Journal of

Mechanics of
Materials and Structures

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Volume 12, No. 1 January 2017

msp

JOURNAL OF MECHANICS OF MATERIALS AND STRUCTURES
Vol. 12, No. 1, 2017
dx.doi.org/10.2140/jomms.2017.12.93

COMPUTER SIMULATION OF THE EFFECTIVE VISCOSITY IN BRINKMAN FILTRATION EQUATION USING THE TREFFTZ METHOD

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This paper presents a determination of the effective viscosity in the Brinkman equation by a numerical simulation of an imaginary physical experiment with a viscometer. The model of a porous medium and the applied method of solution are very simple. In the idealized problem, we consider axial flow through an infinite array of cylindrical rods. Assuming that the flow in such a porous medium is described by the Brinkman filtration equation, the effective viscosity can be calculated as a function of the porosity. In this paper, a relation between the volume fraction and the effective viscosity is given for triangular and square arrays of rods.

1. Introduction

Usually for slow viscous fluid flow in porous media, the Darcy equation,

$$\boldsymbol{q} = -\frac{k}{\mu}\nabla P, \tag{1}$$

is used, where $\boldsymbol{q}$ is the macroscopic velocity [m/s], P is the pressure [Pa], μ is the viscosity of fluid [Pa·s], and k is the permeability of the porous medium [m^2].

For a porous medium with very high porosity in the presence of a free fluid region or wall-bounded porous medium (see Figure 1), sometimes the Brinkman equation is applied [Chen and Wang 2014; Cortez et al. 2010; Hill and Straughan 2009; Parvazinia et al. 2006; Tan and Pillai 2009]:

$$\nabla P = -\frac{\mu}{k}\boldsymbol{q} + \tilde{\mu}\nabla^2\boldsymbol{q}, \tag{2}$$

where $\tilde{\mu}$ is the effective viscosity of the porous medium [Pa·s].

Brinkman [1949] considered the viscous force exerted on a dense swarm of particles by fluid flowing through them. The validity and theoretical justification of the Brinkman equation were presented in the papers [Durlofsky and Brady 1987; Kim and Russel 1985; Lundgren 1972; Rubinstein 1986; Tam 1969; Vafai and Kim 1995]. However, there are also publications in which the authors questioned the application of the Brinkman equation as a proper mathematical description for filtration flow [James and Davis 2001; Nield 1983]. In their opinion, the main problem is related to the effective viscosity (which depends on the flow) and the unknown boundary condition between the porous medium and the pure fluid area.

Almost all authors using the Brinkman equation assumed that the effective viscosity $\tilde{\mu}$ was equal to the pure fluid viscosity μ.

Keywords: porous media, Brinkman equation, effective viscosity, Trefftz method.

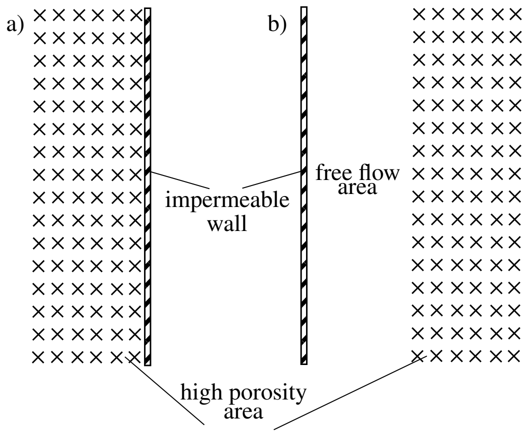

Figure 1. Porous medium with very high porosity in the presence of free fluid region (b) or only wall-bounded (a).

There are a limited number of papers in which the effective viscosity in the Brinkman filtration equation is determined. Brinkman [1949] suggested a possible use of Einstein's formula to describe the viscosity for a suspension, given as

$$\frac{\tilde{\mu}}{\mu} = 1 + 2.5\phi, \tag{3}$$

where ϕ is the volume fraction of the skeleton of the porous medium (the porosity of the medium equals $1-\phi$). According to (3), the effective viscosity is greater than the fluid viscosity. However, it should be noticed that the porous medium differs significantly from the suspension. In the porous medium, the skeleton is fixed; while in the suspension, particles move during fluid motion. The fact that the effective viscosity is greater than the viscosity of the pure fluid is correct for the suspension. It does not have to be correct for the porous media. In the present paper, we show that the effective viscosity of the Brinkman equation is less than the fluid viscosity. However, it is a fact that in the literature some authors obtained an effective viscosity that is greater than the fluid viscosity.

