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AN INTERFACIAL ARC CRACK IN BONDED DISSIMILAR ISOTROPIC LAMINATED PLATES

XU WANG, CUIYING WANG AND PETER SCHIAVONE

We consider an arc-shaped crack lying along the interface of a through-thickness circular elastic inhomogeneity in an infinite isotropic laminated thin plate subjected to bending and stretching within the context of classical Kirchhoff theory. A novel Stroh-type formalism is developed to reduce the original boundary value problem to a nonhomogeneous Riemann–Hilbert problem of vector form. The latter is solved analytically using a matrix diagonalization scheme. Elegant closed-form solutions are obtained for the stress resultants, in-plane displacements and slopes everywhere in the composite when the plate is subjected to remote uniform membrane stress resultants and bending moments. In particular, the surface stress resultants and surface bending moment along the bonded part of the interface, the jump in the generalized displacement vector across the debonded segment of the interface and the two complex intensity factors at each of the two crack tips are given in explicit form.

1. Introduction

Laminated composite plate structures have been widely used in the fields of aerospace, mechanical and civil engineering. The Kirchhoff plate theory is the most celebrated model for describing the bending deformations of a thin plate [Timoshenko and Woinowsky-Krieger 1959; Lekhnitskii 1968; Reddy 1997]. Problems involving cracks lying along the straight interface between two dissimilar plates under in-plane and bending loads have been studied extensively (see, for example, [Williams 1959; England 1965; Rice and Sih 1965; Erdogan 1965; Rice 1988; Suo 1989] for in-plane stretching, [Sih and Rice 1964; Sih 1966] for out-of-plane bending, and [Wang and Schiavone 2013] for coupled stretching and bending). The problem associated with interfacial arc-shaped cracks in bonded dissimilar homogeneous and isotropic elastic plates subjected to in-plane loads has also been considered by various authors (see, for example, [England 1966; Perlman and Sih 1967; Toya 1974; Herrmann 1991; 1994]).

Using the framework of Kirchhoff plate bending theory, we endeavour to study the coupled stretching and bending deformations of an infinite isotropic laminated plate (the matrix) containing a partially bonded isotropic laminated circular inhomogeneity when the matrix is subjected to remote uniform membrane stress resultants and bending moments. In order to solve the problem, we first propose an elegant Stroh-type formalism based on the complex variable formulation in [Wang and Zhou 2014b]. As a result, the original boundary value problem is reduced to a nonhomogeneous Riemann–Hilbert problem of vector form. Through simultaneous diagonalization of two positive definite Hermitian matrices, we arrive at four separate Riemann–Hilbert problems of scalar form which can be solved analytically by evaluating the corresponding Cauchy integrals. Consequently, we are able to determine the four analytic functions

Keywords: isotropic laminated plate, circular inhomogeneity, Kirchhoff plate theory, arc-shaped crack, complex variable method, Stroh-type formalism.

defined in the inhomogeneity as well as the four analytic functions defined in the matrix. The resulting stress resultant and displacement fields can then be obtained from the corresponding analytic functions.

The original Stroh sextic formalism was developed for generalized plane strain deformations of an anisotropic elastic material [Stroh 1958; 1962; Ting 1996]. A new Stroh octet formalism was later developed for coupled stretching and bending deformations of an anisotropic elastic thin plate [Cheng and Reddy 2002; 2005]. Both formalisms become rather cumbersome for mathematically degenerate materials in which the 6×6 fundamental elasticity matrix or the 8×8 fundamental elastic plate matrix becomes nonsemisimple [Ting 1996; Cheng and Reddy 2002]. An isotropic laminated plate is degenerate in the sense that there are only two independent eigenvectors associated with the quadruple roots $p_1 = p_2 = p_3 = p_4 = \mathrm{i}$ [Cheng and Reddy 2002]. As part of our analysis, we develop an elegant Stroh-type formalism to tackle the problem at hand: that of an arc-shaped crack in bonded dissimilar isotropic laminated plates.

Finally, we mention that although laminated plates are typically elastically anisotropic, laminates composed of isotropic layers arise quite frequently in practice. In fact, only a simple modification of the aforementioned complex variable formulation developed by Wang and Zhou [2014b] for an isotropic laminated plate will accommodate the more general case of a transversely isotropic laminated plate with the x_3-axis as axis of symmetry [Cheng and Reddy 2002].

2. Basic formulation

2.1. *Complex variable formulation.* In this section, we undertake a brief review of the complex variable formulation for coupled stretching and bending deformations of an isotropic laminated plate.

We establish a Cartesian coordinate system $\{x_i\}$ $(i = 1, 2, 3)$ in which the reference plane of an undeformed plate of uniform thickness h is located at $x_3 = 0$. The plate is composed of an isotropic, linearly elastic material that can be inhomogeneous and laminated in the thickness direction. In what follows, Greek and Latin indices take the values $1, 2$ and $1, 2, 3$, respectively and we sum over repeated indices.

The displacement field in the Kirchhoff plate theory is assumed to take the form

$$\tilde{u}_\alpha(x_i) = u_\alpha + x_3\vartheta_\alpha, \quad \tilde{u}_3(x_i) = w, \tag{1}$$

where the two in-plane displacements u_α, the deflection w and the slopes $\vartheta_\alpha = -w_{,\alpha}$ on the reference plane are all independent of x_3.

The coordinate system is judiciously chosen such that the two in-plane displacements and the deflection on the reference plane are *decoupled* in the equilibrium equations [Beom and Earmme 1998]. We introduce the integral operator $Q(\cdots) = \int_{-h_0}^{h-h_0}(\cdots)\,\mathrm{d}x_3$ in which h_0 is the distance between the reference plane and the lower surface of the plate. Accordingly, the membrane stress resultants and bending moments defined by $N_{\alpha\beta} = Q\sigma_{\alpha\beta}$ and $M_{\alpha\beta} = Qx_3\sigma_{\alpha\beta}$ (with $\sigma_{\alpha\beta}$ denoting the in-plane stress components), the transverse shearing forces $\Re_\beta = M_{\alpha\beta,\alpha}$, in-plane displacements, deflection and slopes on the reference plane of the plate as well as the four stress functions φ_α and η_α can be expressed concisely in terms of four complex potentials $\phi(z)$, $\psi(z)$, $\Phi(z)$ and $\Psi(z)$ of the complex variable $z = x_1 + \mathrm{i}x_2$ as [Wang and Zhou 2014b]

$$\begin{aligned} N_{11} + N_{22} &= 4\,\mathrm{Re}\{\phi'(z) + B\Phi'(z)\}, \\ N_{22} - N_{11} + 2\mathrm{i}N_{12} &= 2\big[\bar{z}\phi''(z) + \psi'(z) + B\bar{z}\Phi''(z) + B\Psi'(z)\big], \end{aligned} \tag{2}$$

$$
\begin{aligned}
M_{11}+M_{22} &= 4D(1+\nu^D)\,\mathrm{Re}\{\Phi'(z)\}+\frac{B(\kappa^A-1)}{\mu}\,\mathrm{Re}\{\phi'(z)\},\\
M_{22}-M_{11}+2\mathrm{i}M_{12} &= -2D(1-\nu^D)[\bar z\Phi''(z)+\Psi'(z)]-\frac{B}{\mu}[\bar z\phi''(z)+\psi'(z)],\\
\Re_1-\mathrm{i}\Re_2 &= 4D\Phi''(z)+\frac{B(\kappa^A+1)}{2\mu}\phi''(z),
\end{aligned}
\tag{3}
$$

$$
\begin{aligned}
2\mu(u_1+\mathrm{i}u_2) &= \kappa^A\phi(z)-z\overline{\phi'(z)}-\overline{\psi(z)},\\
\vartheta_1+\mathrm{i}\vartheta_2 &= \Phi(z)+z\overline{\Phi'(z)}+\overline{\Psi(z)},\qquad w=-\,\mathrm{Re}\{\bar z\Phi(z)+\gamma(z)\},\\
\varphi_1+\mathrm{i}\varphi_2 &= \mathrm{i}[\phi(z)+z\overline{\phi'(z)}+\overline{\psi(z)}]+\mathrm{i}B[\Phi(z)+z\overline{\Phi'(z)}+\overline{\Psi(z)}],\\
\eta_1+\mathrm{i}\eta_2 &= \mathrm{i}D(1-\nu^D)[\kappa^D\Phi(z)-z\overline{\Phi'(z)}-\overline{\Psi(z)}]+\mathrm{i}\frac{B}{2\mu}[\kappa^A\phi(z)-z\overline{\phi'(z)}-\overline{\psi(z)}],
\end{aligned}
\tag{4}
$$

in which $\Psi(z)=\gamma'(z)$, and

$$
\begin{aligned}
&\mu=\tfrac{1}{2}(A_{11}-A_{12}),\quad B=B_{12},\quad D=D_{11},\quad \nu^A=\frac{A_{12}}{A_{11}},\quad \nu^D=\frac{D_{12}}{D_{11}},\\
&\kappa^A=\frac{3A_{11}-A_{12}}{A_{11}+A_{12}}=\frac{3-\nu^A}{1+\nu^A},\quad \kappa^D=\frac{3D_{11}+D_{12}}{D_{11}-D_{12}}=\frac{3+\nu^D}{1-\nu^D},
\end{aligned}
\tag{5}
$$

with $A_{ij}=QC_{ij}$, $B_{ij}=Qx_3C_{ij}$ and $D_{ij}=Qx_3^2C_{ij}$ $(ij=11,12)$. The parameters C_{11} and C_{12} can be expressed in terms of the Young's modulus $E=E(x_3)$ and Poisson's ratio $\nu=\nu(x_3)$ of the plate as $C_{11}=E/(1-\nu^2)$ and $C_{12}=\nu E/(1-\nu^2)$. The distance h_0 is determined as

$$
h_0=\frac{\int_0^h X_3C_{11}\,\mathrm{d}X_3}{\int_0^h C_{11}\,\mathrm{d}X_3}
$$

with $X_3=x_3+h_0$ being the vertical coordinate of the given point from the lower surface of the plate. Detailed derivations of equations (2)–(5) can be found in [Wang and Zhou 2014b].

