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# FLEXURAL BEHAVIOR OF FUNCTIONALLY GRADED SLENDER BEAMS WITH COMPLEX CROSS-SECTION

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Deflection and stress analyses of functionally graded beams with complex cross-section and general material variation, under transverse loading, were carried out. The elastic-fundamental solution is used to derive equations satisfied by the normal stresses in arbitrary cross-sections of the beam, assuming that the plane sections remain plane and normal to the beam axis. The technique was verified by existing analytical and finite element models. Numerical experiments were then performed where the material properties vary through thickness or width of the beams according to power-law and exponential gradations. It was found that the quality of material gradation affects the deflection, stresses and neutral axis position significantly. It is concluded that the technique is useful for the elastic behavior analysis of FGBs with complex cross-sections and various material gradations.

## 1. Introduction

Over the past decades, composite materials with asymmetric material variation, such as asymmetric smart composites [Sharifishourabi et al. 2014a] and functionally graded materials (FGMs) have received the attention of both theoretical and experimental researchers. FGM is a class of material similar to an advanced composite that has a heterogeneous structure in which the constituent varies smoothly, gradually, and continuously from one surface to another. This gradual variation results also in a gradual change in the mechanical and thermal properties [Suresh and Mortensen 1998]. FGMs have the best properties of both ceramics, such as low density, high strength, high stiffness, and temperature resistance, and of metals, such as toughness, electrical conductivity, and machinability. Due to these outstanding properties, FGMs have attracted much attention in industries in many engineering fields such as aerospace, automotive, and the biomedical fields [Miyamoto et al. 1999]. Over the last decades, along with rapid growth in the use of FGMs, different methods have also been developed for analyzing their mechanical behavior [Menaa et al. 2012; Shahba et al. 2013; Ke et al. 2009].

Beams, as the most common engineering structures, are traditionally used as an example. The first exact elasticity solution for a functionally graded beam (FGB) subjected to transverse loads was developed by Sankar [2001]. He assumed that the Poisson's ratio is constant and the elastic modulus of the FGB varies exponentially across the thickness. He also developed the simple Euler–Bernoulli beam theory for FGBs under transverse loads, which is only applicable for long and slender beams with depthwise and exponential variation of materials. Then, Sankar and Tzeng [2002] obtained an exact elasticity solution by solving the thermoelastic governing equations for FGBs subjected to thermal loads. They

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showed that when the variation of the material properties was opposite to the distribution of temperature, the residual stresses due to the thermal loading were reduced. Chakraborty et al. [2003], based on the theory of first-order shear deformation, developed a new beam finite element for analyzing the thermal and mechanical behavior of FGBs. They solved static, wave propagation, and free vibration problems considering both exponential and power-law variations of the mechanical and thermal properties. An elastic solution for sandwich beams having an FGM core with exponential variation was obtained by Venkataraman and Sankar [2003]. They employed the Euler–Bernoulli beam theory for modeling the face sheets, and plane elasticity equations for analyzing the core. Numerical solutions based on the meshless local Petrov–Galerkin method (MLPG) for two-dimensional FG elastic solids subjected to thermal and mechanical loads were obtained by Ching and Yen [2005]. They also obtained transient thermoelastic deformations for two-dimensional FGBs subjected to a nonuniform heat supply [Ching and Yen 2006]. Effect of material gradation on thermomechanical stresses in functionally graded beams was studied by Sharifishourabi et al. [2012]. They also developed a tensile testing machine for FG specimens [Sharifishourabi et al. 2014b]. Lü et al. [2006], by employing the state space method, presented a two-dimensional solution for the thermoelastic analysis of thick FGBs. Ding et al. [2007] presented an elasticity solution for plane anisotropic FGBs. They assumed that the material variation was according to an arbitrary function of the thickness direction. Kadoli et al. [2008] studied the static stresses and deflection of FGBs under ambient temperature using higher-order shear deformation beam theory. Free vibration analysis of FGBs was also studied in depth, and several solutions have been presented [Aydogdu and Taskin 2007; Sina et al. 2009; Wattanasakulpong et al. 2012]. Ying et al. [2008] studied an FGB with exponential material variation resting on an elastic foundation. They presented exact solutions based on the two-dimensional theory of elasticity for the free vibration and bending of orthotropic FGBs. Zhong and Yu [2007] developed a two-dimensional analytical solution by using the Airy stress function method for a cantilever FGB with arbitrary variations of material under various loads. Li [2008] introduced a new unified method for the static and dynamic analysis of Euler–Bernoulli and Timoshenko FGBs with shear deformation and rotary inertia. An analytical approach for the free vibration response of FGBs in the case of temperature dependence with arbitrary boundary conditions has been introduced by Mahi et al. [2010]. They assumed that the material properties are temperature-dependent and vary according to the exponential or power-law forms along the thickness of the beam. Hamed [2012] and Piovan et al. [2012] studied the buckling response of FGBs. Numerical and analytical approaches were presented for deflections of FGBs subjected to inclined and transverse loading [Rahimi and Davoudinik 2010; Farhatnia et al. 2009].

