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TRANSIENT GROWTH OF A PLANAR CRACK IN THREE DIMENSIONS: MIXED MODE

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Transient growth in 3D of a semi-infinite, plane brittle crack in an isotropic, elastic solid is considered. Growth is mixed-mode, caused by in-plane and normal point forces on each face of an existing semi-infinite crack. An analytic solution is obtained for the case of dynamic similarity, i.e., crack edge speed is subcritical and may vary continuously with direction, but is time-invariant. The dynamic energy release rate criterion, with kinetic energy included, is imposed. A nonlinear differential equation for crack edge speed results, and allows the description of crack contour, i.e., the curve formed by the crack edge in the crack plane. Study indicates that forces of a type that increase rapidly from zero can create a fracture initiation phase in which crack growth rate indeed does not vary with time.

Introduction

In 2D dynamic fracture, the rectilinear crack edge can be defined by an equation of motion for the crack tip [Freund 1972; 1990]. In a 3D study, such an equation must describe the crack contour, i.e., curve formed by the crack edge in the crack plane. For the dynamic steady state case, this goal is considered in [Brock 2015a; 2015b]. A semi-infinite crack, driven by compressive point forces that translate on its surfaces, extends at subcritical constant speed in an unbounded, isotropic solid. Results indicate that the crack edge is rectilinear away from the point force path, but forms a bulge near the forces.

This paper treats a transient version of a similar problem. Stationary point forces are applied at the edge of an initially undisturbed, plane semi-infinite crack. The forces include both normal and in plane components, and mixed-mode, brittle fracture occurs. A (presumably) concave region of new crack surface grows out from the point forces. Region behavior is dynamically similar, i.e., points on its edge move at subcritical speeds that may vary with position, but not with time. The process is governed by dynamic energy release rate [Freund 1990; Brock 2015a; 2015b]. In contrast to [Brock 2015a; 2015b] however, the concept of surface energy density [Freund 1990] is employed. In contrast to [Freund 1990; Brock 2015a], kinetic energy [Gdoutos 2005; Brock 2015b] is included in the energy rate balance.

Problem statement

Closed crack $A_C(x_3^0 = 0, x_1^0 < 0)$, with boundary $C(x_1^0, x_3^0) = 0$, exists in an unbounded elastic solid. Cartesian coordinates are denoted by $\boldsymbol{x}_0 = \boldsymbol{x}_0(x_k^0)$, and $k = (1, 2, 3)$. The solid is at rest for time $t \leq 0$, but point forces (compression and shear) appear for $t > 0$ on both crack faces at points $(x_1^0 = 0-, x_2^0 = 0, x_3^0 = 0\pm)$. Brittle fracture is instantaneous, and the crack extends outward from $\boldsymbol{x}_0 = 0$. The crack now

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occupies region $A_C + \delta A$ and boundary C now includes a concave bulge [Brock 2015a; 2015b] that can be defined as

$$\sqrt{(x_1^0)^2 + (x_2^0)^2} = l(\theta, t), \quad l(\theta, t) = V(\theta)t, \tag{1a}$$

$$0 < V < V_R, \quad \theta = \tan^{-1}\frac{x_2^0}{x_1^0}\big(|\theta| < \tfrac{1}{2}\pi\big). \tag{1b}$$

Equation (1) imposes a dynamically similar geometry on the crack. Equations that govern displacement $\boldsymbol{u}(u_k)$ and traction $\boldsymbol{T}(\sigma_{ik})$ for $t > 0$ are [Achenbach 1973]

$$\nabla\cdot\boldsymbol{T} - \rho\ddot{\boldsymbol{u}} = 0, \tag{2a}$$

$$\frac{1}{\mu}\boldsymbol{T} = \frac{2\nu}{1-2\nu}(\nabla\cdot\boldsymbol{u})\mathbb{1} + \nabla\boldsymbol{u} + \boldsymbol{u}\nabla. \tag{2b}$$

In (2), components $u_k = u_k(\boldsymbol{x}_0, t)$ and $\sigma_{ik} = \sigma_{ik}(\boldsymbol{x}_0, t)$; respectively, $(\nabla, \nabla^2, \mathbb{1})$ are gradient, Laplacian, and identity tensor. Operations $(Df, \dot{f})$ signify differentiation with respect to t in the fixed frame of $\boldsymbol{x}_0$, and (μ, ρ, ν) respectively are shear modulus, mass density and Poisson's ratio. Uncoupling of (2a) gives

$$\boldsymbol{u} = \boldsymbol{u}_S + \boldsymbol{u}_D, \tag{3a}$$

$$V_S^2\nabla^2\boldsymbol{u}_S - \ddot{\boldsymbol{u}}_S = 0, \quad V_D^2\nabla^2\boldsymbol{u}_D - \ddot{\boldsymbol{u}}_D = 0, \tag{3b}$$

$$\nabla\cdot\boldsymbol{u}_S = 0, \qquad \nabla\times\boldsymbol{u}_D = 0. \tag{3c}$$

Equation (1b) indicates that Rayleigh speed $V_R < V_S$ is the bound for subcritical crack extension. In (3b) (V_S, V_D) are speeds of shear and dilatational waves:

$$V_S = \sqrt{\frac{\mu}{\rho}}, \quad V_D = c_D V_S, \quad c_D = \sqrt{2\frac{1-\nu}{1-2\nu}}. \tag{4}$$

For $x_3^0 = 0\pm$ and $(x_1^0, x_2^0) \in A_C + \delta A$ (for $t > 0$),

$$\sigma_{3k} = -P_k\delta(x_1^0)\delta(x_2^0). \tag{5a}$$

For $x_3^0 = 0$ and $(x_1^0, x_2^0) \notin A_C + \delta A$ (for $t > 0$),

$$[u_k] = 0. \tag{5b}$$

In (5), force P_k is a positive constant, $\delta(f)$ denotes Dirac function, and $[f] = f^{(+)} - f^{(-)}$ where $f^{(\pm)} = f(x_1^0, x_2^0, 0\pm, t)$. Also $[u_k]$ in $A_C + \delta A$ must vanish continuously on C, but σ_{3k} for $(x_1^0, x_2^0) \notin A_C + \delta A$ can exhibit (integrable) singular behavior on C. For $t \le 0$, then $(\boldsymbol{u}, \boldsymbol{T}) \equiv 0$, and for finite $t > 0$, then $(\boldsymbol{u}, \boldsymbol{T})$ must be bounded as $|\boldsymbol{x}_0| \to \infty$.

Discontinuity problem

A common, e.g., [Barber 1992], procedure for solving crack problems is to represent the relative motion of crack faces as unknown discontinuities in displacement. To implement this procedure for the present initial/boundary value problem, the related problem of discontinuities in $(\boldsymbol{u}, \boldsymbol{T})$ is now considered. The unbounded solid is again at rest when at time $t = 0$ the discontinuities are imposed in the same region A_C of the $x_1^0 x_2^0$-plane. That is, conditions (5b) still hold, but for $x_3 = 0$ and $(x_1^0, x_2^0) \in A_C + \delta A$ (for $t > 0$):

$$[u_k] = \Delta_k, \quad [\sigma_{3k}] = \Sigma_k. \tag{6}$$

Here (Δ_k, Σ_k) are continuous functions of (x_1^0, x_2^0, t). They are bounded in $A_C + \delta A$ for $\sqrt{(x_1^0)^2 + (x_2^0)^2} \to \infty$ and vanish on C and for $t \le 0$. The requirements for (Δ_k, Σ_k) suggest that conditions for $t \le 0$ and for finite $t > 0$ and $|\boldsymbol{x}_0| \to \infty$ are again satisfied.

