensor represents the mapping of the differential $\mathrm{d}\boldsymbol{\varphi}$ of the rotation vector onto the differential rotation vector $\boldsymbol{\varphi}_{\mathrm{d}}$, which characterizes the tangent space of the rotation [Ibrahimbegović et al. 1995; Borri et al. 2000; Ritto-Corrêa and Camotim 2002; Mäkinen 2008].

4. Differentiation of subexponential maps

Subexponential maps $\exp_m(\boldsymbol{\varphi}\times)$ are power series expansions of the argument itself and as such are differentiable tensor functions. In fact, each differential $\mathrm{d}(\boldsymbol{\varphi}\times^n)$ of a power term in (15) is easily brought to the form [de Souza Neto 2001]

$$\mathrm{d}(\boldsymbol{\varphi}\times^n) = \sum_{k=1}^{n} \boldsymbol{\varphi}\times^{k-1}\,\mathrm{d}\boldsymbol{\varphi}\times\boldsymbol{\varphi}\times^{n-k} \qquad (\forall\, n>0), \tag{20}$$

linear with respect to $\mathrm{d}\boldsymbol{\varphi}\times$. So, the differential of a tensor $\boldsymbol{\Phi}_m$ can be brought to an expression like $\mathrm{d}\boldsymbol{\Phi}_m = \boldsymbol{\Phi}_{m/}^{\mathrm{T}132}\cdot\mathrm{d}\boldsymbol{\varphi}$, linear with the variation of the rotation vector, where $\boldsymbol{\Phi}_{m/}$ is a third-order tensor, a function itself of powers of $\boldsymbol{\varphi}\times$. It follows that tensor $\boldsymbol{\Phi}_{m/}$ is a differentiable tensor itself, and as a consequence tensor $\boldsymbol{\Phi}_m$ is two times differentiable.

Recursively, it can be seen that the subexponential maps are continuously differentiable with respect to their vector argument. However, for present purposes, our interest is in the first two successive, and independent, differentiations, say δ and ∂. The relevant differentials can be cast in the form

$$\begin{aligned} \delta\boldsymbol{\Phi}_m &= \boldsymbol{\Phi}_{m/} : \delta\boldsymbol{\varphi}\otimes\boldsymbol{I}, \qquad \partial\boldsymbol{\Phi}_{m/} = \boldsymbol{\Phi}_{m//}^{1\overline{23}4} : \partial\boldsymbol{\varphi}\otimes\boldsymbol{I}, \\ \partial\delta\boldsymbol{\Phi}_m &= \boldsymbol{\Phi}_{m/} : \partial\delta\boldsymbol{\varphi}\otimes\boldsymbol{I} + \boldsymbol{\Phi}_{m//}^{1\overline{23}4} \;\vdots\; \partial\boldsymbol{\varphi}\otimes\delta\boldsymbol{\varphi}\otimes\boldsymbol{I}, \end{aligned} \tag{21}$$

where $\boldsymbol{\Phi}_{m//}^{1\overline{23}4}$ is a fourth-order tensor, symmetric with respect to the two inner polyadic legs (see (24); refer to the Appendix for notation and rules), and the variations ∂ and δ are of course interchangeable. We may refer to tensors $\boldsymbol{\Phi}_{m/}$ and $\boldsymbol{\Phi}_{m//}^{1\overline{23}4}$ as the *first derivative* and the *second derivative*, respectively, of tensor $\boldsymbol{\Phi}_m$ with respect to the rotation vector.

Equation (21) defines the first two derivatives of the subexponential maps, however successive derivatives can analogously be defined on demands. The recursive forms developed in Section 4.2 pave the way for building successive derivatives virtually up to any order. In this section we derive useful expressions of the first two derivative tensors.

4.1. ***Series expansion form of the derivative tensors.*** Simple series expansion forms of the derivative tensors are easily obtained recalling (20). Differentiation of (15) yields

$$\delta\boldsymbol{\Phi}_m = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \frac{m!}{(1+j+k+m)!}\boldsymbol{\varphi}\times^j\,\delta\boldsymbol{\varphi}\times\boldsymbol{\varphi}\times^k = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \frac{-m!}{(1+j+k+m)!}\boldsymbol{\varphi}\times^j\,\boldsymbol{I}^{\times}\boldsymbol{\varphi}\times^k : \delta\boldsymbol{\varphi}\otimes\boldsymbol{I}, \tag{22}$$

and comparison with $(21)_1$ provides the first derivative tensor in the power series expansion form

$$\boldsymbol{\Phi}_{m/} = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \frac{-m!}{(1+j+k+m)!} \boldsymbol{\varphi}\times^{j} \boldsymbol{I}^{\times}\boldsymbol{\varphi}\times^{k}. \tag{23}$$

Subsequently, differentiation of (23) yields

$$\begin{aligned}\partial\boldsymbol{\Phi}_{m/} &= \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \frac{-m!}{(2+j+k+l+m)!} \boldsymbol{\varphi}\times^{j} (\partial\boldsymbol{\varphi}\times\boldsymbol{\varphi}\times^{k}\boldsymbol{I}^{\times} + \boldsymbol{I}^{\times}\boldsymbol{\varphi}\times^{k}\partial\boldsymbol{\varphi}\times)\boldsymbol{\varphi}\times^{l} \\ &= \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \frac{2m!}{(2+j+k+l+m)!} (\boldsymbol{\varphi}\times^{j}\boldsymbol{I}^{\times}\boldsymbol{\varphi}\times^{k}\boldsymbol{I}^{\times}\boldsymbol{\varphi}\times^{l})^{\mathrm{S}1\underline{23}4} : \partial\boldsymbol{\varphi}\otimes\boldsymbol{I},\end{aligned} \tag{24}$$

and comparison with $(21)_2$ provides the second derivative tensor in the power series expansion form

$$\boldsymbol{\Phi}_{m//}^{1\underline{23}4} = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \frac{2m!}{(2+j+k+l+m)!} (\boldsymbol{\varphi}\times^{j}\boldsymbol{I}^{\times}\boldsymbol{\varphi}\times^{k}\boldsymbol{I}^{\times}\boldsymbol{\varphi}\times^{l})^{\mathrm{S}1\underline{23}4}. \tag{25}$$

Equations (23) and (25) are concise straightforward expressions of the derivative tensors, however they may not be fit for numerical computations. More suitable expressions are derived in next subsections.

4.2. ***Recursive form of the derivative tensors.*** Differential (22) can be worked out using (15), (16), $(17)_1$, and $(17)_2$. After some algebraic manipulations, the following expression is obtained:

$$\begin{aligned}\delta\boldsymbol{\Phi}_m = \frac{1}{m+1}\Big(\boldsymbol{I}^{\times} - (\boldsymbol{I}^{\times}\boldsymbol{\Phi}_{m+1} + \boldsymbol{\Phi}_{m+1}\boldsymbol{I}^{\times}) - \frac{1}{2}\big(\boldsymbol{\varphi}\times\boldsymbol{I}^{\times}(\boldsymbol{\Phi}_{m+1} - \boldsymbol{\Phi}_{m+2}) \\ + (\boldsymbol{\Phi}_{m+1} - \boldsymbol{\Phi}_{m+2})\boldsymbol{I}^{\times}\boldsymbol{\varphi}\times\big)\Big) : \delta\boldsymbol{\varphi}\otimes\boldsymbol{I}.\end{aligned} \tag{26}$$

Recalling (21), subsequent differentiation yields

$$\begin{aligned}\partial\delta\boldsymbol{\Phi}_m = &\frac{1}{m+1}\big(\boldsymbol{I}^{\times} - (\boldsymbol{I}^{\times}\boldsymbol{\Phi}_{m+1} + \boldsymbol{\Phi}_{m+1}\boldsymbol{I}^{\times}) - \frac{1}{2}(\boldsymbol{\varphi}\times\boldsymbol{I}^{\times}(\boldsymbol{\Phi}_{m+1} - \boldsymbol{\Phi}_{m+2}) + (\boldsymbol{\Phi}_{m+1} - \boldsymbol{\Phi}_{m+2})\boldsymbol{I}^{\times}\boldsymbol{\varphi}\times)\big) : \partial\delta\boldsymbol{\varphi}\otimes\boldsymbol{I} \\ &+ \frac{1}{m+1}\Big(\frac{1}{2}\big(\boldsymbol{I}^{\times}\boldsymbol{I}^{\times}(\boldsymbol{\Phi}_{m+1} - \boldsymbol{\Phi}_{m+2}) + (\boldsymbol{\Phi}_{m+1} - \boldsymbol{\Phi}_{m+2})\boldsymbol{I}^{\times}\boldsymbol{I}^{\times}\big) - (\boldsymbol{I}^{\times}\boldsymbol{\Phi}_{m+1/} + \boldsymbol{\Phi}_{m+1/}\boldsymbol{I}^{\times}) \\ &- \frac{1}{2}\big(\boldsymbol{\varphi}\times\boldsymbol{I}^{\times}(\boldsymbol{\Phi}_{m+1/} - \boldsymbol{\Phi}_{m+2/}) + (\boldsymbol{\Phi}_{m+1/} - \boldsymbol{\Phi}_{m+2/})\boldsymbol{I}^{\times}\boldsymbol{\varphi}\times\big)\Big)^{\mathrm{S}1\underline{23}4} \vdots\, \partial\boldsymbol{\varphi}\otimes\delta\boldsymbol{\varphi}\otimes\boldsymbol{I}.\end{aligned} \tag{27}$$

In (26) and (27), new expressions of the derivative tensors $\boldsymbol{\Phi}_{m/}$ and $\boldsymbol{\Phi}_{m//}^{1\underline{23}4}$, defined in (21), are clearly recognized. They are referred to as the *recursive form* of the derivatives of tensor $\boldsymbol{\Phi}_m$ as they are functions of the next subexponential maps and of their first derivative tensors. Such recursive forms are better suited for numerical computations than the series expansion forms.

4.3. ***Compact form of the derivative tensors.*** Other expressions of the derivative tensors can be drawn from (26) and (27) using the compact forms (18) of the subexponential maps. These new expressions stand on five tensorial bases and are called the *compact form* of the derivative tensors:

$$\boldsymbol{\Phi}_{m/} = f_{m0}\boldsymbol{I}^{\times} + f_{m1}(\boldsymbol{\varphi}\times\boldsymbol{I}^{\times} + \boldsymbol{I}^{\times}\boldsymbol{\varphi}\times) + f_{m2}(\boldsymbol{\varphi}\times^2\boldsymbol{I}^{\times} + \boldsymbol{\varphi}\times\boldsymbol{I}^{\times}\boldsymbol{\varphi}\times + \boldsymbol{I}^{\times}\boldsymbol{\varphi}\times^2) + f_{m3}\tfrac{1}{2}(\boldsymbol{\varphi}\times^2\boldsymbol{I}^{\times}\boldsymbol{\varphi}\times + \boldsymbol{\varphi}\times\boldsymbol{I}^{\times}\boldsymbol{\varphi}\times^2) + f_{m4}\boldsymbol{\varphi}\times^2\boldsymbol{I}^{\times}\boldsymbol{\varphi}\times^2, \quad (28)$$

$$\boldsymbol{\Phi}_{m//}^{1234} = \big(g_{m0}\boldsymbol{I}^{\times}\boldsymbol{I}^{\times} + g_{m1}(\boldsymbol{\varphi}\times\boldsymbol{I}^{\times}\boldsymbol{I}^{\times} + \boldsymbol{I}^{\times}\boldsymbol{\varphi}\times\boldsymbol{I}^{\times} + \boldsymbol{I}^{\times}\boldsymbol{I}^{\times}\boldsymbol{\varphi}\times) + g_{m2}(\boldsymbol{\varphi}\times^2\boldsymbol{I}^{\times}\boldsymbol{I}^{\times} + \boldsymbol{I}^{\times}\boldsymbol{\varphi}\times^2\boldsymbol{I}^{\times} + \boldsymbol{I}^{\times}\boldsymbol{I}^{\times}\boldsymbol{\varphi}\times^2 + \boldsymbol{\varphi}\times\boldsymbol{I}^{\times}\boldsymbol{\varphi}\times\boldsymbol{I}^{\times} + \boldsymbol{\varphi}\times\boldsymbol{I}^{\times}\boldsymbol{I}^{\times}\boldsymbol{\varphi}\times + \boldsymbol{I}^{\times}\boldsymbol{\varphi}\times\boldsymbol{I}^{\times}\boldsymbol{\varphi}\times + \varphi^2\boldsymbol{I}^{\times}\boldsymbol{I}^{\times}) + g_{m3}\boldsymbol{\varphi}\times\boldsymbol{I}^{\times}\boldsymbol{\varphi}\times\boldsymbol{I}^{\times}\boldsymbol{\varphi}\times + g_{m4}\boldsymbol{\varphi}\times\boldsymbol{I}^{\times}\boldsymbol{\varphi}\times^2\boldsymbol{I}^{\times}\boldsymbol{\varphi}\times\big)^{\mathrm{S}1\underline{234}}. \quad (29)$$

The coefficient functions in (28) and (29) are defined as follows:

$$f_{m0} = \frac{-1}{m+1}, \qquad g_{m0} = 2b_m,$$
$$f_{m1} = -b_m, \qquad g_{m1} = b_m - b_{m+1},$$
$$f_{m2} = \frac{-1}{m+1}b_{m+1}, \qquad g_{m2} = \frac{1}{m+1}(b_{m+1} - b_{m+2}),$$
$$f_{m3} = \frac{-1}{m+1}(b_{m+1} - b_{m+2}), \qquad g_{m3} = \frac{1}{m+1}(b_{m+1} - b_{m+2}) - \frac{1}{m+2}(b_{m+2} - b_{m+3}),$$
$$f_{m4} = \frac{-1}{(m+1)(m+2)}(b_{m+2} - b_{m+3}), \qquad g_{m4} = \frac{1}{m+1}\left(\frac{1}{m+2}(b_{m+2} - b_{m+3}) - \frac{1}{m+3}(b_{m+3} - b_{m+4})\right).$$

5. Differential maps of the rotation

The differential maps of the rotation transform the multiple differentials of the rotation vector into the differential rotation vectors defined in Section 2. The differentiations of the lowest two subexponential maps, as derived above, are used in this section to develop the expected differential maps and to highlight some important properties.

5.1. ***Explicit notation for the lowest two subexponential maps.*** The first and second subexponential maps play important roles in computational mechanics. As mentioned in Section 3.3, $\boldsymbol{\Phi}_0$ is the exponential map itself (that is, the rotation tensor), whereas $\boldsymbol{\Phi}_1$ is known as its tangent map or the associated differential map [Borri et al. 2000]; in the present context, the latter will be referred to concisely as the *tangent tensor* (of the rotation). Both of these tensors are assigned hereafter specific notations:

$$\begin{aligned}\boldsymbol{\Phi} = \boldsymbol{\Phi}_0 = \exp(\boldsymbol{\varphi}\times) = \sum_{n=0}^{\infty}\frac{1}{n!}\boldsymbol{\varphi}\times^n = \boldsymbol{I} + a\boldsymbol{\varphi}\times + b_0\boldsymbol{\varphi}\times^2,\\ \boldsymbol{\Gamma} = \boldsymbol{\Phi}_1 = \exp_1(\boldsymbol{\varphi}\times) = \sum_{n=0}^{\infty}\frac{1}{(n+1)!}\boldsymbol{\varphi}\times^n = \boldsymbol{I} + b_0\boldsymbol{\varphi}\times + b_1\boldsymbol{\varphi}\times^2.\end{aligned} \quad (30)$$

Note that, $\boldsymbol{\Gamma}$ being the strongest truncation of $\boldsymbol{\Phi}$, the relation $\boldsymbol{\Phi} = \boldsymbol{I} + \boldsymbol{\varphi}\times\boldsymbol{\Gamma}$ holds as derived directly from (16).

The first and second differentiations of tensors $\boldsymbol{\Phi}$ and $\boldsymbol{\Gamma}$ are written, from (21), as

$$\delta\boldsymbol{\Phi}=\boldsymbol{\Phi}_{/}:\delta\boldsymbol{\varphi}\otimes\boldsymbol{I},\qquad \partial\boldsymbol{\Phi}_{/}=\boldsymbol{\Phi}_{//}^{\underline{12}34}:\partial\boldsymbol{\varphi}\otimes\boldsymbol{I},$$
$$\partial\delta\boldsymbol{\Phi}=\boldsymbol{\Phi}_{/}:\partial\delta\boldsymbol{\varphi}\otimes\boldsymbol{I}+\boldsymbol{\Phi}_{//}^{\underline{12}34}\;\vdots\;\partial\boldsymbol{\varphi}\otimes\delta\boldsymbol{\varphi}\otimes\boldsymbol{I},\tag{31}$$

and

$$\delta\boldsymbol{\Gamma}=\boldsymbol{\Gamma}_{/}:\delta\boldsymbol{\varphi}\otimes\boldsymbol{I},\qquad \partial\boldsymbol{\Gamma}_{/}=\boldsymbol{\Gamma}_{//}^{\underline{12}34}:\partial\boldsymbol{\varphi}\otimes\boldsymbol{I},$$
$$\partial\delta\boldsymbol{\Gamma}=\boldsymbol{\Gamma}_{/}:\partial\delta\boldsymbol{\varphi}\otimes\boldsymbol{I}+\boldsymbol{\Gamma}_{//}^{\underline{12}34}\;\vdots\;\partial\boldsymbol{\varphi}\otimes\delta\boldsymbol{\varphi}\otimes\boldsymbol{I}.\tag{32}$$

Expressions of the relevant derivative tensors,

$$\boldsymbol{\Phi}_{/}=\boldsymbol{\Phi}_{0/},\qquad \boldsymbol{\Phi}_{//}^{\underline{12}34}=\boldsymbol{\Phi}_{0//}^{\underline{12}34},$$
$$\boldsymbol{\Gamma}_{/}=\boldsymbol{\Phi}_{1/},\qquad \boldsymbol{\Gamma}_{//}^{\underline{12}34}=\boldsymbol{\Phi}_{1//}^{\underline{12}34},\tag{33}$$

are easily written by setting $m=0$ or $m=1$ in (22)–(29). They are not repeated here.