In addition, the membrane stress resultants, bending moments, transverse shearing forces, and modified Kirchhoff transverse shearing forces $V_1=\Re_1+M_{12,2}$ and $V_2=\Re_2+M_{21,1}$ (which apply exclusively to free edges), can be expressed in terms of the four stress functions φ_α and η_α [Cheng and Reddy 2002] as

$$
N_{\alpha\beta}=-\epsilon_{\beta\omega}\varphi_{\alpha,\omega},\quad M_{\alpha\beta}=-\epsilon_{\beta\omega}\eta_{\alpha,\omega}-\tfrac{1}{2}\epsilon_{\alpha\beta}\eta_{\omega,\omega},\quad \Re_\alpha=-\tfrac{1}{2}\epsilon_{\alpha\beta}\eta_{\omega,\omega\beta},\quad V_\alpha=-\epsilon_{\alpha\omega}\eta_{\omega,\omega\omega}. \tag{6}
$$

Here $\epsilon_{\alpha\beta}$ are the components of the two-dimensional permutation tensor.

In a new coordinate system $\{\hat x_i\}$ $(i=1,2,3)$ in which $\hat x_3=0$ lies on an arbitrary plane parallel to the reference plane and $\hat x_\alpha=x_\alpha$, the in-plane displacements $\hat u_\alpha$ and slopes $\hat\vartheta_\alpha$ on $\hat x_3=0$ and the stress functions $\hat\varphi_\alpha$ and $\hat\eta_\alpha$ in the new coordinate system can be given quite simply as

$$
\begin{aligned}
\hat\vartheta_1+\mathrm{i}\hat\vartheta_2 &= \vartheta_1+\mathrm{i}\vartheta_2, &\qquad \hat u_1+\mathrm{i}\hat u_2 &= u_1+\mathrm{i}u_2-\hat h(\vartheta_1+\mathrm{i}\vartheta_2),\\
\hat\varphi_1+\mathrm{i}\hat\varphi_2 &= \varphi_1+\mathrm{i}\varphi_2, &\qquad \hat\eta_1+\mathrm{i}\hat\eta_2 &= \eta_1+\mathrm{i}\eta_2+\hat h(\varphi_1+\mathrm{i}\varphi_2).
\end{aligned}
\tag{7}
$$

Here,

$$
\hat h=h_1-h_0, \tag{8}
$$

and h_1 is the distance between $\hat{x}_3 = 0$ and the lower surface of the plate (we note that h_1 is positive or negative, respectively, if $\hat{x}_3 = 0$ is above or below the lower surface of the plate). In the new coordinate system, the stress resultants $\widehat{N}_{\alpha\beta} = \widehat{Q}\sigma_{\alpha\beta}$ and $\widehat{M}_{\alpha\beta} = \widehat{Q}\hat{x}_3\sigma_{\alpha\beta}$ with

$$\widehat{Q}(\cdots) = \int_{-h_1}^{h-h_1} (\cdots)\,\mathrm{d}\hat{x}_3,$$

the transverse shearing forces $\widehat{\Re}_\beta = \widehat{M}_{\alpha\beta,\alpha}$, and the modified Kirchhoff transverse shearing forces $\widehat{V}_1 = \widehat{\Re}_1 + \widehat{M}_{12,2}$ and $\widehat{V}_2 = \widehat{\Re}_2 + \widehat{M}_{21,1}$ can also be expressed in terms of the newly introduced stress functions $\hat{\varphi}_\alpha$ and $\hat{\eta}_\alpha$ as

$$\widehat{N}_{\alpha\beta} = -\epsilon_{\beta\omega}\hat{\varphi}_{\alpha,\omega}, \quad \widehat{M}_{\alpha\beta} = -\epsilon_{\beta\omega}\hat{\eta}_{\alpha,\omega} - \tfrac{1}{2}\epsilon_{\alpha\beta}\hat{\eta}_{\omega,\omega}, \quad \widehat{\Re}_\alpha = -\tfrac{1}{2}\epsilon_{\alpha\beta}\hat{\eta}_{\omega,\omega\beta}, \quad \widehat{V}_\alpha = -\epsilon_{\alpha\omega}\hat{\eta}_{\omega,\omega\omega}. \tag{9}$$

2.2. *Statement of the problem.* As shown in Figure 1, we consider the coupled stretching and bending deformations of an infinite isotropic laminated plate containing a partially bonded through-thickness isotropic laminated circular inhomogeneity of radius R. The centre of the circular inhomogeneity is located at the origin and an interface arc crack lies along the arc L_c of the interface. Along the remaining arc L_b of the interface, the inhomogeneity remains perfectly bonded to the surrounding matrix. The centre of the arc L_b lies on the positive x_1-axis and the central angle subtended by the arc L_b is given by $2\theta_0$. The two crack tips are located at $a = Re^{\mathrm{i}\theta_0}$ and $\bar{a} = Re^{-\mathrm{i}\theta_0}$. We represent the matrix by the domain S_2 and assume that the circular inhomogeneity occupies the region S_1. The matrix is subjected to remote uniform membrane stress resultants $(N_{11}^\infty, N_{12}^\infty, N_{22}^\infty)$ and bending moments $(M_{11}^\infty, M_{12}^\infty, M_{22}^\infty)$ measured on its reference plane. In what follows, the subscripts 1 and 2 (or the superscripts (1) and (2)) are used to identify the respective quantities in S_1 and S_2, respectively.

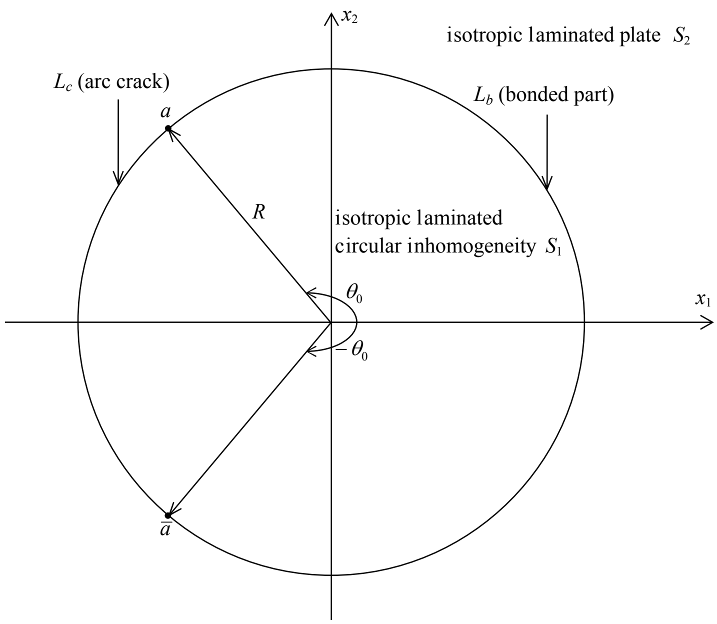

Figure 1. An isotropic laminated circular inhomogeneity partially bonded to an infinite isotropic laminated plate.

The boundary conditions on the circular interface are given specifically by

$$\hat{\varphi}_1^{(1)} = \hat{\varphi}_1^{(2)}, \quad \hat{\varphi}_2^{(1)} = \hat{\varphi}_2^{(2)}, \quad \hat{\eta}_1^{(1)} = \hat{\eta}_1^{(2)}, \quad \hat{\eta}_2^{(1)} = \hat{\eta}_2^{(2)}, \quad z \in L_b; \tag{10a}$$

$$\hat{u}_1^{(1)} = \hat{u}_1^{(2)}, \quad \hat{u}_2^{(1)} = \hat{u}_2^{(2)}, \quad \hat{\vartheta}_1^{(1)} = \hat{\vartheta}_1^{(2)}, \quad \hat{\vartheta}_2^{(1)} = \hat{\vartheta}_2^{(2)}, \quad z \in L_b; \tag{10b}$$

$$\hat{\varphi}_1^{(1)} = \hat{\varphi}_1^{(2)} = 0, \quad \hat{\varphi}_2^{(1)} = \hat{\varphi}_2^{(2)} = 0, \quad \hat{\eta}_1^{(1)} = \hat{\eta}_1^{(2)} = 0, \quad \hat{\eta}_2^{(1)} = \hat{\eta}_2^{(2)} = 0, \quad z \in L_c. \tag{10c}$$

The conditions in equations (10a)–(10b) imply that the stress resultants and displacements are continuous across L_b, whilst equation (10c) indicates that the free edge condition is satisfied on L_c.