Lundgren [1972] determined the ratio $\tilde{\mu}/\mu$ as a function of the volume fraction of the skeleton of the porous medium, which was modeled by the fixed particles. The term $\tilde{\mu}/\mu$ rises slightly above one at the beginning as the volume fraction increases. This ratio reaches a maximum at $\phi = 0.2$. Koplik et al. [1983] obtained the effective viscosity that is less than the pure fluid viscosity (calculating the dissipation energy around the fixed particle):

$$\frac{\tilde{\mu}}{\mu} = 1 - 0.5\phi. \tag{4}$$

Ochoa-Tapia and Whitaker [1995] used the volume averaging method in order to derive the differential equation for flow in the porous medium for the balance of momentum. They obtained the Brinkman equation with the effective viscosity in the form

$$\frac{\tilde{\mu}}{\mu} = \frac{1}{1-\phi}. \tag{5}$$

That means that the effective viscosity is greater than the fluid viscosity. The effective medium theory was used by Freed and Muthukumar [1978]. They obtained the effective viscosity for a swarm of particles in the form

$$\frac{\tilde{\mu}}{\mu} = 1 + \tfrac{5}{2}\phi - \frac{9 \cdot \phi^{3/2}}{2\sqrt{2}}. \tag{6}$$

In this case the effective viscosity is greater than the viscosity of the pure fluid but is smaller than the effective viscosity resulting from (3).

In [Martys et al. 1994], the authors investigated flow of the fluid in the porous medium bounded by the free fluid region using a computer simulation. The porous medium was modeled as spherical particles, whose size and location was determined randomly. As a result, they obtained the ratio $\tilde{\mu}/\mu = 1.9 - 4.2$ at $\phi = 0.2 - 0.5$.

The effective viscosity in the Brinkman–Forchheimer equation was determined experimentally by Givler and Altobelli [1994]. In this work, the filtration equation had the form

$$\nabla P = -\frac{\mu}{k}\boldsymbol{q} - \frac{\rho \cdot c \cdot |\boldsymbol{q}|}{\sqrt{k}}\boldsymbol{q} + \tilde{\mu}\nabla^2 \boldsymbol{q}, \tag{7}$$

where ρ is the density of fluid and c is an experimentally determined constant. The authors used the magnetic resonance in order to determine the velocity profile of flow in the porous medium bounded by the walls. They obtained $\tilde{\mu}/\mu = 7.5$ for a Reynolds number equal to 17 and the volume fraction $\phi = 0.028$.

The effective viscosity of the porous medium modeled by regular displaced particles of the same diameter was determined by Starov and Zhdanov [2001] using the finite difference method. The authors proposed the formula

$$\frac{\tilde{\mu}}{\mu} = \frac{1}{(1-\phi)^{5/2}}. \tag{8}$$

According to the above formula, the effective viscosity is greater than the fluid viscosity.

Beavers and Joseph [1967] published a paper about the boundary condition on the boundary between the porous medium and the fluid. In their work, an experiment was conducted in which the volume flow was measured in two connected channels — one with a pure fluid region and the other with fluid in the porous medium (Figure 1a). On the basis of this experimental data, the following boundary condition was proposed by the authors:

$$\left.\frac{\mathrm{d}u}{\mathrm{d}y}\right|_{y=0} = \frac{\alpha}{\sqrt{k}}(u_i - q), \tag{9}$$

where u is the velocity in the fluid layer, u_i is the slip velocity on the boundary between the porous medium and the free flow area (at $y = 0$), k is the permeability of the porous medium, q is the filtration velocity on the boundary between layers, and α is the dimensionless slip coefficient which depends on the porosity and the structure of the porous medium.

Taylor [1971] noticed that the Beavers–Joseph boundary condition can be related to the Brinkman filtration equation, but he did not develop this idea. Neale and Nader [1974] demonstrated that if the porous layer flow is described by the Brinkman equation and the pure fluid layer flow is described by the Poiseuille law, then the relationship $\alpha^2 = \tilde{\mu}/\mu$ is satisfied. On the basis of this relationship and data from experiments and calculations, the value of the constant α can be used to determine the effective viscosity of the porous medium. In the Beavers–Joseph experiment, the constant α was greater or smaller than one and was dependent on the porous material. That implies that the effective viscosity is greater or smaller than the fluid viscosity.

The purpose of the present paper is to determine the effective viscosity in the Brinkman filtration equation for a fibrous porous medium. This viscosity is determined by the numeric simulation of imaginary physical experiments. The porous medium is modeled by a parallel bundle of straight fibers arranged in regular square or triangular arrays. In this way, the relationship between the effective viscosity and the porosity and the relationship between the effective viscosity and the structure of the porous medium are investigated. In the Brinkman filtration equation, there are two parameters of the porous medium: the permeability and the effective viscosity. Because of this, the determination of these parameters is realized in two steps. In the first step, the permeability is determined by assuming an infinite bundle of fibers. In the second step, the effective viscosity of such a medium is determined by knowing the permeability of the porous medium. The numerical investigations are conducted for a Newtonian fluid in a direction parallel to the fibers.