The compact forms of the derivative tensors (33) are known in the literature. Let us use (28) and (29), together with the coefficient functions (19), to evaluate the following second-order tensors built with tensors (33) and arbitrary vectors $\boldsymbol{u}$ and $\boldsymbol{v}$:

$$\begin{aligned}\boldsymbol{\Phi}_{/}:\boldsymbol{u}\otimes\boldsymbol{I}&=a\boldsymbol{u}\times+b_0(\boldsymbol{u}\otimes\boldsymbol{\varphi}+\boldsymbol{\varphi}\otimes\boldsymbol{u})-a\boldsymbol{\varphi}\cdot\boldsymbol{u}\otimes\boldsymbol{I}\\&\quad-(b_0-b_1)\boldsymbol{\varphi}\cdot\boldsymbol{u}\otimes\boldsymbol{\varphi}\times-(b_1-b_2)\boldsymbol{\varphi}\cdot\boldsymbol{u}\otimes\boldsymbol{\varphi}\otimes\boldsymbol{\varphi},\\ \boldsymbol{\Gamma}_{/}:\boldsymbol{u}\otimes\boldsymbol{I}&=b_0\boldsymbol{u}\times+b_1(\boldsymbol{u}\otimes\boldsymbol{\varphi}+\boldsymbol{\varphi}\otimes\boldsymbol{u})-(b_0-b_1)\boldsymbol{\varphi}\cdot\boldsymbol{u}\otimes\boldsymbol{I}\\&\quad-(b_1-b_2)\boldsymbol{\varphi}\cdot\boldsymbol{u}\otimes\boldsymbol{\varphi}\times-\tfrac{1}{2}(b_2-b_3)\boldsymbol{\varphi}\cdot\boldsymbol{u}\otimes\boldsymbol{\varphi}\otimes\boldsymbol{\varphi},\end{aligned}\tag{34}$$

and

$$\begin{aligned}&\boldsymbol{\Phi}_{//}^{\underline{12}34}\;\vdots\;\boldsymbol{u}\otimes\boldsymbol{v}\otimes\boldsymbol{I}\\&\quad=b_0(\boldsymbol{u}\otimes\boldsymbol{v}+\boldsymbol{v}\otimes\boldsymbol{u})-a\boldsymbol{u}\cdot\boldsymbol{v}\otimes\boldsymbol{I}-(b_0-b_1)(\boldsymbol{\varphi}\cdot\boldsymbol{u}\otimes\boldsymbol{v}\times+\boldsymbol{\varphi}\cdot\boldsymbol{v}\otimes\boldsymbol{u}\times+\boldsymbol{u}\cdot\boldsymbol{v}\otimes\boldsymbol{\varphi}\times)\\&\qquad-(b_1-b_2)\big(\boldsymbol{\varphi}\cdot\boldsymbol{u}\otimes(\boldsymbol{v}\otimes\boldsymbol{\varphi}+\boldsymbol{\varphi}\otimes\boldsymbol{v})+\boldsymbol{\varphi}\cdot\boldsymbol{v}\otimes(\boldsymbol{u}\otimes\boldsymbol{\varphi}+\boldsymbol{\varphi}\otimes\boldsymbol{u})+\boldsymbol{u}\cdot\boldsymbol{v}\otimes\boldsymbol{\varphi}\otimes\boldsymbol{\varphi}\big)\\&\qquad+(b_0-b_1)\boldsymbol{\varphi}\cdot\boldsymbol{v}\otimes\boldsymbol{\varphi}\cdot\boldsymbol{u}\otimes\boldsymbol{I}+\big((b_1-b_2)-\tfrac{1}{2}(b_2-b_3)\big)\boldsymbol{\varphi}\cdot\boldsymbol{v}\otimes\boldsymbol{\varphi}\cdot\boldsymbol{u}\otimes\boldsymbol{\varphi}\times\\&\qquad+\big(\tfrac{1}{2}(b_2-b_3)-\tfrac{1}{3}(b_3-b_4)\big)\boldsymbol{\varphi}\cdot\boldsymbol{v}\otimes\boldsymbol{\varphi}\cdot\boldsymbol{u}\otimes\boldsymbol{\varphi}\otimes\boldsymbol{\varphi},\\&\boldsymbol{\Gamma}_{//}^{\underline{12}34}\;\vdots\;\boldsymbol{u}\otimes\boldsymbol{v}\otimes\boldsymbol{I}\\&\quad=b_1(\boldsymbol{u}\otimes\boldsymbol{v}+\boldsymbol{v}\otimes\boldsymbol{u})-(b_0-b_1)\boldsymbol{u}\cdot\boldsymbol{v}\otimes\boldsymbol{I}\\&\qquad-(b_1-b_2)(\boldsymbol{\varphi}\cdot\boldsymbol{u}\otimes\boldsymbol{v}\times+\boldsymbol{\varphi}\cdot\boldsymbol{v}\otimes\boldsymbol{u}\times+\boldsymbol{u}\cdot\boldsymbol{v}\otimes\boldsymbol{\varphi}\times)-\tfrac{1}{2}(b_2-b_3)\big(\boldsymbol{\varphi}\cdot\boldsymbol{u}\otimes(\boldsymbol{v}\otimes\boldsymbol{\varphi}+\boldsymbol{\varphi}\otimes\boldsymbol{v})\\&\qquad+\boldsymbol{\varphi}\cdot\boldsymbol{v}\otimes(\boldsymbol{u}\otimes\boldsymbol{\varphi}+\boldsymbol{\varphi}\otimes\boldsymbol{u})+\boldsymbol{u}\cdot\boldsymbol{v}\otimes\boldsymbol{\varphi}\otimes\boldsymbol{\varphi}\big)+\big((b_1-b_2)-\tfrac{1}{2}(b_2-b_3)\big)\boldsymbol{\varphi}\cdot\boldsymbol{v}\otimes\boldsymbol{\varphi}\cdot\boldsymbol{u}\otimes\boldsymbol{I}\\&\qquad+\big(\tfrac{1}{2}(b_2-b_3)-\tfrac{1}{3}(b_3-b_4)\big)\boldsymbol{\varphi}\cdot\boldsymbol{v}\otimes\boldsymbol{\varphi}\cdot\boldsymbol{u}\otimes\boldsymbol{\varphi}\times+\tfrac{1}{2}\big(\tfrac{1}{3}(b_3-b_4)-\tfrac{1}{4}(b_4-b_5)\big)\boldsymbol{\varphi}\cdot\boldsymbol{v}\otimes\boldsymbol{\varphi}\cdot\boldsymbol{u}\otimes\boldsymbol{\varphi}\otimes\boldsymbol{\varphi}.\end{aligned}\tag{35}$$

It is seen that $(34)_1$ and $(34)_2$ coincide respectively with the directional derivatives $D\boldsymbol{\Phi}[\boldsymbol{u}]$ and $D\boldsymbol{\Gamma}[\boldsymbol{u}]$ found, respectively, in [Ritto-Corrêa and Camotim 2002, Equations (18) and (19)], once our coefficient functions (19) are converted to those used by them. It is also seen that $(35)_1$ and $(35)_2$ coincide, respectively, with the second directional derivatives $D^2\boldsymbol{\Phi}[\boldsymbol{u},\boldsymbol{v}]$ and $D^2\boldsymbol{\Gamma}[\boldsymbol{u},\boldsymbol{v}]$ [ibid., Equations (20) and (21)].

5.2. ***Lowest three differential maps.*** The differential $(31)_1$ of the rotation tensor is easily obtained from (26) with $m = 0$:

$$\delta\boldsymbol{\Phi} = -\delta\boldsymbol{\varphi}\times + (\delta\boldsymbol{\varphi}\times\boldsymbol{\Phi}_1 + \boldsymbol{\Phi}_1\,\delta\boldsymbol{\varphi}\times) + \tfrac{1}{2}\,(\boldsymbol{\varphi}\times\delta\boldsymbol{\varphi}\times(\boldsymbol{\Phi}_1 - \boldsymbol{\Phi}_2) + (\boldsymbol{\Phi}_1 - \boldsymbol{\Phi}_2)\,\delta\boldsymbol{\varphi}\times\boldsymbol{\varphi}\times).$$

Using (16) with $m = 1$, namely $\boldsymbol{\Phi}_1 = \boldsymbol{I} + \tfrac{1}{2}\boldsymbol{\varphi}\times\boldsymbol{\Phi}_2 = \boldsymbol{I} + \tfrac{1}{2}\boldsymbol{\Phi}_2\boldsymbol{\varphi}\times$, and the tensor identity (A.9), $\delta\boldsymbol{\Phi}$ is rewritten as

$$\delta\boldsymbol{\Phi} = \delta\boldsymbol{\varphi}\times + \tfrac{1}{2}\,(\boldsymbol{\varphi}\otimes\delta\boldsymbol{\varphi}\cdot\boldsymbol{\Phi}_2 + \boldsymbol{\Phi}_2\cdot\delta\boldsymbol{\varphi}\otimes\boldsymbol{\varphi}) - \boldsymbol{\Phi}_1\otimes\boldsymbol{\varphi}\cdot\delta\boldsymbol{\varphi}$$

and is then multiplied by $\boldsymbol{\Phi}_0^{\mathrm{T}}$ to draw an expression for tensor $\delta\boldsymbol{\Phi}\boldsymbol{\Phi}^{\mathrm{T}}$. Using the property $(17)_4$ with $m = 1$ (that is, $\boldsymbol{\Phi}_1\boldsymbol{\Phi}_0^{\mathrm{T}} = \boldsymbol{\Phi}_1^{\mathrm{T}}$) and $m = 2$ (that is, $\boldsymbol{\Phi}_2\boldsymbol{\Phi}_0^{\mathrm{T}} = 2\boldsymbol{\Phi}_1^{\mathrm{T}} - \boldsymbol{\Phi}_2^{\mathrm{T}}$), the tensor identities (A.9) and (A.12), and (16) with $m = 0$ and $m = 1$, the relation

$$\delta\boldsymbol{\Phi}\boldsymbol{\Phi}^{\mathrm{T}} = (\boldsymbol{\Phi}_1\cdot\delta\boldsymbol{\varphi})\times$$

is finally obtained. Comparison with $(2)_1$ and $(30)_2$ provides the sought relation of $\boldsymbol{\varphi}_\delta$ as a linear function of $\delta\boldsymbol{\varphi}$:

$$\boldsymbol{\varphi}_\delta = \boldsymbol{\Gamma}\cdot\delta\boldsymbol{\varphi}, \tag{36}$$

a well-known result involving the tangent tensor $\boldsymbol{\Gamma}$.

The foregoing derivation of the *first differential map* of the rotation is an alternative to other derivations found in the literature, for example, [Borri et al. 1990; Ibrahimbegović et al. 1995; Ritto-Corrêa and Camotim 2002].

Expressions for the next differential vectors are drawn directly from their definitions as elaborated in $(4)_1$ and $(4)_3$, respectively. The *second differential map* is easily obtained using (36) and $(32)_1$:

$$\boldsymbol{\varphi}_{\partial\delta} = \boldsymbol{\Gamma}\cdot\partial\delta\boldsymbol{\varphi} + \boldsymbol{\Gamma}_{/}^{\mathrm{S}1\underline{\underline{23}}} : \partial\boldsymbol{\varphi}\otimes\delta\boldsymbol{\varphi}, \tag{37}$$

where $\boldsymbol{\Gamma}_{/}^{\mathrm{S}1\underline{\underline{23}}} = \tfrac{1}{2}\,(\boldsymbol{\Gamma}_{/} + \boldsymbol{\Gamma}_{/}^{\mathrm{T}132})$ is the right-symmetric part of $\boldsymbol{\Gamma}_{/}$ (that is, symmetric with respect to the two rightmost polyadic legs, see $(\mathrm{A}.2)_1$).

The *third differential map* is obtained, after a more involved derivation, in the form

$$\begin{aligned}\boldsymbol{\varphi}_{\mathrm{d}\partial\delta} = \boldsymbol{\Gamma}\cdot\mathrm{d}\partial\delta\boldsymbol{\varphi} + \boldsymbol{\Gamma}_{/}^{\mathrm{S}1\underline{\underline{23}}} : (\mathrm{d}\boldsymbol{\varphi}\otimes\partial\delta\boldsymbol{\varphi} + \partial\boldsymbol{\varphi}\otimes\delta\mathrm{d}\boldsymbol{\varphi} + \delta\boldsymbol{\varphi}\otimes\mathrm{d}\partial\boldsymbol{\varphi})\\ + \big(\boldsymbol{\Gamma}_{//}^{1\underline{\underline{23}}4} - \tfrac{1}{2}\,(\boldsymbol{I}^{\times}\boldsymbol{\Gamma})^{\mathrm{T}132}\boldsymbol{\Gamma}_{/}^{\mathrm{S}1\underline{\underline{23}}} - \boldsymbol{\Gamma}\otimes\boldsymbol{\Gamma}^{\mathrm{T}}\boldsymbol{\Gamma}\big)^{\mathrm{S}1\underline{\underline{\underline{234}}}} \,\vdots\, \mathrm{d}\boldsymbol{\varphi}\otimes\partial\boldsymbol{\varphi}\otimes\delta\boldsymbol{\varphi},\end{aligned} \tag{38}$$

where $(\;)^{\mathrm{S}1\underline{\underline{\underline{234}}}}$ denotes the full-symmetric part, with respect to the three rightmost polyadic legs, of a fourth-order tensor, see (A.5). Derivation of (38) requires (36), (37), (32), and the property (39).

5.3. ***Properties of the derivative tensors.*** The derivative tensors $\boldsymbol{\Gamma}_{/}$ and $\boldsymbol{\Gamma}_{//}^{1\underline{\underline{23}}4}$ are endowed with useful properties, which descend from the definition of the differential vectors themselves. Let us focus first on the property (7) and develop $\boldsymbol{\Phi}\,\partial(\boldsymbol{\Phi}^{\mathrm{T}}\boldsymbol{\varphi}_\delta)$ from $(7)_1$ and $(7)_2$ separately, using the differential map (36) and the differentiation formula $(32)_1$. Comparison of the results leads to

$$\boldsymbol{\Gamma}_{/}^{\mathrm{T}132} = \boldsymbol{\Gamma}_{/} - (\boldsymbol{I}^{\times}\boldsymbol{\Gamma})^{\mathrm{T}132}\boldsymbol{\Gamma}. \tag{39}$$

Observing that $(\boldsymbol{I}^{\times}\boldsymbol{\Gamma})^{\mathrm{T}132}\boldsymbol{\Gamma}$ is a right-skew-symmetric third-order tensor, see (A.17), (39) entails the right-symmetry of tensor $\boldsymbol{\Gamma}_{/} - \frac{1}{2}(\boldsymbol{I}^{\times}\boldsymbol{\Gamma})^{\mathrm{T}132}\boldsymbol{\Gamma}$, hence the property

$$\boldsymbol{\Gamma}_{/} = \boldsymbol{\Gamma}_{/}^{\mathrm{S}1\underline{23}} + \tfrac{1}{2}(\boldsymbol{I}^{\times}\boldsymbol{\Gamma})^{\mathrm{T}132}\boldsymbol{\Gamma}. \tag{40}$$

Equation (40) represents the decomposition of the tangent-tensor first derivative into the symmetric and skew-symmetric parts of the rightmost polyadic legs. Note that property (7) enables a straightforward derivation of (40), while deriving the latter directly from the expressions of $\boldsymbol{\Gamma}_{/}$ in (33) would be a much more involved task.

Next, let us focus on property (8) and develop $\boldsymbol{\Phi}\,\mathrm{d}(\boldsymbol{\Phi}^{\mathrm{T}}\boldsymbol{\varphi}_{\partial\delta})$ from $(8)_1$ and $(8)_2$ separately, using the differential maps (36) and (37), the differentiation formulae (32), and the property (39). (Alternatively, one might start from property (9).) Comparison of the results leads to a useful property of the symmetric parts of another fourth-order tensor, specifically tensor $\boldsymbol{\Gamma}_{//}^{1\underline{23}4} - (\boldsymbol{I}^{\times}\boldsymbol{\Gamma})^{\mathrm{T}132}\boldsymbol{\Gamma}_{/}$. This property can be written in the following equivalent forms:

$$\begin{aligned}
(\boldsymbol{\Gamma}_{//}^{1\underline{23}4} - (\boldsymbol{I}^{\times}\boldsymbol{\Gamma})^{\mathrm{T}132}\boldsymbol{\Gamma}_{/})^{\mathrm{S}1\underline{23}4} &= (\boldsymbol{\Gamma}_{//}^{1\underline{23}4} - (\boldsymbol{I}^{\times}\boldsymbol{\Gamma})^{\mathrm{T}132}\boldsymbol{\Gamma}_{/})^{\mathrm{T}1342^{\mathrm{S}1\underline{23}4}},\\
(\boldsymbol{\Gamma}_{//}^{1\underline{23}4} - (\boldsymbol{I}^{\times}\boldsymbol{\Gamma})^{\mathrm{T}132}\boldsymbol{\Gamma}_{/})^{\mathrm{S}12\underline{34}} &= (\boldsymbol{\Gamma}_{//}^{1\underline{23}4} - (\boldsymbol{I}^{\times}\boldsymbol{\Gamma})^{\mathrm{T}132}\boldsymbol{\Gamma}_{/})^{\mathrm{T}1423^{\mathrm{S}12\underline{34}}},\\
(\boldsymbol{\Gamma}_{//}^{1\underline{23}4} - (\boldsymbol{I}^{\times}\boldsymbol{\Gamma})^{\mathrm{T}132}\boldsymbol{\Gamma}_{/})^{\mathrm{T}1342^{\mathrm{S}1\underline{2}3\underline{4}}} &= (\boldsymbol{\Gamma}_{//}^{1\underline{23}4} - (\boldsymbol{I}^{\times}\boldsymbol{\Gamma})^{\mathrm{T}132}\boldsymbol{\Gamma}_{/})^{\mathrm{T}1423^{\mathrm{S}1\underline{2}3\underline{4}}}.
\end{aligned} \tag{41}$$

Other remarkable relations descend from the property

$$\boldsymbol{\Gamma} = \boldsymbol{\Gamma}^{\mathrm{T}}\boldsymbol{\Phi} = \boldsymbol{\Phi}\boldsymbol{\Gamma}^{\mathrm{T}}, \tag{42}$$

which is obtained by setting $m = 1$ in $(17)_4$. Equation (42) represents a factorization of the tangent tensor into itself, by means of the rotation tensor $\boldsymbol{\Phi}$.