3. The Stroh-type formalism

It is extremely problematic to apply the complex variable formulation in Section 2.1 directly to solve the boundary value problem associated with the arc-shaped interface crack described in Section 2.2 mainly because of the formidable number of elastic constants involved in the two phases of the composite as well as the expression of the interface conditions in the new coordinate system common to both the inhomogeneity and the matrix. As an alternative, in this section we propose an elegant Stroh-type formalism. We introduce the following analytic function vector $\boldsymbol{f}(z)$ for the circular inhomogeneity and the surrounding matrix

$$\boldsymbol{f}(z) = \begin{bmatrix} \phi(z) \\ \psi(z) + (R^2/z)\phi'(z) \\ \Phi(z) \\ \Psi(z) + (R^2/z)\Phi'(z) \end{bmatrix}. \tag{11}$$

As a result, the generalized displacement vector and the stress function vector along the circular interface $|z| = R$ in the new coordinate system can be concisely and elegantly expressed in terms of $\boldsymbol{f}(z)$ as

$$\hat{\boldsymbol{u}} = \boldsymbol{A}\boldsymbol{f}(z) + \overline{\boldsymbol{A}}\,\overline{\boldsymbol{f}(z)}, \quad \hat{\boldsymbol{\varphi}} = \boldsymbol{B}\boldsymbol{f}(z) + \overline{\boldsymbol{B}}\,\overline{\boldsymbol{f}(z)}, \quad |z| = R, \tag{12}$$

where

$$\hat{\boldsymbol{u}} = \begin{bmatrix} \hat{u}_1 & \hat{u}_2 & \hat{\vartheta}_1 & \hat{\vartheta}_2 \end{bmatrix}^T, \quad \hat{\boldsymbol{\varphi}} = \begin{bmatrix} \hat{\varphi}_1 & \hat{\varphi}_2 & \hat{\eta}_1 & \hat{\eta}_2 \end{bmatrix}^T, \tag{13}$$

$$\boldsymbol{A} = \frac{1}{2}\begin{bmatrix} \boldsymbol{I} & -\hat{h}\boldsymbol{I} \\ \boldsymbol{0} & \boldsymbol{I} \end{bmatrix} \begin{bmatrix} \kappa^A/(2\mu) & -1/(2\mu) & 0 & 0 \\ -\mathrm{i}\kappa^A/(2\mu) & -\mathrm{i}/(2\mu) & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -\mathrm{i} & \mathrm{i} \end{bmatrix}, \tag{14}$$

$$\boldsymbol{B} = \frac{1}{2}\begin{bmatrix} \boldsymbol{I} & \boldsymbol{0} \\ \hat{h}\boldsymbol{I} & \boldsymbol{I} \end{bmatrix} \begin{bmatrix} \mathrm{i} & -\mathrm{i} & \mathrm{i}B & -\mathrm{i}B \\ 1 & 1 & B & B \\ \mathrm{i}B\kappa^A/(2\mu) & \mathrm{i}B/(2\mu) & \mathrm{i}D(3+\nu^D) & \mathrm{i}D(1-\nu^D) \\ B\kappa^A/(2\mu) & -B/(2\mu) & D(3+\nu^D) & -D(1-\nu^D) \end{bmatrix}. \tag{15}$$

It can be deduced that

$$\mathrm{i}\boldsymbol{A}\boldsymbol{B}^{-1}=\boldsymbol{M}^{-1}=\boldsymbol{L}^{-1}-\mathrm{i}\boldsymbol{S}\boldsymbol{L}^{-1}=\begin{bmatrix}\hat{\alpha}_{11} & \mathrm{i}\hat{\alpha}_{12} & \hat{\alpha}_{13} & -\mathrm{i}\hat{\alpha}_{23}\\ -\mathrm{i}\hat{\alpha}_{12} & \hat{\alpha}_{11} & \mathrm{i}\hat{\alpha}_{23} & \hat{\alpha}_{13}\\ \hat{\alpha}_{13} & -\mathrm{i}\hat{\alpha}_{23} & \hat{\alpha}_{33} & -\mathrm{i}\hat{\alpha}_{43}\\ \mathrm{i}\hat{\alpha}_{23} & \hat{\alpha}_{13} & \mathrm{i}\hat{\alpha}_{43} & \hat{\alpha}_{33}\end{bmatrix}, \tag{16}$$

where

$$\begin{gathered}\hat{\alpha}_{11}=\alpha_{11}-2\hat{h}\alpha_{13}+\hat{h}^2\alpha_{33},\quad \hat{\alpha}_{12}=\alpha_{12}+2\hat{h}\alpha_{23}-\hat{h}^2\alpha_{43},\\ \hat{\alpha}_{13}=\alpha_{13}-\hat{h}\alpha_{33},\quad \hat{\alpha}_{23}=\alpha_{23}-\hat{h}\alpha_{43},\quad \hat{\alpha}_{33}=\alpha_{33},\quad \hat{\alpha}_{43}=\alpha_{43},\end{gathered} \tag{17}$$

with α_{ij} given by [Wang and Zhou 2014a] as

$$\begin{aligned}\alpha_{11}&=\frac{2D[2\mu D(1-\nu^D)(3+\nu^D)-B^2(3-\nu^A)]}{[2\mu D(1-\nu^D)-B^2][2\mu D(3+\nu^D)(1+\nu^A)-B^2(3-\nu^A)]},\\ \alpha_{12}&=\frac{D[2\mu D(1-\nu^D)(3+\nu^D)(1-\nu^A)-B^2(1+\nu^D)(3-\nu^A)]}{[2\mu D(1-\nu^D)-B^2][2\mu D(3+\nu^D)(1+\nu^A)-B^2(3-\nu^A)]},\\ \alpha_{13}&=\frac{4B\mu D(\nu^D+\nu^A)}{[2\mu D(1-\nu^D)-B^2][2\mu D(3+\nu^D)(1+\nu^A)-B^2(3-\nu^A)]},\\ \alpha_{23}&=\frac{B[2\mu D(3-\nu^D+\nu^A+\nu^D\nu^A)-B^2(3-\nu^A)]}{[2\mu D(1-\nu^D)-B^2][2\mu D(3+\nu^D)(1+\nu^A)-B^2(3-\nu^A)]},\\ \alpha_{33}&=\frac{4\mu[2\mu D(1+\nu^A)-B^2]}{[2\mu D(1-\nu^D)-B^2][2\mu D(3+\nu^D)(1+\nu^A)-B^2(3-\nu^A)]},\\ \alpha_{43}&=\frac{2\mu[2\mu D(1+\nu^D)(1+\nu^A)-B^2(1-\nu^A)]}{[2\mu D(1-\nu^D)-B^2][2\mu D(3+\nu^D)(1+\nu^A)-B^2(3-\nu^A)]}.\end{aligned} \tag{18}$$

Furthermore, $\boldsymbol{M}^{-1}$ is a positive definite Hermitian matrix. As a result $\boldsymbol{L}$ is real symmetric and positive definite, while $\boldsymbol{S}\boldsymbol{L}^{-1}$ is antisymmetric.

Remark 1. Equations (16) and (17) are obtained by utilizing the following relationship during coordinate translation

$$\boldsymbol{M}^{-1}=\begin{bmatrix}\boldsymbol{I} & -\hat{h}\boldsymbol{I}\\ \boldsymbol{0} & \boldsymbol{I}\end{bmatrix}\boldsymbol{M}_0^{-1}\begin{bmatrix}\boldsymbol{I} & \boldsymbol{0}\\ -\hat{h}\boldsymbol{I} & \boldsymbol{I}\end{bmatrix},$$

where

$$\boldsymbol{I}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix},\quad \boldsymbol{M}_0^{-1}=\begin{bmatrix}\alpha_{11} & \mathrm{i}\alpha_{12} & \alpha_{13} & -\mathrm{i}\alpha_{23}\\ -\mathrm{i}\alpha_{12} & \alpha_{11} & \mathrm{i}\alpha_{23} & \alpha_{13}\\ \alpha_{13} & -\mathrm{i}\alpha_{23} & \alpha_{33} & -\mathrm{i}\alpha_{43}\\ \mathrm{i}\alpha_{23} & \alpha_{13} & \mathrm{i}\alpha_{43} & \alpha_{33}\end{bmatrix}.$$

4. Closed-form solution

The continuity condition of stress resultants across the circular interface $|z| = R$ can then be expressed in terms of $\boldsymbol{f}_1(z)$ and $\boldsymbol{f}_2(z)$ as

$$\boldsymbol{B}_1 \boldsymbol{f}_1^+(z) + \overline{\boldsymbol{B}}_1 \overline{\boldsymbol{f}}_1^-(R^2/z) = \boldsymbol{B}_2 \boldsymbol{f}_2^-(z) + \overline{\boldsymbol{B}}_2 \overline{\boldsymbol{f}}_2^+(R^2/z), \quad |z| = R. \tag{19}$$