2. Imagined physical experiment for the measurement of the effective viscosity

Measurement with a rotational viscometer is one of the methods used to determine the pure fluid viscosity. In this instrument, a fluid layer is located between two cylindrical surfaces. One of the surfaces is immovable and the second one rotates at a constant angular speed. The fluid viscosity is determined by measuring the torque of the rotating cylindrical surface. In this way, it is simple shear flow. It is supposed that the effective viscosity in the Brinkman equation can be determined in a similar way.

Let us consider a layer of porous medium of width l, which is filled by a fluid and located between two flat plates. One of the plates is immovable and the second one moves at a constant velocity U (see Figure 2). The Brinkman equation for one-dimensional shear flow in the absence of a pressure gradient through the porous layer has the form

$$\tilde{\mu}\frac{\mathrm{d}^2 q_z}{\mathrm{d}x^2} - \frac{\mu}{k} q_z = 0, \tag{10}$$

where q_z is the filtration velocity in the direction of the fiber axis [m/s] and k is the permeability of the porous medium [m^2]. The following boundary conditions are formulated for the problem:

$$q_z = 0 \text{ for } x = 0, \quad \text{and} \quad q_z = U \text{ for } x = l. \tag{11}$$

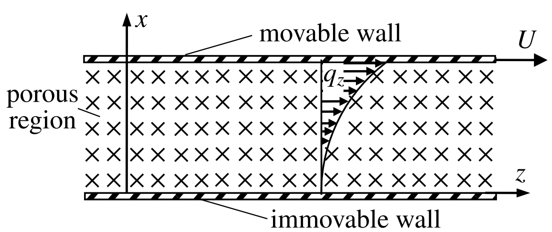

Figure 2. A porous region between two parallel plates.

Let us introduce the nondimensional variables

$$Q = \frac{q_z}{U}, \quad X = \frac{x}{b}, \quad \alpha = \frac{\tilde{\mu}}{\mu}, \quad F = \frac{k}{S}, \quad L = \frac{l}{b}, \tag{12}$$

where $S = \beta b^2$ [m^2] and β is a dimensionless parameter of the porous medium. Then (10) can be written in the dimensionless form

$$\frac{\mathrm{d}^2 Q}{\mathrm{d}X^2} - \frac{1}{\alpha\beta F} Q = 0 \tag{13}$$

with the boundary conditions

$$Q = 0 \text{ for } X = 0, \quad \text{and} \quad Q = 1 \text{ for } X = L. \tag{14}$$

The exact solution of the problem takes the form

$$Q = \frac{\exp(X/\sqrt{\alpha\beta F}) - \exp(-X/\sqrt{\alpha\beta F})}{\exp(L/\sqrt{\alpha\beta F}) - \exp(-L/\sqrt{\alpha\beta F})}. \tag{15}$$

The tangential stress on the movable plate can be expressed as

$$\tau = \tilde{\mu} \frac{\mathrm{d}q_z}{\mathrm{d}x}\bigg|_{x=L} = \tilde{\mu} \frac{U}{b} \frac{1}{\sqrt{\alpha\beta F}} \coth\left(\frac{L}{\sqrt{\alpha\beta F}}\right). \tag{16}$$

If the dimensionless permeability F and the tangential stress τ are known, then the constant α can be obtained from (16).

2.1. ***Determination of the permeability.*** In this section, a porous medium modeled by a regular array of parallel fibers is considered. Triangular ($\beta = 2\sqrt{3}$) and square ($\beta = 4$) arrays of fibers (see Figure 3) are analyzed.

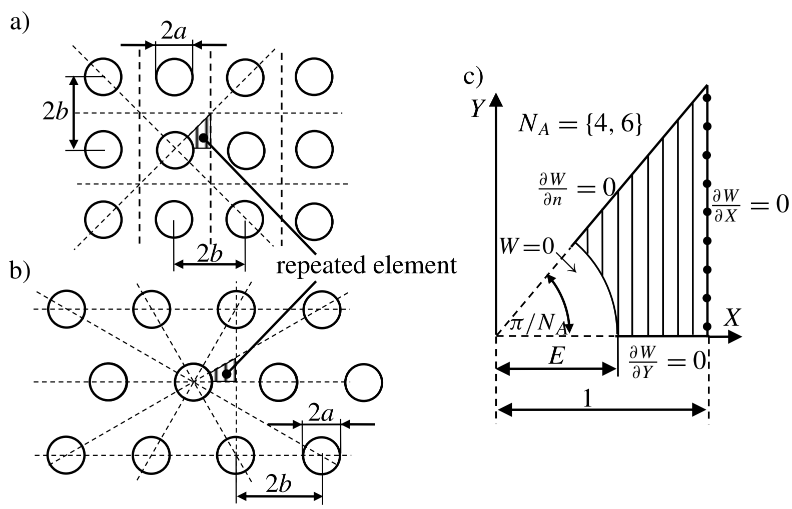

Figure 3. Unbounded porous medium: (a) square array, (b) triangular array, (c) repeated element of the array.