A free vibration analysis of functionally graded spatial curved beams on the basis of first-order shear deformation theory was carried out by Yousefi and Rastgoo [2011]. The nonlinear forced vibration analysis of clamped FGBs was also studied by Shooshtari and Rafiee [2011]. Yaghoobi and Feridoon [2010] investigated the effect of neutral surface location on the deflection of FGBs subjected to a uniform distributed loading. Thai and Vo [2012] presented analytical solutions for the bending and free vibration of FGBs using Hamilton's principle and other higher-order shear deformation beam theories. An experimental work to validate a model based on third-order zigzag theory for the bending and free vibration response of layered FGBs was carried out by Kapuria et al. [2008]. They used the modified rule of mixtures to obtain the effective Young's modulus. Apetre et al. [2008] investigated several existing theories for sandwich beams to determine their appropriateness for sandwich plates with a functionally graded core.


**Figure 1.** Geometry and coordinates of an FGB.

They found good agreement between the results of the finite element method, the higher-order theory, and the Fourier–Galerkin method. The bending of cantilever FGBs subjected to an end force using small and large deformation theories was investigated by Kang and Li [2009]. They investigated the influence of a nonlinearity parameter and Young's modulus on the rotations and deflections. A free and forced vibration analysis for FGBs under a concentrated moving harmonic loading by employing Lagrange's equations and the Euler–Bernoulli beam theory was carried out by Şimşek and Kocatürk [2009]. A free vibration and stability analysis of tapered FGBs with axial gradation of material, based on the Euler–Bernoulli beam theory, was studied by Shahba and Rajasekaran [2012]. A mechanical behavior analysis of FGBs employing the theory of directed curves was carried out by Bîrsan et al. [2012]. They presented a general analytical solution using the effective stiffness properties for beams with arbitrary cross-sectional shape.

Although several analytical solutions are available, the majority of these solutions involve cumbersome calculations to apply them to complex geometries. On the other hand, previous studies only focused on FGBs with material gradation along the thickness direction, while there are many applications of FGBs in which the material properties vary through the width of the beam. This study attempts to use a technique simpler than the currently available ones. For simplicity, the method is compared with applicable models for static analysis of FGBs with complex cross-section and general material gradation along either the thickness or width direction of the beam.

## 2. Problem formulation and solution

Figure 1 shows the geometry and coordinate system of a FGB. The length, width, and thickness of the beam are $L$, $b$, and $h$, respectively. The coordinate system originates at the corner of the cross section of the beam. The material properties vary continuously and gradually across the thickness or width according to arbitrary functions. Two examples of possible material gradation for FGBs are shown in Figure 2. Since the power-law and exponential law are the two most common models, here these material variations will also be considered. The power-law modeling which is introduced by Wakashima et al.

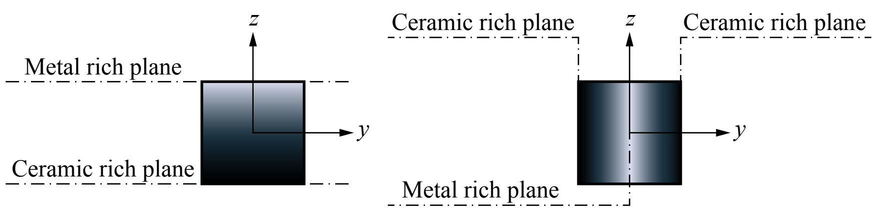



**Figure 2.** Two examples of possible material gradations for FGBs.

[1990] is given by

$$p(z) = (p_m - p_c)\left(\frac{z}{h} + \frac{1}{2}\right)^n + p_c. \tag{1}$$

The exponential law, which is more favorable, is given by

$$p(z) = p_m \exp(-\delta(1 - 2z/h)), \quad \delta = \tfrac{1}{2} \log \frac{p_m}{p_c}. \tag{2}$$

Since this study attempts to use a simple and applied technique for FGBs with complicated geometry and material variation, these basic assumptions were made:

(1) The classical Euler–Bernoulli beam theory was applied.

(2) The Poisson's ratio was held constant.

(3) The normal stresses $\sigma_{zz}$ were assumed to be negligible.

The classical strain-stress relations for a homogenous beam are given by

$$\sigma_x = E\epsilon_x, \tag{3}$$

$$\tau_{xz} = G\gamma_{xz}. \tag{4}$$

The normal strain $\epsilon_x$, based on the assumptions, takes the form

$$\epsilon_x = \epsilon_{x_0} + z\kappa, \tag{5}$$

where $\epsilon_{x_0}$, $\kappa$, and $z$ are the middle plane strain, the curvature, and the distance from the neutral axis of the beam. The axial force ($N$), bending moment ($M$), and shear force ($V$) resultants, based on the classical beam theory, are

$$N = \int_0^h \sigma_x \, dA, \tag{6}$$

$$M = \int_0^h z\sigma_x \, dA, \tag{7}$$

$$V = \int_0^h \tau_{xz} \, dA. \tag{8}$$

Since no assumption was made regarding the material of the beam in deriving equations (5)–(8), they are still valid for FGBs. While (8) is typically neglected due to its insignificant value, (3) and (4) for FGBs become the following equations, given by Sankar [2001]:

$$\sigma_x = E(z)\epsilon_x, \tag{9}$$

$$\tau_{xz} = G(z)\gamma_{xz}. \tag{10}$$

The axial force ($N$) and bending moment ($M$) resultants for a discretized beam cross-section can be derived, based on the classical beam theory, as

$$N = \sum_{i=1}^{m} \int_{\frac{h}{m}(i-1)}^{\frac{h}{m}i} \sigma_i \, dA, \tag{11}$$

$$M = \sum_{i=1}^{m} \int_{\frac{h}{m}(i-1)}^{\frac{h}{m}i} z\sigma_i \, dA. \tag{12}$$

In these equations, $m$ indicates the total numbers of sublayers (see Figure 3). By substituting (9) in (11) and (12), the resultant force and moment expressions under bending are

$$N = \sum_{i=1}^{m} \int_{\frac{h}{m}(i-1)}^{\frac{h}{m}i} E_i b_i \epsilon_x \, dz = 0, \tag{13}$$

$$M = \sum_{i=1}^{m} \int_{\frac{h}{m}(i-1)}^{\frac{h}{m}i} z E_i b_i \epsilon_x \, dz, \tag{14}$$

where $E_i$ and $b_i$ denote the values of the Young's modulus and the width in the $i$-th sublayer. By substituting (5) in (13) and (14), we get the system of equations

$$\begin{cases} \displaystyle\sum_{i=1}^{m} \int_{\frac{h}{m}(i-1)}^{\frac{h}{m}i} E_i b_i (\epsilon_{x_0} + z\kappa) \, dz = 0, \\ \displaystyle\sum_{i=1}^{m} \int_{\frac{h}{m}(i-1)}^{\frac{h}{m}i} z E_i b_i (\epsilon_{x_0} + z\kappa) \, dz = M. \end{cases} \tag{15}$$

This system of equations can be written in the short form

$$\begin{cases} \tilde{A}\epsilon_{x_0} + \tilde{Q}\kappa = 0, \\ \tilde{Q}\epsilon_{x_0} + \tilde{I}\kappa = M. \end{cases} \tag{16}$$

Using definitions (17)–(19), the values of $\epsilon_{x_0}$ and $\kappa$ can be obtained as

$$\tilde{I} = \sum_{i=1}^{m} \int_{\frac{h}{m}(i-1)}^{\frac{h}{m}i} z^2 E_i b_i \, dz, \tag{17}$$

$$\tilde{A} = \sum_{i=1}^{m} \int_{\frac{h}{m}(i-1)}^{\frac{h}{m}i} E_i b_i \, dz, \tag{18}$$

$$\tilde{Q} = \sum_{i=1}^{m} \int_{\frac{h}{m}(i-1)}^{\frac{h}{m}i} z E_i b_i \, dz, \tag{19}$$

$$\epsilon_{x_0} = \frac{-\tilde{Q}M}{(-\tilde{Q}^2 + \tilde{A}\tilde{I})}, \tag{20}$$

$$\kappa = \frac{\tilde{A}M}{(-\tilde{Q}^2 + \tilde{A}\tilde{I})}. \tag{21}$$

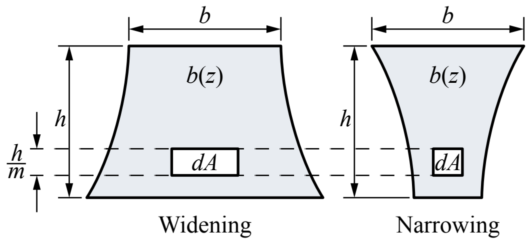



**Figure 3.** Schematic of discretized graded beams.

Furthermore, one can see that the position of the neutral axis is located at

$$a = \frac{\tilde{Q}}{\tilde{A}}. \tag{22}$$

Then, substituting (20), (21) and (5) into (9), the depthwise axial stresses in a discretized graded beam subjected to pure bending can be obtained as

$$\sigma_x(x, z) = E(z)(\epsilon_{x_0} + z\kappa) = \frac{M(x)(z\tilde{A} - \tilde{Q})}{-\tilde{Q}^2 + \tilde{A}\tilde{I}} E(z). \tag{23}$$

Shear stress in the FGB can be easily obtained from the famous differential equation of equilibrium as

$$\tau_{xz}(x, z) = \int_0^z \frac{\partial \sigma_x(x, z)}{\partial x} dz. \tag{24}$$

Substituting (3) and (5) into (7) leads to

$$\kappa = \frac{M(x)}{D} = \frac{d^2 w}{dx^2}. \tag{25}$$

The bending rigidity $D$ can be obtained as

$$D = E_h(\tilde{I} - \tilde{A}\tilde{z}^2), \tag{26}$$

where $E_h$ is the Young's modulus of the surface with higher modulus. By integrating both sides of (25) with respect to $x$ and applying the loads and boundary conditions, the deflections along the length of the FGB ($w(x)$) will be obtained. The boundary conditions for a simply supported beam are

$$w(0) = 0, \quad w(L) = 0. \tag{27}$$

The boundary conditions for a cantilever beam are

$$w(0) = 0, \quad \frac{dw}{dx}(0) = 0. \tag{28}$$

## 3. Results and discussion

After validating the technique, numerical solutions are applied using the above equations for static analysis of FGBs with complex cross-section and material variation.