By applying the generalized Liouville's theorem, we arrive at the relationship

$$\overline{\boldsymbol{B}}_2 \overline{\boldsymbol{f}}_2(R^2/z) - \boldsymbol{B}_1 \boldsymbol{f}_1(z) = \overline{\boldsymbol{B}}_1 \overline{\boldsymbol{f}}_1(R^2/z) - \boldsymbol{B}_2 \boldsymbol{f}_2(z) = \boldsymbol{g}(z), \tag{20}$$

where $\boldsymbol{g}(z)$ is given by

$$\boldsymbol{g}(z) = \left[\overline{\boldsymbol{B}}_1 (\boldsymbol{i}_2 \overline{\phi_1'(0)} + \boldsymbol{i}_4 \overline{\Phi_1'(0)}) - \boldsymbol{B}_2 \boldsymbol{k}\right] z + R^2 \left[\overline{\boldsymbol{B}}_2 \bar{\boldsymbol{k}} - \boldsymbol{B}_1 (\boldsymbol{i}_2 \phi_1'(0) + \boldsymbol{i}_4 \Phi_1'(0))\right] z^{-1}, \tag{21}$$

$\boldsymbol{i}_K$ is a unit 4-vector defined by $(\boldsymbol{i}_K)_L = \delta_{KL}$ and the vector $\boldsymbol{k}$ is related to the remote uniform membrane stress resultants and bending moments through

$$\boldsymbol{k} = \begin{bmatrix} \dfrac{\mu_2 D_2 (1+\nu_2^D)(N_{11}^\infty + N_{22}^\infty) - B_2 \mu_2 (M_{11}^\infty + M_{22}^\infty)}{4\mu_2 D_2 (1+\nu_2^D) - B_2^2 (\kappa_2^A - 1)} \\ \dfrac{\mu_2 D_2 (1-\nu_2^D)(N_{22}^\infty - N_{11}^\infty + 2\mathrm{i} N_{12}^\infty) + B_2 \mu_2 (M_{22}^\infty - M_{11}^\infty + 2\mathrm{i} M_{12}^\infty)}{2\mu_2 D_2 (1-\nu_2^D) - B_2^2} \\ \dfrac{4\mu_2 (M_{11}^\infty + M_{22}^\infty) - B_2 (\kappa_2^A - 1)(N_{11}^\infty + N_{22}^\infty)}{16\mu_2 D_2 (1+\nu_2^D) - 4B_2^2 (\kappa_2^A - 1)} \\ \dfrac{-2\mu_2 (M_{22}^\infty - M_{11}^\infty + 2\mathrm{i} M_{12}^\infty) - B_2 (N_{22}^\infty - N_{11}^\infty + 2\mathrm{i} N_{12}^\infty)}{4\mu_2 D_2 (1-\nu_2^D) - 2B_2^2} \end{bmatrix}. \tag{22}$$

Remark 2. Equation (20) can be obtained by noting that $\boldsymbol{f}_1(z) \cong R^2[\boldsymbol{i}_2 \phi_1'(0) + \boldsymbol{i}_4 \Phi_1'(0)] z^{-1} + O(1)$ as $z \to 0$ and $\boldsymbol{f}_2(z) \cong \boldsymbol{k} z + O(1)$ as $|z| \to \infty$.

The continuity condition for the generalized displacement vector across the bonded section of the interface can be expressed in terms of $\boldsymbol{f}_1(z)$ and $\boldsymbol{f}_2(z)$ by

$$\boldsymbol{A}_1 \boldsymbol{f}_1^+(z) + \overline{\boldsymbol{A}}_1 \overline{\boldsymbol{f}}_1^-(R^2/z) = \boldsymbol{A}_2 \boldsymbol{f}_2^-(z) + \overline{\boldsymbol{A}}_2 \overline{\boldsymbol{f}}_2^+(R^2/z), \quad z \in L_b. \tag{23}$$

Eliminating $\overline{\boldsymbol{f}}_1^-(R^2/z)$ and $\overline{\boldsymbol{f}}_2^+(R^2/z)$ from equation (23) using the relationship in equation (20) yields

$$\boldsymbol{M}_* \boldsymbol{B}_1 \boldsymbol{f}'^+_1(z) - \overline{\boldsymbol{M}}_* \boldsymbol{B}_2 \boldsymbol{f}'^-_2(z) = (\overline{\boldsymbol{M}}_1^{-1} - \overline{\boldsymbol{M}}_2^{-1}) \boldsymbol{g}'(z), \quad z \in L_b, \tag{24}$$

where

$$\boldsymbol{M}_* = \boldsymbol{M}_1^{-1} + \overline{\boldsymbol{M}}_2^{-1} = \boldsymbol{L}_1^{-1} + \boldsymbol{L}_2^{-1} - \mathrm{i}(\boldsymbol{S}_1 \boldsymbol{L}_1^{-1} - \boldsymbol{S}_2 \boldsymbol{L}_2^{-1}). \tag{25}$$

For convenience, we write the positive definite Hermitian matrix $\boldsymbol{M}_*$ in the component form as

$$\boldsymbol{M}_* = \begin{bmatrix} Y_{11} & -\mathrm{i}Y_{21} & Y_{13} & -\mathrm{i}Y_{23} \\ \mathrm{i}Y_{21} & Y_{11} & \mathrm{i}Y_{23} & Y_{13} \\ Y_{13} & -\mathrm{i}Y_{23} & Y_{33} & -\mathrm{i}Y_{43} \\ \mathrm{i}Y_{23} & Y_{13} & \mathrm{i}Y_{43} & Y_{33} \end{bmatrix}, \tag{26}$$

where

$$
\begin{aligned}
&Y_{11}=\hat{\alpha}_{11}^{(1)}+\hat{\alpha}_{11}^{(2)}, \quad Y_{13}=\hat{\alpha}_{13}^{(1)}+\hat{\alpha}_{13}^{(2)}, \quad Y_{33}=\hat{\alpha}_{33}^{(1)}+\hat{\alpha}_{33}^{(2)},\\
&Y_{21}=\hat{\alpha}_{12}^{(2)}-\hat{\alpha}_{12}^{(1)}, \quad Y_{23}=\hat{\alpha}_{23}^{(1)}-\hat{\alpha}_{23}^{(2)}, \quad Y_{43}=\hat{\alpha}_{43}^{(1)}-\hat{\alpha}_{43}^{(2)}.
\end{aligned} \tag{27}
$$

Considering equation (24), we introduce the sectionally holomorphic function vector $\boldsymbol{h}(z)$ as

$$
\begin{aligned}
\boldsymbol{h}(z)=\boldsymbol{B}_1\boldsymbol{f}_1'(z)-\boldsymbol{M}_*^{-1}\big[(\overline{\boldsymbol{M}}_1^{-1}-\overline{\boldsymbol{M}}_2^{-1})\overline{\boldsymbol{B}}_1(\boldsymbol{i}_2\overline{\phi_1'(0)}+\boldsymbol{i}_4\overline{\Phi_1'(0)})+2\boldsymbol{L}_2^{-1}\boldsymbol{B}_2\boldsymbol{k}\big]&\\
+R^2\boldsymbol{B}_1(\boldsymbol{i}_2\phi_1'(0)+\boldsymbol{i}_4\Phi_1'(0))z^{-2},& \qquad |z|<R,
\end{aligned} \tag{28a}
$$

$$
\begin{aligned}
\boldsymbol{h}(z)=\boldsymbol{M}_*^{-1}\overline{\boldsymbol{M}}_*\boldsymbol{B}_2\boldsymbol{f}_2'(z)-\boldsymbol{M}_*^{-1}\overline{\boldsymbol{M}}_*\boldsymbol{B}_2\boldsymbol{k}&\\
-R^2\boldsymbol{M}_*^{-1}\big[(\overline{\boldsymbol{M}}_1^{-1}-\overline{\boldsymbol{M}}_2^{-1})\overline{\boldsymbol{B}}_2\bar{\boldsymbol{k}}-2\boldsymbol{L}_1^{-1}\boldsymbol{B}_1(\boldsymbol{i}_2\phi_1'(0)+\boldsymbol{i}_4\Phi_1'(0))\big]z^{-2},& \qquad |z|>R.
\end{aligned} \tag{28b}
$$

It is not difficult to check that the newly introduced function $\boldsymbol{h}(z)$ is analytic in the regions $|z|<R$ and $|z|>R$ and is continuous across the bonded section of the interface. In addition, $\boldsymbol{h}(z)\cong O(z^{-2})$ as $|z|\to\infty$. The traction-free condition on the debonded portion of the interface will yield the nonhomogeneous Riemann–Hilbert problem of vector form as

$$
\begin{aligned}
\overline{\boldsymbol{M}}_*\boldsymbol{h}^+(z)+\boldsymbol{M}_*\boldsymbol{h}^-(z)=\boldsymbol{v}(z), &\quad z\in L_c,\\
\boldsymbol{h}^+(z)-\boldsymbol{h}^-(z)=\boldsymbol{0}, &\quad z\in L_b.
\end{aligned} \tag{29}
$$