One of the basic methods for determining the permeability of the porous medium is an experiment in which the flow rate (the filtration velocity at the known pressure gradient) is measured. Based on this measurement, the permeability can be calculated.

Various authors considered longitudinal flow with respect to the fibers [Banerjee and Hadaller 1973; DeValve and Pitchumani 2012; Drummond and Tahir 1984; Gebart 1992; Happel 1959; Larson and Higdon 1986; Sparrow and Loeffler 1959; Wang 2002]. In the present paper, the same problem is considered but the solution is obtained using the special-purpose Trefftz functions [Mierzwiczak and Kołodziej 2012]. The method is semianalytical. That means that application of the method for the problem considered in the paper gives the analytical form of the dimensionless permeability of the porous medium, in which only the unknown coefficients of the solution are obtained numerically.

Let us consider steady, fully developed, laminar, isothermal flow of an incompressible viscous fluid driven by a constant pressure in a system of regular parallel fibers. The flow is longitudinal with respect to the fibers which are arranged in a regular square (Figure 3a) and triangular (Figure 3b) arrays. The radius of the fibers is equal to a, and the distance between the fibers is equal to $2b$. The fluid domain is the whole space $\mathbb{R}^3$. In this case, the equation of motion is reduced to a single partial differential equation in the form (in the polar coordinate system on the xy-plane)

$$\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\frac{\partial w}{\partial r}+\frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}=\frac{1}{\mu}\frac{\mathrm{d}p}{\mathrm{d}z} \quad \text{in } \Omega_R, \tag{17}$$

where w is the velocity component in the z-axis direction [m/s], $\mathrm{d}p/\mathrm{d}z$ is the constant pressure gradient [Pa/m], μ is the viscosity of the fluid [Pa·s], and Ω_R is the repeated element of array (Figure 3). It is convenient to introduce the dimensionless variables

$$W=-\frac{w}{(b^2/\mu)(\mathrm{d}p/\mathrm{d}z)}, \quad R=\frac{r}{b}, \quad E=\frac{a}{b}. \tag{18}$$

Now the governing equation (17) has the dimensionless form

$$\frac{\partial^2 W}{\partial R^2}+\frac{1}{R}\frac{\partial W}{\partial R}+\frac{1}{R^2}\frac{\partial^2 W}{\partial \theta^2}=-1, \tag{19}$$

with the boundary conditions

$$W=0 \quad \text{for } R=E, \tag{20}$$

$$\frac{\partial W}{\partial \theta}=0 \quad \text{for } \theta=\begin{cases}0,\\ \pi/N_A,\end{cases} \tag{21}$$

$$\frac{\partial W}{\partial X}=0 \quad \text{for } X=1, \tag{22}$$

where W is the dimensionless axial velocity, $N_A=6$ for a triangular array and $N_A=4$ for a square array.

The fiber volume fraction is given by

$$\phi=\frac{A_P}{A_T}=\frac{\pi\cdot E^2}{N_A\cdot\tan(\pi/N_A)}, \tag{23}$$

where $A_T=0.5\tan(\pi/N_A)$ is the total area of the repeated element (areas of the fluid and the fiber together) and $A_P=0.5E^2\cdot\pi/N_A$ is the area of the fiber in the repeated element.

The exact solution of (19) can be expressed using the special-purpose Trefftz functions in the form

$$W(R,\theta) = -\tfrac{1}{4}(R^2 - E^2) + \frac{1}{2\cdot\phi_{\mathrm{MAX}}}\ln(R/E) + \sum_{k=1}^{N} B_k\left(R^{N_A k} - \frac{E^{2N_A k}}{R^{N_A k}}\right)\cos(N_A k\theta), \tag{24}$$

where ϕ_{MAX} is the maximal volume fraction (obtained when neighboring fibers are in contact with each other, $E = 1:\ \phi_{\mathrm{MAX}} = \pi\sqrt{3}/6$ for a triangular array $N_A = 6$ and $\phi_{\mathrm{MAX}} = \pi/4$ for a square array $N_A = 4$).