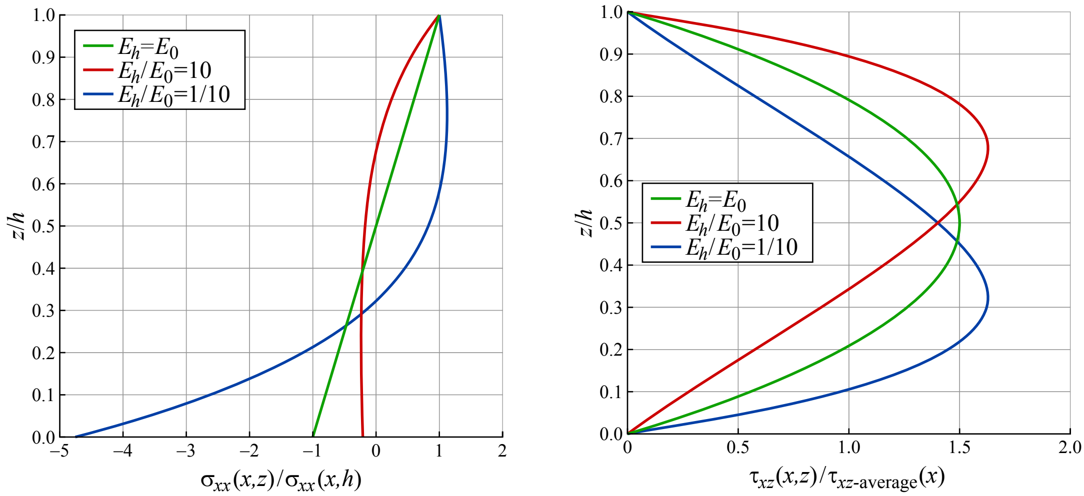
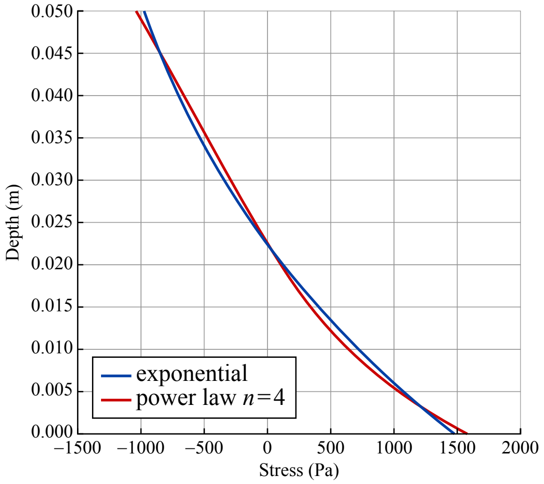



**Figure 4.** Left: distribution of depthwise normalized axial stresses $\sigma_{xx}$ for the simply supported FGB under transverse distributed loads. Right: distribution of depthwise normalized shear stresses $\tau_{xz}$ for the simply supported FGB under transverse distributed loads.

**3.1.** ***Validation.*** To evaluate the accuracy of this method, the same beam problem solved analytically by Sankar [2001] was studied again using the presented technique ($m = 100$). Figure 4 shows the depthwise normalized axial and transverse shear stress distributions for a simply supported FGB subjected to transverse distributed loads. The Young's modulus of the beam was assumed to vary exponentially along the thickness from $E_0$ at the bottom to $E_h$ at the topmost surface. The axial stresses were normalized by dividing by the corresponding stress on the top surface and the shear stresses were normalized with respect to the average shear stress at the same cross-section. Since the present solution and that of Sankar [2001] were based on the same assumptions, the results were obviously the same.

Furthermore, to find out the accuracy range of this method, the same cantilever FGB studied by Chakraborty et al. [2003] was solved again. An FGB with unit width and length of $L = 0.5$ m is subjected to a unit transverse load at the tip. Steel and alumina are considered as the topmost and bottom material of the FGB. Figure 5 shows the depthwise axial and shear stress distributions, using the presented technique, for an FGB with exponential and power-law gradation through the thickness. By comparing the results to the finite element solution based on first-order shear deformation theory developed by Chakraborty et al. [2003], it is found that for long, slender FGBs, the axial stress distributions were in excellent agreement. But since these two kinds of solutions were based on different theories, the shear stress distributions were obviously different.