Here, the superscripts "+" and "−" denote the limiting values as we approach the circular interface from S_1 and S_2 respectively, while the vector $\boldsymbol{v}(z)$ is defined by

$$
\begin{aligned}
\boldsymbol{v}(z)=-2\overline{\boldsymbol{M}}_*\boldsymbol{M}_*^{-1}\big[\boldsymbol{L}_2^{-1}\boldsymbol{B}_2\boldsymbol{k}+\boldsymbol{L}_1^{-1}\overline{\boldsymbol{B}}_1(\boldsymbol{i}_2\overline{\phi_1'(0)}+\boldsymbol{i}_4\overline{\Phi_1'(0)})\big]&\\
+2R^2\big[\boldsymbol{L}_2^{-1}\overline{\boldsymbol{B}}_2\bar{\boldsymbol{k}}+\boldsymbol{L}_1^{-1}\boldsymbol{B}_1(\boldsymbol{i}_2\phi_1'(0)+\boldsymbol{i}_4\Phi_1'(0))\big]z^{-2}.&
\end{aligned} \tag{30}
$$

In order to solve the nonhomogeneous Riemann–Hilbert problem in equation (29), we consider the eigenvalue problem

$$
\overline{\boldsymbol{M}}_*\boldsymbol{w}=\mathrm{e}^{2\pi\epsilon}\boldsymbol{M}_*\boldsymbol{w}. \tag{31}
$$

In view of the fact that both Hermitian matrices $\boldsymbol{M}_*$ and $\overline{\boldsymbol{M}}_*$ are positive definite, we can identify four distinct eigenpairs $(\epsilon_1,\boldsymbol{w}_1)$, $(-\epsilon_1,\overline{\boldsymbol{w}}_1)$, $(\epsilon_3,\boldsymbol{w}_3)$, $(-\epsilon_3,\overline{\boldsymbol{w}}_3)$ with ϵ_1, ϵ_3 two real numbers and $\boldsymbol{w}_1$, $\boldsymbol{w}_3$ two complex vectors such that

$$
\overline{\boldsymbol{M}}_*\boldsymbol{w}_1=\mathrm{e}^{2\pi\epsilon_1}\boldsymbol{M}_*\boldsymbol{w}_1, \quad \overline{\boldsymbol{M}}_*\boldsymbol{w}_3=\mathrm{e}^{2\pi\epsilon_3}\boldsymbol{M}_*\boldsymbol{w}_3. \tag{32}
$$

The four eigenvectors are orthogonal in the sense that

$$
\boldsymbol{w}_i^T\boldsymbol{M}_*\boldsymbol{w}_j=\overline{\boldsymbol{w}}_k^T\boldsymbol{M}_*\boldsymbol{w}_l=\boldsymbol{w}_k^T\boldsymbol{M}_*\overline{\boldsymbol{w}}_l=0, \quad (i,j,k,l=1,3 \text{ and } k\neq l). \tag{33}
$$

The two oscillatory indices ϵ_1 and ϵ_3 are given explicitly by [Wang and Schiavone 2013]:

$$
\epsilon_1=\frac{1}{2\pi}\ln\frac{1+\beta_1}{1-\beta_1}, \quad \epsilon_3=\frac{1}{2\pi}\ln\frac{1+\beta_2}{1-\beta_2}, \tag{34}
$$

where

$$
\beta_1,\beta_2=\sqrt{\xi}\big(\big[\tfrac{1}{2}(\rho+1)\big]^{\frac{1}{2}}\pm\big[\tfrac{1}{2}(\rho-1)\big]^{\frac{1}{2}}\big)>0, \tag{35}
$$

with

$$\xi = \sqrt{|\breve{\boldsymbol{S}}|} > 0, \quad \rho = -\frac{\operatorname{tr}(\breve{\boldsymbol{S}}^2)}{4\sqrt{|\breve{\boldsymbol{S}}|}} \geq 1, \tag{36}$$

and

$$\breve{\boldsymbol{S}} = (\boldsymbol{L}_1^{-1} + \boldsymbol{L}_2^{-1})^{-1}(\boldsymbol{S}_1\boldsymbol{L}_1^{-1} - \boldsymbol{S}_2\boldsymbol{L}_2^{-1}). \tag{37}$$

Furthermore, the two parameters ξ and ρ are found explicitly as

$$\xi = \frac{|Y_{21}Y_{43} - Y_{23}^2|}{Y_{11}Y_{33} - Y_{13}^2}, \quad \rho = \frac{(Y_{11}Y_{43} + Y_{33}Y_{21} - 2Y_{13}Y_{23})^2 - 2(Y_{11}Y_{33} - Y_{13}^2)(Y_{21}Y_{43} - Y_{23}^2)}{2(Y_{11}Y_{33} - Y_{13}^2)|Y_{21}Y_{43} - Y_{23}^2|}. \tag{38}$$

Now $\boldsymbol{h}(z)$ can be represented as a linear combination of the four eigenvectors as

$$\boldsymbol{h}(z) = h_1(z)\boldsymbol{w}_1 + h_2(z)\bar{\boldsymbol{w}}_1 + h_3(z)\boldsymbol{w}_3 + h_4(z)\bar{\boldsymbol{w}}_3. \tag{39}$$

Premultiplying equation $(29)_1$ by $[\bar{\boldsymbol{w}}_1 \;\; \boldsymbol{w}_1 \;\; \bar{\boldsymbol{w}}_3 \;\; \boldsymbol{w}_3]^T$ and using the representation in equation (39) with the orthogonal relations in equation (33), we obtain the decoupled form of the equations as

$$\left.\begin{aligned} h_1^+(z) + \mathrm{e}^{-2\pi\epsilon_1}h_1^-(z) &= \frac{\bar{\boldsymbol{w}}_1^T\boldsymbol{v}(z)}{\bar{\boldsymbol{w}}_1^T\overline{\boldsymbol{M}}_*\boldsymbol{w}_1}, \\ h_2^+(z) + \mathrm{e}^{2\pi\epsilon_1}h_2^-(z) &= \frac{\boldsymbol{w}_1^T\boldsymbol{v}(z)}{\boldsymbol{w}_1^T\overline{\boldsymbol{M}}_*\bar{\boldsymbol{w}}_1}, \\ h_3^+(z) + \mathrm{e}^{-2\pi\epsilon_3}h_3^-(z) &= \frac{\bar{\boldsymbol{w}}_3^T\boldsymbol{v}(z)}{\bar{\boldsymbol{w}}_3^T\overline{\boldsymbol{M}}_*\boldsymbol{w}_3}, \\ h_4^+(z) + \mathrm{e}^{2\pi\epsilon_3}h_4^-(z) &= \frac{\boldsymbol{w}_3^T\boldsymbol{v}(z)}{\boldsymbol{w}_3^T\overline{\boldsymbol{M}}_*\bar{\boldsymbol{w}}_3}, \end{aligned}\right\} \quad z \in L_c,$$
$$h_j^+(z) - h_j^-(z) = 0, \quad j = 1, 2, 3, 4, \quad z \in L_b. \tag{40}$$

The solutions to the four decoupled nonhomogeneous Riemann–Hilbert problems of scalar form in equation (40) are conveniently given by [Muskhelishvili 1953]:

$$\begin{aligned} h_1(z) &= \frac{1}{\bar{\boldsymbol{w}}_1^T\overline{\boldsymbol{M}}_*\boldsymbol{w}_1}\frac{\chi_1(z)}{2\pi\mathrm{i}}\int_{L_c}\frac{\bar{\boldsymbol{w}}_1^T\boldsymbol{v}(t)\,\mathrm{d}t}{\chi_1^+(t)(t-z)}, &\quad h_3(z) &= \frac{1}{\bar{\boldsymbol{w}}_3^T\overline{\boldsymbol{M}}_*\boldsymbol{w}_3}\frac{\chi_3(z)}{2\pi\mathrm{i}}\int_{L_c}\frac{\bar{\boldsymbol{w}}_3^T\boldsymbol{v}(t)\,\mathrm{d}t}{\chi_3^+(t)(t-z)}, \\ h_2(z) &= \frac{1}{\boldsymbol{w}_1^T\overline{\boldsymbol{M}}_*\bar{\boldsymbol{w}}_1}\frac{\chi_2(z)}{2\pi\mathrm{i}}\int_{L_c}\frac{\boldsymbol{w}_1^T\boldsymbol{v}(t)\,\mathrm{d}t}{\chi_2^+(t)(t-z)}, &\quad h_4(z) &= \frac{1}{\boldsymbol{w}_3^T\overline{\boldsymbol{M}}_*\bar{\boldsymbol{w}}_3}\frac{\chi_4(z)}{2\pi\mathrm{i}}\int_{L_c}\frac{\boldsymbol{w}_3^T\boldsymbol{v}(t)\,\mathrm{d}t}{\chi_4^+(t)(t-z)}, \end{aligned} \tag{41}$$

where

$$\begin{aligned} \chi_1(z) &= (z-a)^{-\frac{1}{2}-\mathrm{i}\epsilon_1}(z-\bar{a})^{-\frac{1}{2}+\mathrm{i}\epsilon_1}, &\quad \chi_3(z) &= (z-a)^{-\frac{1}{2}-\mathrm{i}\epsilon_3}(z-\bar{a})^{-\frac{1}{2}+\mathrm{i}\epsilon_3}, \\ \chi_2(z) &= (z-a)^{-\frac{1}{2}+\mathrm{i}\epsilon_1}(z-\bar{a})^{-\frac{1}{2}-\mathrm{i}\epsilon_1}, &\quad \chi_4(z) &= (z-a)^{-\frac{1}{2}+\mathrm{i}\epsilon_3}(z-\bar{a})^{-\frac{1}{2}-\mathrm{i}\epsilon_3}. \end{aligned} \tag{42}$$