Differentiating (42) and developing the identity $\delta\boldsymbol{\Gamma} = \delta(\boldsymbol{\Phi}\boldsymbol{\Gamma}^{\mathrm{T}})$ using $(32)_1$, $(2)_1$, (36), and (39), the relation $\boldsymbol{\Gamma}_{/}^{\mathrm{T}132} = \boldsymbol{\Phi}\boldsymbol{\Gamma}_{/}^{\mathrm{T}321}$ is obtained; hence the factorization of the first derivative of the tangent tensor follows in the forms

$$\boldsymbol{\Gamma}_{/} = \boldsymbol{\Gamma}_{/}^{\mathrm{T}231}\boldsymbol{\Phi} = \boldsymbol{\Phi}\boldsymbol{\Gamma}_{/}^{\mathrm{T}312}. \tag{43}$$

Differentiating (42) further and developing the identity $\partial\delta\boldsymbol{\Gamma} = \partial\delta(\boldsymbol{\Phi}\boldsymbol{\Gamma}^{\mathrm{T}})$ using (32), (2), (36), (37), and (39)–(43), a factorization of the second derivative of the tangent tensor is obtained in the form

$$\boldsymbol{\Gamma}_{//}^{1\underline{23}4} = (\boldsymbol{\Gamma}_{//}^{1\underline{23}4} - (\boldsymbol{I}^{\times}\boldsymbol{\Gamma})^{\mathrm{T}132}\boldsymbol{\Gamma}_{/})^{\mathrm{T}3241^{\mathrm{S}1\underline{23}4}}\boldsymbol{\Phi}. \tag{44}$$

Finally, factorization formulae of the derivative tensors $\boldsymbol{\Phi}_{/}$ and $\boldsymbol{\Phi}_{//}^{1\underline{23}4}$ are easily obtained from (31) using (2), (36) and (37), and (42) and (43):

$$\boldsymbol{\Phi}_{/} = (\boldsymbol{I}^{\times}\boldsymbol{\Gamma})^{\mathrm{T}132}\boldsymbol{\Phi} = \boldsymbol{\Phi}(\boldsymbol{I}^{\times}\boldsymbol{\Gamma}^{\mathrm{T}})^{\mathrm{T}132}, \tag{45}$$

and

$$\boldsymbol{\Phi}_{//}^{1\underline{23}4} = \big((\boldsymbol{I}^{\times}\boldsymbol{\Gamma}_{/})^{\mathrm{T}1342} + (\boldsymbol{I}^{\times}\boldsymbol{\Gamma})^{\mathrm{T}132}(\boldsymbol{I}^{\times}\boldsymbol{\Gamma})^{\mathrm{T}132}\big)^{\mathrm{S}1\underline{23}4}\boldsymbol{\Phi}. \tag{46}$$

All the above properties will be exploited in the following sections.

6. Slender beam variational mechanics

As an application of the formulation discussed so far to computational elastostatics, we address the finite-element modeling of space-curved slender beams for geometrically nonlinear problems. In the context of a variational approach and a typical Newton–Raphson solution procedure, we are concerned with three independent variations, namely virtual variations, incremental variations, and field derivatives on a one-coordinate domain (the ordinary derivatives with respect to beam abscissa). The simple form taken by the spatial variations in this problem is the main motivation for the choice of this particular application, as it allows us to focus on the significance of correctly differentiating the rotation up to third order.

Within the domain of one-dimensional intrinsic continuum mechanics, we identify the beam cross-sections with material particles aligned along a curvilinear arc-length s. The particle position and orientation are independent configuration variables and allow for shear-deformable beam response. Customary hypotheses such as linear elastic material, small strains and small stresses with respect to the elastic moduli, constant cross-sections, and rigid-section deformations are understood. In these circumstances, we may refer to the beam theory proposed by Reissner [1973], reformulated by Simo [1985], and discussed in [Ritto-Corrêa and Camotim 2002] as Reissner–Simo beam theory (see also [Simo and Vu-Quoc 1986]). The essentials of the beam variational mechanics are addressed in this section, whereas a beam element is formulated in the next section. For our present purposes, we may focus on just the variational term arising from the internal virtual work.

6.1. *Nonlinear mechanics setup.*

Kinematics. Nonrigid beam deformation produces a one-dimensional strain that can be identified by the rotational and translational strain measures

$$\boldsymbol{\omega} = \boldsymbol{k}' - \boldsymbol{\Phi}\boldsymbol{k}, \qquad \boldsymbol{\chi} = \boldsymbol{x}'_{,s} - \boldsymbol{\Phi}\boldsymbol{x}_{,s}\,, \tag{47}$$

also referred to as the angular and linear strain vectors, respectively. Here, $(\,)_{,s}$ denotes a derivative with respect to the beam abscissa and the appended prime $(\,)'$ distinguishes the current configuration from the reference one. The position vector $\boldsymbol{x}$ and the orthogonal orientation tensor $\boldsymbol{\alpha}$ of an orthonormal triad define the section configuration. Vector $\boldsymbol{k}$ is the angular curvature defined by $\boldsymbol{\alpha}^{\mathrm{T}}\boldsymbol{\alpha}_{,s} = (\boldsymbol{\alpha}^{\mathrm{T}}\boldsymbol{k})\times$ as for $(2)_1$, and $\boldsymbol{x}_{,s}$ is the tangent vector to the beam axis. Tensor $\boldsymbol{\Phi}$ is the section rotation from $\boldsymbol{\alpha}$ to $\boldsymbol{\alpha}' = \boldsymbol{\Phi}\boldsymbol{\alpha}$. It can be seen (see $(58)_1$ and $(62)_1$) that the expression of $\boldsymbol{\omega}$ in (47) coincides with the angular strain induced by a variable rotation along the beam and defined by $\boldsymbol{\Phi}^{\mathrm{T}}\boldsymbol{\Phi}_{,s} = (\boldsymbol{\Phi}^{\mathrm{T}}\boldsymbol{\omega})\times$ as for $(2)_1$. The *kinematical strain vectors* $\boldsymbol{\omega}$ and $\boldsymbol{\chi}$ in (47) represent differences of curvature and tangent vectors, that are made comparable thanks to a forward rotation of the reference values by $\boldsymbol{\Phi}$; they vanish in the case of rigid deformation.

Statics. The differential equilibrium equations

$$\boldsymbol{M}'_{,s} + \boldsymbol{x}'_{,s} \times \boldsymbol{T}' + \boldsymbol{c}' = \boldsymbol{0}, \qquad \boldsymbol{T}'_{,s} + \boldsymbol{f}' = \boldsymbol{0},$$

involve a one-dimensional stress state identified by the internal couple $\boldsymbol{M}'$ and force $\boldsymbol{T}'$ in the current configuration, also referred to as the angular and linear *stress resultants*. Current external loads $\boldsymbol{c}'$ and $\boldsymbol{f}'$ are couples and forces per unit length.

Constitutive law. Assuming a hyperelastic model, there exists a strain energy $w(\boldsymbol{\beta}, \boldsymbol{\varepsilon})$, a function of the angular ($\boldsymbol{\beta}$) and linear ($\boldsymbol{\varepsilon}$) *strain parameters*, whose first derivatives with respect to the strain parameters define the conjugate angular and linear *stress parameters*,

$$\hat{\boldsymbol{M}} = w_{/\boldsymbol{\beta}}, \qquad \hat{\boldsymbol{T}} = w_{/\boldsymbol{\varepsilon}}. \tag{48}$$

The second derivatives define the *elastic tensors* $\hat{\boldsymbol{E}}_{\beta\beta} = w_{/\beta\beta}$, $\hat{\boldsymbol{E}}_{\beta\varepsilon} = \hat{\boldsymbol{E}}_{\varepsilon\beta}^{\mathrm{T}} = w_{/\beta\varepsilon}$, and $\hat{\boldsymbol{E}}_{\varepsilon\varepsilon} = w_{/\varepsilon\varepsilon}$, which allow us to write the tangent map that transforms strain-parameter variations into stress-parameter variations:

$$\begin{Bmatrix} \partial\hat{\boldsymbol{M}} \\ \partial\hat{\boldsymbol{T}} \end{Bmatrix} = \begin{bmatrix} \hat{\boldsymbol{E}}_{\beta\beta} & \hat{\boldsymbol{E}}_{\beta\varepsilon} \\ \hat{\boldsymbol{E}}_{\varepsilon\beta} & \hat{\boldsymbol{E}}_{\varepsilon\varepsilon} \end{bmatrix} \cdot \begin{Bmatrix} \partial\boldsymbol{\beta} \\ \partial\boldsymbol{\varepsilon} \end{Bmatrix}. \tag{49}$$

Constitutive-to-mechanical variables connection. The parameters governing the constitutive model are connected to the corresponding mechanical variables defined above by the relations (cf. [Ritto-Corrêa and Camotim 2002])

$$\begin{aligned} \boldsymbol{\beta} &= \boldsymbol{\Phi}^{\mathrm{T}}\boldsymbol{\omega}, & \hat{\boldsymbol{M}} &= \boldsymbol{\Phi}^{\mathrm{T}}\boldsymbol{M}', \\ \boldsymbol{\varepsilon} &= \boldsymbol{\Phi}^{\mathrm{T}}\boldsymbol{\chi}, & \hat{\boldsymbol{T}} &= \boldsymbol{\Phi}^{\mathrm{T}}\boldsymbol{T}'. \end{aligned} \tag{50}$$

Thus, the strain parameters are the back-rotated versions of the kinematical strains and the stress parameters are the back-rotated versions of the stress resultants. By analogy with three-dimensional elasticity, we may say that $\boldsymbol{M}'$ and $\boldsymbol{T}'$ are stress vectors of the first Piola–Kirchhoff kind, whereas $\hat{\boldsymbol{M}}$ and $\hat{\boldsymbol{T}}$ are stress vectors of the Biot kind.

The formulation summarized above matches the Reissner–Simo beam theory as described in [Ritto-Corrêa and Camotim 2002]. In that paper, the two representations of either strains or stresses are referred to as the spatial and the material representations, the latter being the back-rotated version of the former. Moreover, in that formulation (but not in the implementation, see [ibid., Section 5.2]) the cross-section in the material representation is oriented as the absolute reference frame, hence what they call section rotation corresponds to our section orientation ($\boldsymbol{\alpha}$ in the reference configuration and $\boldsymbol{\alpha}'$ in the current configuration), and the pull-back operation from the spatial to the material representation is performed by the orientation $\boldsymbol{\alpha}'$. This difference does not impair the equivalence of our formulation to theirs. However, it must be pointed out that tensors $\bar{\boldsymbol{E}}_{\beta\beta}$, $\bar{\boldsymbol{E}}_{\beta\varepsilon} = \bar{\boldsymbol{E}}_{\varepsilon\beta}^{\mathrm{T}}$, and $\bar{\boldsymbol{E}}_{\varepsilon\varepsilon}$, as built with the customary matrices of the section elastic properties, are meant to be defined in the absolute reference frame and must be rotated by $\boldsymbol{\alpha}$ to build the elastic tensors in (49), that is, $\hat{\boldsymbol{E}}_{\beta\beta} = \boldsymbol{\alpha}\bar{\boldsymbol{E}}_{\beta\beta}\boldsymbol{\alpha}^{\mathrm{T}}$, etc.

The foregoing relations, together with the essential and natural boundary conditions, allow us to set up a variational functional $\Pi_\delta = \int_s \pi_\delta \, \mathrm{d}s$ and state the principle of virtual work for the beam as $\Pi_\delta = 0$. In nonlinear elasticity, we are usually concerned with the linearized form of the principle, $\Pi_\delta + \partial\Pi_\delta = 0$. In the forthcoming discussion, we focus on the internal work of stresses and address only the contributions $\pi_{\mathrm{int}\,\delta}$ and $\partial\pi_{\mathrm{int}\,\delta}$ to the virtual work per unit length; recalling the constitutive equations (48) and the relevant tangent map (49), they are given by

$$\begin{aligned} \pi_{\mathrm{int}\,\delta} &= \begin{Bmatrix} \delta\boldsymbol{\beta} \\ \delta\boldsymbol{\varepsilon} \end{Bmatrix}^{\mathrm{T}} \cdot \begin{Bmatrix} \hat{\boldsymbol{M}} \\ \hat{\boldsymbol{T}} \end{Bmatrix}, \\ \partial\pi_{\mathrm{int}\,\delta} &= \partial\pi_{\mathrm{int\,G}\delta} + \partial\pi_{\mathrm{int\,E}\delta} = \begin{Bmatrix} \partial\delta\boldsymbol{\beta} \\ \partial\delta\boldsymbol{\varepsilon} \end{Bmatrix}^{\mathrm{T}} \cdot \begin{Bmatrix} \hat{\boldsymbol{M}} \\ \hat{\boldsymbol{T}} \end{Bmatrix} + \begin{Bmatrix} \delta\boldsymbol{\beta} \\ \delta\boldsymbol{\varepsilon} \end{Bmatrix}^{\mathrm{T}} \cdot \begin{bmatrix} \hat{\boldsymbol{E}}_{\beta\beta} & \hat{\boldsymbol{E}}_{\beta\varepsilon} \\ \hat{\boldsymbol{E}}_{\varepsilon\beta} & \hat{\boldsymbol{E}}_{\varepsilon\varepsilon} \end{bmatrix} \cdot \begin{Bmatrix} \partial\boldsymbol{\beta} \\ \partial\boldsymbol{\varepsilon} \end{Bmatrix}. \end{aligned}$$

The incremental term $\partial\pi_{\mathrm{int}\,\delta}$ is made of a geometric contribution, that accounts for the current stress state, and an elastic contribution, that depends on the strain increments. Using (50), the contributions to the linearized internal virtual work are rewritten in terms of current mechanical variables as

$$\begin{aligned} \pi_{\mathrm{int}\,\delta} &= \begin{Bmatrix} \boldsymbol{\Phi}\,\delta(\boldsymbol{\Phi}^{\mathrm{T}}\boldsymbol{\omega}) \\ \boldsymbol{\Phi}\,\delta(\boldsymbol{\Phi}^{\mathrm{T}}\boldsymbol{\chi}) \end{Bmatrix}^{\mathrm{T}} \cdot \begin{Bmatrix} \boldsymbol{M}' \\ \boldsymbol{T}' \end{Bmatrix}, \qquad \partial\pi_{\mathrm{int\,G}\delta} = \begin{Bmatrix} \boldsymbol{\Phi}\,\partial\delta(\boldsymbol{\Phi}^{\mathrm{T}}\boldsymbol{\omega}) \\ \boldsymbol{\Phi}\,\partial\delta(\boldsymbol{\Phi}^{\mathrm{T}}\boldsymbol{\chi}) \end{Bmatrix}^{\mathrm{T}} \cdot \begin{Bmatrix} \boldsymbol{M}' \\ \boldsymbol{T}' \end{Bmatrix}, \\ \partial\pi_{\mathrm{int\,E}\delta} &= \begin{Bmatrix} \boldsymbol{\Phi}\,\delta(\boldsymbol{\Phi}^{\mathrm{T}}\boldsymbol{\omega}) \\ \boldsymbol{\Phi}\,\delta(\boldsymbol{\Phi}^{\mathrm{T}}\boldsymbol{\chi}) \end{Bmatrix}^{\mathrm{T}} \cdot \begin{bmatrix} \boldsymbol{\Phi}\hat{\boldsymbol{E}}_{\beta\beta}\boldsymbol{\Phi}^{\mathrm{T}} & \boldsymbol{\Phi}\hat{\boldsymbol{E}}_{\beta\varepsilon}\boldsymbol{\Phi}^{\mathrm{T}} \\ \boldsymbol{\Phi}\hat{\boldsymbol{E}}_{\varepsilon\beta}\boldsymbol{\Phi}^{\mathrm{T}} & \boldsymbol{\Phi}\hat{\boldsymbol{E}}_{\varepsilon\varepsilon}\boldsymbol{\Phi}^{\mathrm{T}} \end{bmatrix} \cdot \begin{Bmatrix} \boldsymbol{\Phi}\,\partial(\boldsymbol{\Phi}^{\mathrm{T}}\boldsymbol{\omega}) \\ \boldsymbol{\Phi}\,\partial(\boldsymbol{\Phi}^{\mathrm{T}}\boldsymbol{\chi}) \end{Bmatrix}. \end{aligned} \tag{51}$$

The development of the corotational variations of the kinematical strain that appear in (51) can be carried out in various ways. The proper choice mainly depends on the interpolating model one chooses to set up a beam element. Two different approaches are discussed in the next subsections.