The solution (24) satisfies exactly the boundary conditions (20)–(21) and the balance of the pressure and shear stress. The unknown coefficients $B_k (k = 1, \ldots, N)$ are determined by solving the system of linear equations that results from satisfying the boundary condition (22) using the boundary collocation technique. Imposing these boundary conditions at N collocation points, the following set of equations is to be solved:

$$\sum_{j=1}^{N} A_{kj}\cdot B_k = C_k, \qquad k = 1, 2, \ldots, N,$$

$$A_{kj} = j\cdot N_A\cdot(\cos\theta_k)^{1-N_A j}\left\{\cos[(N_A j - 1)\theta_k] + (\phi/\phi_{\mathrm{MAX}})^{N_A j}\cdot(\cos\theta_k)^{2N_A j}\cos[(N_A j + 1)\theta_k]\right\},$$

$$C_k = \tfrac{1}{2} - \frac{1}{2\cdot\phi_{\mathrm{MAX}}}\cos^2\theta_k, \qquad \theta_k = \frac{\pi(k-1)}{N_A(N-1)}. \tag{25}$$

Using Darcy's law (1), the longitudinal permeability can be related to the average velocity through the repeated element of the fiber system. The longitudinal component of the filtration velocity has the form

$$q_z = \frac{1}{A_T}\iint_{A_F} w\cdot \mathrm{d}A_F, \tag{26}$$

where $A_F = A_T - A_P$ is the region occupied by the fluid in the repeated element.

Using the definition of the nondimensional velocity given by (18), and introducing the fiber volume fraction, the longitudinal component of the filtration velocity can be expressed as

$$q_z = -\frac{b^2}{\mu}F(\phi)\frac{\mathrm{d}p}{\mathrm{d}z}, \tag{27}$$

where

$$F(\phi) = \frac{\iint_{A_F} W(R,\theta)\,\mathrm{d}A_F}{\beta\cdot A_T} \tag{28}$$

is the nondimensional component of the permeability tensor in the direction parallel to the fibers.

After analytical integration in (28), the dimensionless permeability becomes a function of the number of collocation points N and can be calculated from

$$F = \frac{1}{4\pi}\left\{\ln\frac{1}{\phi} + C + 2\phi - \frac{\phi^2}{2} + D\cdot\sum_{k=1}^{N} B_k\left[\frac{H_k}{N_A k + 2} + \left(\frac{\phi}{\phi_{\mathrm{MAX}}}\right)^{N_A k}\frac{G_k}{N_A k - 2}\right]\right\}, \tag{29}$$

where

$$H_k = \frac{\sin\left[(N_A k + 1)\pi/N_A\right]}{(N_A k + 1)\left[\cos(\pi/N_A)\right]^{N_A k + 1}}, \qquad G_k = \frac{\sin\left[(1 - N_A k)\pi/N_A\right]}{(1 - N_A k)\left[\cos(\pi/N_A)\right]^{1 - N_A k}}. \tag{30}$$

2.2. ***Determination of the effective viscosity.*** After determining the permeability of the porous medium, microstructural flow in the layer of porous medium modeled by a bundle of regular arranged (triangular or square) arrays of fibers can be considered.

Let us consider a layer of the porous medium located between two plates — the first one is fixed and the second one is movable (Figure 4). Because the flow is driven only by the movable wall (the pressure gradient equals zero), it is described by the Laplace equation,

$$\frac{\partial^2 W}{\partial R^2}+\frac{1}{R}\frac{\partial W}{\partial R}+\frac{1}{R^2}\frac{\partial^2 W}{\partial \theta^2}=0, \tag{31}$$

with no-slip boundary conditions: $W=0$ on the immovable wall and fibers and $W=1$ on the movable wall.

Since the array of fibers is streaked and periodic in one direction, it is sufficient to consider the problem only in one repeated strip; depicted in Figure 5 — for the square (Figure 5a) and triangular (Figure 5b) arrays. In both cases, the repeated strip is divided into smaller elements associated with each of the fibers, which are called large finite elements in rest of the paper.

The flow problem is solved by means of the Trefftz method using the special-purpose Trefftz functions. These functions are associated with the large finite elements (Figure 5). For each large finite element, the approximate solution is expressed as

$$W(R,\theta)=C_1\ln(R/E)+\sum_{k=2}^{N}C_k\left[R^{(k-1)}-\frac{E^{2(k-1)}}{R^{(k-1)}}\right]\cos[(k-1)\theta], \tag{32}$$

which satisfies exactly the governing equation (31) and some of the boundary conditions (see Figure 6). For each large finite element, the origin of the coordinates (R,θ) is placed at the center of fiber. The unknown coefficients $C_k(k=1,2,\dots,N)$ are determined using the boundary collocation technique by satisfying the remaining boundary conditions (in particular, the splitting boundary conditions between the large finite elements).

