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NONLOCAL PROBLEMS WITH LOCAL DIRICHLET AND NEUMANN BOUNDARY CONDITIONS

BURAK AKSOYLU AND FATIH CELIKER

We present novel governing operators in the theory of peridynamics (PD) which will allow the extension of PD to applications that require local boundary conditions (BC). Due to its nonlocal nature, the original PD governing operator uses nonlocal BC. The novel operators agree with the original PD operator in the bulk of the domain and simultaneously enforce local Dirichlet or Neumann BC. Our construction is straightforward and easily accessible. The main ingredients are antiperiodic and periodic extensions of kernel functions together with even and odd parts of functions. We also present governing operators that enforce antiperiodic and periodic BC and the corresponding compatibility conditions for the right-hand side function in a given operator equation. Finally, we present the basic idea in extending the 1D construction to 2D.

1. Introduction

We present novel governing operators in the theory of peridynamics (PD), a nonlocal extension of continuum mechanics developed by Silling [2000]. We consider problems in 1D and choose the domain $\Omega := [-1, 1]$. By suppressing the dependence of u on the time variable t, the original bond based PD governing operator is given as

$$\mathcal{L}_{\text{orig}} u(x) := \int_\Omega \widehat{C}(x'-x)u(x)\,\mathrm{d}x' - \int_\Omega \widehat{C}(x'-x)u(x')\,\mathrm{d}x', \quad x \in \Omega. \tag{1-1}$$

Due to its nonlocal nature, the operator $\mathcal{L}_{\text{orig}}$ uses nonlocal boundary conditions (BC); see [Silling 2000, p. 201]. We define the operator that is closely related to $\mathcal{L}_{\text{orig}}$ as

$$\mathcal{L}u(x) := cu(x) - \int_\Omega \widehat{C}(x'-x)u(x')\,\mathrm{d}x', \quad x \in \Omega, \tag{1-2}$$

where $c := \int_\Omega C(x')\,\mathrm{d}x'$. We will prove that the two operators agree in the bulk. As the main contribution, we present novel governing operators that agree with $\mathcal{L}$ in the bulk of Ω, and, at the same time, enforce local Dirichlet or Neumann BC.

Since PD is a nonlocal theory, one might expect only the appearance of nonlocal BC. Indeed, so far the concept of local BC does not apply to PD. Instead, external forces must be supplied through the loading force density [Silling 2000]. On the other hand, we demonstrate that the anticipation of local BC being incompatible with nonlocal operators is not quite correct. Hence, our novel operators present an alternative to nonlocal BC and we hope that the ability to enforce local BC will provide a remedy for

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surface effects seen in PD; see [Madenci and Oterkus 2014, Chapters 4, 5, 7, and 12] and [Kilic 2008; Mitchell et al. 2015].

For $x, x' \in [-1,1]$, it follows that $x'-x \in [-2,2]$. Hence, in (1-1), the kernel function $C(x)$ needs to be extended from Ω to the domain of $\widehat{C}(x'-x)$, which is $\widehat{\Omega} := [-2,2]$. The default extension is the zero extension defined by

$$\widehat{C}(x) := \begin{cases} 0 & \text{if } x \in [-2,-1), \\ C(x) & \text{if } x \in [-1,1], \\ 0 & \text{if } x \in (1,2]. \end{cases}$$

Furthermore, the kernel function $C(x)$ is assumed to be even. Namely, $C(-x) = C(x)$. An important first choice of $C(x)$ is the *canonical* kernel function $\chi_\delta(x)$ whose only role is the representation of the nonlocal neighborhood, called the *horizon*, by a characteristic function. More precisely, for $x \in \Omega$,

$$\chi_\delta(x) := \begin{cases} 1 & \text{if } x \in (-\delta,\delta), \\ 0 & \text{otherwise.} \end{cases} \tag{1-3}$$

The size of nonlocality is determined by δ and we assume $\delta < 1$. Since the horizon is constructed by $\chi_\delta(x)$, a kernel function used in practice is in the form

$$C(x) = \chi_\delta(x)\mu(x), \tag{1-4}$$

where $\mu(x) \in L^2(\Omega)$ is even.

We define the periodic and antiperiodic extensions of $C(x)$ from Ω to $\widehat{\Omega}$, respectively, as follows

$$\widehat{C}_{\mathrm{a}}(x) := \begin{cases} -C(x+2) & \text{if } x \in [-2,-1), \\ C(x) & \text{if } x \in [-1,1], \\ -C(x-2) & \text{if } x \in (1,2], \end{cases} \qquad \widehat{C}_{\mathrm{p}}(x) := \begin{cases} C(x+2) & \text{if } x \in [-2,-1), \\ C(x) & \text{if } x \in [-1,1], \\ C(x-2) & \text{if } x \in (1,2]. \end{cases} \tag{1-5}$$

Even for smooth $\mu(x)$, note that $\widehat{C}(x)$, $\widehat{C}_{\mathrm{a}}(x)$, and $\widehat{C}_{\mathrm{p}}(x)$ are not necessarily smooth; see Figure 1.