6.2. ***Vectorial parameterization of motion.*** The first approach is more suitable for beam elements based on a linear interpolating model of the beam motion. According to (2), three variation variables ($\boldsymbol{\varphi}_\delta$, $\boldsymbol{\varphi}_\partial$, and $\boldsymbol{\varphi}_{\partial\delta}$) characterize the virtual and incremental variations of the section rotation,

$$\begin{aligned} &\boldsymbol{\Phi}^{\mathrm{T}}\,\delta\boldsymbol{\Phi} = (\boldsymbol{\Phi}^{\mathrm{T}}\boldsymbol{\varphi}_\delta)\times, \qquad \boldsymbol{\Phi}^{\mathrm{T}}\,\partial\boldsymbol{\Phi} = (\boldsymbol{\Phi}^{\mathrm{T}}\boldsymbol{\varphi}_\partial)\times, \\ &\boldsymbol{\Phi}^{\mathrm{T}}\,\partial\delta\boldsymbol{\Phi} = (\boldsymbol{\Phi}^{\mathrm{T}}\boldsymbol{\varphi}_{\partial\delta})\times + \tfrac{1}{2}((\boldsymbol{\Phi}^{\mathrm{T}}\boldsymbol{\varphi}_\partial)\times(\boldsymbol{\Phi}^{\mathrm{T}}\boldsymbol{\varphi}_\delta)\times + (\boldsymbol{\Phi}^{\mathrm{T}}\boldsymbol{\varphi}_\delta)\times(\boldsymbol{\Phi}^{\mathrm{T}}\boldsymbol{\varphi}_\partial)\times). \end{aligned} \tag{52}$$

Thus, recalling the properties (7) and (9), one immediately writes

$$\begin{aligned} &\boldsymbol{\Phi}\,\delta(\boldsymbol{\Phi}^{\mathrm{T}}\boldsymbol{\omega}) = \boldsymbol{\varphi}_\delta,_s\,, \qquad \boldsymbol{\Phi}\,\partial(\boldsymbol{\Phi}^{\mathrm{T}}\boldsymbol{\omega}) = \boldsymbol{\varphi}_\partial,_s\,, \\ &\boldsymbol{\Phi}\,\partial\delta(\boldsymbol{\Phi}^{\mathrm{T}}\boldsymbol{\omega}) = \boldsymbol{\varphi}_{\partial\delta},_s - \tfrac{1}{2}(\boldsymbol{\varphi}_\partial\times\boldsymbol{\varphi}_\delta,_s + \boldsymbol{\varphi}_\delta\times\boldsymbol{\varphi}_\partial,_s). \end{aligned} \tag{53}$$

Moreover, recalling (47) and using (52), one easily obtains

$$\begin{aligned} &\boldsymbol{\Phi}\,\delta(\boldsymbol{\Phi}^{\mathrm{T}}\boldsymbol{\chi}) = \delta\boldsymbol{x}',_s - \boldsymbol{\varphi}_\delta\times\boldsymbol{x}',_s\,, \qquad \boldsymbol{\Phi}\,\partial(\boldsymbol{\Phi}^{\mathrm{T}}\boldsymbol{\chi}) = \partial\boldsymbol{x}',_s - \boldsymbol{\varphi}_\partial\times\boldsymbol{x}',_s\,, \\ &\boldsymbol{\Phi}\,\partial\delta(\boldsymbol{\Phi}^{\mathrm{T}}\boldsymbol{\chi}) = \partial\delta\boldsymbol{x}',_s - \boldsymbol{\varphi}_{\partial\delta}\times\boldsymbol{x}',_s - (\boldsymbol{\varphi}_\partial\times\delta\boldsymbol{x}',_s + \boldsymbol{\varphi}_\delta\times\partial\boldsymbol{x}',_s) + \tfrac{1}{2}(\boldsymbol{\varphi}_\partial\times\boldsymbol{\varphi}_\delta\times + \boldsymbol{\varphi}_\delta\times\boldsymbol{\varphi}_\partial\times)\boldsymbol{x}',_s\,. \end{aligned} \tag{54}$$

The corotational variations of the kinematical strain vectors in (53) and (54) are expressed as functions of the differential rotation vectors $\boldsymbol{\varphi}_\delta$, $\boldsymbol{\varphi}_\partial$ (the spins, in the terminology of [Ritto-Corrêa and Camotim 2002]), and $\boldsymbol{\varphi}_{\partial\delta}$; their derivatives with respect to the beam abscissa; and the derivatives of the virtual and incremental variations $\delta\boldsymbol{x}'$, $\partial\boldsymbol{x}'$, and $\partial\delta\boldsymbol{x}'$ of the current position vector — that is, the tangent vector virtual and incremental variations. The motivation for retaining, at this level, the mixed virtual-incremental variations $\partial\delta\boldsymbol{x}'$ will become clear later on.

Substituting (53) and (54) in (51), the virtual work contributions are rewritten in terms of the differential rotation and position vectors and their derivatives. In particular, contributions $\pi_{\mathrm{int}\,\delta}$ and $\partial\pi_{\mathrm{int\,G}\delta}$ take the very neat expressions

$$\begin{aligned}
\pi_{\text{int}\,\delta} &= \begin{Bmatrix} \boldsymbol{\varphi}_{\delta},_s \\ \delta\boldsymbol{x}',_s \\ \boldsymbol{\varphi}_{\delta} \end{Bmatrix}^{\mathrm{T}} \cdot \begin{Bmatrix} \boldsymbol{M}' \\ \boldsymbol{T}' \\ -\boldsymbol{x}',_s \times \boldsymbol{T}' \end{Bmatrix}, \\
\partial\pi_{\text{int}\,\mathrm{G}\delta} &= \begin{Bmatrix} \boldsymbol{\varphi}_{\partial\delta},_s \\ \partial\delta\boldsymbol{x}',_s \\ \boldsymbol{\varphi}_{\partial\delta} \end{Bmatrix}^{\mathrm{T}} \cdot \begin{Bmatrix} \boldsymbol{M}' \\ \boldsymbol{T}' \\ -\boldsymbol{x}',_s \times \boldsymbol{T}' \end{Bmatrix} + \begin{Bmatrix} \boldsymbol{\varphi}_{\delta},_s \\ \delta\boldsymbol{x}',_s \\ \boldsymbol{\varphi}_{\delta} \end{Bmatrix}^{\mathrm{T}} \cdot \begin{bmatrix} \mathbf{0} & \mathbf{0} & \frac{1}{2}\boldsymbol{M}'\times^{\mathrm{T}} \\ \mathbf{0} & \mathbf{0} & \boldsymbol{T}'\times^{\mathrm{T}} \\ \frac{1}{2}\boldsymbol{M}'\times & \boldsymbol{T}'\times & (\boldsymbol{x}',_s \times \boldsymbol{T}'\times)^{\mathrm{S}} \end{bmatrix} \cdot \begin{Bmatrix} \boldsymbol{\varphi}_{\partial},_s \\ \partial\boldsymbol{x}',_s \\ \boldsymbol{\varphi}_{\partial} \end{Bmatrix}.
\end{aligned} \tag{55}$$

It is worth stressing that the virtual work contributions developed here through (55), are true and natural expressions in nonlinear beam variational mechanics, independently of any particular parameterization of the rotation one may choose to solve the elastic problem numerically. However, it is also worth noting that the multiplier $\boldsymbol{\varphi}_{\partial\delta},_s$ is the derivative of a (second) differential rotation vector, not the characteristic differential rotation vector of the third differentiation of the rotation tensor.

Let us introduce now the vectorial parameterization of the rotation tensor defined in (10). Recalling the differential maps (36) and (37) and the derivative tensors defined in (32), the space-derivatives of the differential rotation vectors in (55) are expressed as functions of the variations of the rotation vector:

$$\begin{aligned}
\boldsymbol{\varphi}_{\delta},_s &= \boldsymbol{\Gamma}\cdot\delta\boldsymbol{\varphi},_s + (\boldsymbol{\varphi},_s \cdot \boldsymbol{\Gamma}_{/}^{\mathrm{T}213})\cdot\delta\boldsymbol{\varphi}, \\
\boldsymbol{\varphi}_{\partial},_s &= \boldsymbol{\Gamma}\cdot\partial\boldsymbol{\varphi},_s + (\boldsymbol{\varphi},_s \cdot \boldsymbol{\Gamma}_{/}^{\mathrm{T}213})\cdot\partial\boldsymbol{\varphi}, \\
\boldsymbol{\varphi}_{\partial\delta},_s &= \boldsymbol{\Gamma}\cdot\partial\delta\boldsymbol{\varphi},_s + (\boldsymbol{\varphi},_s \cdot \boldsymbol{\Gamma}_{/}^{\mathrm{T}213})\cdot\partial\delta\boldsymbol{\varphi} \\
&\qquad + \boldsymbol{\Gamma}_{/}^{\mathrm{S}1\underline{23}} : (\partial\boldsymbol{\varphi},_s \otimes\delta\boldsymbol{\varphi} + \partial\boldsymbol{\varphi}\otimes\delta\boldsymbol{\varphi},_s) + \big(\boldsymbol{\varphi},_s \cdot \boldsymbol{\Gamma}_{//}^{1\underline{23}4\,\mathrm{T}2134}\big)^{\mathrm{S}1\underline{23}} : \partial\boldsymbol{\varphi}\otimes\delta\boldsymbol{\varphi}.
\end{aligned} \tag{56}$$

Then, the differential maps (36) and (37), the space-derivatives (56), and the relations (50) are introduced within (55). Note that the first term of the expression of $\partial\pi_{\text{int}\,\mathrm{G}\delta}$ splits into a term working for the mixed double variations ($\partial\delta$) and a term working for the single virtual (δ) and incremental (∂) variations. The latter adds to the second term in the right-hand-side of $(55)_2$ and the final result follows in the form

$$\begin{aligned}
\pi_{\text{int}\,\delta} &= \begin{Bmatrix} \delta\boldsymbol{\varphi},_s \\ \delta\boldsymbol{x}',_s \\ \delta\boldsymbol{\varphi} \end{Bmatrix}^{\mathrm{T}} \cdot \begin{Bmatrix} \boldsymbol{\Gamma}\hat{\boldsymbol{M}} \\ \boldsymbol{\Phi}\hat{\boldsymbol{T}} \\ \boldsymbol{\varphi},_s \cdot \boldsymbol{\Gamma}_{/}\hat{\boldsymbol{M}} + \boldsymbol{x}',_s \cdot \boldsymbol{\Phi}_{/}\hat{\boldsymbol{T}} \end{Bmatrix}, \\
\partial\pi_{\text{int}\,\mathrm{G}\delta} &= \begin{Bmatrix} \partial\delta\boldsymbol{\varphi},_s \\ \partial\delta\boldsymbol{x}',_s \\ \partial\delta\boldsymbol{\varphi} \end{Bmatrix}^{\mathrm{T}} \cdot \begin{Bmatrix} \boldsymbol{\Gamma}\hat{\boldsymbol{M}} \\ \boldsymbol{\Phi}\hat{\boldsymbol{T}} \\ \boldsymbol{\varphi},_s \cdot \boldsymbol{\Gamma}_{/}\hat{\boldsymbol{M}} + \boldsymbol{x}',_s \cdot \boldsymbol{\Phi}_{/}\hat{\boldsymbol{T}} \end{Bmatrix} \\
&\quad + \begin{Bmatrix} \delta\boldsymbol{\varphi},_s \\ \delta\boldsymbol{x}',_s \\ \delta\boldsymbol{\varphi} \end{Bmatrix}^{\mathrm{T}} \cdot \begin{bmatrix} \mathbf{0} & \mathbf{0} & \boldsymbol{\Gamma}_{/}\hat{\boldsymbol{M}} \\ \mathbf{0} & \mathbf{0} & \boldsymbol{\Phi}_{/}\hat{\boldsymbol{T}} \\ (\boldsymbol{\Gamma}_{/}\hat{\boldsymbol{M}})^{\mathrm{T}} & (\boldsymbol{\Phi}_{/}\hat{\boldsymbol{T}})^{\mathrm{T}} & \boldsymbol{\varphi},_s \cdot \boldsymbol{\Gamma}_{//}^{1\underline{23}4}\hat{\boldsymbol{M}} + \boldsymbol{x}',_s \cdot \boldsymbol{\Phi}_{//}^{1\underline{23}4}\hat{\boldsymbol{T}} \end{bmatrix} \cdot \begin{Bmatrix} \partial\boldsymbol{\varphi},_s \\ \partial\boldsymbol{x}',_s \\ \partial\boldsymbol{\varphi} \end{Bmatrix}.
\end{aligned} \tag{57}$$

Derivation of (57) is carried out with the help of the properties (40), (42)–(46), and $(\mathrm{A}.16)_2$.

It is seen that $(57)_1$ and the second term in $(57)_2$ coincide respectively with the integrands found in [Ritto-Corrêa and Camotim 2002, Equations (36) and (40)]. The first term of $\partial\pi_{\text{int}\,\mathrm{G}\delta}$ is lacking in that paper, as configuration-independent interpolations of both the displacement and the rotation are assumed there, so the mixed double variations hold null in their formulation. This is not the case in our design of a nonlinear beam element, as it will be shown in the next section.

The foregoing development of the kinematical strain corotational variations is suitable for beam elements relying on a linear interpolation of the parameters of the section motion. This was a common approach in the nineties [Cardona and Géradin 1988; Ibrahimbegović 1995; Ibrahimbegović et al. 1995]. The beam element developed by Ritto-Corrêa and Camotim [2002] relies on the interpolation of the total position and rotation vectors as for $\boldsymbol{x} = \sum_{J=1}^{N} W_J \boldsymbol{x}_J$ and $\boldsymbol{\varphi} = \sum_{J=1}^{N} W_J \boldsymbol{\varphi}_J$ (whence $\boldsymbol{x}_{,s} = \sum_{J=1}^{N} W_{J,s}\, \boldsymbol{x}_J$ and $\boldsymbol{\varphi}_{,s} = \sum_{J=1}^{N} W_{J,s}\, \boldsymbol{\varphi}_J$). These very simple formulae are linear with the nodal unknowns $\boldsymbol{x}_J$ and $\boldsymbol{\varphi}_J$, and substituting them in (57) leads to a linear beam element. However, as Ritto-Corrêa and Camotim themselves point out, that element was not frame invariant.

6.3. ***Consistent account of the rotation variations.*** The second approach is more suitable for a nonlinear interpolating model of the section orientation and allows us to build nonlinear beam elements that are frame invariant and path independent at the same time. Equation (2) is now written for all combinations of the virtual, incremental, and spatial independent variations of the section rotation; they provide seven equations, those in (52) and, in addition,

$$\begin{aligned}
\boldsymbol{\Phi}^{\mathrm{T}}\boldsymbol{\Phi}_{,s} &= (\boldsymbol{\Phi}^{\mathrm{T}}\boldsymbol{\omega})\times,\\
\boldsymbol{\Phi}^{\mathrm{T}}\,\delta\boldsymbol{\Phi}_{,s} &= (\boldsymbol{\Phi}^{\mathrm{T}}\boldsymbol{\omega}_\delta)\times - \tfrac{1}{2}(\boldsymbol{\Phi}^{\mathrm{T}}\boldsymbol{\varphi}_\delta\times\boldsymbol{\omega})\times + (\boldsymbol{\Phi}^{\mathrm{T}}\boldsymbol{\varphi}_\delta)\times(\boldsymbol{\Phi}^{\mathrm{T}}\boldsymbol{\omega})\times,\\
\boldsymbol{\Phi}^{\mathrm{T}}\,\partial\boldsymbol{\Phi}_{,s} &= (\boldsymbol{\Phi}^{\mathrm{T}}\boldsymbol{\omega}_\partial)\times - \tfrac{1}{2}(\boldsymbol{\Phi}^{\mathrm{T}}\boldsymbol{\varphi}_\partial\times\boldsymbol{\omega})\times + (\boldsymbol{\Phi}^{\mathrm{T}}\boldsymbol{\varphi}_\partial)\times(\boldsymbol{\Phi}^{\mathrm{T}}\boldsymbol{\omega})\times,\\
\boldsymbol{\Phi}^{\mathrm{T}}\,\partial\delta\boldsymbol{\Phi}_{,s} &= (\boldsymbol{\Phi}^{\mathrm{T}}\boldsymbol{\omega}_{\partial\delta})\times - \tfrac{1}{2}(\boldsymbol{\Phi}^{\mathrm{T}}\boldsymbol{\varphi}_{\partial\delta}\times\boldsymbol{\omega})\times + (\boldsymbol{\Phi}^{\mathrm{T}}\boldsymbol{\varphi}_{\partial\delta})\times(\boldsymbol{\Phi}^{\mathrm{T}}\boldsymbol{\omega})\times\\
&\quad - \tfrac{1}{2}(\boldsymbol{\Phi}^{\mathrm{T}}\boldsymbol{\varphi}_\partial\times\boldsymbol{\omega}_\delta + \boldsymbol{\Phi}^{\mathrm{T}}\boldsymbol{\varphi}_\delta\times\boldsymbol{\omega}_\partial)\times + (\boldsymbol{\Phi}^{\mathrm{T}}\boldsymbol{\varphi}_\partial)\times(\boldsymbol{\Phi}^{\mathrm{T}}\boldsymbol{\omega}_\delta)\times + (\boldsymbol{\Phi}^{\mathrm{T}}\boldsymbol{\varphi}_\delta)\times(\boldsymbol{\Phi}^{\mathrm{T}}\boldsymbol{\omega}_\partial)\times\,.
\end{aligned}\tag{58}$$

Equations (52) and (58) actually define seven variation variables: the virtual, incremental, and mixed virtual-incremental rotation vectors $\boldsymbol{\varphi}_\delta$, $\boldsymbol{\varphi}_\partial$, and $\boldsymbol{\varphi}_{\partial\delta}$, the finite angular strain vector $\boldsymbol{\omega}$, and the virtual, incremental, and mixed virtual-incremental angular strain vectors $\boldsymbol{\omega}_\delta$, $\boldsymbol{\omega}_\partial$, and $\boldsymbol{\omega}_{\partial\delta}$. Using (6), the angular strain corotational variations are given the expressions

$$\begin{aligned}
\boldsymbol{\Phi}\,\delta(\boldsymbol{\Phi}^{\mathrm{T}}\boldsymbol{\omega}) &= \boldsymbol{\omega}_\delta - \tfrac{1}{2}\boldsymbol{\varphi}_\delta\times\boldsymbol{\omega}, \qquad \boldsymbol{\Phi}\,\partial(\boldsymbol{\Phi}^{\mathrm{T}}\boldsymbol{\omega}) = \boldsymbol{\omega}_\partial - \tfrac{1}{2}\boldsymbol{\varphi}_\partial\times\boldsymbol{\omega},\\
\boldsymbol{\Phi}\,\partial\delta(\boldsymbol{\Phi}^{\mathrm{T}}\boldsymbol{\omega}) &= \boldsymbol{\omega}_{\partial\delta} - \tfrac{1}{2}\boldsymbol{\varphi}_{\partial\delta}\times\boldsymbol{\omega} - \tfrac{1}{2}(\boldsymbol{\varphi}_\partial\times\boldsymbol{\omega}_\delta + \boldsymbol{\varphi}_\delta\times\boldsymbol{\omega}_\partial) + (\boldsymbol{\varphi}_\partial\otimes\boldsymbol{\varphi}_\delta)^{\mathrm{S}}\cdot\boldsymbol{\omega}.
\end{aligned}\tag{59}$$

Note that, in (58) and (59), $\boldsymbol{\omega}_{\partial\delta}$ is a *third* characteristic differential rotation vector as defined in $(2)_3$.