Transform solution

An effective procedure, e.g., [Brock and Achenbach 1973], for 2D transient study of semi-infinite crack extension at constant speed employs coordinates that translate with the crack edge, and unilateral temporal and bilateral spatial integral transforms [Sneddon 1972]. In view of (1), translating base $\boldsymbol{x}$ is defined as

$$x_1 = x_1^0 - [c(\theta)\cos\theta]s, \quad x_2 = x_2^0 - [c(\theta)\sin\theta]s, \quad x_3 = x_3^0, \tag{7a}$$

$$s = V_S t, \quad c(\theta) = \frac{V(\theta)}{V_S}, \tag{7b}$$

$$\dot{f} = Df = V_S[\partial_S f - c(\theta)(\partial_1 f \cos\theta + \partial_2 f \sin\theta)], \tag{7c}$$

$$\partial_S f = \frac{\partial f}{\partial s}, \quad \partial_k f = \frac{\partial f}{\partial x_k} \quad k = (1, 2, 3). \tag{7d}$$

The temporal Laplace transform operation is

$$L(f) = \hat{f}(p) = \int f(s)\exp(-ps)\,\mathrm{d}s. \tag{8a}$$

Integration is over positive real s and $\mathrm{Re}(p) > 0$. A double spatial integral transform is now required. The operation in terms of (x_1, x_2) and the corresponding inverse operation are [Sneddon 1972]

$$\tilde{f}(p, q_1, q_2) = \iint \hat{f}(p, x_1, x_2)\exp[-p(q_1 x_1 + q_2 x_2)]\,\mathrm{d}x_1\,\mathrm{d}x_2, \tag{8b}$$

$$\hat{f}(p, x_1, x_2) = \left(\frac{p}{2\pi i}\right)^2 \iint \tilde{f}(p, q_1, q_2)\exp[p(q_1 x_1 + q_2 x_2)]\,\mathrm{d}q_1\,\mathrm{d}q_2. \tag{8c}$$

Integration in (8b) is over real (x_1, x_2); integration in (8c) is along the imaginary (q_1, q_2)-axes. It is noted that $(\boldsymbol{x}, s)$ have dimensions of length, p has dimensions of inverse length, and (q_1, q_2) are dimensionless. Because (1) involves a speed that varies with direction, application of (8b) to (2)–(4), (5b) and (6) is complicated. Despite the role of θ, the problem is not axially symmetric. Nevertheless, 3D studies of sliding contact [Brock 2012] and crack growth [Brock 2015a; 2015b] suggest transformations

$$\mathrm{Im}(q_1) = \mathrm{Im}(q)\cos\psi, \quad \mathrm{Im}(q_2) = \mathrm{Im}(q)\sin\psi, \tag{9a}$$

$$x_1 = x\cos\psi, \qquad x_2 = x\sin\psi. \tag{9b}$$

Here $\mathrm{Re}(q) = 0+$ and $|\mathrm{Im}(q), x| < \infty$ and $|\psi| < \frac{1}{2}\pi$. Parameters (q, ψ) and (x, ψ) resemble quasipolar coordinates, i.e.,

$$\mathrm{d}x_1\,\mathrm{d}x_2 = |x|\,\mathrm{d}x\,\mathrm{d}\psi, \quad \mathrm{d}q_1\,\mathrm{d}q_2 = |q|\,\mathrm{d}q\,\mathrm{d}\psi. \tag{9c}$$

The uncoupling effect of (9) leads to the combination

$$\tilde{f}(p, q_1, q_2) \rightarrow \bar{f}(p, q, \psi), \tag{10a}$$

$$\hat{f}(p, x, \psi) = -\frac{p^2}{2\pi} \int \frac{|q|}{q} \bar{f}(p, q, \psi) \exp(pqx) \, \mathrm{d}q. \tag{10b}$$

Integration is along the positive ($\mathrm{Re}(q) = 0+$) side of the $\mathrm{Im}(q)$-axis. In view of (7), (9) and (10a) the transform of differential equations (3b) take the respective form

$$[\partial_3^2 - p^2(1 - cq)^2]\bar{\boldsymbol{u}}_S = 0, \tag{11a}$$

$$[c_D^2 \partial_3^2 - p^2(1 - cq)^2]\bar{\boldsymbol{u}}_D = 0. \tag{11b}$$

In view of (11), (3c) and conditions for $s \leq 0$ and finite $s > 0$ and $|\boldsymbol{x}| \rightarrow \infty$, the displacement transforms for $x_3 > 0(+)$ and $x_3 < 0(-)$ are

$$\bar{u}_1^{(\pm)} = q \cos\psi \, U_{\pm} \exp(-pA|x_3|) + U_1^{\pm} \exp(-pB|x_3|), \tag{12a}$$

$$\bar{u}_2^{(\pm)} = q \sin\psi \, U_{\pm} \exp(-pA|x_3|) + U_2^{\pm} \exp(-pB|x_3|), \tag{12b}$$

$$\bar{u}_3^{(\pm)} = (\mp) A U_{\pm} \exp(-pA|x_3|)(\pm)\frac{q}{B}(U_1^{\pm} \cos\psi + U_2^{\pm} \sin\psi) \exp(-pB|x_3|). \tag{12c}$$

In (11) and (12), $c = c(\psi)$ and coefficients $(U_{\pm}, U_1^{\pm}, U_2^{\pm})$ are functions of (p, q, ψ). Terms (A, B) are

$$A = \sqrt{\frac{1}{c_D^2}(1 - cq)^2 - q^2}, \qquad B = \sqrt{(1 - cq)^2 - q^2}. \tag{13a}$$

Therefore, bounded behavior for $\hat{u}_k$ as $|x_3| \rightarrow \infty$ requires that $\mathrm{Re}(A) > 0$ in the q-plane with branch cut

$$\mathrm{Im}(q) = 0, \qquad \frac{-1}{c_D - c} < \mathrm{Re}(q) < \frac{1}{c_D + c}. \tag{13b}$$

Similarly $\mathrm{Re}(B) > 0$ in the q-plane with branch cut

$$\mathrm{Im}(q) = 0, \qquad -\frac{1}{1 - c} < \mathrm{Re}(q) < \frac{1}{1 + c}. \tag{13c}$$

Restriction (1b) guarantees that $c_D - c > 1 - c > 0$. Equations (2b), (5b) and (6) for $x_3^0 = x_3 = 0$ can be operated upon with (8) in light of (7) and (9)–(11). Coefficients $(U_{\pm}, U_1^{\pm}, U_2^{\pm})$ in (12) can then be found in terms of transforms $(\bar{\Delta}_k, \bar{\Sigma}_k)$. Results lead to six homogeneous equations that relate $(\bar{u}_k, \bar{\sigma}_{3k})$ for $x_3 = 0\pm$ and $(\bar{\Delta}_k, \bar{\Sigma}_k)$ (Appendix A).