After determining the unknown coefficients $C_k(k=1,2,\dots,N)$ of the approximate solution, the tangential stress on the movable wall can be determined from

$$\tau=\mu\frac{U}{b}\int_0^{\tan(\pi/N_A)}\frac{\mathrm{d}W}{\mathrm{d}X}\,\mathrm{d}Y, \tag{33}$$

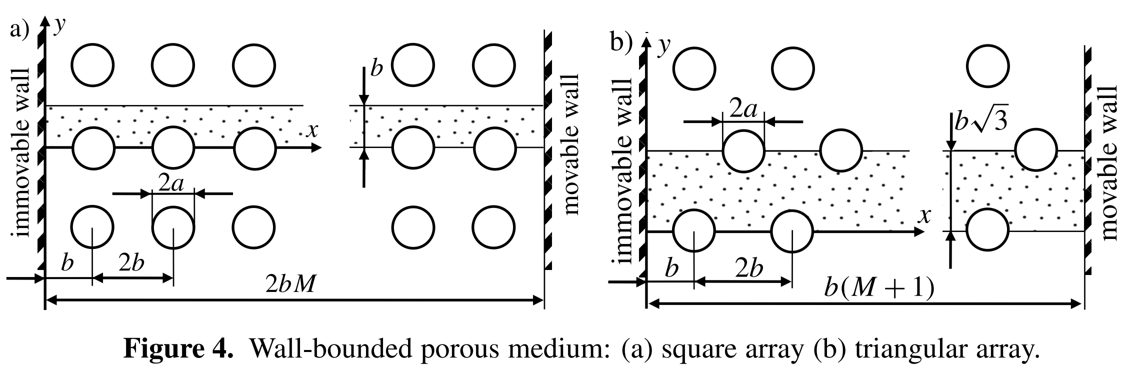

Figure 4. Wall-bounded porous medium: (a) square array (b) triangular array.

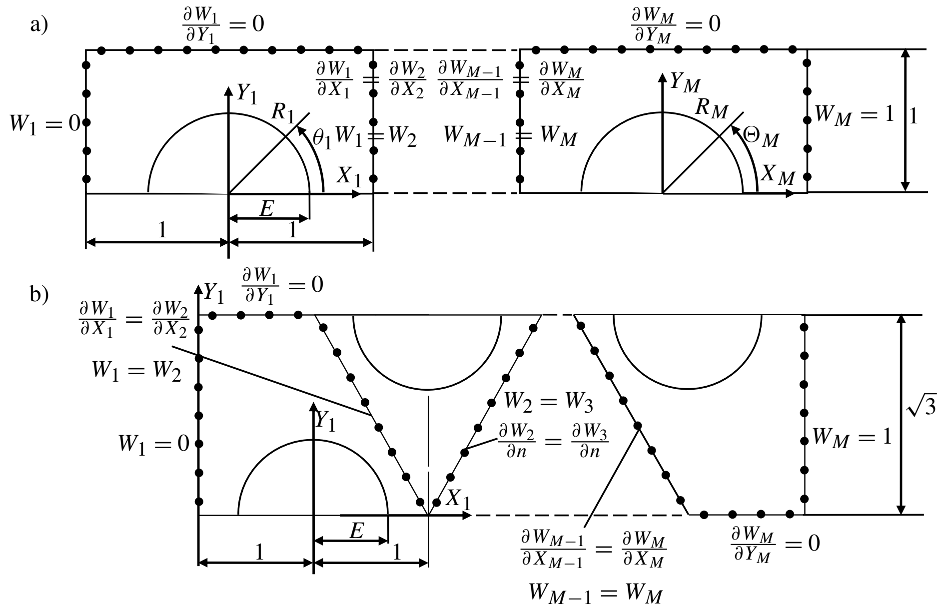

Figure 5. The symmetry lane of porous medium: square (a) and triangular array (b).

where the derivative $\mathrm{d}W/\mathrm{d}X$ can be easily obtained from the formula

$$\frac{\mathrm{d}W}{\mathrm{d}X} = \frac{\partial W}{\partial R}\cos(\theta) - \frac{1}{R}\frac{\partial W}{\partial \theta}\sin(\theta). \tag{34}$$

Comparing (16) and (33), the following relationship can be written:

$$\alpha = \frac{c(\phi, M, N)\cdot\sqrt{\alpha\beta F}}{\coth(L/\sqrt{\alpha\beta F})}, \tag{35}$$

where $c(\phi, M, N) = \int_0^{\tan(\pi/N_A)} (\mathrm{d}W/\mathrm{d}X)\,\mathrm{d}X$ and M denotes the number of fiber rows.

3. Numerical results and discussion

The numerical calculations were performed for the fibrous porous medium with cylindrical fibers arranged according to the square and triangular arrays. The value of the viscosity and the dimensionless effective permeability as a function of the fiber volume fraction are presented in Figures 7a and 7b, respectively. The ratio of the effective viscosity to the viscosity of the pure fluid decreases as the volume fraction of fibers increases, and for the square array, the ratio is greater than for the triangular array. Permeability F for both types of fiber arrays is similar and substantially different for $\phi > 0.3$. The problem was solved for $N = 10$ collocation points on the boundary. The dimensionless effective viscosity α was calculated for 5 rows of fibers for the square array and 8 rows for the triangular array.