The branch cuts for the Plemelj functions $\chi_1(z)$, $\chi_2(z)$, $\chi_3(z)$ and $\chi_4(z)$ are chosen to be the debonded part of the interface; i.e., $z \in L_c$ such that $\chi_1(z)$, $\chi_2(z)$, $\chi_3(z)$, $\chi_4(z) \cong z^{-1}$ as $|z| \to \infty$. We note here that the solutions given by equation (41) are unique in view of the fact that $\boldsymbol{h}(z) \cong O(z^{-2})$ as $|z| \to \infty$ (or equivalently $h_j(z) \cong O(z^{-2}),\ j = 1, \dots, 4$ as $|z| \to \infty$). By evaluating the Cauchy integrals in equation (41) and noting equation (39), we obtain the expression for $\boldsymbol{h}(z)$ as

$$\begin{aligned}\boldsymbol{h}(z) = &-2\sum_{j=1}^{4} \frac{\boldsymbol{w}_j \bar{\boldsymbol{w}}_j^T \overline{\boldsymbol{M}}_* \boldsymbol{M}_*^{-1}\big[\boldsymbol{L}_2^{-1}\boldsymbol{B}_2\boldsymbol{k}+\boldsymbol{L}_1^{-1}\overline{\boldsymbol{B}}_1(\boldsymbol{i}_2\overline{\phi_1'(0)}+\boldsymbol{i}_4\overline{\Phi_1'(0)})\big]}{\bar{\boldsymbol{w}}_j^T \overline{\boldsymbol{M}}_* \boldsymbol{w}_j(1+\mathrm{e}^{-2\pi\epsilon_j})}\big[1-\chi_j(z)[z-\mathrm{Re}\{a(1+2\mathrm{i}\epsilon_j)\}]\big]\\ &+2R^2\sum_{j=1}^{4} \frac{\boldsymbol{w}_j \bar{\boldsymbol{w}}_j^T \big[\boldsymbol{L}_2^{-1}\overline{\boldsymbol{B}}_2\bar{\boldsymbol{k}}+\boldsymbol{L}_1^{-1}\boldsymbol{B}_1(\boldsymbol{i}_2\phi_1'(0)+\boldsymbol{i}_4\Phi_1'(0))\big]}{\bar{\boldsymbol{w}}_j^T \overline{\boldsymbol{M}}_* \boldsymbol{w}_j(1+\mathrm{e}^{-2\pi\epsilon_j})}\left[\frac{1}{z^2}-\frac{\chi_j(z)}{\chi_j(0)z^2}+\frac{\chi_j(z)\chi_j'(0)}{[\chi_j(0)]^2 z}\right], \end{aligned} \tag{43}$$

where $\boldsymbol{w}_2 = \bar{\boldsymbol{w}}_1$, $\boldsymbol{w}_4 = \bar{\boldsymbol{w}}_3$, $\epsilon_2 = -\epsilon_1$, $\epsilon_4 = -\epsilon_3$.

There remain two complex constants $\phi'(0)$ and $\Phi'(0)$ to be determined in the expression for $\boldsymbol{h}(z)$. The consistency conditions for $\phi'(0)$ and $\Phi'(0)$ can be derived from equation (28a) as

$$\begin{aligned}&\phi_1'(0)-R^2\boldsymbol{i}_1^T\boldsymbol{B}_1^{-1}\sum_{j=1}^{4}\frac{\boldsymbol{w}_j\bar{\boldsymbol{w}}_j^T\big[2[\chi_j'(0)]^2-\chi_j(0)\chi_j''(0)\big]}{\bar{\boldsymbol{w}}_j^T\overline{\boldsymbol{M}}_*\boldsymbol{w}_j(1+\mathrm{e}^{-2\pi\epsilon_j})[\chi_j(0)]^2}\boldsymbol{L}_1^{-1}\boldsymbol{B}_1(\boldsymbol{i}_2\phi_1'(0)+\boldsymbol{i}_4\Phi_1'(0))\\ &\quad+\boldsymbol{i}_1^T\boldsymbol{B}_1^{-1}\left[\boldsymbol{M}_*^{-1}(\overline{\boldsymbol{M}}_2^{-1}-\overline{\boldsymbol{M}}_1^{-1})+2\sum_{j=1}^{4}\frac{\boldsymbol{w}_j\bar{\boldsymbol{w}}_j^T\big[1+\chi_j(0)\,\mathrm{Re}\{a(1+2\mathrm{i}\epsilon_j)\}\big]}{\bar{\boldsymbol{w}}_j^T\overline{\boldsymbol{M}}_*\boldsymbol{w}_j(1+\mathrm{e}^{-2\pi\epsilon_j})}\overline{\boldsymbol{M}}_*\boldsymbol{M}_*^{-1}\boldsymbol{L}_1^{-1}\right]\\ &\qquad\qquad\times\overline{\boldsymbol{B}}_1(\boldsymbol{i}_2\overline{\phi_1'(0)}+\boldsymbol{i}_4\overline{\Phi_1'(0)})\\ &=2\boldsymbol{i}_1^T\boldsymbol{B}_1^{-1}\left[\boldsymbol{I}-\sum_{j=1}^{4}\frac{\boldsymbol{w}_j\bar{\boldsymbol{w}}_j^T\big[1+\chi_j(0)\,\mathrm{Re}\{a(1+2\mathrm{i}\epsilon_j)\}\big]}{\bar{\boldsymbol{w}}_j^T\overline{\boldsymbol{M}}_*\boldsymbol{w}_j(1+\mathrm{e}^{-2\pi\epsilon_j})}\overline{\boldsymbol{M}}_*\right]\boldsymbol{M}_*^{-1}\boldsymbol{L}_2^{-1}\boldsymbol{B}_2\boldsymbol{k}\\ &\qquad+R^2\boldsymbol{i}_1^T\boldsymbol{B}_1^{-1}\sum_{j=1}^{4}\frac{\boldsymbol{w}_j\bar{\boldsymbol{w}}_j^T\big[2[\chi_j'(0)]^2-\chi_j(0)\chi_j''(0)\big]}{\bar{\boldsymbol{w}}_j^T\overline{\boldsymbol{M}}_*\boldsymbol{w}_j(1+\mathrm{e}^{-2\pi\epsilon_j})[\chi_j(0)]^2}\boldsymbol{L}_2^{-1}\overline{\boldsymbol{B}}_2\bar{\boldsymbol{k}}, \end{aligned}\tag{44}$$

$$\begin{aligned}&\Phi_1'(0)-R^2\boldsymbol{i}_3^T\boldsymbol{B}_1^{-1}\sum_{j=1}^{4}\frac{\boldsymbol{w}_j\bar{\boldsymbol{w}}_j^T\big[2[\chi_j'(0)]^2-\chi_j(0)\chi_j''(0)\big]}{\bar{\boldsymbol{w}}_j^T\overline{\boldsymbol{M}}_*\boldsymbol{w}_j(1+\mathrm{e}^{-2\pi\epsilon_j})[\chi_j(0)]^2}\boldsymbol{L}_1^{-1}\boldsymbol{B}_1(\boldsymbol{i}_2\phi_1'(0)+\boldsymbol{i}_4\Phi_1'(0))\\ &\quad+\boldsymbol{i}_3^T\boldsymbol{B}_1^{-1}\left[\boldsymbol{M}_*^{-1}(\overline{\boldsymbol{M}}_2^{-1}-\overline{\boldsymbol{M}}_1^{-1})+2\sum_{j=1}^{4}\frac{\boldsymbol{w}_j\bar{\boldsymbol{w}}_j^T\big[1+\chi_j(0)\,\mathrm{Re}\{a(1+2\mathrm{i}\epsilon_j)\}\big]}{\bar{\boldsymbol{w}}_j^T\overline{\boldsymbol{M}}_*\boldsymbol{w}_j(1+\mathrm{e}^{-2\pi\epsilon_j})}\overline{\boldsymbol{M}}_*\boldsymbol{M}_*^{-1}\boldsymbol{L}_1^{-1}\right]\\ &\qquad\qquad\times\overline{\boldsymbol{B}}_1\big(\boldsymbol{i}_2\overline{\phi_1'(0)}+\boldsymbol{i}_4\overline{\Phi_1'(0)}\big)\\ &=2\boldsymbol{i}_3^T\boldsymbol{B}_1^{-1}\left[\boldsymbol{I}-\sum_{j=1}^{4}\frac{\boldsymbol{w}_j\bar{\boldsymbol{w}}_j^T\big[1+\chi_j(0)\,\mathrm{Re}\{a(1+2\mathrm{i}\epsilon_j)\}\big]}{\bar{\boldsymbol{w}}_j^T\overline{\boldsymbol{M}}_*\boldsymbol{w}_j(1+\mathrm{e}^{-2\pi\epsilon_j})}\overline{\boldsymbol{M}}_*\right]\boldsymbol{M}_*^{-1}\boldsymbol{L}_2^{-1}\boldsymbol{B}_2\boldsymbol{k}\\ &\qquad+R^2\boldsymbol{i}_3^T\boldsymbol{B}_1^{-1}\sum_{j=1}^{4}\frac{\boldsymbol{w}_j\bar{\boldsymbol{w}}_j^T\big[2[\chi_j'(0)]^2-\chi_j(0)\chi_j''(0)\big]}{\bar{\boldsymbol{w}}_j^T\overline{\boldsymbol{M}}_*\boldsymbol{w}_j(1+\mathrm{e}^{-2\pi\epsilon_j})[\chi_j(0)]^2}\boldsymbol{L}_2^{-1}\overline{\boldsymbol{B}}_2\bar{\boldsymbol{k}}. \end{aligned}\tag{45}$$

The two complex constants $\phi'(0)$ and $\Phi'(0)$ can then be uniquely determined by solving the coupled linear algebraic equations in (44) and (45).