The deflection of a cantilever FGB under a unit concentrated force at the tip was also studied using the present method. The results were compared with those available in the literature, as shown in Tables 1 and 2. Table 1 compares the maximum deflection obtained for various $L/h$ to the finite element method (FEM) based on higher-order shear deformation theory (HSDT) [Kadoli et al. 2008]. From Table 1 it is found that the method for FGBs with bigger values of $L/h$ is more valid, while for short beams it is not applicable. Table 2 gives the one-dimensional maximum deflection for different material gradations according to power-law modeling for $n = 0.5$, 1, and 2. Despite some differences between the present



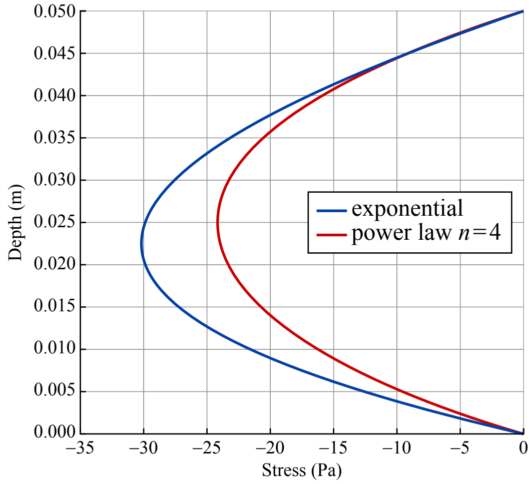




**Figure 5.** Left: depthwise axial stress distributions for the cantilever FGB subjected to unit transverse load at the tip. Right: depthwise shear stress distributions for the cantilever FGB subjected to unit transverse load at the tip.

method and the FEM [Kadoli et al. 2008], excellent agreement was found with the results from the analytical solution based on the Euler–Bernoulli beam theory discussed in [Yaghoobi and Feridoon 2010].

**3.2.** ***Numerical experiments.*** Numerical solutions to determine the deflections of FGBs composed of steel ($E = 210\,\text{GPa}$) and alumina ($E = 390\,\text{GPa}$) have been obtained. The distribution of the stresses

| $L$ | $h$ | FEM-HSDT | Present Method | % Error |
|---|---|---|---|---|
| 160 | 12 | 32.822 | 32.65 | 0.52 |
| 80 | 12 | 4.1567 | 4.081 | 1.82 |
| 12 | 12 | 0.239307 | 0.01377 | 94.24 |

**Table 1.** Comparison of maximum deflection obtained for various $L/h$. FEM-HSDT results from [Kadoli et al. 2008].

| $n$ | FEM-HSDT | Beam theory | Present method |
|---|---|---|---|
| Ceramic | 2.436 | 2.576 | 2.576 |
| 0.5 | 2.785 | 2.960 | 2.962 |
| 1.0 | 2.942 | 3.176 | 3.179 |
| 2 | 3.067 | 3.323 | 3.326 |
| Metal | 3.605 | 4 | 4 |

**Table 2.** Nondimensional maximum deflections obtained for different material gradations. FEM-HSDT results from [Kadoli et al. 2008]; beam theory results from [Yaghoobi and Feridoon 2010].

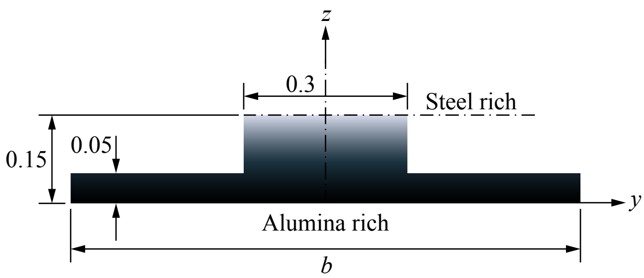



**Figure 6.** Cross-section of the FGB and quality of material gradation along the thickness.

when the Young's modulus of the beam varies according to the power-law or exponential law through the thickness or width has also been obtained.

**3.2.1.** *Depthwise varying FGB under a unit transverse distributed load.* Using the present technique, a static analysis of an alumina-steel FGB with length of $L = 5$ m under a unit transverse distributed load will now be carried out. Both exponential and power-law ($n = 1, 2, 3$) gradations of the material along the thickness will be studied. The geometry of the cross-section and the quality of the material gradation are shown in Figure 6.

The longitudinal deflection distributions for the cantilever and simply supported FGBs are shown in Figure 7. As may be seen, an increasing value of $n$ results in an increased value of the deflection. This is due to the fact that a combination of a beam with a bigger value of $n$ is closer to a combination of a homogeneous steel beam.