In view of the interpolating model that will be introduced in the next section, we also address the variations of the section orientation in the current configuration, $\boldsymbol{\alpha}' = \boldsymbol{\Phi}\boldsymbol{\alpha}$. Since the orientations are assumed to be orthogonal tensors, we may write, as for (52) and (58),

$$\begin{aligned}
\boldsymbol{\alpha}'^{\mathrm{T}}\,\delta\boldsymbol{\alpha}' &= (\boldsymbol{\alpha}'^{\mathrm{T}}\boldsymbol{\varphi}_\delta)\times, \qquad \boldsymbol{\alpha}'^{\mathrm{T}}\,\partial\boldsymbol{\alpha}' = (\boldsymbol{\alpha}'^{\mathrm{T}}\boldsymbol{\varphi}_\partial)\times,\\
\boldsymbol{\alpha}'^{\mathrm{T}}\,\partial\delta\boldsymbol{\alpha}' &= (\boldsymbol{\alpha}'^{\mathrm{T}}\boldsymbol{\varphi}_{\partial\delta})\times + \tfrac{1}{2}\big((\boldsymbol{\alpha}'^{\mathrm{T}}\boldsymbol{\varphi}_\partial)\times(\boldsymbol{\alpha}'^{\mathrm{T}}\boldsymbol{\varphi}_\delta)\times + (\boldsymbol{\alpha}'^{\mathrm{T}}\boldsymbol{\varphi}_\delta)\times(\boldsymbol{\alpha}'^{\mathrm{T}}\boldsymbol{\varphi}_\partial)\times\big),
\end{aligned}\tag{60}$$

and

$$\begin{aligned}
\boldsymbol{\alpha}'^{\mathrm{T}}\boldsymbol{\alpha}',_s &= (\boldsymbol{\alpha}'^{\mathrm{T}}\boldsymbol{k}')\times,\\
\boldsymbol{\alpha}'^{\mathrm{T}}\,\delta\boldsymbol{\alpha}',_s &= (\boldsymbol{\alpha}'^{\mathrm{T}}\boldsymbol{k}'_\delta)\times -\tfrac{1}{2}(\boldsymbol{\alpha}'^{\mathrm{T}}\boldsymbol{\varphi}_\delta\times\boldsymbol{k}')\times+(\boldsymbol{\alpha}'^{\mathrm{T}}\boldsymbol{\varphi}_\delta)\times(\boldsymbol{\alpha}'^{\mathrm{T}}\boldsymbol{k}')\times,\\
\boldsymbol{\alpha}'^{\mathrm{T}}\,\partial\boldsymbol{\alpha}',_s &= (\boldsymbol{\alpha}'^{\mathrm{T}}\boldsymbol{k}'_\partial)\times -\tfrac{1}{2}(\boldsymbol{\alpha}'^{\mathrm{T}}\boldsymbol{\varphi}_\partial\times\boldsymbol{k}')\times+(\boldsymbol{\alpha}'^{\mathrm{T}}\boldsymbol{\varphi}_\partial)\times(\boldsymbol{\alpha}'^{\mathrm{T}}\boldsymbol{k}')\times,\\
\boldsymbol{\alpha}'^{\mathrm{T}}\,\partial\delta\boldsymbol{\alpha}',_s &= (\boldsymbol{\alpha}'^{\mathrm{T}}\boldsymbol{k}'_{\partial\delta})\times -\tfrac{1}{2}(\boldsymbol{\alpha}'^{\mathrm{T}}\boldsymbol{\varphi}_{\partial\delta}\times\boldsymbol{k}')\times+(\boldsymbol{\alpha}'^{\mathrm{T}}\boldsymbol{\varphi}_{\partial\delta})\times(\boldsymbol{\alpha}'^{\mathrm{T}}\boldsymbol{k}')\times\\
&\quad -\tfrac{1}{2}(\boldsymbol{\alpha}'^{\mathrm{T}}\boldsymbol{\varphi}_\partial\times\boldsymbol{k}'_\delta+\boldsymbol{\alpha}'^{\mathrm{T}}\boldsymbol{\varphi}_\delta\times\boldsymbol{k}'_\partial)\times+(\boldsymbol{\alpha}'^{\mathrm{T}}\boldsymbol{\varphi}_\partial)\times(\boldsymbol{\alpha}'^{\mathrm{T}}\boldsymbol{k}'_\delta)\times+(\boldsymbol{\alpha}'^{\mathrm{T}}\boldsymbol{\varphi}_\delta)\times(\boldsymbol{\alpha}'^{\mathrm{T}}\boldsymbol{k}'_\partial)\times .
\end{aligned}\tag{61}$$

The differential vectors that characterize the virtual and incremental orientation variations in (60) are easily seen to coincide with vectors $\boldsymbol{\varphi}_\delta$, $\boldsymbol{\varphi}_\partial$, and $\boldsymbol{\varphi}_{\partial\delta}$. Equation (61), instead, defines the current curvature vector $\boldsymbol{k}'$ and three differential curvature vectors, $\boldsymbol{k}'_\delta$, $\boldsymbol{k}'_\partial$, and $\boldsymbol{k}'_{\partial\delta}$ (again, a *third* characteristic differential rotation vector).

The finite and differential strain vectors can be linked to the finite and differential current curvature vectors by differentiating the relation $\boldsymbol{\alpha}'=\boldsymbol{\Phi}\boldsymbol{\alpha}$ and recalling the definition of the reference curvature, $\boldsymbol{\alpha}^{\mathrm{T}}\boldsymbol{\alpha},_s=(\boldsymbol{\alpha}^{\mathrm{T}}\boldsymbol{k})\times$. This yields

$$\begin{aligned}
\boldsymbol{\omega} &= \boldsymbol{k}'-\boldsymbol{\Phi}\boldsymbol{k},\\
\boldsymbol{\omega}_\delta &= \boldsymbol{k}'_\delta-\tfrac{1}{2}\boldsymbol{\varphi}_\delta\times\boldsymbol{\Phi}\boldsymbol{k},\\
\boldsymbol{\omega}_\partial &= \boldsymbol{k}'_\partial-\tfrac{1}{2}\boldsymbol{\varphi}_\partial\times\boldsymbol{\Phi}\boldsymbol{k},\\
\boldsymbol{\omega}_{\partial\delta} &= \boldsymbol{k}'_{\partial\delta}-\tfrac{1}{2}\boldsymbol{\varphi}_{\partial\delta}\times\boldsymbol{\Phi}\boldsymbol{k}+\tfrac{1}{2}(\boldsymbol{\varphi}_\partial\otimes\boldsymbol{\varphi}_\delta+\boldsymbol{\varphi}_\partial\cdot\boldsymbol{\varphi}_\delta\otimes\boldsymbol{I})^{\mathrm{S}}\cdot\boldsymbol{\Phi}\boldsymbol{k}.
\end{aligned}\tag{62}$$

Finally, using (62) within (59), the angular strain corotational variations are brought to the following expressions, which coincide with the curvature corotational variations:

$$\begin{aligned}
\boldsymbol{\Phi}\,\delta(\boldsymbol{\Phi}^{\mathrm{T}}\boldsymbol{\omega}) &= \boldsymbol{\alpha}'\,\delta(\boldsymbol{\alpha}'^{\mathrm{T}}\boldsymbol{k}')=\boldsymbol{k}'_\delta-\tfrac{1}{2}\boldsymbol{\varphi}_\delta\times\boldsymbol{k}',\\
\boldsymbol{\Phi}\,\partial(\boldsymbol{\Phi}^{\mathrm{T}}\boldsymbol{\omega}) &= \boldsymbol{\alpha}'\,\partial(\boldsymbol{\alpha}'^{\mathrm{T}}\boldsymbol{k}')=\boldsymbol{k}'_\partial-\tfrac{1}{2}\boldsymbol{\varphi}_\partial\times\boldsymbol{k}',\\
\boldsymbol{\Phi}\,\partial\delta(\boldsymbol{\Phi}^{\mathrm{T}}\boldsymbol{\omega}) &= \boldsymbol{\alpha}'\,\partial\delta(\boldsymbol{\alpha}'^{\mathrm{T}}\boldsymbol{k}')=\boldsymbol{k}'_{\partial\delta}-\tfrac{1}{2}\boldsymbol{\varphi}_{\partial\delta}\times\boldsymbol{k}'-\tfrac{1}{2}(\boldsymbol{\varphi}_\partial\times\boldsymbol{k}'_\delta+\boldsymbol{\varphi}_\delta\times\boldsymbol{k}'_\partial)+(\boldsymbol{\varphi}_\partial\otimes\boldsymbol{\varphi}_\delta)^{\mathrm{S}}\cdot\boldsymbol{k}'.
\end{aligned}\tag{63}$$

The linear strain corotational variations are much easier to write and are seen to coincide with the tangent vector corotational variations. From $(47)_2$, recalling (52), one obtains

$$\begin{aligned}
\boldsymbol{\Phi}\,\delta(\boldsymbol{\Phi}^{\mathrm{T}}\boldsymbol{\chi}) &= \boldsymbol{\alpha}'\,\delta(\boldsymbol{\alpha}'^{\mathrm{T}}\boldsymbol{x}',_s)=\delta\boldsymbol{x}',_s-\boldsymbol{\varphi}_\delta\times\boldsymbol{x}',_s,\\
\boldsymbol{\Phi}\,\partial(\boldsymbol{\Phi}^{\mathrm{T}}\boldsymbol{\chi}) &= \boldsymbol{\alpha}'\,\partial(\boldsymbol{\alpha}'^{\mathrm{T}}\boldsymbol{x}',_s)=\partial\boldsymbol{x}',_s-\boldsymbol{\varphi}_\partial\times\boldsymbol{x}',_s,\\
\boldsymbol{\Phi}\,\partial\delta(\boldsymbol{\Phi}^{\mathrm{T}}\boldsymbol{\chi}) &= \boldsymbol{\alpha}'\,\partial\delta(\boldsymbol{\alpha}'^{\mathrm{T}}\boldsymbol{x}',_s)=\partial\delta\boldsymbol{x}',_s-\boldsymbol{\varphi}_{\partial\delta}\times\boldsymbol{x}',_s-(\boldsymbol{\varphi}_\partial\times\delta\boldsymbol{x}',_s+\boldsymbol{\varphi}_\delta\times\partial\boldsymbol{x}',_s)+(\boldsymbol{\varphi}_\partial\times\boldsymbol{\varphi}_\delta\times)^{\mathrm{S}}\boldsymbol{x}',_s,
\end{aligned}\tag{64}$$

which also coincides with (54).

The corotational variations of the kinematical strain vectors are expressed in (63) and (64) as functions of true characteristic differential vectors: the differential rotation vectors $\boldsymbol{\varphi}_\delta$, $\boldsymbol{\varphi}_\partial$, and $\boldsymbol{\varphi}_{\partial\delta}$, the differential curvature vectors $\boldsymbol{k}'_\delta$, $\boldsymbol{k}'_\partial$, and $\boldsymbol{k}'_{\partial\delta}$, and the differential tangent vectors $\delta\boldsymbol{x}',_s$, $\partial\boldsymbol{x}',_s$, and $\partial\delta\boldsymbol{x}',_s$. The interested reader could substitute (63) and (64) within (51) to write expressions for $\pi_{\mathrm{int}\,\delta}$ and $\partial\pi_{\mathrm{int}\,\mathrm{G}\delta}$ that are the counterparts of (55) in terms of true characteristic differential vectors. However, let us address now the issue of a consistent interpolation along a beam element.

7. Slender beam element

Ensuring path-independence and frame-invariance is an important asset for an interpolating mechanism in finite element approximations. In nonlinear problems, the interpolation of the total rotation vector on a Euclidean vector space is path independent but not frame invariant [Crisfield and Jelenić 1999]. Interpolating either the incremental or the iterative rotation vector impairs path-independence, however Ibrahimbegović and Taylor [2002] proved that it may achieve frame-invariance. These issues were remarked on by Ritto-Corrêa and Camotim [2002] and were also discussed in [Merlini and Morandini 2004b] with reference to multicoordinate domains. It was ascertained that interpolating the orientations instead of their motions is the key to ensuring path-independence, and that averaging relative orientations in a given configuration is the key to ensuring frame-invariance. An interpolation scheme for a beam element in line with these concepts and ensuring frame-invariance and path-independence was proposed by Jelenić and Crisfield [1999].

In Section 7.1 we discuss the basic features that an interpolating model fit for rotational kinematics should conform to, and in Section 7.2 we outline our general interpolation scheme that ensures path-independence and frame-invariance.

7.1. ***Interpolating model.*** A consistent interpolation for a beam element should provide a section orientation (function of the beam abscissa s) consistent with the special manifold the rotations belong to. Accordingly, any interpolated orientation $\boldsymbol{\alpha}$ in the reference configuration must be allowed to be regarded as the result of a relative rotation $\tilde{\boldsymbol{\Phi}}_J(s)$ from the orientation $\boldsymbol{\alpha}_J$ of whichever node J (with $J = 1, 2, \dots, N$ and N the number of nodes). The same must be true for the current configuration, so multiplicative expressions for the interpolated orientations,

$$\boldsymbol{\alpha} = \tilde{\boldsymbol{\Phi}}_J \boldsymbol{\alpha}_J \quad \text{and} \quad \boldsymbol{\alpha}' = \tilde{\boldsymbol{\Phi}}'_J \boldsymbol{\alpha}'_J, \tag{65}$$

must hold. In (65) the relative rotations are transcendental functions $\tilde{\boldsymbol{\Phi}}_J(s)$ and $\tilde{\boldsymbol{\Phi}}'_J(s)$ of the beam abscissa. The derivatives of these functions along the beam, namely $\tilde{\boldsymbol{\Phi}}_J^{\mathrm{T}} \tilde{\boldsymbol{\Phi}}_{J,s} = (\tilde{\boldsymbol{\Phi}}_J^{\mathrm{T}} \tilde{\boldsymbol{\omega}}_J)\times$ and $\tilde{\boldsymbol{\Phi}}_J'^{\mathrm{T}} \tilde{\boldsymbol{\Phi}}'_{J,s} = (\tilde{\boldsymbol{\Phi}}_J'^{\mathrm{T}} \tilde{\boldsymbol{\omega}}'_J)\times$, define relative angular strains $\tilde{\boldsymbol{\omega}}_J(s)$ and $\tilde{\boldsymbol{\omega}}'_J(s)$. The interpolated curvatures $\boldsymbol{k}$ and $\boldsymbol{k}'$ are immediately found to coincide with these relative strains:

$$\boldsymbol{k} = \tilde{\boldsymbol{\omega}}_J \quad \text{and} \quad \boldsymbol{k}' = \tilde{\boldsymbol{\omega}}'_J. \tag{66}$$

An interpolation scheme provides the relative-rotation functions $\tilde{\boldsymbol{\Phi}}_J(s)$ and $\tilde{\boldsymbol{\Phi}}'_J(s)$ and the characteristic vectors $\tilde{\boldsymbol{\omega}}_J(s)$ and $\tilde{\boldsymbol{\omega}}'_J(s)$ of their derivatives. Once these quantities become known and orientations and curvatures are obtained in both the reference and the current configurations, the section rotation and the angular strain can immediately be recovered as $\boldsymbol{\Phi} = \boldsymbol{\alpha}' \boldsymbol{\alpha}^{\mathrm{T}}$ and $\boldsymbol{\omega} = \boldsymbol{k}' - \boldsymbol{\Phi} \boldsymbol{k}$ (Figure 2). This way, the correct equivalent of an isoparametric element for rotational kinematics is achieved, in the sense that the reference and the current orientation fields (not the rotation field) are approximated using the same interpolating model [Merlini and Morandini 2011b].

The interpolating model is completed with a linearization procedure that yields the appropriate variation variables (virtual, incremental, and mixed virtual-incremental variations). In the current configuration, the reference nodal orientations $\boldsymbol{\alpha}_J$ are rotated by the unknowns $\boldsymbol{\Phi}_J$:

$$\boldsymbol{\alpha}'_J = \boldsymbol{\Phi}_J \boldsymbol{\alpha}_J. \tag{67}$$

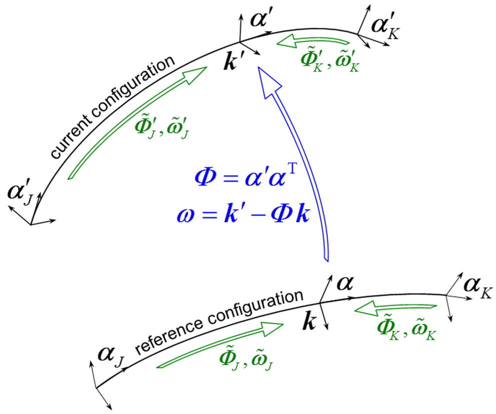

Figure 2. Interpolating model: recovering rotations and angular strains.