Throughout the paper, we assume that

$$u(x) \in L^2(\Omega) \cap C^1(\partial\Omega). \tag{1-6}$$

Even and odd parts of the function u are used in the novel governing operators. Here we provide their definitions. We denote the orthogonal projections that give the even and odd parts, respectively, of a function by $P_e, P_o : L^2(\Omega) \to L^2(\Omega)$, whose definitions are

$$P_e u(x) := \tfrac{1}{2}\big(u(x) + u(-x)\big), \quad P_o u(x) := \tfrac{1}{2}\big(u(x) - u(-x)\big). \tag{1-7}$$

Theorem 1.1 (Main Theorem). *Let $c = \int_\Omega C(x')\,\mathrm{d}x'$. The following operators $\mathcal{M}_{\mathrm{D}}$ and $\mathcal{M}_{\mathrm{N}}$ defined by*

$$(\mathcal{M}_{\mathrm{D}} - c)u(x) := -\int_\Omega \big[\widehat{C}_{\mathrm{a}}(x'-x)P_e u(x') + \widehat{C}_{\mathrm{p}}(x'-x)P_o u(x')\big]\,\mathrm{d}x',$$
$$(\mathcal{M}_{\mathrm{N}} - c)u(x) := -\int_\Omega \big[\widehat{C}_{\mathrm{p}}(x'-x)P_e u(x') + \widehat{C}_{\mathrm{a}}(x'-x)P_o u(x')\big]\,\mathrm{d}x'$$

agree with $\mathcal{L}u(x)$ in the bulk, i.e., for $x \in (-1+\delta, 1-\delta)$. Furthermore, the operators $\mathcal{M}_{\mathrm{D}}$ and $\mathcal{M}_{\mathrm{N}}$ enforce homogeneous Dirichlet and Neumann BC, respectively. More precisely, for $u(\pm 1) = 0$ and $u'(\pm 1) = 0$, we obtain $\mathcal{M}_{\mathrm{D}}u(\pm 1) = 0$ and $\mathcal{M}_{\mathrm{N}}u'(\pm 1) = 0$, respectively.

Related work and structure of the paper. In [Beyer et al. 2016], one of our major results was the finding that, in $\mathbb{R}$, the PD governing operator is a function of the governing operator of (local) classical elasticity. This result opened the path to the introduction of local boundary conditions into PD theory. Building on [Beyer et al. 2016], we generalized the results in $\mathbb{R}$ to bounded domains, a critical feature for all practical applications. In [Aksoylu et al. 2017b], we laid the theoretical foundations and in [Aksoylu et al. 2017a], we applied the foundations to prominent BC such as Dirichlet and Neumann, as well as presented numerical implementation of the corresponding wave propagation. We carried out numerical experiments by utilizing $\mathcal{M}_{\mathtt{D}}$ and $\mathcal{M}_{\mathtt{N}}$ as governing operators in the previously cited work. In [Aksoylu and Celiker 2016], we studied other related governing operators. In [Aksoylu et al. 2017c], we presented the extension of the novel operators to 2D. In [Aksoylu and Kaya 2017], we studied the condition numbers of the novel governing operators. Therein, we proved that the modifications made to the operator $\mathcal{L}_{\text{orig}}$ to obtain the novel operators are minor as far as the condition numbers are concerned.

The rest of the article is structured as follows. In Section 2, we present the main observation that leads to the construction of the novel operators that enforce Dirichlet and Neumann BC. In Section 3, we give the proof of the main theorem. In Section 4, we show how to obtain the operators that enforce antiperiodic and periodic BC by choosing suitable combinations of kernel functions. In Section 5, when an equation using the governing operators is solved, we show that the right-hand side function should satisfy the same the BC enforced by the governing operator. In Section 6, we provide the highlights of the extension from the 1D construction to 2D. We conclude in Section 7.

2. The main observation and the construction

Let us study the definition of $\widehat{C}_{\mathtt{a}}(x)$ given in (1-5) by explicitly writing the expression of the kernel in (1-4) as follows:

$$\widehat{C}_{\mathtt{a}}(x) = \begin{cases} -\chi_\delta(x+2)\mu(x+2), & x \in [-2,-1), \\ \chi_\delta(x)\mu(x), & x \in [-1,1], \\ -\chi_\delta(x-2)\mu(x-2), & x \in (1,2]. \end{cases}$$

Let us closely look at the first expression in the above definition of $\widehat{C}_{\mathtt{a}}(x)$:

$$\widehat{C}_{\mathtt{a}}(x)|_{x\in[-2,-1)} = -\chi_\delta(x+2)\mu(x+2). \tag{2-1}$$