The effect of the number of rows of the fibers $M = \{1, 2, 3, 5\}$ on the value of the effective permeability for the square array was examined. In this experiment, 10 collocation points on the boundary were used. Table 1 shows the numerical results. For a small value of the volume fraction ϕ, the influence of the number of fiber rows is significant. For $\phi = 0.1$ with 2 or more rows of fibers, the same value of α was obtained. For $\phi = 0.01$ the same value of α was obtained for 3 or more rows of fibers. The calculation

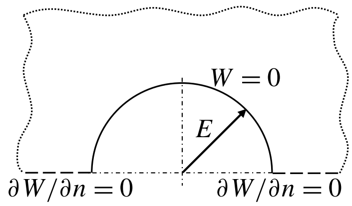

Figure 6. Boundary conditions satisfied by the special-purpose Trefftz functions applied in Equation (32).

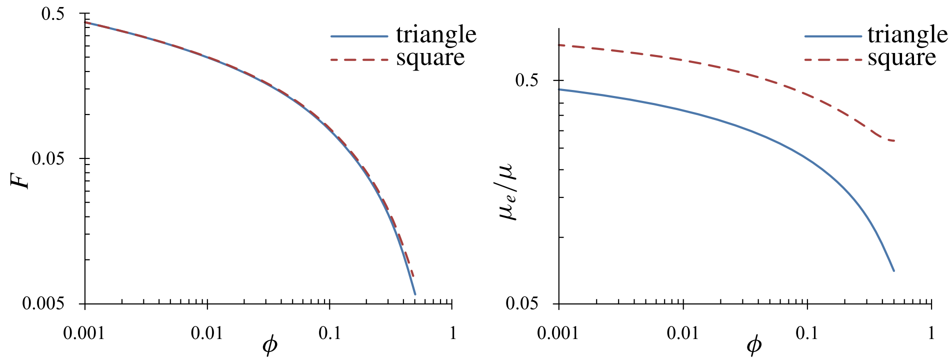

Figure 7. Values of the dimensionless effective viscosity α and the dimensionless permeability F, versus the values of the volume fraction ϕ for the square and triangular array.

$\phi \downarrow M \rightarrow$	1	2	3	5
$1.0 \cdot 10^{-8}$	1.8182	1.0103	0.9103	0.8906
$1.0 \cdot 10^{-7}$	1.6394	0.9649	0.8886	0.8754
$1.0 \cdot 10^{-6}$	1.4551	0.9173	0.8633	0.8554
$1.0 \cdot 10^{-5}$	1.2664	0.8659	0.8319	0.8280
$1.0 \cdot 10^{-4}$	1.0729	0.8068	0.7892	0.7878
$1.0 \cdot 10^{-3}$	0.8722	0.7306	0.7246	0.7243
$1.0 \cdot 10^{-2}$	0.6604	0.6170	0.6163	0.6163
$1.0 \cdot 10^{-1}$	0.4329	0.4312	0.4312	0.4312

Table 1. Influence of the number of the rows of fibers {1, 2, 3, 5} for the square array on the values of the dimensionless effective viscosity $\alpha = \mu_e/\mu$ for different values of the volume fraction ϕ.

of the effective permeability of the porous medium for $\phi \geq 0.01$ can be reduced to the numerical model with two rows of cylindrical fibers.

Since some of the boundary conditions are satisfied in an approximate sense for the collocation points located on the edge of the considered small repeated element, in the next step the effect of the number of these points on the value of the permeability F was checked. Table 2 shows the results for various values of the fiber volume fraction $\phi = \{0.0001, 0.001, 0.01, 0.1\}$ for the square array of fibers. Whatever

$N \downarrow \phi \rightarrow$	0.0001	0.001	0.01	0.1
2	0.61551076	0.43242005	0.25061461	0.08131019
3	0.61546913	0.43237843	0.25057299	0.08126870
4	0.61546816	0.43237745	0.25057202	0.08126775
5	0.61546826	0.43237756	0.25057212	0.08126785
6	0.61546825	0.43237755	0.25057211	0.08126784
7	0.61546825	0.43237755	0.25057211	0.08126784
8	0.61546825	0.43237755	0.25057211	0.08126784

Table 2. Influence the number N of collocation points (square array) on the dimensionless permeability F, for different values of the volume fraction ϕ.

$N \downarrow \phi \rightarrow$	$1.0 \cdot 10^{-8}$	$1.0 \cdot 10^{-6}$	0.0001	0.001	0.01	0.1
2	0.410243	0.468108	0.565460	0.646570	0.783514	1.150752
3	0.411038	0.468957	0.566328	0.647407	0.784279	1.151888
4	0.410907	0.468816	0.566182	0.647263	0.784143	1.151706
5	0.410930	0.468841	0.566207	0.647288	0.784166	1.151726
6	0.410926	0.468836	0.566202	0.647283	0.784162	1.151724
7	0.410926	0.468837	0.566203	0.647284	0.784162	1.151724
8	0.410926	0.468837	0.566203	0.647284	0.784162	1.151724
9	0.410926	0.468837	0.566203	0.647284	0.784162	1.151724

Table 3. Values of $c(\phi, M, N)$ for the square array, versus the number N of collocation points, for different values of the volume fraction ϕ.

the value of ϕ, the converged results were obtained for $N \geq 6$ collocation points. The present method requires a small number of collocation points; hence, it is very effective and is not time consuming.