Application to crack growth problem

Study of (5a) shows that Σ_k is either not specified or is required to vanish. Therefore, $\bar{\Sigma}_k$ can be dropped from the analysis. Equations (A.2) reduce to

$$\bar{\tau}_1^+ + \bar{\tau}_{1P} + \mu p \tfrac{1}{2} B \bar{\Delta}_1 - \mu p \frac{q^2 M \cos\psi}{2B(1-cq)^2}(\bar{\Delta}_1 \cos\psi + \bar{\Delta}_2 \sin\psi) = 0, \tag{14a}$$

$$\bar{\tau}_2^+ + \bar{\tau}_{2P} + \mu p \tfrac{1}{2} B \bar{\Delta}_2 - \mu p \frac{q^2 M \sin\psi}{2B(1-cq)^2}(\bar{\Delta}_1 \cos\psi + \bar{\Delta}_2 \sin\psi) = 0, \tag{14b}$$

$$\bar{\tau}_3^+ + \bar{\tau}_{3P} + \frac{\mu p M_R \bar{\Delta}_3}{2A(1-cq)^2} = 0. \tag{14c}$$

Here τ_{kP} is the point-force contribution to σ_{3k}. In view of (5) and (8)–(10),

$$\bar{\tau}_{kP} = -\frac{P_k}{p(1-cq)}, \qquad \mathrm{Re}(1-cq) > 0. \tag{15}$$

The contribution to σ_{3k} for $x > 0$ is $\tau_k^+(x, \psi, s)$, which is generated behind wave front $c_D s - x - cs > 0$ and is unknown. In view of (8)–(10) therefore, transform $\bar{\tau}_k^+$ exists for $\mathrm{Re}(q) > -1/(c_D - c)$. Term $\Delta_k(x, \psi, s) = 0$ (for $x > 0$) and is an unknown function for $x < 0$ that is generated behind wave front $c_D s + x + cs > 0$. Therefore transform $\bar{\Delta}_k$ exists for $\mathrm{Re}(q) < 1/(c_D + c)$.

Equations (14a) and (14b) are coupled, and are considered first. Elimination of the M-term and use of (15) gives

$$\left[\bar{\tau}_1^+ - \frac{P_1}{p(1-cq)}\right]\sin\psi - \left[\bar{\tau}_2^+ - \frac{P_2}{p(1-cq)}\right]\cos\psi + \mu p \tfrac{1}{2} B(\bar{\Delta}_2 \cos\psi - \bar{\Delta}_1 \sin\psi) = 0. \tag{16}$$

Equation (16) is of the Wiener–Hopf type [Morse and Feshbach 1953; Achenbach 1973] and can be solved as follows: (A, B) are written as products (A_+A_-, B_+B_-), where

$$A_\pm = \sqrt{\frac{1}{c_D} \pm q\left(1 \mp \frac{c}{c_D}\right)}, \qquad B_\pm = \sqrt{1 \pm q(1 \mp c)}. \tag{17}$$

In (17) $A_\pm$ are analytic in, respectively, overlapping half-planes $\mathrm{Re}(q) > -1/(c_D - c)$ and $\mathrm{Re}(q) < 1/(c_D + c)$. In (17) $B_\pm$ are analytic in, respectively, overlapping half-planes $\mathrm{Re}(q) > -1/(1-c)$ and $\mathrm{Re}(q) < 1/(1+c)$. Equation (16) can be rearranged as

$$\frac{2}{\mu B_+}(\bar{\tau}_1^+ \sin\psi - \bar{\tau}_2^+ \cos\psi) + \frac{2}{\mu p(1-cq)}\left(\frac{1}{B_+} - \sqrt{c}\right)(P_2 \cos\psi - P_1 \sin\psi)$$
$$= pB_-(\bar{\Delta}_1 \sin\psi - \bar{\Delta}_2 \cos\psi) - \frac{2\sqrt{c}}{\mu p(1-cq)}(P_2 \cos\psi - P_1 \sin\psi), \tag{18a}$$

$$A_+(1/c) = B_+(1/c) = 1/\sqrt{c}. \tag{18b}$$

The left-hand and right-hand sides of (18a) are analytic, respectively, in overlapping half-plane $\mathrm{Re}(q) > -1/(c_D - c)$ and $\mathrm{Re}(q) < 1/(c_D + c)$, so that each side is an analytic continuation of the same entire function. In connection with (6), Δ_k must vanish continuously on C for $x \to 0-$. Equations (8a) and (10b) therefore require that $pq\bar{\Delta}_k$, and also the right-hand side of (18a), vanish for $|q| \to \infty$. The entire function itself must then in light of Liouville's theorem [Morse and Feshbach 1953] vanish, and (18a) yields

$$\bar{\tau}_1^+ \sin\psi - \bar{\tau}_2^+ \cos\psi = \frac{1}{p(1-cq)}(\sqrt{c}B_+ - 1)(P_2\cos\psi - P_1\sin\psi), \tag{19a}$$

$$\bar{\Delta}_1 \sin\psi - \bar{\Delta}_2 \cos\psi = \frac{2\sqrt{c}}{\mu p^2(1-cq)B_-}(P_2\cos\psi - P_1\sin\psi). \tag{19b}$$

In (A.3) it can be shown that

$$B^2(1-cq)^2 - q^2 M = M_R. \tag{20}$$

Use of (15), (19a) and (20) in (14a) gives

$$\bar{\tau}_1^+ - \frac{P_1}{p(1-cq)} - \frac{\sqrt{c}B_+ \sin\psi}{p(1-cq)}(P_2\cos\psi - P_1\sin\psi) = \frac{\mu p M_R \cos\psi}{2B(1-cq)^2}(\bar{\Delta}_1\cos\psi + \bar{\Delta}_2\sin\psi). \tag{21}$$

Equation (21) is also of the Wiener–Hopf type, and it is noted that M_R in (A.3a) is the Rayleigh function in (q, ψ)-space. Its branch points on the Re(q)-axis are ascertained in (13a). It also exhibits two roots on the Re(q)-axis, $-1/(c_R - c)$ and $1/(c_R - c)$. Here $V_R = c_R V_S$ is the Rayleigh wave speed in the solid, and $c_R < 1 < c_D$. In view of (1b) and (A.3a),

$$\frac{-1}{c_R - c} < \frac{-1}{1-c} < \frac{-1}{c_D - c} < 0 < \frac{1}{c_D + c} < \frac{1}{1+c} < \frac{1}{c_R + c} < \frac{1}{c}, \tag{22a}$$

$$M_R \approx -Rq^4 (|q| \to \infty), \tag{22b}$$

$$R = 4ab - K^2, \quad R(\pm c_R) = 0, \tag{22c}$$

$$a(c) = \sqrt{1 - \frac{c^2}{c_D^2}}, \quad b(c) = \sqrt{1-c^2}, \quad K(c) = c^2 - 2. \tag{22d}$$

In view of (22), function $G(q, \psi)$ with the property $G \to 1$ (as $|q| \to \infty$) is defined as

$$G = \frac{c^2(c_R^2 - c^2)}{R(1-cq)^2} \frac{M_R}{(1-cq)^2 - c_R^2 q^2}. \tag{23a}$$

One can write $G = G_+ G_-$, where $G_\pm$ respectively are analytic in overlapping half-planes Re$(q) > -1/(c_D - c)$ and Re$(q) < 1/(c_D + c)$:

$$G_\pm = \exp\left(\frac{1}{\pi}\int \tan^{-1}\frac{4\sqrt{u^2-1}\sqrt{c_D^2 - u^2}}{c_D(u^2-2)^2} \frac{du}{(u \mp c)[q(u \mp c) \pm 1]}\right). \tag{23b}$$

Integration is over range $1 < u < c_D$. This result and the behavior discussed in connection with (17) allows (21) to be treated in the same fashion as (16), and here solved for $\bar{\tau}_1^+$ and $\bar{\Delta}_1 \cos\psi + \bar{\Delta}_2 \sin\psi$.