Remark 3. Equations (44) and (45) can be obtained in a straightforward manner by evaluating $\boldsymbol{h}(0)$ from the expression for $\boldsymbol{h}(z)$ in equation (43) and by noting that

$$\boldsymbol{i}_1^T \boldsymbol{B}_1^{-1}\boldsymbol{h}(0) = \phi_1'(0) - \boldsymbol{i}_1^T \boldsymbol{B}_1^{-1}\boldsymbol{M}_*^{-1}\big[(\overline{\boldsymbol{M}}_1^{-1} - \overline{\boldsymbol{M}}_2^{-1})\overline{\boldsymbol{B}}_1(\boldsymbol{i}_2\overline{\phi_1'(0)} + \boldsymbol{i}_4\overline{\Phi_1'(0)}) + 2\boldsymbol{L}_2^{-1}\boldsymbol{B}_2\boldsymbol{k}\big],$$
$$\boldsymbol{i}_3^T \boldsymbol{B}_1^{-1}\boldsymbol{h}(0) = \Phi_1'(0) - \boldsymbol{i}_3^T \boldsymbol{B}_1^{-1}\boldsymbol{M}_*^{-1}\big[(\overline{\boldsymbol{M}}_1^{-1} - \overline{\boldsymbol{M}}_2^{-1})\overline{\boldsymbol{B}}_1(\boldsymbol{i}_2\overline{\phi_1'(0)} + \boldsymbol{i}_4\overline{\Phi_1'(0)}) + 2\boldsymbol{L}_2^{-1}\boldsymbol{B}_2\boldsymbol{k}\big].$$

In addition, $\int \boldsymbol{h}(z)\,\mathrm{d}z$ can also be found exactly as

$$\int \boldsymbol{h}(z)\,\mathrm{d}z = -2\sum_{j=1}^{4} \frac{\boldsymbol{w}_j \overline{\boldsymbol{w}}_j^T \overline{\boldsymbol{M}}_* \boldsymbol{M}_*^{-1}\big[\boldsymbol{L}_2^{-1}\boldsymbol{B}_2\boldsymbol{k} + \boldsymbol{L}_1^{-1}\overline{\boldsymbol{B}}_1(\boldsymbol{i}_2\overline{\phi_1'(0)} + \boldsymbol{i}_4\overline{\Phi_1'(0)})\big]}{\overline{\boldsymbol{w}}_j^T \overline{\boldsymbol{M}}_* \boldsymbol{w}_j(1+\mathrm{e}^{-2\pi\epsilon_j})}[z - \mathrm{X}_j(z)]$$
$$-2R^2\sum_{j=1}^{4} \frac{\boldsymbol{w}_j \overline{\boldsymbol{w}}_j^T \big[\boldsymbol{L}_2^{-1}\overline{\boldsymbol{B}}_2\bar{\boldsymbol{k}} + \boldsymbol{L}_1^{-1}\boldsymbol{B}_1(\boldsymbol{i}_2\phi_1'(0) + \boldsymbol{i}_4\Phi_1'(0))\big]}{\overline{\boldsymbol{w}}_j^T \overline{\boldsymbol{M}}_* \boldsymbol{w}_j(1+\mathrm{e}^{-2\pi\epsilon_j})}\left[\frac{1}{z} - \frac{\mathrm{X}_j(z)}{\mathrm{X}_j(0)z}\right], \quad (46)$$

where

$$\mathrm{X}_j(z) = (z-a)^{\frac{1}{2}-\mathrm{i}\epsilon_j}(z-\bar{a})^{\frac{1}{2}+\mathrm{i}\epsilon_j}, \quad j = 1,2,3,4. \quad (47)$$

The branch cuts for $\mathrm{X}_j(z)$ are again chosen along the arc crack L_c such that $\mathrm{X}_j(z) \cong z$ as $|z| \to \infty$.

5. The elastic field along the interface

Using equations (43), (46) and (28), we can obtain expressions for the two sets of four analytic functions $\phi_1(z)$, $\psi_1(z)$, $\Phi_1(z)$, $\Psi_1(z)$ and $\phi_2(z)$, $\psi_2(z)$, $\Phi_2(z)$, $\Psi_2(z)$ together with their derivatives defined in the inhomogeneity and in the matrix, respectively. Consequently, we can find the stress resultants, in-plane displacements and slopes everywhere in the inhomogeneity and in the matrix. In particular, the surface membrane shear stress resultant $\widehat{N}_{\theta r}$, surface membrane normal stress resultant $\widehat{N}_{rr}$ and surface bending moment $\widehat{M}_{rr}$ along the bonded portion of the circular interface are given by

$$\boldsymbol{t}_r = [-\widehat{N}_{\theta r} \;\; \widehat{N}_{rr} \;\; * \;\; \widehat{M}_{rr}]^T = \boldsymbol{\Omega}(\theta)\boldsymbol{t}, \quad z = R\mathrm{e}^{\mathrm{i}\theta}, \quad -\theta_0 < \theta < \theta_0, \quad (48)$$

where $\boldsymbol{t} = -\mathrm{d}\hat{\boldsymbol{\varphi}}_1/(R\,\mathrm{d}\theta)$ is given explicitly by the expression

$$\boldsymbol{t} = \frac{4}{R}\mathrm{Im}\left\{\sum_{j=1}^{4} \frac{\mathrm{e}^{2\pi\epsilon_j}\boldsymbol{w}_j \overline{\boldsymbol{w}}_j^T \big[\boldsymbol{L}_2^{-1}\boldsymbol{B}_2\boldsymbol{k} + \boldsymbol{L}_1^{-1}\overline{\boldsymbol{B}}_1(\boldsymbol{i}_2\overline{\phi_1'(0)} + \boldsymbol{i}_4\overline{\Phi_1'(0)})\big]}{\overline{\boldsymbol{w}}_j^T \overline{\boldsymbol{M}}_* \boldsymbol{w}_j} z\chi_j(z)[z - \mathrm{Re}\{a(1+2\mathrm{i}\epsilon_j)\}]\right\}, \quad (49)$$

and the 4×4 orthogonal matrix $\boldsymbol{\Omega}(\theta)$ is defined by

$$\boldsymbol{\Omega}(\theta) = \begin{bmatrix} \sin\theta & -\cos\theta & 0 & 0 \\ \cos\theta & \sin\theta & 0 & 0 \\ 0 & 0 & \sin\theta & -\cos\theta \\ 0 & 0 & \cos\theta & \sin\theta \end{bmatrix}. \quad (50)$$

In addition, the jump in the generalized displacement vector across the debonded part of the interface is given by

$$\hat{\boldsymbol{u}}_1 - \hat{\boldsymbol{u}}_2 = 4\,\mathrm{Im}\left\{\sum_{j=1}^{4} \frac{\mathrm{e}^{4\pi\epsilon_j}\boldsymbol{M}_*\boldsymbol{w}_j \overline{\boldsymbol{w}}_j^T \big[\boldsymbol{L}_2^{-1}\boldsymbol{B}_2\boldsymbol{k} + \boldsymbol{L}_1^{-1}\overline{\boldsymbol{B}}_1(\boldsymbol{i}_2\overline{\phi_1'(0)} + \boldsymbol{i}_4\overline{\Phi_1'(0)})\big]}{\overline{\boldsymbol{w}}_j^T \overline{\boldsymbol{M}}_* \boldsymbol{w}_j}\mathrm{X}_j^+(z)\right\}, \quad z \in L_c. \quad (51)$$