Figure 8 shows the depthwise axial stress distributions for a cantilever FGB at the fixed end. As may be seen, the variation of the material affects the neutral axis position, changing it from a centroid axis at

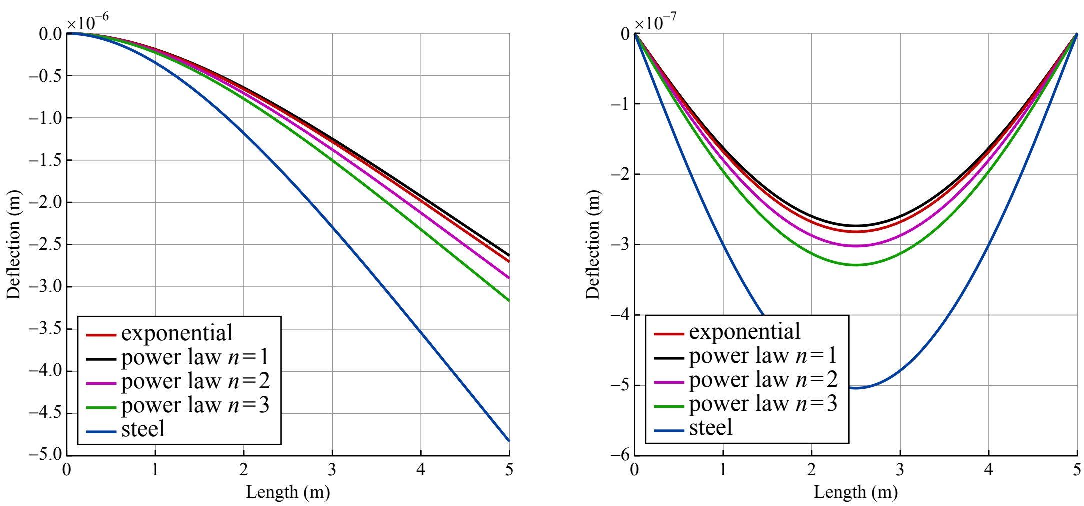



**Figure 7.** Left: deflection distributions of the cantilever FGB along the length axis. Right: deflection distributions of the simply supported FGB along the length axis.

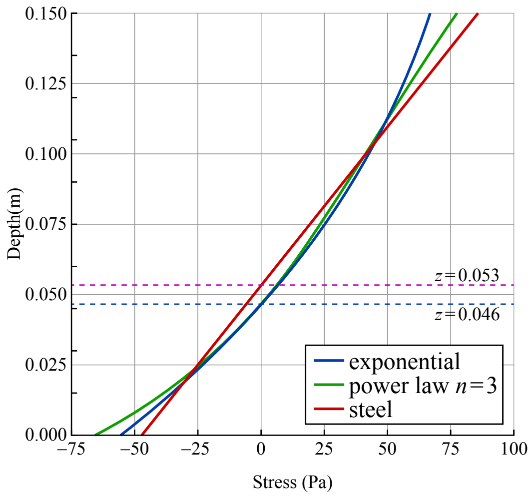



**Figure 8.** Depthwise axial stress distributions of the cantilever FG beam at the fixed end.

$z = 0.0531$ for the homogenous steel beam to $z = 0.0464$ and $z = 0.0468$ for exponential and power-law ($n = 3$) variations.

**3.2.2.** *Widthwise varying FGB under unit transverse distributed loads.* The distributions of the axial stresses and deflections are also obtained for an FGB with widthwise material variation, subjected to a unit transverse distributed load. The beam has length of $L = 5\,\text{m}$, width of $b = 0.1\,\text{m}$, and height of $h = 0.05\,\text{m}$. Two kinds of variations of materials according to a power-law modeling for $n = 2$ are considered. Schematic views of the widthwise material gradation are shown in Figure 9. The widthwise distributions of Young's modulus for ceramic-metal-ceramic (CMC) and metal-ceramic-metal (MCM) gradations are also shown in Figures 10 and 11.

For the discretized FGBs with material gradation along the width, the axial force and bending moment resultants can be written as

$$N = \sum_{i=1}^{m} \int_{\frac{-h}{2}}^{\frac{h}{2}} \sigma_i \, dA, \tag{29}$$

$$M = \sum_{i=1}^{m} \int_{\frac{-h}{2}}^{\frac{h}{2}} z\sigma_i \, dA. \tag{30}$$

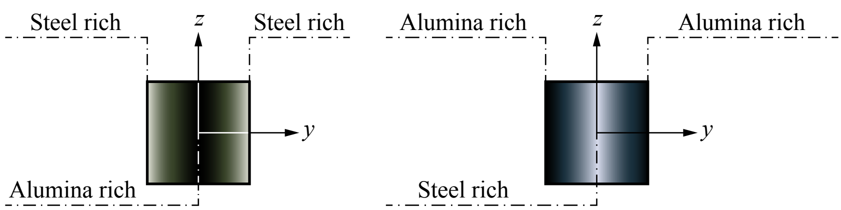



**Figure 9.** Schematic views for two kind of widthwise gradation of material.

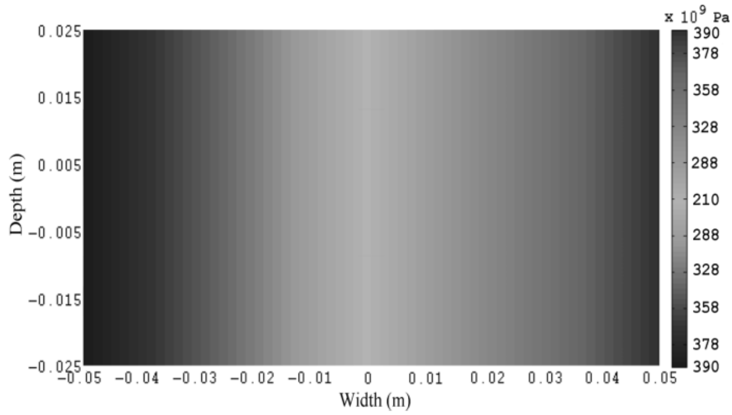



**Figure 10.** Widthwise Young's modulus distribution for ceramic-metal-ceramic gradation.

Therefore, the definitions (31)–(32) take the forms

$$\tilde{I} = \sum_{i=1}^{m} \int_{\frac{-h}{2}}^{\frac{h}{2}} z^2 E_i b_i \, dz, \tag{31}$$