Combining the latter result with (19b) and (23b) then gives

$$\vec{\Delta}_1 = \frac{2\sqrt{c}\sin\psi}{\mu p^2(1-cq)B_-}(P_2\cos\psi - P_1\sin\psi) - \frac{2\sqrt{c}\cos\psi}{\mu p^2 c_R}\frac{B_-G_+}{M_R g_+}(1-cq)[1+q(c_R-c)](P_1\cos\psi + P_2\sin\psi), \tag{24a}$$

$$\vec{\Delta}_2 = \frac{2\sqrt{c}\cos\psi}{\mu p^2(1-cq)B_-}(P_1\sin\psi - P_2\cos\psi) - \frac{2\sqrt{c}\sin\psi}{\mu p^2 c_R}\frac{B_-G_+}{M_R g_+}(1-cq)[1+q(c_R-c)](P_2\sin\psi + P_1\cos\psi), \tag{24b}$$

$$g_+ = G_+(1/c). \tag{24c}$$

It is noted that identical results follow upon substitution of (19b) into (14b). Equation (14c) is studied in the same fashion as (14a) and (14b), and solved for $(\bar{\tau}_3^+, \bar{\Delta}_3)$, e.g.,

$$\bar{\Delta}_3 = \frac{2\sqrt{c}}{\mu p^2 c_R}\frac{A_-G_+}{M_R g_+}(1-cq)[1+q(c_R-c)]P_3. \tag{24d}$$

Transform inversion valid on crack plane near *C*

In view of (7b), (7c) and (9),

$$\dot{\Delta}_k = V_S(\partial_S - c\partial)\Delta_k, \quad \partial f = \frac{\partial f}{\partial x}. \tag{25}$$

Therefore, (8a), (10) and (24c) give for $x < 0$

$$L(\dot{\Delta}_3) = -\frac{P_3}{\mu\pi}\frac{\sqrt{c}V_S}{g_+ c_R}p\int \frac{|q|}{qM_R}\,\mathrm{d}q\, G_+A_-(1-cq)^2[1+q(c_R-c)]\exp(pqx). \tag{26a}$$

For $x < 0$, Cauchy theory is used to transform the integration to the upper ($\mathrm{Im}(q) = 0+$) side of the positive real q-axis. An expression valid for $x \to 0-$ is then extracted:

$$L(\dot{\Delta}_3) \approx \frac{2P_3}{\mu\pi}\sqrt{1+\frac{c}{c_D}}\left(1-\frac{c}{c_R}\right)\frac{\sqrt{c}V_S}{g_+R}\frac{p}{\sqrt{-x}}\int \frac{\mathrm{d}u}{\sqrt{u}}\exp(-pu). \tag{26b}$$

Integration is over the entire positive u-axis and, in fact, gives $\sqrt{\pi/p}$ [Gradshteyn and Ryzhik 2014]. However $p\exp(-pu)$ is the transform of function $\partial_S\delta(s-u)$ [Abramowitz and Stegun 1972]. Point-force loading (5a) represents a step-function in time. For generality therefore, we consider the case

$$P_k = P_k(V_S t), \quad P_k(0) = 0. \tag{27}$$

Solution behavior is more discernible if points in the $x_1^0x_2^0$-plane are defined with respect to $\boldsymbol{x}_0 = 0$. Therefore, upon inversion [Abramowitz and Stegun 1972; Sneddon 1972] of the modified (26b), coordinates (x_0, ψ, s), where $x_0 = x + cs$, are introduced, and for $\left(s > 0, x_0 \to cs-, |\psi| < \tfrac{1}{2}\pi\right)$,

$$\dot{\Delta}_3 \approx \frac{2V_S}{\mu\pi}\sqrt{1+\frac{c}{c_D}}\frac{\sqrt{c}K_{\mathrm{I}}}{\sqrt{cs-x_0}}, \quad K_{\mathrm{I}} = \left(1-\frac{c}{c_R}\right)\frac{c^2\partial_S}{Rg_+}\int_0^s \frac{\partial_S P_3}{\sqrt{s-u}}\,\mathrm{d}u. \tag{28}$$

A similar procedure for (24a) and (24b) give for ($s > 0$, $x_0 \to cs-$)

$$\dot{\Delta}_1 \approx \frac{2V_S}{\mu\pi}\frac{\sqrt{c}}{\sqrt{cs-x_0}}\left(-\frac{\sin\psi}{\sqrt{1+c}}K_{\mathrm{III}} + \cos\psi\sqrt{1+c}\,K_{\mathrm{II}}\right), \tag{29a}$$

$$\dot{\Delta}_2 \approx \frac{2V_S}{\mu\pi}\frac{\sqrt{c}}{\sqrt{cs-x_0}}\left(\frac{\cos\psi}{\sqrt{1+c}}K_{\mathrm{III}} + \sin\psi\sqrt{1+c}\,K_{\mathrm{II}}\right), \tag{29b}$$

$$K_{\mathrm{II}} = \left(1-\frac{c}{c_R}\right)\frac{c^2\partial_S}{Rg_+}\int_0^s \frac{\mathrm{d}u}{\sqrt{s-u}}\partial_S(\cos\psi\,P_1 + \sin\psi\,P_2), \tag{29c}$$

$$K_{\mathrm{III}} = \partial_S\int_0^s \frac{\mathrm{d}u}{\sqrt{s-u}}\partial_S(\sin\psi\,P_1 - \cos\psi\,P_2). \tag{29d}$$

Subscripts (I, II, III) indicate that the K-terms are, based on the role of (c_D, c_R, g_+), related to what are referred to [Freund 1990], respectively, as the opening, in-plane and antiplane modes of fracture. The K-terms are finite for all $0 < V(\psi) < c_R V_S$, e.g.,

$$\frac{c^2}{R} \to \frac{2c_D^2}{c_D^2-1}\ (c=0), \qquad \left(1-\frac{c}{c_R}\right)\frac{1}{R} \to \frac{1}{F_R}\left(\frac{c_D}{2c_R}\right)^2\ (c=c_R), \tag{30a}$$

$$F_R = c_R^2 - 2 + \frac{1}{c_D}\left(\sqrt{\frac{c_D^2-c_R^2}{1-c_R^2}} + \sqrt{\frac{1-c_R^2}{c_D^2-c_R^2}}\right) > 0. \tag{30b}$$

The analogous results for σ_{3k} when ($s > 0$, $x_3^0 = 0$, $x_0 \to cs+$) are

$$\sigma_{31}^0 \approx \frac{1}{\pi\sqrt{c}\sqrt{x_0-cs}}\left(-\sin\psi\sqrt{1-c}\,K_{\mathrm{III}} + \frac{\cos\psi}{\sqrt{1-c}}\frac{R}{c^2}K_{\mathrm{II}}\right), \tag{31a}$$

$$\sigma_{32}^0 \approx \frac{1}{\pi\sqrt{c}\sqrt{x_0-cs}}\left(\cos\psi\sqrt{1-c}\,K_{\mathrm{III}} + \frac{\sin\psi}{\sqrt{1-c}}\frac{R}{c^2}K_{\mathrm{II}}\right), \tag{31b}$$

$$\sigma_{33}^0 \approx \frac{\sqrt{c_D}}{\pi\sqrt{c}\sqrt{c_D-c}}\frac{R}{c^2}\frac{K_{\mathrm{I}}}{\sqrt{x_0-cs}}. \tag{31c}$$