The expression in (2-1) is equivalent to

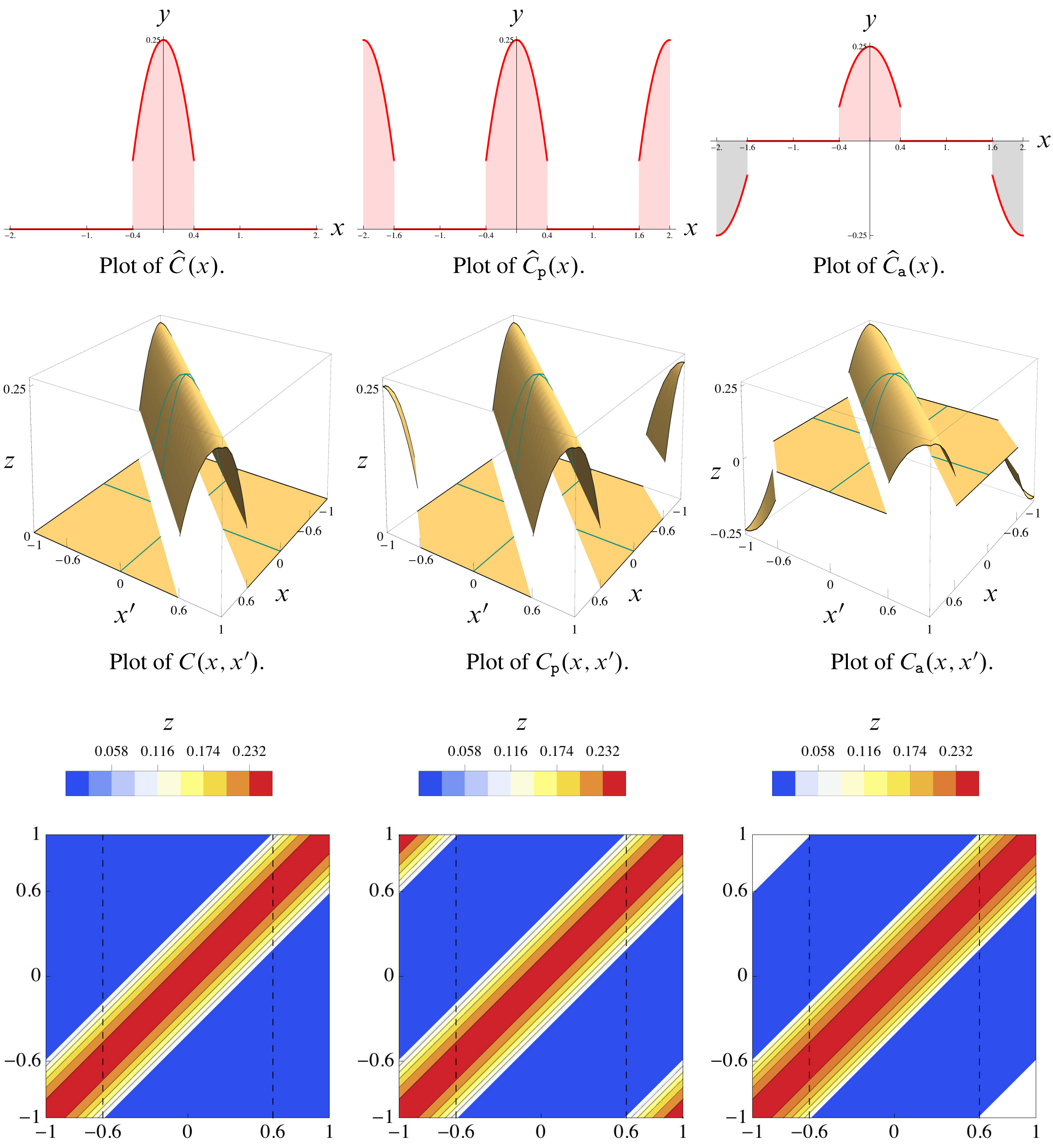
$$\widehat{C}_{\mathtt{a}}(x)|_{x\in[-2,-1)} = \begin{cases} -\mu(x+2) & \text{if } x+2 \in (-\delta,\delta) \text{ and } x \in [-2,-1), \\ 0 & \text{if } x+2 \notin (-\delta,\delta) \text{ and } x \in [-2,-1). \end{cases} \tag{2-2}$$

Due to the set equivalence

$$\{x : x+2 \in (-\delta,\delta) \text{ and } x \in [-2,-1)\} = \{x : x \in [-2-\delta,-2+\delta) \cap [-2,-1) = [-2,-2+\delta)\},$$

the expression in (2-2) reduces to

$$\widehat{C}_{\mathtt{a}}(x)|_{x\in[-2,-1)} = \begin{cases} -\mu(x+2) & \text{if } x \in [-2,-2+\delta), \\ 0 & \text{if } x \in [-2+\delta,-1). \end{cases} \tag{2-3}$$



Plot of $\widehat{C}(x)$. Plot of $\widehat{C}_{\mathrm{p}}(x)$. Plot of $\widehat{C}_{\mathrm{a}}(x)$.

Plot of $C(x, x')$. Plot of $C_{\mathrm{p}}(x, x')$. Plot of $C_{\mathrm{a}}(x, x')$.

Contour plot of $C(x, x')$. Contour plot of $C_{\mathrm{p}}(x, x')$. Contour plot of nonnegative part of $C_{\mathrm{a}}(x, x')$.

Figure 1. The kernel function $C(x) = \chi_\delta(x)\mu(x)$ with $\chi_\delta(x)$ given in (1-3), $\delta = 0.4$, and $\mu(x) = 0.25 - x^2$. The zero, periodic, and antiperiodic extensions of $C(x)$ are denoted by $\widehat{C}(x)$, $\widehat{C}_p(x)$, and $\widehat{C}_a(x)$, respectively. For plotting, we employ bivariate versions of $\widehat{C}(x' - x)$, $\widehat{C}_{\mathrm{p}}(x' - x)$, and $\widehat{C}_{\mathrm{a}}(x' - x)$ defined by $C(x, x') := \widehat{C}(x' - x)$, $C_{\mathrm{p}}(x, x') := \widehat{C}_{\mathrm{p}}(x' - x)$, and $C_{\mathrm{a}}(x, x') := \widehat{C}_{\mathrm{a}}(x' - x)$.