$$\tilde{A} = \sum_{i=1}^{m} \int_{\frac{-h}{2}}^{\frac{h}{2}} E_i b_i \, dz, \tag{32}$$

$$\tilde{Q} = \sum_{i=1}^{m} \int_{\frac{-h}{2}}^{\frac{h}{2}} z E_i b_i \, dz. \tag{33}$$

Using definitions (31)–(33) and equations (20), (21) and (5) the distribution of axial stress at the cross-section of the FGB can be obtained as

$$\sigma_x(x, y, z) = E(y)(\epsilon_{x_0} + z\kappa) = \frac{M(x)(z\tilde{A} - \tilde{Q})}{-\tilde{Q}^2 + \tilde{A}\tilde{I}} E(y). \tag{34}$$

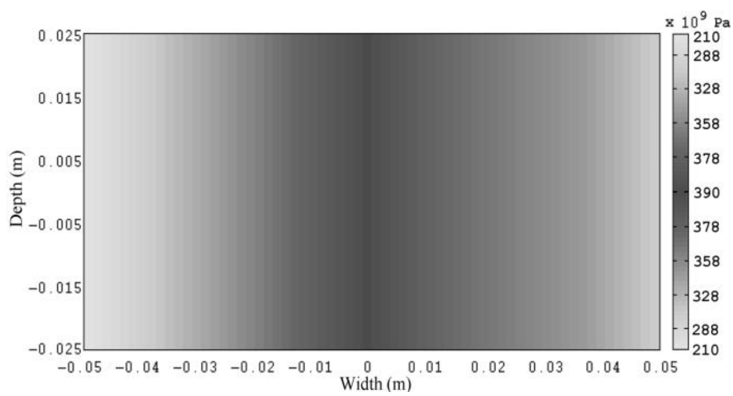



**Figure 11.** Widthwise Young's modulus distribution for metal-ceramic-metal gradation.

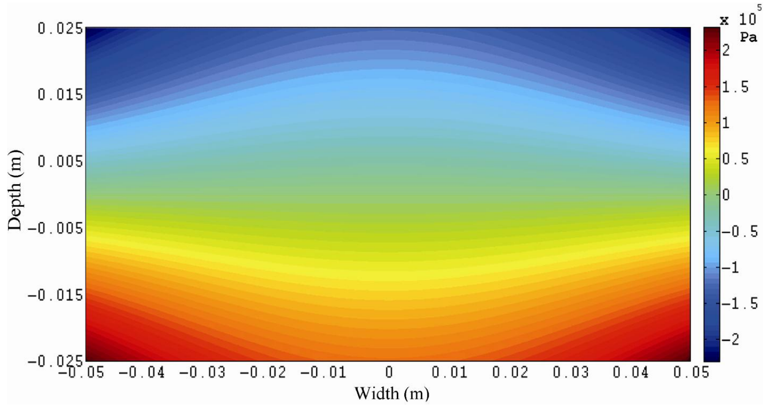



**Figure 12.** Depthwise axial stress distribution for the FGB with ceramic-metal-ceramic gradation.

The depthwise axial stress distribution at the fixed end of the cantilever FGB for the CMC and MCM gradations are shown in Figures 12 and 13. The figures reveal that the maximum stress occurs at the regions with the biggest values of Young's modulus and the maximum distance from the neutral axis.

Figures 14 and 15 show the distributions of the longitudinal deflection for a cantilever and a simply supported FGB with CMC and MCM gradations. From the figures it can be observed that the deflections for the MCM are more than those for the CMC. This is due to the fact that Young's modulus of alumina is higher than that of steel. Consequently, the bending rigidity of the beam with the CMC gradation is higher than that of MCM.