For a rectilinear crack edge, e.g., [Freund 1972; 1990], an orthogonal basis is chosen so that the crack plane, crack edge and normal to the crack edge can each be defined in terms of one coordinate direction. Here the outwardly directed normal to C forms angle φ with respect to the positive (x, x_0)-direction, where

$$\varphi = \tan^{-1}\frac{c'}{c}\left(|\varphi| < \tfrac{1}{2}\pi\right), \qquad f' = \frac{\mathrm{d}f}{\mathrm{d}\psi}. \tag{32}$$

Therefore for each point ($x_0 = cs$, ψ) on C traction ($\sigma_{3\tau}^0$, $\sigma_{3\nu}^0$, σ_{33}^0) and velocity discontinuity ($\dot{\Delta}_\tau$, $\dot{\Delta}_\nu$, $\dot{\Delta}_3$) set can be defined in terms of a local coordinate system (τ, ν, $x_3^0 = x_3$). Here τ is the tangent to C, taken in the clockwise sense, and ν is the outwardly directed normal to C. Equation (28) is still valid, and the other members of the set are listed in Appendix B.

Criterion: dynamic energy release rate

A standard criterion for brittle fracture, e.g., [Freund 1972], equates the rate at which surface energy is released to the rate of work associated with traction and relative displacements in the fracture zone $\Im$. If kinetic energy is included [Gdoutos 2005; Brock 2015b], the equation takes the form

$$D \iint_{\delta A} e_F \, dx_1^0 \, dx_2^0 = \iint_{\Im} \sigma_{3k}^0 \dot{\Delta}_k \, dx_1^0 \, dx_2^0 + D \iiint_{123} \tfrac{1}{2} \rho \dot{u}_k \dot{u}_k \, dx_1^0 \, dx_2^0 \, dx_3^0. \tag{33}$$

Here e_F is the surface energy per unit area, and is generally assumed to be constant [deBoer et al. 1988; Skriver and Rosengaard 1992]. Fracture zone $\Im$ of course is a strip of infinitesimal thickness in the $x_1^0 x_2^0$-plane that straddles the portion of C that borders δA. Subscript 123 signifies integration over the unbounded solid. Use of Green's theorem [Malvern 1969] and translating basis $\boldsymbol{x}$ expressed in terms of $(x_0, \psi, x_3 = 0)$ gives for the left-hand side of (33)

$$V e_F s \int_{\Psi} d\psi \, c\sqrt{c^2 + (c')^2}. \tag{34a}$$

Here Ψ denotes integration over range $|\psi| < \frac{1}{2}\pi$. Use of the translating basis for the integration over $\Im$ in (33) gives

$$\int_{\Psi} d\psi \int_{cs-}^{cs+} |x_0| \sigma_{3k}^0 \dot{\Delta}_k \, dx_0. \tag{34b}$$

Use of (28)–(30) in (34b) gives rise to Dirac function $\delta(x_0 - cs)$ [Freund 1972]. It is also recognized [Achenbach and Brock 1973] that the linear behavior in s displayed by (34a) places a restriction on the behavior of $\partial_S P(s)$. That is, $V(\psi)$ in general must vary with time. One case, however, for which time-invariance is valid is

$$\partial_S P_k(s) = p_k \sqrt{s}. \tag{35a}$$

Equation (34b) then gives

$$\pi \frac{s}{\mu} \int_{\Psi} d\psi \, V \left[\frac{R}{c^2} \left(K_{\mathrm{I}}^2 \sqrt{\frac{c_D + c}{c_D - c}} + K_{\mathrm{II}}^2 \sqrt{\frac{1+c}{1-c}} \right) + K_{\mathrm{III}}^2 \sqrt{\frac{1-c}{1+c}} \right]. \tag{35b}$$

In light of (28), (29c) and (29d),

$$K_{\mathrm{I}} = \left(1 - \frac{c}{c_R}\right) \frac{c^2 p_3}{R g_+}, \tag{36a}$$

$$K_{\mathrm{II}} = \left(1 - \frac{c}{c_R}\right) \frac{c^2}{R g_+} (p_1 \cos\psi + p_2 \sin\psi), \tag{36b}$$

$$K_{\mathrm{III}} = p_1 \sin\psi - p_2 \cos\psi. \tag{36c}$$

In view of (7c) and (9) the transforms of $\dot{u}_k$ are $p(1 - cq)\bar{u}_k$, where $\bar{u}_k$ and coefficients $(U_\pm, U_1^\pm, U_2^\pm)$ are given by (12) and (C.1). Therefore the last integral in (33) requires inversion for $x_3 \neq 0$, and a more explicit version of (10) is useful:

$$\tilde{f}(p, q_1, q_2, x_3) \to f_\Psi(p, q, \psi) \exp(-p\Omega|x_3|), \qquad \Omega = (A, B), \tag{37a}$$

$$\hat{f}(p, x, \psi, x_3) = -\frac{p^2}{2\pi} \int \frac{|q|}{q} f_\Psi(p, q, \psi) \exp[p(qx - \Omega|x_3|)] \, dq. \tag{37b}$$

Derivation of (28)–(30) is based on changing the integration path in (10b) to the $\mathrm{Re}(q)$-axis. After [Achenbach 1973], the integration path in (37b) can be changed, via Cauchy theory, to a contour defined in the complex q-plane such that the exponential term takes the form $\exp(-pu)$, where u is real and positive. In carrying out this procedure, it is recognized that the singular behavior displayed by $\dot{\Delta}_k$ in (28) and (29) is manifest in $\dot{\boldsymbol{u}}$ near C. Therefore, Green's theorem [Malvern 1969] is invoked to write the last term in (33) as an integral over the surface of a tube of radius $r_C \to 0$ that encloses the portion of C that borders δA. Translating basis $\boldsymbol{x}$ is employed, but with local coordinates (r, ψ, ϕ), centered on C, where

$$r = \sqrt{x^2 + x_3^2}, \quad \phi = \tan^{-1}\frac{x_3}{x} \; (|\phi| < \pi). \tag{38}$$

Equations (C.2) and (C.3) define the integration contour functions, parameterized by (38) and $u > 0$. For $r_C \to 0$, however, asymptotic forms (C.4) can be used, and it can then be shown that (37b) gives a linear combination of terms defined as the real or imaginary parts of (compare (26b))

$$\frac{p}{\sqrt{r Q_\Omega^\pm}} \int \frac{\mathrm{d}u}{\sqrt{u}} \exp(-pu), \tag{39a}$$

$$\frac{1}{\sqrt{Q_A^\pm}} = \frac{1}{\sqrt{2}}(A_\Phi^+ \pm i A_\Phi^-), \quad A_\Phi^\pm = \sqrt{1 \pm \frac{\cos\phi}{A_\Phi}}, \tag{39b}$$

$$\frac{1}{\sqrt{Q_B^\pm}} = \frac{1}{\sqrt{2}}(B_\Phi^+ \pm i B_\Phi^-), \quad B_\Phi^\pm = \sqrt{1 \pm \frac{\cos\phi}{B_\Phi}}. \tag{39c}$$