Similarly, for $x \in (1, 2]$, we have

$$\widehat{C}_{\mathrm{a}}(x)|_{x\in(1,2]} = \begin{cases} 0 & \text{if } x \in (1, 2-\delta], \\ -\mu(x-2) & \text{if } x \in (2-\delta, 2]. \end{cases} \tag{2-4}$$

Combining (2-3) and (2-4), for $x \in [-2, 2]$, we obtain

$$\widehat{C}_{\mathrm{a}}(x) = \begin{cases} -\mu(x-2) & \text{if } x \in [-2, -2+\delta), \\ \mu(x) & \text{if } x \in (-\delta, \delta), \\ -\mu(x+2) & \text{if } x \in (2-\delta, 2], \\ 0 & \text{otherwise.} \end{cases} \tag{2-5}$$

Similarly, we obtain the following expression for the periodic extension:

$$\widehat{C}_{\mathrm{p}}(x) = \begin{cases} \mu(x-2) & \text{if } x \in [-2, -2+\delta), \\ \mu(x) & \text{if } x \in (-\delta, \delta), \\ \mu(x+2) & \text{if } x \in (2-\delta, 2], \\ 0 & \text{otherwise.} \end{cases} \tag{2-6}$$

Lemma 2.1. *Let the kernel function $C(x)$ be in the form*

$$C(x) = \chi_\delta(x)\mu(x),$$

where $\mu(x) \in L^2(\Omega)$ is even. Let $\widehat{C}(x)$, $\widehat{C}_{\mathrm{a}}(x)$, and $\widehat{C}_{\mathrm{p}}(x)$ denote the zero, antiperiodic, and periodic extensions of $C(x)$ to $\widehat{\Omega} := [-2, 2]$, respectively. Then,

$$\widehat{C}(x) = \widehat{C}_{\mathrm{a}}(x) = \widehat{C}_{\mathrm{p}}(x), \quad x \in (-2+\delta, 2-\delta). \tag{2-7}$$

Furthermore, we have the following agreement in the bulk. For $x \in (-1+\delta, 1-\delta)$,

$$\widehat{C}(x'-x) = \widehat{C}_{\mathrm{a}}(x'-x) = \widehat{C}_{\mathrm{p}}(x'-x), \quad x' \in [-1, 1]. \tag{2-8}$$

Proof. By the definition of the functions $\widehat{C}_{\mathrm{a}}(x)$ and $\widehat{C}_{\mathrm{p}}(x)$ in (2-5) and (2-6), respectively, they differ from $\widehat{C}(x)$ only on $[-2, -2+\delta) \cup (2-\delta, 2]$. Also, see Figure 1. Hence, $\widehat{C}(x)$, $\widehat{C}_{\mathrm{a}}(x)$, and $\widehat{C}_{\mathrm{p}}(x)$ coincide on $[-2+\delta, 2-\delta)$, i.e., (2-7) holds.

Since for x in the bulk, i.e., $x \in (-1+\delta, 1-\delta)$ and x' in the range of integration, i.e., $x' \in [-1, 1]$, we have $x'-x \in (-2+\delta, 2-\delta)$. From (2-7), we conclude (2-8). □

Remark 2.2. The kernel $\widehat{C}(x) : [-2, 2] \to \mathbb{R}$ is a univariate function. The operator $\mathcal{L}_{\text{orig}}$ utilizes $\widehat{C}(x'-x)$. In order to visualize (2-7), it is more useful to define bivariate versions of $\widehat{C}(x'-x)$, $\widehat{C}_{\mathrm{a}}(x'-x)$, and $\widehat{C}_{\mathrm{p}}(x'-x)$, respectively, as follows:

$$C,\ C_{\mathrm{a}},\ C_{\mathrm{p}} : [-1, 1] \times [-1, 1] \to \mathbb{R}.$$

For brevity, with a slight abuse of notation, we represent the bivariate functions using the same name of the univariate function $C(\cdot)$, i.e., $C(x, x') := \widehat{C}(x'-x)$, $C_{\mathrm{a}}(x, x') := \widehat{C}_{\mathrm{a}}(x'-x)$, and $C_{\mathrm{p}}(x, x') := \widehat{C}_{\mathrm{p}}(x'-x)$. This notation is also used in Figure 1. Hence, analogous to (2-7), in the bulk (i.e., $x \in (-1+\delta, 1-\delta)$), kernel functions coincide

$$C(x, x') = C_{\mathrm{a}}(x, x') = C_{\mathrm{p}}(x, x'). \tag{2-9}$$

We first prove that the operators $\mathcal{L}$ and $\mathcal{L}_{\text{orig}}$ agree in the bulk.

Lemma 2.3. $$\mathcal{L}u(x) = \mathcal{L}_{\text{orig}}u(x), \quad x \in (-1+\delta, 1-\delta).$$

Proof. For x in the bulk, we have $(x-\delta, x+\delta) \cap \Omega = (x-\delta, x+\delta)$. Hence,

$$\begin{aligned}\int_\Omega \widehat{C}(x'-x)\,\mathrm{d}x' &= \int_\Omega \widehat{\chi}_\delta(x'-x)\widehat{\mu}(x'-x)\,\mathrm{d}x' \\ &= \int_{(x-\delta,x+\delta)\cap\Omega} \widehat{\mu}(x'-x)\,\mathrm{d}x' \\ &= \int_{(x-\delta,x+\delta)} \widehat{\mu}(x'-x)\,\mathrm{d}x' \\ &= \int_{(-\delta,\delta)} \mu(x')\,\mathrm{d}x' \\ &= \int_\Omega \chi_\delta(x')\mu(x')\,\mathrm{d}x' \\ &= \int_\Omega C(x')\,\mathrm{d}x'.\end{aligned}$$

The result follows. □

3. Dirichlet and Neumann BC, and differentiation under the integral sign

Imposing Neumann (also antiperiodic and periodic) BC requires differentiation. Thus, we present technical details regarding differentiation under the integral sign which are provided in Lemma 3.1. The proof of Lemma 3.1, which is omitted here, is by the Lebesgue dominated convergence theorem. Similarly, the limit in the definition of the Dirichlet BC can be interchanged with the integral, again by the Lebesgue dominated convergence theorem.