The interpolated orientation is a multiplicative combination of two successive rotations, $\boldsymbol{\Phi}_J$ in (67) followed by $\tilde{\boldsymbol{\Phi}}'_J$ in $(65)_2$, resulting in $\boldsymbol{\alpha}' = \tilde{\boldsymbol{\Phi}}'_J \boldsymbol{\Phi}_J \boldsymbol{\alpha}_J$. A careful linearization of the compound rotation $\tilde{\boldsymbol{\Phi}}'_J \boldsymbol{\Phi}_J$ is required in order to derive consistent orientation variations.

According to (2), the variations of either $\boldsymbol{\Phi}_J$ or $\tilde{\boldsymbol{\Phi}}'_J$ are characterized by specific differential vectors. The virtual, incremental and mixed virtual-incremental variations of the discrete rotation $\boldsymbol{\Phi}_J$ (or alternatively the discrete orientation $\boldsymbol{\alpha}'_J$) define, as usual, the differential rotation vectors $\boldsymbol{\varphi}_{J\delta}$, $\boldsymbol{\varphi}_{J\partial}$, and $\boldsymbol{\varphi}_{J\partial\delta}$. The relevant formulae are not displayed here, but are easily written from (52) (or alternatively (60)), simply by replacing $\boldsymbol{\Phi}$ with $\boldsymbol{\Phi}_J$ (or $\boldsymbol{\alpha}'$ with $\boldsymbol{\alpha}'_J$). In a similar way, variations of the relative-rotation field $\tilde{\boldsymbol{\Phi}}'_J$ define the virtual, incremental, and mixed virtual-incremental relative rotation vectors $\tilde{\boldsymbol{\varphi}}'_{J\delta}$, $\tilde{\boldsymbol{\varphi}}'_{J\partial}$, and $\tilde{\boldsymbol{\varphi}}'_{J\partial\delta}$ and relative angular differential strains $\tilde{\boldsymbol{\omega}}'_{J\delta}$, $\tilde{\boldsymbol{\omega}}'_{J\partial}$, and $\tilde{\boldsymbol{\omega}}'_{J\partial\delta}$. The relevant formulae can be written from (52) and (58) after replacing $\boldsymbol{\Phi}$ with $\tilde{\boldsymbol{\Phi}}'_J$.

Differentiating $\boldsymbol{\alpha}' = \tilde{\boldsymbol{\Phi}}'_J \boldsymbol{\Phi}_J \boldsymbol{\alpha}_J$ yields a relationship between the local variation variables ($\boldsymbol{\varphi}_\delta$, $\boldsymbol{\varphi}_\partial$, $\boldsymbol{\varphi}_{\partial\delta}$, $\boldsymbol{k}'_\delta$, $\boldsymbol{k}'_\partial$, $\boldsymbol{k}'_{\partial\delta}$), the relative variation variables ($\tilde{\boldsymbol{\varphi}}'_{J\delta}$, $\tilde{\boldsymbol{\varphi}}'_{J\partial}$, $\tilde{\boldsymbol{\varphi}}'_{J\partial\delta}$, $\tilde{\boldsymbol{\omega}}'_{J\delta}$, $\tilde{\boldsymbol{\omega}}'_{J\partial}$, $\tilde{\boldsymbol{\omega}}'_{J\partial\delta}$), and the nodal variation variables ($\boldsymbol{\varphi}_{J\delta}$, $\boldsymbol{\varphi}_{J\partial}$, $\boldsymbol{\varphi}_{J\partial\delta}$). After cumbersome algebraic manipulations (detailed in [Merlini 2002]), one obtains

$$\begin{aligned}
\boldsymbol{\varphi}_\delta &= \tilde{\boldsymbol{\varphi}}'_{J\delta} + \tilde{\boldsymbol{\Phi}}'_J \boldsymbol{\varphi}_{J\delta}, \\
\boldsymbol{\varphi}_\partial &= \tilde{\boldsymbol{\varphi}}'_{J\partial} + \tilde{\boldsymbol{\Phi}}'_J \boldsymbol{\varphi}_{J\partial}, \\
\boldsymbol{\varphi}_{\partial\delta} &= \tilde{\boldsymbol{\varphi}}'_{J\partial\delta} + \tilde{\boldsymbol{\Phi}}'_J \boldsymbol{\varphi}_{J\partial\delta} - \tfrac{1}{2}\big((\tilde{\boldsymbol{\Phi}}'_J \boldsymbol{\varphi}_{J\partial}) \times \tilde{\boldsymbol{\varphi}}'_{J\delta} + (\tilde{\boldsymbol{\Phi}}'_J \boldsymbol{\varphi}_{J\delta}) \times \tilde{\boldsymbol{\varphi}}'_{J\partial}\big),
\end{aligned} \tag{68}$$

and

$$\begin{aligned}
\boldsymbol{k}'_\delta &= \tilde{\boldsymbol{\omega}}'_{J\delta} - \tfrac{1}{2}(\tilde{\boldsymbol{\Phi}}'_J \boldsymbol{\varphi}_{J\delta}) \times \tilde{\boldsymbol{\omega}}'_J, \\
\boldsymbol{k}'_\partial &= \tilde{\boldsymbol{\omega}}'_{J\partial} - \tfrac{1}{2}(\tilde{\boldsymbol{\Phi}}'_J \boldsymbol{\varphi}_{J\partial}) \times \tilde{\boldsymbol{\omega}}'_J, \\
\boldsymbol{k}'_{\partial\delta} &= \tilde{\boldsymbol{\omega}}'_{J\partial\delta} - \tfrac{1}{2}(\tilde{\boldsymbol{\Phi}}'_J \boldsymbol{\varphi}_{J\partial\delta}) \times \tilde{\boldsymbol{\omega}}'_J - \tfrac{1}{2}\big((\tilde{\boldsymbol{\Phi}}'_J \boldsymbol{\varphi}_{J\partial}) \times \tilde{\boldsymbol{\omega}}'_{J\delta} + (\tilde{\boldsymbol{\Phi}}'_J \boldsymbol{\varphi}_{J\delta}) \times \tilde{\boldsymbol{\omega}}'_{J\partial}\big) \\
&\quad + \tfrac{1}{4}\big((\tilde{\boldsymbol{\Phi}}'_J \boldsymbol{\varphi}_{J\partial}) \times (\tilde{\boldsymbol{\varphi}}'_{J\delta} + \tilde{\boldsymbol{\Phi}}'_J \boldsymbol{\varphi}_{J\delta}) \times + (\tilde{\boldsymbol{\Phi}}'_J \boldsymbol{\varphi}_{J\delta}) \times (\tilde{\boldsymbol{\varphi}}'_{J\partial} + \tilde{\boldsymbol{\Phi}}'_J \boldsymbol{\varphi}_{J\partial}) \times\big)\tilde{\boldsymbol{\omega}}'_J \\
&\qquad - (\tilde{\boldsymbol{\Phi}}'_J \boldsymbol{\varphi}_{J\partial} \otimes \tilde{\boldsymbol{\varphi}}'_{J\delta} + \tilde{\boldsymbol{\Phi}}'_J \boldsymbol{\varphi}_{J\delta} \otimes \tilde{\boldsymbol{\varphi}}'_{J\partial} + \tilde{\boldsymbol{\Phi}}'_J \boldsymbol{\varphi}_{J\partial} \otimes \tilde{\boldsymbol{\Phi}}'_J \boldsymbol{\varphi}_{J\delta})^{\mathrm{S}} \cdot \tilde{\boldsymbol{\omega}}'_J.
\end{aligned} \tag{69}$$

The reader should notice the relevance of considering the third characteristic differential vector defined in $(2)_3$, which is present here as $\boldsymbol{k}'_{\partial\delta}$ and $\tilde{\boldsymbol{\omega}}'_{J\partial\delta}$.

7.2. ***A multiplicative interpolation scheme.*** It is worth noting that the findings in the previous subsection are perfectly independent of any particular parameterization of the rotation tensor and of any particular interpolation scheme. They are just the natural premise for an interpolation to be consistent with the traits of the manifold the rotations belong to. In this subsection instead, we introduce the algorithm we use to implement a frame-invariant and path-independent multiplicative interpolation scheme for rotational kinematics.

The idea underlying the interpolation scheme is quite simple. The sought orientation is a weighted average between the nodal orientations, in the sense that the relative orientations with respect to each node are weighted in conformity with the position of the section within the beam element. This concept is achieved by satisfying a condition of multiplicative nature,

$$\sum_{J=1}^{N} W_J \ln(\boldsymbol{\alpha}\boldsymbol{\alpha}_J^{\mathrm{T}}) = \mathbf{0},$$

where $W_J(s)$ are (normalized) weight functions and the skew-symmetric tensors $\ln(\boldsymbol{\alpha}\boldsymbol{\alpha}_J^{\mathrm{T}}) = \tilde{\boldsymbol{\varphi}}_J\times$ (recall $(65)_1$) are built with the relative rotation vectors from each node. Note that this condition is the exact transposition of the condition of additive nature $\sum_{J=1}^{N} W_J(\boldsymbol{x}-\boldsymbol{x}_J) = \mathbf{0}$, which underlies the commonly used interpolation on the Euclidean position-vector space and simply results in $\boldsymbol{x} = \sum_{J=1}^{N} W_J\boldsymbol{x}_J$. In contrast, the interpolation condition for rotational kinematics is a nonlinear implicit equation that cannot be solved in general in closed form, and therefore requires an iterative procedure. This interpolation scheme, together with a fast algorithm to solve the nonlinear condition, has been proposed in [Merlini and Morandini 2004b] with reference to multicoordinate domains.

In the current configuration, the interpolation algorithm reads

$$\sum_{J=1}^{N} W_J \ln(\boldsymbol{\alpha}'\boldsymbol{\alpha}_J'^{\mathrm{T}}) = \mathbf{0}, \qquad \sum_{J=1}^{N} W_J(\boldsymbol{x}'-\boldsymbol{x}_J') = \mathbf{0}. \tag{70}$$

The linearization of the multiplicative condition $(70)_1$ is a quite subtle task. A short account is outlined below, but any details can be recovered from [Merlini and Morandini 2004b] and references therein. The reader should notice that so far, in the whole formulation aimed at building a beam element, we have not yet referred to any parameterization of the rotation tensor. Only now is it time to introduce a parameterization. We resort to the natural vectorial parameterization (10) and its linearization discussed in Section 5, where the differential maps (36)–(38) are obtained. Recalling $(65)_2$ and the exponential map $\tilde{\boldsymbol{\Phi}}_J' = \exp(\tilde{\boldsymbol{\varphi}}_J'\times)$, $(70)_1$ can be rewritten as a vectorial equation as

$$\sum_{J=1}^{N} W_J\tilde{\boldsymbol{\varphi}}_J' = \mathbf{0}. \tag{71}$$

Differentiating (71) up to third order yields seven algebraic equations for the unknowns $\delta\tilde{\boldsymbol{\varphi}}_J'$, $\partial\tilde{\boldsymbol{\varphi}}_J'$, $\partial\delta\tilde{\boldsymbol{\varphi}}_J'$, $\tilde{\boldsymbol{\varphi}}_J'{}_{,s}$, $\delta\tilde{\boldsymbol{\varphi}}_J'{}_{,s}$, $\partial\tilde{\boldsymbol{\varphi}}_J'{}_{,s}$, and $\partial\delta\tilde{\boldsymbol{\varphi}}_J'{}_{,s}$. After inverting (36)–(38) for $\delta\boldsymbol{\varphi}$, $\partial\delta\boldsymbol{\varphi}$, and $\mathrm{d}\partial\delta\boldsymbol{\varphi}$ and using the results within the seven equations, the unknowns of the equation set are turned to the relative variation variables $\tilde{\boldsymbol{\varphi}}_{J\delta}'$, $\tilde{\boldsymbol{\varphi}}_{J\partial}'$, $\tilde{\boldsymbol{\varphi}}_{J\partial\delta}'$, $\tilde{\boldsymbol{\omega}}_J'$, $\tilde{\boldsymbol{\omega}}_{J\delta}'$, $\tilde{\boldsymbol{\omega}}_{J\partial}'$, and $\tilde{\boldsymbol{\omega}}_{J\partial\delta}'$. Finally, after solving (68), $(66)_2$, and (69) for $\tilde{\boldsymbol{\varphi}}_{J\delta}'$, $\tilde{\boldsymbol{\varphi}}_{J\partial}'$, $\tilde{\boldsymbol{\varphi}}_{J\partial\delta}'$, $\tilde{\boldsymbol{\omega}}_J'$, $\tilde{\boldsymbol{\omega}}_{J\delta}'$, $\tilde{\boldsymbol{\omega}}_{J\partial}'$, and $\tilde{\boldsymbol{\omega}}_{J\partial\delta}'$ and substituting the results within the seven equations, the unknowns are changed to the local variation variables $\boldsymbol{\varphi}_\delta$, $\boldsymbol{\varphi}_\partial$, $\boldsymbol{\varphi}_{\partial\delta}$, $\boldsymbol{k}'$, $\boldsymbol{k}_\delta'$, $\boldsymbol{k}_\partial'$, and $\boldsymbol{k}_{\partial\delta}'$.

The equations are solved in cascade and yield an expression for the curvature,

$$\boldsymbol{k}' = -\Big(\sum_{K=1}^{N} W_K \tilde{\boldsymbol{\Gamma}}_K'^{-1}\Big)^{-1} \sum_{J=1}^{N} W_{J,s}\, \tilde{\boldsymbol{\varphi}}_J', \tag{72}$$

and six expressions for the local variation variables,

$$\begin{aligned}\boldsymbol{\varphi}_\delta &= \sum_{J=1}^{N} \tilde{\boldsymbol{Y}}_J' \cdot \boldsymbol{\varphi}_{J\delta}, \qquad \boldsymbol{\varphi}_\partial = \sum_{K=1}^{N} \tilde{\boldsymbol{Y}}_K' \cdot \boldsymbol{\varphi}_{K\partial},\\ \boldsymbol{\varphi}_{\partial\delta} &= \sum_{J=1}^{N} \tilde{\boldsymbol{Y}}_J' \cdot \boldsymbol{\varphi}_{J\partial\delta} + \sum_{J=1}^{N}\sum_{K=1}^{N} \tilde{\mathfrak{Y}}_{JK}' : \boldsymbol{\varphi}_{J\delta} \otimes \boldsymbol{\varphi}_{K\partial},\end{aligned} \tag{73}$$

and

$$\begin{aligned}\boldsymbol{k}_\delta' &= \sum_{J=1}^{N} \tilde{\mathfrak{X}}_J' : \boldsymbol{\varphi}_{J\delta} \otimes \boldsymbol{I}, \qquad \boldsymbol{k}_\partial' = \sum_{K=1}^{N} \tilde{\mathfrak{X}}_K' : \boldsymbol{\varphi}_{K\partial} \otimes \boldsymbol{I},\\ \boldsymbol{k}_{\partial\delta}' &= \sum_{J=1}^{N} \tilde{\mathfrak{X}}_J' : \boldsymbol{\varphi}_{J\partial\delta} \otimes \boldsymbol{I} + \sum_{J=1}^{N}\sum_{K=1}^{N} \tilde{\mathbb{Z}}_{JK}' \,\vdots\, \boldsymbol{\varphi}_{J\delta} \otimes \boldsymbol{\varphi}_{K\partial} \otimes \boldsymbol{I}.\end{aligned} \tag{74}$$

Equations (73) and (74) are interpolating functions, linear in the nodal variation variables $\boldsymbol{\varphi}_{J\delta}$, $\boldsymbol{\varphi}_{J\partial}$, and $\boldsymbol{\varphi}_{J\partial\delta}$. The tensor-valued coefficients $\tilde{\boldsymbol{Y}}_J'$, $\tilde{\mathfrak{Y}}_{JK}'$, $\tilde{\mathfrak{X}}_J'$, and $\tilde{\mathbb{Z}}_{JK}'$ (see [Merlini and Morandini 2004b]) are nonlinear functions of the current nodal unknowns and thus need to be computed dynamically in the solution process. The curvature in the reference configuration is computed once for all as in (72).

The linearization of translational kinematics descends straightforwardly from $(70)_2$. Since no double variations $\partial\delta\boldsymbol{x}_J'$ of free variables make sense, and the interpolation is linear with the free variables $\boldsymbol{x}_J'$, it follows

$$\delta\boldsymbol{x}'_{,s} = \sum_{J=1}^{N} W_{J,s}\, \delta\boldsymbol{x}_J', \qquad \partial\boldsymbol{x}'_{,s} = \sum_{J=1}^{N} W_{J,s}\, \partial\boldsymbol{x}_J', \tag{75}$$

and $\partial\delta\boldsymbol{x}'_{,s}$ is discarded.