Table 3 presents the effect of the number of collocation points N on the constant $c(\phi, M, N)$ for different values of the volume fraction of fibers ϕ for the square array with five rows of fibers. As in the previous case, the converged solutions for the constant c are obtained for $N \geq 6$.

Comparison of the results obtained in this study by means of the Trefftz method with other works is presented in Figure 9. The results clearly show that the ratio $\tilde{\mu}/\mu$ is less than one, which means that the effective viscosity $\tilde{\mu}$ is less than the fluid viscosity μ.

4. Conclusions

In the Brinkman filtration equation there are two parameters of the porous medium: the permeability and the effective viscosity. These two parameters were determined by numerical simulation of an imaginary physical experiment. The porous medium was modeled by a parallel bundle of straight fibers arranged in a regular triangular or square array. In order to determine the permeability, the flow driven by the pressure gradient in an unbounded porous medium was considered. To determine the effective viscosity,

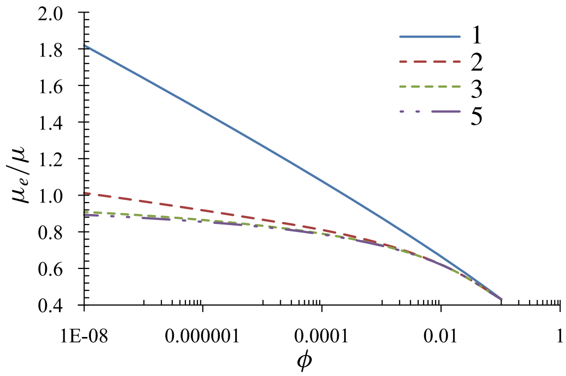

Figure 8. Influence of the number of the rows of fibers {1, 2, 3, 5} for the square array on the values of the dimensionless effective viscosity α for different values of the volume fraction ϕ.

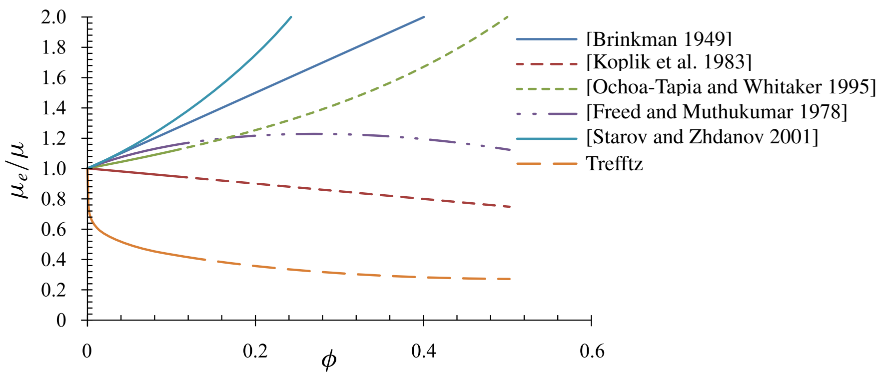

Figure 9. Comparison of the effective viscosity obtained in the present study with the others results.

the shear flow in a flat layer of porous medium was considered. In both numerical simulations, the Trefftz method with the special-purpose Trefftz functions was applied.

In all considered cases: the square and the triangular array for different porosity, in the Brinkman filtration equation the effective viscosity is lower than the viscosity of the pure fluid. The ratio of the effective viscosity to the viscosity of the pure fluid decreases as the volume fraction of fibers increases (decreasing porosity). For the same volume fraction of the fibers, both the permeability and the effective viscosity for the triangular array is smaller than for the square array. This difference is not significant in the permeability case, but is very important for the effective viscosity.

Parameters of the numerical method (for example the number of collocation points or the number of rows of fibers in the layer) have inconsiderable effect on the simulation results. To determine the permeability, a few collocation points are enough to get an accurate result ($N = 6$). For the effective viscosity, the result does not change with the number of rows greater than four.

The obtained results clearly show that the effective viscosity of the porous medium should be less than the fluid viscosity.

Acknowledgments

This work was supported by the grant 2012/07/B/ST8/03449 from the National Science Center, Poland.

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Received 29 Feb 2016. Revised 22 Aug 2016. Accepted 25 Sep 2016.

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Journal of Mechanics of Materials and Structures

Volume 12, No. 1 January 2017

Special issue on
Coupled Field Problems and Multiphase Materials

1559-3959(2017)12:1;1-4