Lemma 3.1. *Suppose that the function $k : \Omega_x \times \Omega_{x'} \to \mathbb{R}$ satisfies the following conditions.*

(1) *The function $k(x, x')$ is measurable with respect to x' for each $x \in \Omega_x$.*

(2) *For almost every $x' \in \Omega_{x'}$, the derivative $\partial k/\partial x(x, x')$ exists for all $x \in \Omega_x$.*

(3) *There is an integrable function $\ell : \Omega_{x'} \to \mathbb{R}$ such that $|\partial k/\partial x(x, x')| \le \ell(x')$ for all $x \in \Omega_x$.*

Then,

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_{\Omega_{x'}} k(x, x')\,\mathrm{d}x' = \int_{\Omega_{x'}} \frac{\partial k}{\partial x}(x, x')\,\mathrm{d}x'.$$

We use Lemma 3.1 to check if the operator $\mathcal{M}_{\mathtt{N}}$ enforces homogeneous Neumann BC. First, we want to identify the integrand associated to $\mathcal{M}_{\mathtt{N}}$. We start with writing P_e and P_o explicitly and utilizing a simple change of variable as follows:

$$\begin{aligned}(\mathcal{M}_{\mathtt{N}} - c)u(x) &= -\int_\Omega \bigl[\widehat{C}_{\mathtt{p}}(x'-x)\tfrac{1}{2}\bigl(u(x') + u(-x')\bigr) + \widehat{C}_{\mathtt{a}}(x'-x)\tfrac{1}{2}\bigl(u(x') - u(-x')\bigr)\bigr]\,\mathrm{d}x' \\ &= -\int_\Omega K_{\mathtt{N}}(x, x')u(x')\,\mathrm{d}x',\end{aligned} \tag{3-1}$$

where

$$K_{\mathrm{N}}(x,x') := \tfrac{1}{2}\big\{[\widehat{C}_{\mathrm{a}}(x'-x) - \widehat{C}_{\mathrm{a}}(x'+x)] + [\widehat{C}_{\mathrm{p}}(x'-x) + \widehat{C}_{\mathrm{p}}(x'+x)]\big\}.$$

Analogous to the construction given in [Aksoylu et al. 2017a], we assume that $C(x) \in L^2(\Omega)$, and hence,

$$\widehat{C}(x),\ \widehat{C}_{\mathrm{a}}(x),\ \widehat{C}_{\mathrm{p}}(x) \in L^2(\widehat{\Omega}). \tag{3-2}$$

We are now in a position to determine the necessary conditions needed to apply Lemma 3.1. First, we set $\Omega_x = \Omega_{x'} = \Omega$ and

$$k(x,x') = K_{\mathrm{N}}(x,x')u(x').$$

Considering the jumps and the fact that the BC is enforced at the boundary, we assume that $\widehat{C}_{\mathrm{a}}(x)$ and $\widehat{C}_{\mathrm{p}}(x)$ are piecewise continuously differentiable in $\widehat{\Omega}$ and continuously differentiable functions up to $\partial\widehat{\Omega}$. Hence, the first two conditions of Lemma 3.1 are satisfied. To satisfy the third condition, we define

$$\ell(x') := \operatorname*{ess\,sup}_{x\in\Omega_x} \left| \frac{\partial K_{\mathrm{N}}}{\partial x}(x,x') \right| |u(x')|,$$

and assume that

$$\operatorname*{ess\,sup}_{x\in\Omega_x} \left| \frac{\partial K_{\mathrm{N}}}{\partial x}(x,x') \right| \in L^2(\Omega_{x'}). \tag{3-3}$$

The integrability of $\ell(x')$ is sufficient to satisfy the third condition. We could choose any $L^p(\Omega_{x'})$ space. We choose the space $L^2(\Omega_{x'})$ in (3-3) in order to align with the construction given in [Aksoylu et al. 2017a]. Since $u(x') \in L^2(\Omega_{x'})$, we obtain $\ell(x') \in L^2(\Omega_{x'})$.

We are now ready to prove our Main Theorem.

Proof of Theorem 1.1. We exploit (2-9) in constructing the governing operators that enforce Neumann and Dirichlet BCs by rewriting the $\mathcal{L}$ operator in the following way. For $x \in (-1+\delta, 1-\delta)$, we have

$$\begin{aligned}
(\mathcal{L}-c)u(x) &= -\int_\Omega \widehat{C}(x'-x)u(x')\,\mathrm{d}x' \\
&= -\int_\Omega \widehat{C}(x'-x)(P_e+P_o)u(x')\,\mathrm{d}x' \\
&= -\int_\Omega \big[\widehat{C}(x'-x)P_e u(x') + \widehat{C}(x'-x)P_o u(x')\big]\,\mathrm{d}x' \\
&= -\int_\Omega \big[\widehat{C}_{\mathrm{p}}(x'-x)P_e u(x') + \widehat{C}_{\mathrm{a}}(x'-x)P_o u(x')\big]\,\mathrm{d}x' \\
&= (\mathcal{M}_{\mathrm{N}}-c)u(x).
\end{aligned}$$

Similarly, for $x \in (-1+\delta, 1-\delta)$,

$$\begin{aligned}
(\mathcal{L}-c)u(x) &= -\int_\Omega \big[\widehat{C}(x'-x)P_e u(x') + \widehat{C}(x'-x)P_o u(x')\big]\,\mathrm{d}x' \\
&= -\int_\Omega \big[\widehat{C}_{\mathrm{a}}(x'-x)P_e u(x') + \widehat{C}_{\mathrm{p}}(x'-x)P_o u(x')\big]\,\mathrm{d}x' \\
&= (\mathcal{M}_{\mathrm{D}}-c)u(x).
\end{aligned}$$

Next, we show that $\mathcal{M}_{\mathtt{N}}$ and $\mathcal{M}_{\mathtt{D}}$ enforce homogeneous Neumann and Dirichlet BC, respectively.