In view of (12), (27)–(29), (35a) and (39) the asymptotic forms of $\dot{u}_k$ are given listed in Appendix D. Use of (D.1) and (D.2) gives for the last term in (33)

$$-\frac{s}{\mu}\int_\Psi \mathrm{d}\psi\, V\sqrt{c^2 + (c')^2}\left[\left(1 + \frac{c}{c_D}\right)\frac{K_\mathrm{I}^2}{c^3}E_\mathrm{I} + (1+c)\frac{K_\mathrm{II}^2}{c^3}E_\mathrm{II} + \frac{K_\mathrm{III}^2}{c(1+c)}E_\mathrm{III}\right]. \tag{40}$$

In (40) coefficients $(E_\mathrm{I}, E_\mathrm{II}, E_\mathrm{III})$ are functions of $c(\psi)$ and defined by

$$E_\mathrm{I} = \int_\Phi \cos\phi[(Q_3^+)^2 + (Q_3^-)^2]\,\mathrm{d}\phi, \tag{41a}$$

$$E_\mathrm{II} = \int_\Phi \cos\phi[(Q_{12}^+)^2 + (Q_{12}^-)^2]\,\mathrm{d}\phi, \tag{41b}$$

$$E_\mathrm{III} = \int_\Phi \cos\phi\left(\tfrac{1}{2}B_\Phi^-\right)^2 \mathrm{d}\phi. \tag{41c}$$

Here Φ signifies integration over range $|\phi| < \pi$, and it can also be shown for $c \to 0$ that $(E_\mathrm{I}, E_\mathrm{II}) \approx O(c^4)$ and $E_\mathrm{III} \approx O(c^2)$. Equations (34a), (35b) and (40) all involve integration with respect to ψ, and thus (33) reduces to the nonlinear differential equation for $c(\psi)$:

$$\mu e_F\sqrt{c^2 + (c')^2} = \left(1 + \frac{c}{c_D}\right)K_\mathrm{I}^2\left[\frac{\pi R}{c^2 a} - \frac{E_\mathrm{I}}{c^3}\sqrt{c^2 + (c')^2}\right] + (1+c)K_\mathrm{II}^2\left[\frac{\pi R}{c^2 b} - \frac{E_\mathrm{II}}{c^3}\sqrt{c^2 + (c')^2}\right] + \frac{K_\mathrm{III}^2}{1+c}\left[\pi b - \frac{E_\mathrm{III}}{c}\sqrt{c^2 + (c')^2}\right]. \tag{42}$$

Special cases: aspects of solution behavior

Knowledge of $c(\psi)$ provides a contour function $x_0 = V(\psi)t$ (where $t > 0$, $|\psi| < \frac{1}{2}\pi$) that defines C for the extending portion of crack A_C. Extension of each crack edge point is of course $V(\psi)t\cos\psi$ in the positive x_1^0-direction. Analysis of (42) must deal with its nonlinear form and ψ-dependent coefficient $(K_{\mathrm{I}}^2, K_{\mathrm{II}}^2, K_{\mathrm{III}}^2)$. Nevertheless experience [Brock 2015a; 2015b] suggests that some aspects of solution behavior can be determined from (42). In particular, Rayleigh limit case $c(\psi) \to c_R$ arises only when

$$e_F + \frac{\pi}{\mu c_R^3}\left(\frac{p_3}{4F_R g_R^+}\right)^2\left(1+\frac{c_R}{c_D}\right)E_{\mathrm{I}} = 0, \qquad g_R^+ = G_+\left(\frac{1}{c_R}\right). \tag{43}$$

On the other hand, a low-speed assumption $c(\psi) \ll c_R$ and expansions in c^2 give (42) the more explicit form

$$\frac{\mu e_F}{\pi}\sqrt{c^2+(c')^2} \approx \frac{c_D^2}{2(c_D^2-1)}p_3^2[1+E_{\mathrm{I}}c\sqrt{c^2+(c')^2}] + (p_1\sin\psi - p_2\cos\psi)^2[1+E_{\mathrm{III}}c\sqrt{c^2+(c')^2}]$$
$$+\frac{c_D^2}{2(c_D^2-1)}(p_1\cos\psi + p_2\sin\psi)^2[1+E_{\mathrm{II}}c\sqrt{c^2+(c')^2}], \tag{44a}$$

$$E_{\mathrm{I}} = \frac{5}{32}\left(1+\frac{1}{c_D^2}\right), \quad E_{\mathrm{II}} = \frac{1}{32}\left(3+\frac{8}{c_D^2}\right), \quad E_{\mathrm{III}} = \frac{1}{4}. \tag{44b}$$

Crack contour behavior for the low-speed assumption

We treat two cases governed by (44):

Case A: $p_3 = p_A$ (pure compression).

Case B: $p_2 = 0$, $p_1 = p_3 = p_B$ (combined loading).

Problem symmetry exists with respect to the $x_2^0x_3^0$-plane for both cases, and the effect of kinetic energy is considered for each.

Case A. For $c(\psi) < 0.5$, Equation (44) for Case A reduces to

$$\sqrt{z^2+(z')^2}(1-c_A^2E_{\mathrm{I}}z) = 1, \tag{45a}$$

$$z = \frac{c}{c_A}, \quad c_A = \frac{c_D^2}{2(c_D^2-1)}\frac{\pi p_A^2}{\mu e_F}. \tag{45b}$$

When kinetic energy is, respectively, neglected ($E_{\mathrm{I}} = 0$) and included, (45) gives

$$c(\psi) = c_A, \tag{46a}$$

$$c(\psi) = \frac{1}{2E_{\mathrm{I}}c_A}(1-\sqrt{1-\Gamma_A}), \quad \Gamma_A = \frac{(2c_A)^2}{E_{\mathrm{I}}}. \tag{46b}$$

Equation (46) describes semicircular crack edge extension zone contours. To illustrate, consider a generic metal with properties [deBoer et al. 1988; Skriver and Rosengaard 1992; Brock 2015a; 2015b]

$$\mu = 79\,\mathrm{GPa}, \quad e_F = 2.2\,\mathrm{J/m^2}, \quad V_S = 3094\,\mathrm{m/s}, \quad c_D = 2, \quad c_R = 0.933.$$

α_A	5	10	14.4	15	20
c_A	0.03013	0.12051	0.25	0.27114	0.48202
$c(0)$	0.04163	0.13581	0.2703	0.29212	0.50889

Table 1. Case A: parameters c_A and $c(0)$ for values of $p_A = \alpha_A(10^4)$ N/m$^{3/2}$.

Calculations for c_A and $c(0)$ are presented in Table 1 for various values of p_A. It is noted that $c(0) > c_A$. Therefore neglect of kinetic energy under-predicts crack extension speed $V \cos\psi$. This effect decreases with increasing p_A, however. For example, c_A and $c(0)$ for $p_A = 5(10^4)$ N/m$^{3/2}$ in Table 1 differ by 38.2%, but for $p_A = 15(10^4)$ N/m$^{3/2}$ the difference is 7.7%.