## 4. Conclusions

Stress and deflection analyses of functionally graded beams with complex cross-section and different material variations, subjected to transverse loads, were carried out using a simplified technique. The accuracy of the technique was evaluated. Numerical investigations were then performed. From the results it can be concluded that:

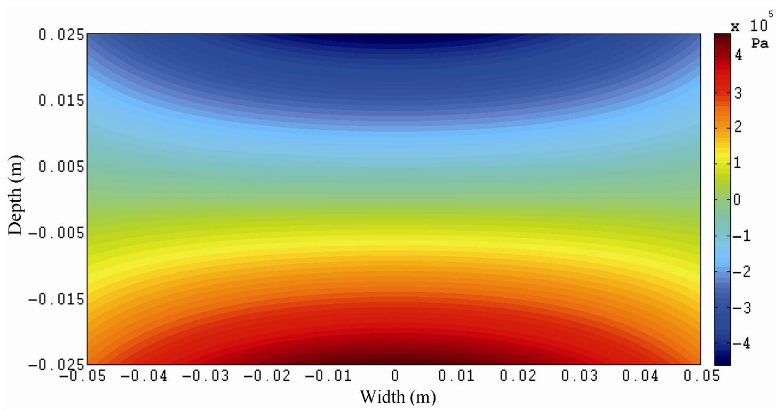



**Figure 13.** Depthwise axial stress distribution for the FGB with metal-ceramic-metal gradation.

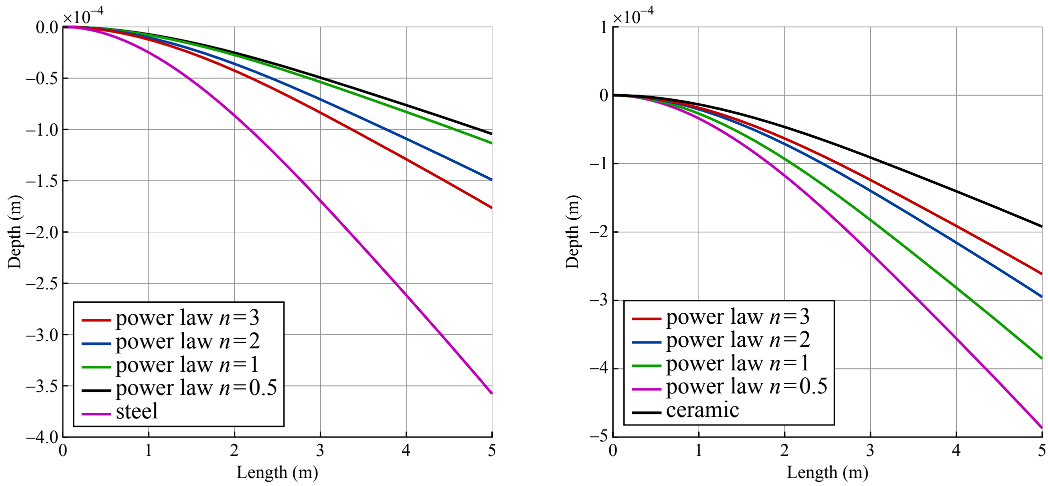



**Figure 14.** Longitudinal deflection distributions for a cantilever FGB. Left: ceramic-metal-ceramic gradation. Right: metal-ceramic-metal gradation.

(1) Quality of material gradation affects the deflection, stresses and neutral axis position significantly.

(2) The maximum axial stress occurs at the regions with the biggest values of Young's modulus and the maximum distance from the neutral axis (for ceramic-metal-ceramic gradation at vertices, and for metal-ceramic-metal at midpoints of top and bottom edges).

(3) The bending rigidity of FGBs with the ceramic-metal-ceramic gradation is higher than metal-ceramic-metal.

(4) The technique is useful for the static analysis of long, slender FGBs with complex cross-sections and various material gradations.

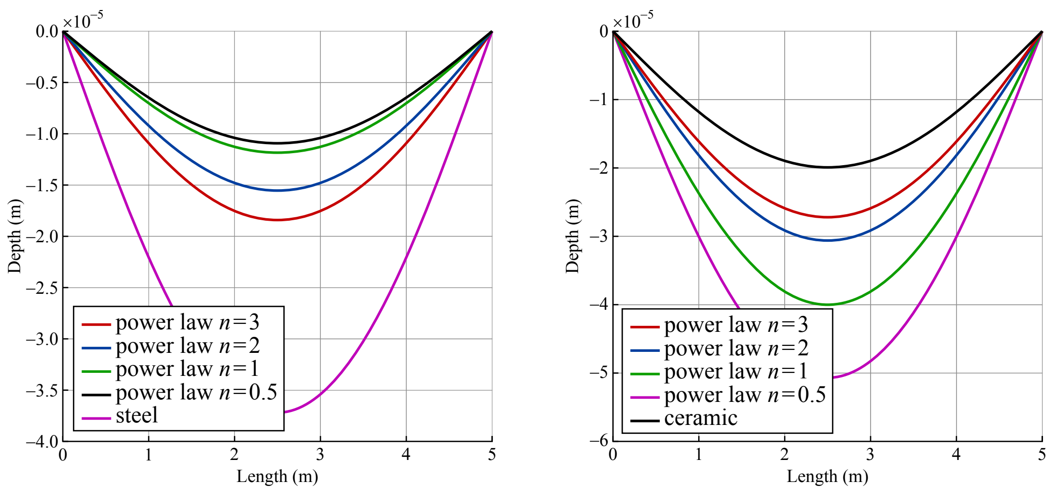



**Figure 15.** Longitudinal deflection distributions for a simply supported FGB. Left: ceramic-metal-ceramic gradation. Right: metal-ceramic-metal gradation.

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