The operator $\mathcal{M}_{\mathtt{N}}$: First we remove the points at which the derivative of $K_{\mathtt{N}}(x,x')$ does not exist from the set of integration. Note that such points form a set of measure zero, and hence, do not affect the value of the integral. We differentiate both sides of (3-1) and apply Lemma 3.1 to interchange the differentiation with the integral. We can differentiate the integrand $K_{\mathtt{N}}(x,x')$ in a piecewise fashion and obtain

$$\frac{\mathrm{d}}{\mathrm{d}x}[(\mathcal{M}_{\mathtt{N}}-c)u](x) = -\int_{\Omega}\frac{\partial K_{\mathtt{N}}}{\partial x}(x,x')u(x')\,\mathrm{d}x', \tag{3-4}$$

where

$$\frac{\partial K_{\mathtt{N}}}{\partial x}(x,x') = \tfrac{1}{2}\big\{[\widehat{C}'_{\mathtt{a}}(x'-x)-\widehat{C}'_{\mathtt{a}}(x'+x)]+[\widehat{C}'_{\mathtt{p}}(x'-x)+\widehat{C}'_{\mathtt{p}}(x'+x)]\big\}.$$

We check the boundary values by plugging $x=\pm 1$ in (3-4):

$$\frac{\mathrm{d}}{\mathrm{d}x}[(\mathcal{M}_{\mathtt{N}}-c)u](\pm 1) = -\int_{\Omega}\frac{\partial K_{\mathtt{N}}}{\partial x}(\pm 1,x')u(x')\,\mathrm{d}x'. \tag{3-5}$$

The functions $\widehat{C}'_{\mathtt{a}}$ and $\widehat{C}'_{\mathtt{p}}$ are 2-antiperiodic and 2-periodic because they are the derivatives of 2-antiperiodic and 2-periodic functions, respectively. Hence,

$$\widehat{C}'_{\mathtt{a}}(\pm 1+x') = -\widehat{C}'_{\mathtt{a}}(\mp 1+x') \quad \text{and} \quad \widehat{C}'_{\mathtt{p}}(\mp 1+x') = \widehat{C}'_{\mathtt{p}}(\pm 1+x'). \tag{3-6}$$

Hence, the integrand in (3-5) vanishes, i.e.,

$$\frac{\partial K_{\mathtt{N}}}{\partial x}(\pm 1,x') = 0.$$

Therefore, we arrive at

$$\frac{\mathrm{d}}{\mathrm{d}x}\mathcal{M}_{\mathtt{N}}u(\pm 1) = cu'(\pm 1). \tag{3-7}$$

Since we assume that u satisfies homogeneous Neumann BC, i.e., $u'(\pm 1)=0$, we conclude that the operator $\mathcal{M}_{\mathtt{N}}$ enforces homogeneous Neumann BC as well.

The operator $\mathcal{M}_{\mathtt{D}}$: In order to check if the operator $\mathcal{M}_{\mathtt{D}}$ enforces homogeneous Dirichlet BC, we start again with writing P_e and P_o explicitly and utilizing a simple change of variables as follows:

$$\begin{aligned}(\mathcal{M}_{\mathtt{D}}-c)u(x) &= -\int_{\Omega}\widehat{C}_{\mathtt{a}}(x'-x)\tfrac{1}{2}(u(x')+u(-x'))\,\mathrm{d}x' - \int_{\Omega}\widehat{C}_{\mathtt{p}}(x'-x)\tfrac{1}{2}(u(x')-u(-x'))\,\mathrm{d}x' \\ &= -\int_{\Omega}K_{\mathtt{D}}(x,x')u(x')\,\mathrm{d}x',\end{aligned}$$

where

$$K_{\mathtt{D}}(x,x') := \tfrac{1}{2}\big\{[\widehat{C}_{\mathtt{a}}(x'-x)+\widehat{C}_{\mathtt{a}}(x'+x)]+[\widehat{C}_{\mathtt{p}}(x'-x)-\widehat{C}_{\mathtt{p}}(x'+x)]\big\}. \tag{3-8}$$

By the Lebesgue dominated convergence theorem, the limit in the definition of the Dirichlet BC can be interchanged with the integral. Now, we check the boundary values by plugging $x=\pm 1$ in (3-8):

$$(\mathcal{M}_{\mathtt{D}}-c)u(\pm 1) = -\int_{\Omega}K_{\mathtt{D}}(\pm 1,x')u(x')\,\mathrm{d}x'. \tag{3-9}$$

Since $\widehat{C}_{\mathtt{a}}$ and $\widehat{C}_{\mathtt{p}}$ are 2-antiperiodic and 2-periodic, respectively, we have

$$\widehat{C}_{\mathtt{a}}(\mp 1 + x') = -\widehat{C}_{\mathtt{a}}(\pm 1 + x') \quad \text{and} \quad \widehat{C}_{\mathtt{p}}(\mp 1 + x') = \widehat{C}_{\mathtt{p}}(\pm 1 + x'). \tag{3-10}$$