7.3. ***Element implementation.*** Equations (72)–(75) are now used within (63)–(64) to provide appropriate interpolating functions for the kinematical strain corotational variations, and the latter are finally used within (51). After integration over the element domain, the internal virtual work contribution from each element to the linearized functional is obtained in the form

$$\begin{aligned}\Pi_{\mathrm{int}\,\delta}^e &= \int_{s^e} \pi_{\mathrm{int}\,\delta}\, \mathrm{d}s^e = \begin{Bmatrix}\boldsymbol{\varphi}_{J\delta}\\ \delta\boldsymbol{x}_J'\end{Bmatrix}^{\mathrm{T}} \cdot \begin{Bmatrix}\boldsymbol{F}_{\varphi J}^e\\ \boldsymbol{F}_{xJ}^e\end{Bmatrix},\\ \partial\Pi_{\mathrm{int}\,\delta}^e &= \int_{s^e} \partial\pi_{\mathrm{int}\,\delta}\, \mathrm{d}s^e = \boldsymbol{\varphi}_{J\partial\delta} \cdot \boldsymbol{F}_{\varphi J}^e + \begin{Bmatrix}\boldsymbol{\varphi}_{J\delta}\\ \delta\boldsymbol{x}_J'\end{Bmatrix}^{\mathrm{T}} \cdot \begin{bmatrix}\boldsymbol{K}_{\varphi\varphi JK}^e & \boldsymbol{K}_{\varphi x JK}^e\\ \boldsymbol{K}_{\varphi x JK}^{e\mathrm{T}} & \boldsymbol{K}_{xx JK}^e\end{bmatrix} \cdot \begin{Bmatrix}\boldsymbol{\varphi}_{K\partial}\\ \partial\boldsymbol{x}_K'\end{Bmatrix}.\end{aligned} \tag{76}$$

However, (76) cannot be assembled within the global structure model in the present form. The nodal mixed multipliers $\boldsymbol{\varphi}_{J\partial\delta}$ do not have a correspondence in finite element formulations, and must be solved for the separate virtual and incremental nodal variation variables $\boldsymbol{\varphi}_{J\delta}$ and $\boldsymbol{\varphi}_{J\partial}$. Actually, this resolution is feasible at each single node, where the rotation vector $\boldsymbol{\varphi}_J$ is a truly free variable, so that $\partial\delta\boldsymbol{\varphi}_J$ does not exist at all. By applying the differential maps (36) and (37) to the total nodal rotation $\boldsymbol{\Phi}_J$ and discarding the term in $\partial\delta\boldsymbol{\varphi}_J$, the resolution formula

$$\boldsymbol{\varphi}_{J\partial\delta} = \cancel{\boldsymbol{\Gamma}_J \cdot \partial\delta\boldsymbol{\varphi}_J} + \boldsymbol{\Gamma}_{J/}^{\mathrm{S}\underline{123}} : \boldsymbol{\Gamma}_J^{-1}\boldsymbol{\varphi}_{J\partial} \otimes \boldsymbol{\Gamma}_J^{-1}\boldsymbol{\varphi}_{J\delta} \tag{77}$$

is made available. (Note that $\boldsymbol{\Gamma}_J$ and $\boldsymbol{\Gamma}_{J/}$ in (77) are global tensors, built with the total nodal rotation vector $\boldsymbol{\varphi}_J$.) The first term of $\partial\Pi^e_{\mathrm{int}\,\delta}$ can then be unfolded into a form like the second term and added to the latter. Thus, a workable expression for the increment of the internal virtual work is obtained; it provides a symmetrical and fully nonlinear tangent matrix, built with the interpolating functions (72)–(74), which are nonlinear functions of the current nodal orientations. The global tangent matrix can be assembled and the linearized problem is solved for the incremental unknowns $\boldsymbol{\varphi}_{K\partial}$ and $\partial\boldsymbol{x}'_K$. The nodal variables are then updated consistently as for $\boldsymbol{\alpha}'_K \leftarrow \exp(\boldsymbol{\varphi}_{K\partial}\times)\boldsymbol{\alpha}'_K$ and $\boldsymbol{x}'_K \leftarrow \boldsymbol{x}'_K + \partial\boldsymbol{x}'_K$.

It may be worth stressing that the double variation $\partial\delta\boldsymbol{\varphi}$ of the local rotation vector cannot be discarded as if $\boldsymbol{\varphi}$ were a free variable. This follows from the differential maps (36) and (37), since the local variation variables $\boldsymbol{\varphi}_\delta$, $\boldsymbol{\varphi}_\partial$, and $\boldsymbol{\varphi}_{\partial\delta}$ are computed — consistently with the assumed interpolating model — from (73).

The framework of this nonlinear slender beam element, and in particular the interpolation methodology, are the same as in our solid element [Merlini and Morandini 2004b; 2005] and shell element [Merlini and Morandini 2011a]. The frame-invariance and path-independence of our interpolation scheme was discussed and proved in those papers. In those papers, actually, we adopted a particular modeling of the continuum — called *helicoidal modeling* — where the angular and linear kinematic fields are coupled into a single field of orthogonal dual tensors, to which however the multiplicative interpolation discussed above applies as is. Incidentally, a working slender beam element based on helicoidal modeling is ready and will be published separately.

8. Numerical tests

A two-node and a three-node beam element have been implemented in our own finite-element code, formerly developed for nonlinear solid and shell elements. In the two-node case, the interpolation discussed in Section 7 provides an element having a constant curvature $\boldsymbol{k}$ along the element length. In this case, the linearization of the interpolation is much simpler, and manageable expressions for the corotational variations (63) and (64) can be obtained in the form of linear functions of the nodal variation variables (similarly to (73)–(75)), without the need of the third differential map (38). However, both the two-node and the three-node elements are built with the general form of the interpolation discussed above. Standard Lagrange polynomials are used for the weight functions $W_J(s)$. As a common practice in beam element technology, shear locking is avoided by resorting to reduced integration; we adopt Gaussian quadrature with one integration point for the two-node element and two integration points for the three-node element.

Two benchmark problems in nonlinear beam analysis illustrate the performance of the proposed element. In all computations, the convergence of the Newton–Raphson iterations is checked against a tolerance of 10^{-2} on the maximum absolute value of the residual. No explicit units are reported here, but it is understood that all measures are associated with a coherent system, for example, SI.

8.1. ***Bending of 45° curved cantilever.*** The cantilever bend was introduced by Bathe and Bolourchi [1979] and then repeated by almost every author of new nonlinear beam elements. The circular arc of radius 100 spans 45° in the horizontal plane and is loaded at the free tip by a vertical force up to the final value 600 (Figure 3). The beam has a square cross-section of size 1×1 and elastic material with Young's modulus $E = 10^7$ and Poisson's ratio $\nu = 0$. After [Ritto-Corrêa and Camotim 2002], two sets of cross-section elastic properties are considered, according to a different choice of the effective shear areas and torsional constant. The set denoted by AJ takes unchanged areas $A_2 = A_3 = 1$ for shear and polar moment of inertia $J = J_2 + J_3$ for torsion; the set denoted by SK takes effective shear areas $A_2 = A_3 = 5/6$ and torsional constant $J = 0.141$; the values of both sets are listed in Figure 3.

Several meshes are analyzed, from coarse ones (4 elements) to refined ones (128 elements). All analyses are requested to seek convergence in a single load-step, but an algorithm of automatic step control makes the step size shrink or stretch dynamically. The resulting numbers of steps and iterations, together with the coordinates of the free tip at the final load 600, are listed in Tables 1 and 2. Some values available from recent literature are also compared in the tables, while earlier results — for which, however, it is not always certain which cross-section elastic properties were used — can be found in the cited papers.

8.2. ***Cantilever beam twisted to a helical form.*** A clamped slender beam is bent to a helical shape by a force and coaxial couple at the free tip, as illustrated in Figure 4 for a couple-to-force ratio $M/F = 4\pi$. At increasing load, the beam coils into narrower and narrower circles while the tip crosses the circles planes alternately from one side to the opposite side. At the final load $M = 200\pi$ and $F = 50$, the helix develops in 10 circles and the out-of-plane displacement is opposite to the applied force. This example was introduced by Ibrahimbegović [1997] to test analyses with space rotations exceeding 2π, and was then used by other authors [Zupan and Saje 2003; 2004; Mäkinen 2007; Zupan et al. 2009].

Five meshes with two-node and three-node elements are considered (Table 3). The load is applied in 100 equal steps, but an automatic step control may halve a step if necessary; this happens three to five times in any of the computations. However, the computations converge quite quickly, as can be seen

Section properties	"AJ"	"SK"	
EA	1	1	$\times 10^7$
GA_2, GA_3	6/12	5/12	$\times 10^7$
EJ_2, EJ_3	1/12	1/12	$\times 10^7$
GJ	1/12	0.0705	$\times 10^7$

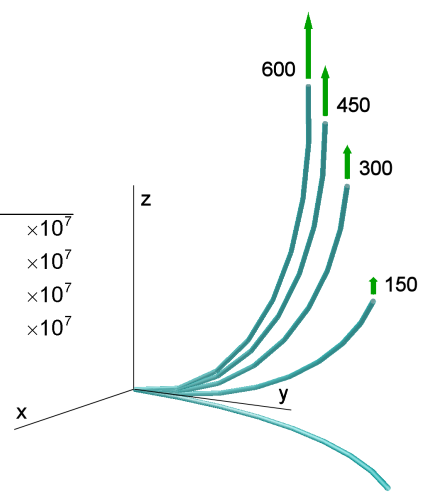

Figure 3. 45° bend: reference and deformed configurations of the SK beam model with eight two-node elements.

Element type	# of elements	# of steps	# of iterations	Tip coordinates at load 600		
				x	y	z
2-node[1]	8	6 equal	38	15.7426	47.2606	53.3730
2-node[2]	8	6 equal	54	15.67	47.29	53.37
2-node	8	6	41	15.7423	47.2600	53.3742
2-node	16	2	26	15.6991	47.1779	53.4495
2-node	32	6	41	15.6884	47.1573	53.4685
2-node	64	6	40	15.6857	47.1522	53.4733
2-node	128	6	40	15.6850	47.1509	53.4745
3-node	4	8	51	15.6837	47.1553	53.4671
3-node	8	6	42	15.6848	47.1507	53.4744
3-node	16	6	40	15.6848	47.1504	53.4748
3-node	32	6	41	15.6848	47.1504	53.4749
3-node	64	5	35	15.6848	47.1504	53.4749

Table 1. 45° bend: AJ model computation data. Data from ([1]) [Ritto-Corrêa and Camotim 2002] and ([2]) [Ghosh and Roy 2009].

from Table 3. To check the relevance of the resolution (77) of the nodal mixed variation variables, a test was run with the recovering of the first term of $\partial \Pi^e_{\text{int}\,\delta}$ in $(76)_2$ disabled, but it failed untimely at 3% of the final load.

The displacement of the free tip in the direction of the applied force is plotted in Figure 5 versus the fraction of the total load. Among the five meshes, only the response of the coarse mesh of twenty four

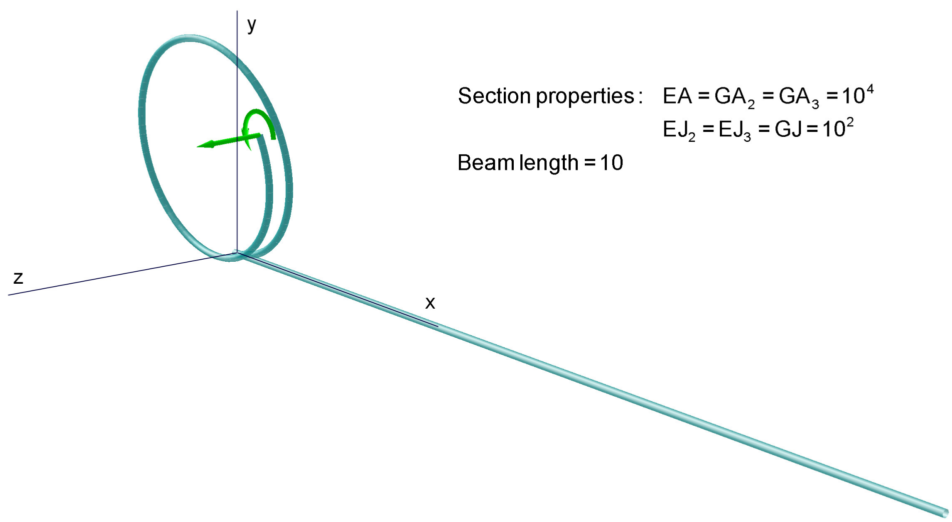

Figure 4. Helical beam: reference and deformed configuration of the model with ninety six two-node elements at 13% of the final load.

Element type	# of elements	# of steps	# of Iterations	Tip coordinates at load 600 x	y	z
2-node[1]	8	6 equal	38	15.6213	47.0142	53.4980
4-node[3]	1	4 equal	7–13 /step	15.57	46.95	53.51
4-node[3]	2	4 equal	7–13 /step	15.56	46.89	53.61
4-node[3]	8	4 equal	7–13 /step	15.56	46.89	53.61
3-int-pt[4]	8	6 equal	30	15.61	46.89	53.60
2-node	8	6	41	15.6211	47.0137	53.4991
2-node	16	2	26	15.5754	46.9259	53.5786
2-node	32	6	41	15.5640	46.9038	53.5988
2-node	64	6	40	15.5613	46.8982	53.6038
2-node	128	6	40	15.5606	46.8968	53.6050
3-node	4	8	51	15.5251	46.8361	53.6350
3-node	8	6	42	15.5410	46.8592	53.6295
3-node	16	6	40	15.5502	46.8766	53.6190
3-node	32	6	41	15.5551	46.8863	53.6125
3-node	64	5	35	15.5577	46.8913	53.6091

Table 2. 45° bend: SK model computation data. Data from ([1]) [Ritto-Corrêa and Camotim 2002], ([3]) [Kapania and Li 2003], and ([4]) [Zupan et al. 2009].

Element type	# of elements	# of steps	# of iterations	Average iterations per step
2-node	24	105	600	5.7
2-node	48	103	548	5.3
2-node	96	104	519	5.0
3-node	24	104	575	5.5
3-node	48	104	537	5.2

Table 3. Helical beam: computation data.

two-node elements is clearly distinguishable. These results are in good agreement with the papers cited above: though data for a tabular comparison are not available, a graphical superposition confirms that the curves in the published plots lie within the band between the curves in Figure 5. A series of coordinates of the free tip, as computed at completion of each successive coil, is listed in Table 4.

It is worth stressing that the computations are insensitive to rotations exceeding 2π. This is due to the formulation implemented, which does not need to store and update total rotation vectors, hence eludes the discontinuities arising at rotation angles which are multiples of 2π; instead, the nodal orientation tensors are stored and updated via the incremental rotation tensors at each iteration. Of course, the proposed formulation is not able to handle the extreme case of relative nodal orientations exceeding 2π within a single element; this causes the computation to stall, as punctually happened at just 95% of the final load when testing a mesh of twelve three-node elements.

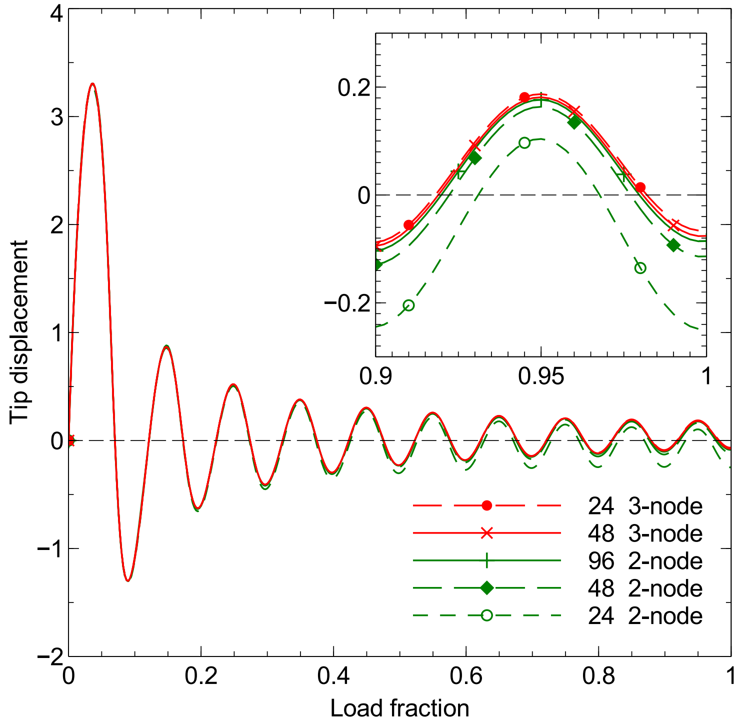

Figure 5. Helical beam: tip displacement in the direction of the applied force.

9. Closing remarks

When tackling nonlinear problems of computational solid mechanics involving rotation fields, sooner or later a parameterization of the rotation tensor is introduced to allow a numerical solution. Thus, we may discern two subsequent stages: a first stage where an intrinsic, parameterization-free formulation is

Load fraction	x	y	z
0.0	10	0	0
0.1	0.413441	52.67×10^{-3}	-1.078310
0.2	0.115254	8.381×10^{-3}	-0.600272
0.3	0.052562	2.606×10^{-3}	-0.402270
0.4	0.029900	1.122×10^{-3}	-0.296987
0.5	0.018846	589.0×10^{-6}	-0.230692
0.6	0.013505	344.8×10^{-6}	-0.185107
0.7	0.009904	219.0×10^{-6}	-0.151404
0.8	0.007593	147.4×10^{-6}	-0.124996
0.9	0.005902	101.7×10^{-6}	-0.103637
1.0	0.005318	78.40×10^{-6}	-0.085524

Table 4. Helical beam: tip coordinates of the beam model with ninety six two-node elements.

developed, hopefully in a consistent way with the properties of the special orthogonal group the rotations belong to; and a second, parameterization-driven stage. In the sample beam problem we consider in this paper, an effort is made to keep intrinsic the formulation for as long as possible and to introduce a parameterization as late as possible.