Case B. For $c(\psi) < 0.5$, Equation (44) now reduces to

$$2\sqrt{z^2 + (z')^2}\Big[1 - \big(\tfrac{1}{8}c_B\big)^2 z(E_0 + E_\Psi \cos^2\psi)\Big] = A_0 + \frac{\cos^2\psi}{c_D^2}, \tag{47a}$$

$$z = \frac{c}{c_B}, \quad c_B = \frac{c_D^2}{c_D^2 - 1}\frac{\pi p_B^2}{\mu e_F}, \quad A_0 = 2 - \frac{1}{c_D^2}. \tag{47b}$$

Coefficients $(E_0, E_\Psi) = 0$ when kinetic energy is neglected. If it is included,

$$E_0 = 13 - \frac{3}{c_D^2}, \quad E_\Psi = \frac{16}{c_D^2} - 5. \tag{47c}$$

Explicit ψ-dependence of (47a) implies that the crack extension zone contour is not circular, and that obtaining an analytical solution for $c(\psi)$ may be difficult. The form of (47a) suggests use of the series representation:

$$z(\psi) = \alpha\left[1 + \sum_1^N a_{2j}\cos^{2j}\psi\right]. \tag{48}$$

Substitution into (47a) and equating coefficients of terms $\cos^{2j}\psi$ (for $j \geq 0$) when kinetic energy is included gives recursive equations for (dimensionless) coefficients (α, a_{2j}). Terms (α, a_2) are the solutions to quadratic equations

$$\alpha = \frac{A_0}{1+\Omega}, \qquad a_2 = \frac{1}{4}\left[\sqrt{1 + 8\left(\frac{\omega E_0}{1+\Omega} + \frac{1}{c_D^2 A_0}\right)} - 1\right], \tag{49a}$$

$$\Omega = \sqrt{1 - \omega A_0 E_0}, \quad \omega = \frac{1}{2}\left(\frac{1}{4}c_B\right)^2. \tag{49b}$$

Terms $(a_4, a_6, \ldots)$ satisfy linear equations, but are complicated. Some simplicity is achieved by not expressing them completely in explicit form, e.g.,

$$a_4 = \frac{1}{2A_0^2(1+8a_2)}\left[\frac{1}{c_D^4} + a_2^2 A_0(3A_0 + 8\alpha\omega E_0) + 6\alpha\omega A_0 E_0 a_2 - (\omega E_\Psi a_2^2)^2\right]. \tag{50}$$

Kinetic energy neglected

ψ	$0°$	$15°$	$30°$	$45°$	$60°$	$75°$	$90°$
c	0.148	0.138	0.116	0.095	0.081	0.074	0.071
c'	0	−0.043	−0.059	−0.51	−0.035	−0.018	0
$c\cos\psi$	0.148	0.133	0.101	0.067	0.041	0.019	0

Kinetic energy included

ψ	$0°$	$15°$	$30°$	$45°$	$60°$	$75°$	$90°$
c	0.169	0.164	0.152	0.139	0.13	0.126	0.125
c'	0	−0.038	−0.052	−0.043	−0.024	−0.009	0
$c\cos\psi$	0.169	0.158	0.132	0.098	0.015	0.033	0

Table 2. Case B: parameters c, c' and $c\cos\psi$ for $p_B = 7.7(10^4)\ \mathrm{N/m^{3/2}}$ and various ψ.

The coefficients exhibit simpler forms, of course, when kinetic energy is neglected, e.g.,

$$\alpha = \frac{1}{2}A_0, \quad a_2 = \frac{1}{4}\left[\sqrt{1+\frac{8}{c_D^2 A_0}} - 1\right], \quad a_4 = \frac{1}{2A_0^2(1+8a_2)}\left[\frac{1}{c_D^4} + 3a_2^2 A_0^2\right]. \tag{51}$$

Calculations of (c, c') and dimensionless crack extension speed $c\cos\psi$ for $0 < \psi < \frac{1}{2}\pi$ based on (48) are given in Table 2 for the generic metal featured in Table 1. In view of the observations concerning p_A, only the single value $p_B = 7.7(10^4)\ \mathrm{N/m^{3/2}}$ is considered. This corresponds to a dimensionless speed parameter $c_B = 0.1429$ and $\omega = 6.378(10^{-4})$, so that use of zero-order expansions of the kinetic energy-dependent coefficients $(\alpha, a_2, a_4, a_6, \dots)$ generates negligible error. For clarity, ψ-values in Table 2 are given in degrees.

Table 2 entries indicate that combined loading creates an elliptical crack contour for which the maximum rate of crack extension into the solid exceeds the rate at which new crack surface spreads along the original, semi-infinite crack contour. Neglect of kinetic energy exaggerates this effect while it (see Case A) under-predicts both rates.

Some observations

This paper extends the range of studies for 3D dynamic crack growth [Brock 2015a; 2015b] by considering a transient problem with mixed-mode loading. The crack is initially a closed, semi-infinite slit, with a rectilinear edge, and point forces applied on the surfaces just behind the edge. The dynamically similar case is treated, i.e., crack edge extension rate is constant in time, but can vary with direction. Unilateral temporal and bilateral spatial Laplace transforms are employed. However, the latter, and their inverses, make use of variable transformations based on quasipolar coordinates. An equation associated with crack-opening mode, and two coupled equations associated with shear mode, are produced in transform space. Both sets are put in Wiener–Hopf form [Morse and Feshbach 1953] and solved exactly.

Inversion is carried out analytically, and results subjected to a dynamic energy release rate criterion, with kinetic energy included. As is predictable [Achenbach and Brock 1973], the assumption of dynamic similarity restricts the time-variation of the point forces. Under this restriction, examples of pure compression loading (Case A) and mixed-mode loading (Case B) are examined. In Case A, the extending crack edge is semicircular, and inclusion of kinetic energy gives larger crack extension speeds. A similar effect on speed is noted for Case B. However, the edge is not circular, with the maximum rate of crack extension into the solid being greater than the expansion rate of new crack surface along the original crack contour.

The point force behavior considered here involves extremely rapid, but not instantaneous, growth in time. Thus, the results presented here suggest that dynamic similarity can exist, at least in a brief fracture initiation phase.

Appendix A

$$2\bar{u}_1 \mp \bar{\Delta}_1 + \frac{q\cos\psi}{(1-cq)^2}\frac{N}{A}\bar{\Delta}_3 + \frac{\bar{\Sigma}_1}{\mu B} = 0, \tag{A.1a}$$

$$2\bar{u}_2 \mp \bar{\Delta}_2 + \frac{q\sin\psi}{(1-cq)^2}\frac{N}{A}\bar{\Delta}_3 + \frac{\bar{\Sigma}_2}{\mu B} = 0, \tag{A.1b}$$

$$2\bar{u}_3 \mp \bar{\Delta}_3 - \frac{q}{(1-cq)^2}\frac{N}{B}(\bar{\Delta}_1\cos\psi + \bar{\Delta}_2\sin\psi) + \frac{q^2+AB}{(1-cq)^2}\frac{\bar{\Sigma}_3}{\mu B} = 0. \tag{A.1c}$$

$$2\bar{\sigma}_{31} \mp \bar{\Sigma}_1 + \mu p B\bar{\Delta}_1 - \frac{pq\cos\psi}{B(1-cq)^2}P = 0, \tag{A.2a}$$

$$2\bar{\sigma}_{32} \mp \bar{\Sigma}_2 + \mu p B\bar{\Delta}_2 - \frac{pq\sin\psi}{B(1-cq)^2}P = 0, \tag{A.2b}$$

$$2\bar{\sigma}_{33} \mp \bar{\Sigma}_3 + \frac{p}{A(1-cq)^2}\big[Nq(\bar{\Sigma}_1\cos\psi + \bar{\Sigma}_2\sin\psi) + \mu M_R\bar{\Delta}_3\big] = 0. \tag{A.2c}$$