Hence, the integrand in (3-9) vanishes, i.e., $K_{\mathtt{D}}(\pm 1, x') = 0$. Therefore, we arrive at

$$\mathcal{M}_{\mathtt{D}} u(\pm 1) = c u(\pm 1). \tag{3-11}$$

Since we assume that u satisfies homogeneous Dirichlet BC, i.e., $u(\pm 1) = 0$, we conclude that the operator $\mathcal{M}_{\mathtt{D}}$ enforces homogeneous Dirichlet BC as well. □

Remark 3.2. We have defined $\mathcal{M}_{\mathtt{D}}$ and $\mathcal{M}_{\mathtt{N}}$ in a way that they are linear bounded operators. More precisely, $\mathcal{M}_{\mathtt{D}}, \mathcal{M}_{\mathtt{N}} \in L(X, X)$ where $X = L^2(\Omega) \cap C^1(\partial\Omega)$. For $\mathcal{M}_{\mathtt{D}}$, the choice of X can be relaxed as $L^2(\Omega) \cap C^0(\partial\Omega)$. This choice is implied when we study $\mathcal{M}_{\mathtt{D}}$. Boundedness of $\mathcal{M}_{\mathtt{D}}$ and $\mathcal{M}_{\mathtt{N}}$ follows from the choice of (1-6) and (3-2). In addition, since $\mathcal{M}_{\mathtt{D}}$ and $\mathcal{M}_{\mathtt{N}}$ are both integral operators, their self-adjointness follows easily from the fact that the corresponding kernels are symmetric (due to evenness of C), i.e., $K_{\mathtt{D}}(x, x') = K_{\mathtt{D}}(x', x)$ and $K_{\mathtt{N}}(x, x') = K_{\mathtt{N}}(x', x)$.

4. Other possible boundary conditions

The construction employed to satisfy local BC is based on the following decomposition of $u(x')$:

$$u(x') = P_e u(x') + P_o u(x'),$$

and the agreement of $\widehat{C}(x'-x)$ with $\widehat{C}_{\mathtt{a}}(x'-x)$ and $\widehat{C}_{\mathtt{p}}(x'-x)$ in the bulk; see (2-8). By replacing $\widehat{C}(x'-x)$ with either $\widehat{C}_{\mathtt{a}}(x'-x)$ or $\widehat{C}_{\mathtt{p}}(x'-x)$, we have the following four combinations for the integrand of $\mathcal{L}$:

$$\widehat{C}(x'-x)u(x') = \{\widehat{C}_{\mathtt{a}}(x'-x), \widehat{C}_{\mathtt{p}}(x'-x)\} P_e u(x') + \{\widehat{C}_{\mathtt{a}}(x'-x), \widehat{C}_{\mathtt{p}}(x'-x)\} P_o u(x').$$

Denoting the choice of $\widehat{C}_{\mathtt{a}}(x'-x)$ and $\widehat{C}_{\mathtt{p}}(x'-x)$ by a and p, respectively, the combinations ap and pa give rise to Dirichlet and Neumann BC, respectively; see Theorem 1.1. Namely,

$$\begin{aligned}
\text{Dirichlet (ap):} \quad & \widehat{C}(x'-x)u(x') = \widehat{C}_{\mathtt{a}}(x'-x) P_e u(x') + \widehat{C}_{\mathtt{p}}(x'-x) P_o u(x'), \\
\text{Neumann (pa):} \quad & \widehat{C}(x'-x)u(x') = \widehat{C}_{\mathtt{p}}(x'-x) P_e u(x') + \widehat{C}_{\mathtt{a}}(x'-x) P_o u(x').
\end{aligned}$$

We show that the combinations aa and pp give rise to antiperiodic and periodic BC, respectively. Namely,

$$\begin{aligned}
\text{Antiperiodic (aa):} \quad & \widehat{C}(x'-x)u(x') = \widehat{C}_{\mathtt{a}}(x'-x) P_e u(x') + \widehat{C}_{\mathtt{a}}(x'-x) P_o u(x') = \widehat{C}_{\mathtt{a}}(x'-x) u(x'), \\
\text{Periodic (pp):} \quad & \widehat{C}(x'-x)u(x') = \widehat{C}_{\mathtt{p}}(x'-x) P_e u(x') + \widehat{C}_{\mathtt{p}}(x'-x) P_o u(x') = \widehat{C}_{\mathtt{p}}(x'-x) u(x').
\end{aligned}$$

Then, the operators $\mathcal{M}_{\mathtt{a}}$ and $\mathcal{M}_{\mathtt{p}}$ are defined by

$$\begin{aligned}
(\mathcal{M}_{\mathtt{a}} - c)u(x) &:= -\int_\Omega \widehat{C}_{\mathtt{a}}(x'-x)u(x')\,dx', \\
(\mathcal{M}_{\mathtt{p}} - c)u(x) &:= -\int_\Omega \widehat{C}_{\mathtt{p}}(x'-x)u(x')\,dx'.
\end{aligned}$$

We recall the space of functions used to enforce antiperiodic and periodic BC, respectively [Aksoylu et al. 2017a]:

$$\big\{u \in L^2(\Omega)\cap C^1(\partial\Omega) : \quad \lim_{x\to -1} u(x) = -\lim_{x\to 1} u(x), \quad \lim_{x\to -1} u'(x) = -\lim_{x\to 1} u'(x)\big\},$$
$$\big\{u \in L^2(\Omega)\cap C^1(\partial\Omega) : \quad \lim_{x\to -1} u(x) = \lim_{x\to 1} u(x), \quad \lim_{x\to -1} u'(x) = \lim_{x\to 1} u'(x)\big\}.$$