It is further deduced from equations (48) and (49) that the surface membrane stress resultants and surface bending moment are singularly distributed near the crack tips as

$$\begin{aligned} \boldsymbol{t}_r &= (2\pi r)^{-\frac{1}{2}}\boldsymbol{\Omega}(+\theta_0)[K_1 r^{\mathrm{i}\epsilon_1}\boldsymbol{w}_1+\bar{K}_1 r^{-\mathrm{i}\epsilon_1}\bar{\boldsymbol{w}}_1+K_3 r^{\mathrm{i}\epsilon_3}\boldsymbol{w}_3+\bar{K}_3 r^{-\mathrm{i}\epsilon_3}\bar{\boldsymbol{w}}_3], \quad r=|z-a|\to 0, \quad z\in L_b; \\ \boldsymbol{t}_r &= (2\pi r)^{-\frac{1}{2}}\boldsymbol{\Omega}(-\theta_0)[K_1 r^{\mathrm{i}\epsilon_1}\boldsymbol{w}_1+\bar{K}_1 r^{-\mathrm{i}\epsilon_1}\bar{\boldsymbol{w}}_1+K_3 r^{\mathrm{i}\epsilon_3}\boldsymbol{w}_3+\bar{K}_3 r^{-\mathrm{i}\epsilon_3}\bar{\boldsymbol{w}}_3], \quad r=|z-\bar{a}|\to 0, \quad z\in L_b, \end{aligned} \tag{52}$$

where K_1 and K_3 are two complex intensity factors derived as

$$\begin{aligned} K_1 &= 2(1+2\mathrm{i}\epsilon_1)(\pi R)^{\frac{1}{2}}(2R)^{-\mathrm{i}\epsilon_1}(\sin\theta_0)^{\frac{1}{2}-\mathrm{i}\epsilon_1} \\ &\quad\times\Bigg(\frac{\boldsymbol{w}_1^T e^{\epsilon_1(\pi+\theta_0)-\frac{\theta_0}{2}\mathrm{i}}\big[\boldsymbol{L}_2^{-1}\bar{\boldsymbol{B}}_2\bar{\boldsymbol{k}}+\boldsymbol{L}_1^{-1}\boldsymbol{B}_1(\boldsymbol{i}_2\phi_1'(0)+\boldsymbol{i}_4\Phi_1'(0))\big]}{\bar{\boldsymbol{w}}_1^T\bar{\boldsymbol{M}}_*\boldsymbol{w}_1} \\ &\qquad\qquad+\frac{\boldsymbol{w}_1^T e^{\epsilon_1(\pi-\theta_0)+\frac{\theta_0}{2}\mathrm{i}}\big[\boldsymbol{L}_2^{-1}\boldsymbol{B}_2\boldsymbol{k}+\boldsymbol{L}_1^{-1}\bar{\boldsymbol{B}}_1(\boldsymbol{i}_2\overline{\phi_1'(0)}+\boldsymbol{i}_4\overline{\Phi_1'(0)})\big]}{\bar{\boldsymbol{w}}_1^T\bar{\boldsymbol{M}}_*\boldsymbol{w}_1}\Bigg), \\ K_3 &= 2(1+2\mathrm{i}\epsilon_3)(\pi R)^{\frac{1}{2}}(2R)^{-\mathrm{i}\epsilon_3}(\sin\theta_0)^{\frac{1}{2}-\mathrm{i}\epsilon_3} \\ &\quad\times\Bigg(\frac{\boldsymbol{w}_3^T e^{\epsilon_3(\pi+\theta_0)-\frac{\theta_0}{2}\mathrm{i}}\big[\boldsymbol{L}_2^{-1}\bar{\boldsymbol{B}}_2\bar{\boldsymbol{k}}+\boldsymbol{L}_1^{-1}\boldsymbol{B}_1(\boldsymbol{i}_2\phi_1'(0)+\boldsymbol{i}_4\Phi_1'(0))\big]}{\bar{\boldsymbol{w}}_3^T\bar{\boldsymbol{M}}_*\boldsymbol{w}_3} \\ &\qquad\qquad+\frac{\boldsymbol{w}_3^T e^{\epsilon_3(\pi-\theta_0)+\frac{\theta_0}{2}\mathrm{i}}\big[\boldsymbol{L}_2^{-1}\boldsymbol{B}_2\boldsymbol{k}+\boldsymbol{L}_1^{-1}\bar{\boldsymbol{B}}_1(\boldsymbol{i}_2\overline{\phi_1'(0)}+\boldsymbol{i}_4\overline{\Phi_1'(0)})\big]}{\bar{\boldsymbol{w}}_3^T\bar{\boldsymbol{M}}_*\boldsymbol{w}_3}\Bigg) \end{aligned} \tag{53}$$

at the upper crack tip $z=a$, and

$$\begin{aligned} K_1 &= -2(1+2\mathrm{i}\epsilon_1)(\pi R)^{\frac{1}{2}}(2R)^{-\mathrm{i}\epsilon_1}(\sin\theta_0)^{\frac{1}{2}-\mathrm{i}\epsilon_1} \\ &\quad\times\Bigg(\frac{\bar{\boldsymbol{w}}_1^T e^{\epsilon_1(\pi-\theta_0)+\frac{\theta_0}{2}\mathrm{i}}\big[\boldsymbol{L}_2^{-1}\bar{\boldsymbol{B}}_2\bar{\boldsymbol{k}}+\boldsymbol{L}_1^{-1}\boldsymbol{B}_1(\boldsymbol{i}_2\phi_1'(0)+\boldsymbol{i}_4\Phi_1'(0))\big]}{\bar{\boldsymbol{w}}_1^T\bar{\boldsymbol{M}}_*\boldsymbol{w}_1} \\ &\qquad\qquad+\frac{\bar{\boldsymbol{w}}_1^T e^{\epsilon_1(\pi+\theta_0)-\frac{\theta_0}{2}\mathrm{i}}\big[\boldsymbol{L}_2^{-1}\boldsymbol{B}_2\boldsymbol{k}+\boldsymbol{L}_1^{-1}\bar{\boldsymbol{B}}_1\big(\boldsymbol{i}_2\overline{\phi_1'(0)}+\boldsymbol{i}_4\overline{\Phi_1'(0)}\big)\big]}{\bar{\boldsymbol{w}}_1^T\bar{\boldsymbol{M}}_*\boldsymbol{w}_1}\Bigg), \\ K_3 &= -2(1+2\mathrm{i}\epsilon_3)(\pi R)^{\frac{1}{2}}(2R)^{-\mathrm{i}\epsilon_3}(\sin\theta_0)^{\frac{1}{2}-\mathrm{i}\epsilon_3} \\ &\quad\times\Bigg(\frac{\bar{\boldsymbol{w}}_3^T e^{\epsilon_3(\pi-\theta_0)+\frac{\theta_0}{2}\mathrm{i}}\big[\boldsymbol{L}_2^{-1}\bar{\boldsymbol{B}}_2\bar{\boldsymbol{k}}+\boldsymbol{L}_1^{-1}\boldsymbol{B}_1(\boldsymbol{i}_2\phi_1'(0)+\boldsymbol{i}_4\Phi_1'(0))\big]}{\bar{\boldsymbol{w}}_3^T\bar{\boldsymbol{M}}_*\boldsymbol{w}_3} \\ &\qquad\qquad+\frac{\bar{\boldsymbol{w}}_3^T e^{\epsilon_3(\pi+\theta_0)-\frac{\theta_0}{2}\mathrm{i}}\big[\boldsymbol{L}_2^{-1}\boldsymbol{B}_2\boldsymbol{k}+\boldsymbol{L}_1^{-1}\bar{\boldsymbol{B}}_1(\boldsymbol{i}_2\overline{\phi_1'(0)}+\boldsymbol{i}_4\overline{\Phi_1'(0)})\big]}{\bar{\boldsymbol{w}}_3^T\bar{\boldsymbol{M}}_*\boldsymbol{w}_3}\Bigg) \end{aligned} \tag{54}$$

at the lower crack tip $z=\bar{a}$.

If the crack propagates along the circular interface, the energy release rate can then be conveniently obtained as [Wang and Schiavone 2013]

$$G = \frac{\bar{\boldsymbol{w}}_1^T(\boldsymbol{L}_1^{-1}+\boldsymbol{L}_2^{-1})\boldsymbol{w}_1|K_1|^2}{2\cosh^2\pi\epsilon_1} + \frac{\bar{\boldsymbol{w}}_3^T(\boldsymbol{L}_1^{-1}+\boldsymbol{L}_2^{-1})\boldsymbol{w}_3|K_3|^2}{2\cosh^2\pi\epsilon_3}. \tag{55}$$

6. Conclusions

We undertake a rigorous analysis of a challenging problem associated with a circular inhomogeneity partially bonded to an infinite isotropic laminated thin plate subjected to remote uniform membrane stress resultants and bending moments. A novel Stroh-type formalism is used to obtain a closed-form solution by reducing the boundary value problem to a nonhomogeneous Riemann–Hilbert problem of vector form which is solved analytically using a decoupling method and by evaluating the ensuing Cauchy integrals. The stress resultants, in-plane displacements and slopes everywhere in the composite are determined from the resulting analytic functions. The pertinent elastic fields on the circular interface such as the surface membrane shear stress resultant, surface membrane normal stress resultant and surface bending moment along the bonded part of the interface, the jump in the generalized displacement vector across the debonded section of the interface and the complex intensity factors at the two crack tips are all determined explicitly.

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