After completing a consistent formulation and linearization of the beam variational mechanics, a consistent nonlinear, but properly linearized interpolating model provides an objective finite element approximation. Differential rotation vectors up to third order are used along these steps and lead to an approximate linearized virtual functional that includes the term $\boldsymbol{\varphi}_{J\partial\delta} \cdot \boldsymbol{F}^{e}_{\boldsymbol{\varphi}J}$, which is unusual for the finite element method. All these steps are performed without resorting to any parameterization, so the outlined formulation is consistent and intrinsic. The parameterization of the rotation tensor is resorted to only twice in the discrete approximation: when linearizing the algorithm that interpolates the section orientation among the nodal orientations, and when solving the nodal mixed multipliers $\boldsymbol{\varphi}_{J\partial\delta}$ while assembling the beam elements.

The proposed differentiations of the rotation tensor are applied here to the one-dimensional case of slender beams just as an example. However, they are also valid for other continuum mechanics problems, and proved to be a valuable tool for shells and solids, where two and three-dimensional gradients are involved.

Appendix: Tensor reference guide

A short reference guide, in the form of a bare collection of tensor notations and rules, is appended here to help in following the mathematical developments in the paper. Hereafter, $\boldsymbol{a}$ denotes a vector, $\boldsymbol{A}$ a second-order tensor, $\mathscr{A}$ a third-order tensor, and $\mathbb{A}$ a fourth-order tensor. A three-dimensional space is understood, with $\boldsymbol{g}_j$ and $\boldsymbol{g}^j$ $(j = 1, 2, 3)$ reciprocal triads of base vectors. Repeated covariant and contravariant indexes entail summation from 1 to 3 (Einsteinian rule).

Polyadic representation. For example:

$$\mathbb{A} = \mathbb{A}_l \otimes \boldsymbol{g}^l = \mathbb{A}_{kl} \otimes \boldsymbol{g}^k \otimes \boldsymbol{g}^l = \mathbb{A}_{jkl} \otimes \boldsymbol{g}^j \otimes \boldsymbol{g}^k \otimes \boldsymbol{g}^l = \mathbb{A}_{ijkl}\boldsymbol{g}^i \otimes \boldsymbol{g}^j \otimes \boldsymbol{g}^k \otimes \boldsymbol{g}^l,$$

where components $\mathbb{A}_l$, $\mathbb{A}_{kl}$, $\mathbb{A}_{jkl}$, and $\mathbb{A}_{ijkl}$ are third-order tensors, second-order tensors, vectors, and scalars, respectively.

Dot operators. Multiple dots saturate the neighboring polyadic legs in order left-to-right. For example:

$$\mathbb{A} \vdots \mathscr{B} = \mathbb{A}_{jkl} \otimes \boldsymbol{g}^j \otimes \boldsymbol{g}^k \otimes \boldsymbol{g}^l \vdots \mathscr{B}^{pqr}\boldsymbol{g}_p \otimes \boldsymbol{g}_q \otimes \boldsymbol{g}_r$$

produces vector $\mathbb{A}_{jkl}\mathscr{B}^{jkl}$. A single dot is often understood.

Transpose tensors. For example:

$$\boldsymbol{A} = \boldsymbol{A}_{ij}\boldsymbol{g}^i \otimes \boldsymbol{g}^j, \qquad \boldsymbol{A}^{\mathrm{T}} = \boldsymbol{A}_{ij}\boldsymbol{g}^j \otimes \boldsymbol{g}^i;$$

$$\mathscr{A} = \mathscr{A}_{ijk}\boldsymbol{g}^i \otimes \boldsymbol{g}^j \otimes \boldsymbol{g}^k = \mathscr{A}^{\mathrm{T312}^{\mathrm{T231}}} = \mathscr{A}^{\mathrm{T231}^{\mathrm{T312}}},$$

$$\mathscr{A}^{\mathrm{T231}} = \mathscr{A}_{ijk}\boldsymbol{g}^j \otimes \boldsymbol{g}^k \otimes \boldsymbol{g}^i = \mathscr{A}^{\mathrm{T312}^{\mathrm{T312}}}, \qquad \mathscr{A}^{\mathrm{T312}} = \mathscr{A}_{ijk}\boldsymbol{g}^k \otimes \boldsymbol{g}^i \otimes \boldsymbol{g}^j = \mathscr{A}^{\mathrm{T231}^{\mathrm{T231}}},$$

$$\mathscr{A}^{\mathrm{T132}} = \mathscr{A}_{ijk}\boldsymbol{g}^i \otimes \boldsymbol{g}^k \otimes \boldsymbol{g}^j, \qquad \mathscr{A}^{\mathrm{T321}} = \mathscr{A}_{ijk}\boldsymbol{g}^k \otimes \boldsymbol{g}^j \otimes \boldsymbol{g}^i, \qquad \mathscr{A}^{\mathrm{T213}} = \mathscr{A}_{ijk}\boldsymbol{g}^j \otimes \boldsymbol{g}^i \otimes \boldsymbol{g}^k.$$

Symmetric tensors. Symmetric second-order tensor:

$$\boldsymbol{A}^{\underline{12}} \quad \text{if} \quad \boldsymbol{A} = \boldsymbol{A}^{\mathrm{T}}.$$

Simple-symmetric third-order tensors:

$$\mathscr{A}^{1\underline{23}} \quad \text{if} \quad \mathscr{A} = \mathscr{A}^{\mathrm{T}132}, \qquad \mathscr{A}^{\underline{1}2\underline{3}} \quad \text{if} \quad \mathscr{A} = \mathscr{A}^{\mathrm{T}321}, \qquad \mathscr{A}^{\underline{12}3} \quad \text{if} \quad \mathscr{A} = \mathscr{A}^{\mathrm{T}213};$$

double-symmetric third-order tensor:

$$\mathscr{A}^{\underline{123}} \quad \text{if} \quad \mathscr{A} = \mathscr{A}^{\mathrm{T}231} = \mathscr{A}^{\mathrm{T}312};$$

and full-symmetric third-order tensor:

$$\mathscr{A}^{\underline{\underline{123}}} \quad \text{if} \quad \mathscr{A} = \mathscr{A}^{\mathrm{T}231} = \mathscr{A}^{\mathrm{T}312} = \mathscr{A}^{\mathrm{T}132} = \mathscr{A}^{\mathrm{T}321} = \mathscr{A}^{\mathrm{T}213}.$$

Such symmetries are used as well for the rightmost polyadic legs of fourth-order tensors.

Tensor additive decompositions. Second-order tensor decomposition with a symmetric part:

$$\boldsymbol{A} = \boldsymbol{A}^{\mathrm{S}} + \boldsymbol{A}^{\mathrm{A}}, \qquad \boldsymbol{A}^{\mathrm{S}} = \tfrac{1}{2}(\boldsymbol{A} + \boldsymbol{A}^{\mathrm{T}}), \qquad \boldsymbol{A}^{\mathrm{A}} = \tfrac{1}{2}(\boldsymbol{A} - \boldsymbol{A}^{\mathrm{T}}) = \boldsymbol{a}\times, \tag{A.1}$$

where $\boldsymbol{a} = \operatorname{ax} \boldsymbol{A} = \frac{1}{2}\boldsymbol{g}^j \times \boldsymbol{A}\boldsymbol{g}_j = \frac{1}{2}\boldsymbol{I}^{\times} : \boldsymbol{A}$ is the axial vector of $\boldsymbol{A}$. Third-order tensor decompositions with simple-symmetric parts:

$$\begin{aligned}
&\mathscr{A} = \mathscr{A}^{\mathrm{S}1\underline{23}} + \mathscr{A}^{\mathrm{A}1\underline{23}}, &&\mathscr{A}^{\mathrm{S}1\underline{23}} = \tfrac{1}{2}(\mathscr{A} + \mathscr{A}^{\mathrm{T}132}), &&\mathscr{A}^{\mathrm{A}1\underline{23}} = \tfrac{1}{2}(\mathscr{A} - \mathscr{A}^{\mathrm{T}132}),\\
&\mathscr{A} = \mathscr{A}^{\mathrm{S}\underline{1}2\underline{3}} + \mathscr{A}^{\mathrm{A}\underline{1}2\underline{3}}, &&\mathscr{A}^{\mathrm{S}\underline{1}2\underline{3}} = \tfrac{1}{2}(\mathscr{A} + \mathscr{A}^{\mathrm{T}321}), &&\mathscr{A}^{\mathrm{A}\underline{1}2\underline{3}} = \tfrac{1}{2}(\mathscr{A} - \mathscr{A}^{\mathrm{T}321}),\\
&\mathscr{A} = \mathscr{A}^{\mathrm{S}\underline{12}3} + \mathscr{A}^{\mathrm{A}\underline{12}3}, &&\mathscr{A}^{\mathrm{S}\underline{12}3} = \tfrac{1}{2}(\mathscr{A} + \mathscr{A}^{\mathrm{T}213}), &&\mathscr{A}^{\mathrm{A}\underline{12}3} = \tfrac{1}{2}(\mathscr{A} - \mathscr{A}^{\mathrm{T}213});
\end{aligned} \tag{A.2}$$

third-order tensor decomposition with a double-symmetric part:

$$\mathscr{A} = \mathscr{A}^{\mathrm{S}\underline{123}} + \mathscr{A}^{\mathrm{A}\underline{123}}, \qquad \begin{cases} \mathscr{A}^{\mathrm{S}\underline{123}} = \tfrac{1}{3}(\mathscr{A} + \mathscr{A}^{\mathrm{T}231} + \mathscr{A}^{\mathrm{T}312}),\\ \mathscr{A}^{\mathrm{A}\underline{123}} = \tfrac{1}{3}(2\mathscr{A} - \mathscr{A}^{\mathrm{T}231} - \mathscr{A}^{\mathrm{T}312}); \end{cases} \tag{A.3}$$

and third-order tensor decomposition with a full-symmetric part:

$$\mathscr{A} = \mathscr{A}^{\mathrm{S}\underline{\underline{123}}} + \mathscr{A}^{\mathrm{A}\underline{\underline{123}}}, \qquad \begin{cases} \mathscr{A}^{\mathrm{S}\underline{\underline{123}}} = \tfrac{1}{6}(\mathscr{A} + \mathscr{A}^{\mathrm{T}231} + \mathscr{A}^{\mathrm{T}312} + \mathscr{A}^{\mathrm{T}132} + \mathscr{A}^{\mathrm{T}321} + \mathscr{A}^{\mathrm{T}213}),\\ \mathscr{A}^{\mathrm{A}\underline{\underline{123}}} = \tfrac{1}{6}(5\mathscr{A} - \mathscr{A}^{\mathrm{T}231} - \mathscr{A}^{\mathrm{T}312} - \mathscr{A}^{\mathrm{T}132} - \mathscr{A}^{\mathrm{T}321} - \mathscr{A}^{\mathrm{T}213}). \end{cases} \tag{A.4}$$

Such decompositions are used as well for the rightmost polyadic legs of fourth-order tensors. For example:

$$\mathbb{A} = \mathbb{A}^{\mathrm{S}1\underline{\underline{234}}} + \mathbb{A}^{\mathrm{A}1\underline{\underline{234}}}, \qquad \begin{cases} \mathbb{A}^{\mathrm{S}1\underline{\underline{234}}} = \tfrac{1}{6}(\mathbb{A} + \mathbb{A}^{\mathrm{T}1342} + \mathbb{A}^{\mathrm{T}1423} + \mathbb{A}^{\mathrm{T}1243} + \mathbb{A}^{\mathrm{T}1432} + \mathbb{A}^{\mathrm{T}1324}),\\ \mathbb{A}^{\mathrm{A}1\underline{\underline{234}}} = \tfrac{1}{6}(5\mathbb{A} - \mathbb{A}^{\mathrm{T}1342} - \mathbb{A}^{\mathrm{T}1423} - \mathbb{A}^{\mathrm{T}1243} - \mathbb{A}^{\mathrm{T}1432} - \mathbb{A}^{\mathrm{T}1324}). \end{cases} \tag{A.5}$$

Unitary tensors. Second-order tensor identity:

$$\boldsymbol{I} = \boldsymbol{g}_j \otimes \boldsymbol{g}^j.$$

Third-order Ricci's tensor:

$$\boldsymbol{I}^\times = \boldsymbol{g}_j \times \otimes \boldsymbol{g}^j = \boldsymbol{I}^{\times\mathrm{T}312} = \boldsymbol{g}_j \otimes \boldsymbol{g}^j \times = \boldsymbol{I}^{\times\mathrm{T}231} = -\boldsymbol{I}^{\times\mathrm{T}213} = -\boldsymbol{I}^{\times\mathrm{T}321} = -\boldsymbol{I}^{\times\mathrm{T}132}. \tag{A.6}$$

Fourth-order unitary tensors:

$$\begin{aligned} \mathbb{I} &= \boldsymbol{g}_j \otimes \boldsymbol{g}_k \otimes \boldsymbol{g}^j \otimes \boldsymbol{g}^k = \breve{\mathbb{I}}^{\mathrm{T}1243} = (\boldsymbol{I} \otimes \boldsymbol{I})^{\mathrm{T}1324}, \\ \breve{\mathbb{I}} &= \boldsymbol{g}_j \otimes \boldsymbol{g}_k \otimes \boldsymbol{g}^k \otimes \boldsymbol{g}^j = \mathbb{I}^{\mathrm{T}1243} = (\boldsymbol{I} \otimes \boldsymbol{I})^{\mathrm{T}1342}, \\ \boldsymbol{I} \otimes \boldsymbol{I} &= \boldsymbol{g}_j \otimes \boldsymbol{g}^j \otimes \boldsymbol{g}_k \otimes \boldsymbol{g}^k = \mathbb{I}^{\mathrm{T}1324} = \breve{\mathbb{I}}^{\mathrm{T}1423}, \end{aligned}$$

where, according to the standards introduced by Del Piero [1979], the fourth-order identity also writes $\mathbb{I} = \boldsymbol{I} \boxtimes \boldsymbol{I}$, and $\breve{\mathbb{I}}$ is the transposer $\mathbb{T}$ (the tensor product $\boxtimes$ between second-order tensors converts, in our notation, as $\boldsymbol{A} \boxtimes \boldsymbol{B} = (\boldsymbol{A} \otimes \boldsymbol{B})^{\mathrm{T}1324}$); other useful fourth-order unitary tensors are the symmetrizer $\mathbb{S} = \frac{1}{2}(\mathbb{I} + \breve{\mathbb{I}})$ and the skew-symmetrizer

$$\mathbb{W} = \tfrac{1}{2}(\mathbb{I} - \breve{\mathbb{I}}) = \tfrac{1}{2}\boldsymbol{I}^\times \boldsymbol{I}^\times = \tfrac{1}{2}\boldsymbol{g}_j \times \otimes \boldsymbol{g}^j \times .$$

Some useful tensor identities.

$$\boldsymbol{a}\times = \boldsymbol{I}^\times \boldsymbol{a} = \boldsymbol{a} \cdot \boldsymbol{I}^\times, \tag{A.7}$$

$$\boldsymbol{a} \times \boldsymbol{b} = \boldsymbol{I}^\times \boldsymbol{a} \cdot \boldsymbol{b} = \boldsymbol{a} \cdot \boldsymbol{I}^\times \boldsymbol{b} = \boldsymbol{I}^\times : \boldsymbol{b} \otimes \boldsymbol{a}, \tag{A.8}$$

$$\boldsymbol{a} \times \boldsymbol{b} \times = \boldsymbol{b} \otimes \boldsymbol{a} - \boldsymbol{a} \cdot \boldsymbol{b} \otimes \boldsymbol{I}, \tag{A.9}$$

$$\boldsymbol{a}\times : \boldsymbol{b}\times = 2\boldsymbol{a} \cdot \boldsymbol{b}, \tag{A.10}$$

$$\boldsymbol{a} \times \boldsymbol{a} \times \boldsymbol{b} \times \boldsymbol{a} \times = \boldsymbol{a} \times \boldsymbol{b} \times \boldsymbol{a} \times \boldsymbol{a} \times, \tag{A.11}$$

$$(\boldsymbol{a} \times \boldsymbol{b})\times = \boldsymbol{I}^\times \boldsymbol{I}^\times : \boldsymbol{b} \otimes \boldsymbol{a} = \boldsymbol{b} \otimes \boldsymbol{a} - \boldsymbol{a} \otimes \boldsymbol{b} = \boldsymbol{a} \times \boldsymbol{b} \times - \boldsymbol{b} \times \boldsymbol{a} \times, \tag{A.12}$$

$$\boldsymbol{I}^\times \boldsymbol{a} \times - \boldsymbol{a} \times \boldsymbol{I}^\times = \boldsymbol{I} \otimes \boldsymbol{a} - \boldsymbol{a} \otimes \boldsymbol{I}, \tag{A.13}$$

$$\boldsymbol{a} \times \boldsymbol{a} \times \boldsymbol{I}^\times \boldsymbol{a} \times = \boldsymbol{a} \times \boldsymbol{I}^\times \boldsymbol{a} \times \boldsymbol{a} \times, \tag{A.14}$$

$$\boldsymbol{a} \times \boldsymbol{c} \times \boldsymbol{b} \times + \boldsymbol{b} \times \boldsymbol{c} \times \boldsymbol{a} \times = -((\boldsymbol{a} \otimes \boldsymbol{b} + \boldsymbol{b} \otimes \boldsymbol{a}) \cdot \boldsymbol{c})\times, \tag{A.15}$$

$$\boldsymbol{I}^\times \boldsymbol{A} = (\boldsymbol{A}^\mathrm{T} \boldsymbol{I}^\times)^{\mathrm{T}231}, \qquad \boldsymbol{A}^\mathrm{T} \boldsymbol{I}^\times = (\boldsymbol{I}^\times \boldsymbol{A})^{\mathrm{T}312}, \tag{A.16}$$

$$((\boldsymbol{I}^\times \boldsymbol{A})^{\mathrm{T}132} \boldsymbol{A})^{\mathrm{T}132} = -(\boldsymbol{I}^\times \boldsymbol{A})^{\mathrm{T}132} \boldsymbol{A}. \tag{A.17}$$

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