$$P = N\bar{\Sigma}_3 + \mu Mq(\bar{\Delta}_1\cos\psi + \bar{\Delta}_2\sin\psi), \tag{A.3a}$$

$$N = T - 2AB, \quad M = 2N - (1-cq)^2, \quad M_R = 4q^2AB + T^2, \tag{A.3b}$$

$$T = (1-cq)^2 - 2q^2. \tag{A.3c}$$

Appendix B

$$\sigma^0_{3\tau} \approx \frac{1}{\pi\sqrt{c}\sqrt{x_0 - cs}}\left(-K_{\rm III}\sqrt{1-c}\cos\varphi + K_{\rm II}\frac{R}{c^2}\frac{\sin\varphi}{\sqrt{1-c}}\right), \tag{B.1a}$$

$$\sigma^0_{3\nu} \approx \frac{1}{\pi\sqrt{c}\sqrt{x_0 - cs}}\left(\Sigma_{\rm III}\sqrt{1-c}\sin\varphi + K_{\rm II}\frac{R}{c^2}\frac{\cos\varphi}{\sqrt{1-c}}\right). \tag{B.1b}$$

$$\dot{\Delta}_\tau \approx \frac{2V_S\sqrt{c}}{\mu\pi\sqrt{cs - x_0}}\left(-K_{\rm III}\frac{\cos\varphi}{\sqrt{1+c}} + K_{\rm II}\sqrt{1+c}\sin\varphi\right), \tag{B.2a}$$

$$\dot{\Delta}_\nu \approx \frac{2V_S\sqrt{c}}{\mu\pi\sqrt{cs - x_0}}\left(K_{\rm III}\frac{\sin\varphi}{\sqrt{1+c}} + K_{\rm II}\sqrt{1+c}\cos\varphi\right). \tag{B.2b}$$

$$\sin\varphi = \frac{c'}{\sqrt{c^2+(c')^2}}, \quad \cos\varphi = \frac{c}{\sqrt{c^2+(c')^2}}, \quad f' = \frac{\mathrm{d}f}{\mathrm{d}\psi}. \tag{B.3}$$

Appendix C

$$U_{\pm} = \pm\frac{q}{(1-cq)^2}(\bar{\Delta}_1\cos\psi + \bar{\Delta}_2\sin\psi) - \frac{T\bar{\Delta}_3}{2A(1-cq)^2}, \tag{C.1a}$$

$$U_1^{\pm} = \pm\tfrac{1}{2}\bar{\Delta}_1 \mp \frac{q^2\cos\psi}{(1-cq)^2}(\bar{\Delta}_1\cos\psi + \bar{\Delta}_2\sin\psi) + \frac{qB\cos\psi}{(1-cq)^2}\bar{\Delta}_3, \tag{C.1b}$$

$$U_2^{\pm} = \pm\tfrac{1}{2}\bar{\Delta}_2 \mp \frac{q^2\sin\psi}{(1-cq)^2}(\bar{\Delta}\cos\psi + \bar{\Delta}_2\sin\psi) + \frac{qB\sin\psi}{(1-cq)^2}\bar{\Delta}_3. \tag{C.1c}$$

The contour function q for the case $\Omega = A$ in (37) can be written in terms of parameter u and local coordinates (r, ψ, ϕ) as

$$q = \frac{-1}{A_\Phi^2}\left[\frac{u}{r}\cos\phi + \frac{c}{c_D^2} \pm i\frac{\sin\phi}{c_D}\sqrt{\left(c_D\frac{u}{r}\right)^2 - \sin^2\phi - \left(c\frac{u}{r}+\cos\phi\right)^2}\,\right] \ (u > u_A), \tag{C.2a}$$

$$A_\Phi = \sqrt{1-\frac{c^2}{c_D^2}\sin^2\phi}, \quad u_A = \frac{r}{c_D a^2}\left(\frac{c}{c_D}\cos\phi - A_\Phi^2\right). \tag{C.2b}$$

For case $\Omega = B$ in (37),

$$q = -\frac{1}{B_\Phi^2}\left[\frac{u}{r}\cos\phi + c\sin^2\phi \pm i\sin\phi\sqrt{\left(\frac{u}{r}\right)^2 - \sin^2\phi - \left(c\frac{u}{r}+\cos\phi\right)^2}\,\right] \ (u > u_B), \tag{C.3a}$$

$$B_\Phi = \sqrt{1-c^2\sin^2\phi}, \quad u_B = \frac{r}{b^2}(c\cos\phi - B_\Phi^2). \tag{C.3b}$$

Equations (C.2a) and (C.3a) respectively behave for $r \to 0$ as

$$q \approx -\frac{u}{rQ_A^{\pm}} \ (u>0), \quad Q_A^{\pm} = \cos\phi \mp ia\sin\phi, \tag{C.4a}$$

$$q \approx -\frac{u}{rQ_B^{\pm}} \ (u>0), \quad Q_B^{\pm} = \cos\phi \mp ib\sin\phi. \tag{C.4b}$$

Appendix D

$$\dot{u}_1 \approx \frac{\sqrt{2c}}{\mu\sqrt{r}}\left[\frac{Q_{12}^-}{c^2}\cos\psi\sqrt{1+c}K_{\mathrm{II}} + B_\Phi^-\sin\psi\frac{K_{\mathrm{III}}}{\sqrt{1+c}}\right]\mathrm{sgn}(\phi) + \frac{\sqrt{2c}}{\mu\sqrt{r}}\frac{Q_3^+}{c^2}\cos\psi\sqrt{1+\frac{c}{c_D}}K_{\mathrm{I}}, \tag{D.1a}$$

$$\dot{u}_2 \approx \frac{\sqrt{2c}}{\mu\sqrt{r}}\left[\frac{Q_{12}^-}{c^2}\sin\psi\sqrt{1+c}K_{\mathrm{II}} - B_\Phi^-\cos\psi\frac{K_{\mathrm{III}}}{\sqrt{1+c}}\right]\mathrm{sgn}(\phi) + \frac{\sqrt{2c}}{\mu\sqrt{r}}\frac{Q_3^+}{c^2}\sin\psi\sqrt{1+\frac{c}{c_D}}K_{\mathrm{I}}, \tag{D.1b}$$

$$\dot{u}_3 \approx \frac{\sqrt{2c}}{\mu\sqrt{r}}\frac{Q_{12}^+}{c^2}\sqrt{1+c}K_{\mathrm{II}} - \frac{\sqrt{2c}}{\mu\sqrt{r}}\frac{Q_3^-}{c^2}\sqrt{1+\frac{c}{c_D}}K_{\mathrm{I}}\,\mathrm{sgn}(\phi). \tag{D.1c}$$

$$Q_{12}^{+} = aA_{\Phi}^{+} + \frac{K}{2b}B_{\Phi}^{+}, \quad Q_{12}^{-} = A_{\Phi}^{-} + \tfrac{1}{2}KB_{\Phi}^{-}, \tag{D.2a}$$

$$Q_{3}^{+} = \frac{K}{2a}A_{\Phi}^{+} + bB_{\Phi}^{+}, \quad Q_{3}^{-} = \tfrac{1}{2}KA_{\Phi}^{-} + B_{\Phi}^{-}. \tag{D.2b}$$

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