Since $\widehat{C}_{\mathtt{a}}$ and $\widehat{C}'_{\mathtt{a}}$ are 2-antiperiodic and $\widehat{C}_{\mathtt{p}}$ and $\widehat{C}'_{\mathtt{p}}$ are 2-periodic, similar to (3-10) and (3-6), we have

$$\widehat{C}_{\mathtt{a}}(-1-x') = -\widehat{C}_{\mathtt{a}}(1-x'), \quad \widehat{C}_{\mathtt{p}}(-1-x') = \widehat{C}_{\mathtt{p}}(1-x'),$$
$$\widehat{C}'_{\mathtt{a}}(-1-x') = -\widehat{C}'_{\mathtt{a}}(1-x'), \quad \widehat{C}'_{\mathtt{p}}(-1-x') = \widehat{C}'_{\mathtt{p}}(1-x').$$

Consequently,

$$(\mathcal{M}_{\mathtt{a}} - c)u(-1) = -(\mathcal{M}_{\mathtt{a}} - c)u(1), \tag{4-1}$$
$$(\mathcal{M}_{\mathtt{p}} - c)u(-1) = (\mathcal{M}_{\mathtt{p}} - c)u(1). \tag{4-2}$$

In addition, by applying Lemma 3.1, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}x}[(\mathcal{M}_{\mathtt{a}} - c)u](-1) = -\frac{\mathrm{d}}{\mathrm{d}x}[(\mathcal{M}_{\mathtt{a}} - c)u](1), \tag{4-3}$$
$$\frac{\mathrm{d}}{\mathrm{d}x}[(\mathcal{M}_{\mathtt{p}} - c)u](-1) = \frac{\mathrm{d}}{\mathrm{d}x}[(\mathcal{M}_{\mathtt{p}} - c)u](1). \tag{4-4}$$

These imply that the operators $\mathcal{M}_{\mathtt{a}}$ and $\mathcal{M}_{\mathtt{p}}$ enforce antiperiodic and periodic BC, respectively.

5. Compatibility conditions

When we solve an equation using the operators $\mathcal{M}_{\mathtt{BC}}$ where $\mathtt{BC} \in \{\mathtt{D}, \mathtt{N}, \mathtt{a}, \mathtt{p}\}$, i.e.,

$$\mathcal{M}_{\mathtt{BC}} u = f_{\mathtt{BC}},$$

we want to identify the conditions imposed on $f_{\mathtt{BC}}$. Since the operator $\mathcal{M}_{\mathtt{BC}}$ enforces the corresponding BC, we observe that the same BC is imposed on $f_{\mathtt{BC}}$. To see this, we start by assuming that u satisfies the corresponding BC. Then, we choose $f_{\mathtt{BC}}$ from the same space to which u belongs, i.e.,

$$f_{\mathtt{BC}} \in L^2(\Omega)\cap C^1(\partial\Omega).$$

From (4-1) and (4-2), respectively, we immediately see that

$$f_{\mathtt{a}}(-1) = -f_{\mathtt{a}}(1), \quad f_{\mathtt{p}}(-1) = f_{\mathtt{p}}(1).$$

From (4-3) and (4-4), respectively, we also get

$$f'_{\mathtt{a}}(-1) = -f'_{\mathtt{a}}(1), \quad f'_{\mathtt{p}}(-1) = f'_{\mathtt{p}}(1).$$

In addition, from (3-11) and (3-7), respectively, we obtain

$$f_{\mathtt{D}}(\pm 1) = 0, \quad \frac{\mathrm{d} f_{\mathtt{N}}}{\mathrm{d}x}(\pm 1) = 0.$$

6. The extension to a 2D problem

In this section, we present the extension of the present work to 2D problems. The main idea of this extension relies on our 1D construction but it is nontrivial. Its proof requires a significant amount of technical detail. Here, we provide only a small part of the results without proof.

We choose the domain in 2D to be $\Omega = [-1, 1] \times [-1, 1]$. There are various combinations of BC one can enforce. Here, we report only pure Dirichlet and pure Neumann BC, the 2D analogues of the ones presented in Theorem 1.1. The proofs, a comprehensive discussion, and numerical results are provided in [Aksoylu et al. 2017c].

In 2D, the governing operator in (1-1) takes the form

$$\mathcal{L}_{\text{orig}} u(x, y) := \iint_{\Omega} \widehat{C}(x'-x, y'-y) u(x, y) \, \mathrm{d}x \, \mathrm{d}x' - \iint_{\Omega} \widehat{C}(x'-x, y'-y) u(x', y') \, \mathrm{d}x' \, \mathrm{d}y'.$$

Similar to (1-2), we define the operator that is closely related to $\mathcal{L}_{\text{orig}}$ as

$$\mathcal{L}u(x, y) := c u(x, y) - \iint_{\Omega} \widehat{C}(x'-x, y'-y) u(x', y') \, \mathrm{d}x' \, \mathrm{d}y', \quad (x, y) \in \Omega,$$

where $c = \iint_{\Omega} C(x', y') \, \mathrm{d}x' \, \mathrm{d}y'$. The kernel function $C(x, y)$ is assumed to be even. Namely,

$$C(-x, -y) = C(x, y).$$

Similar to the 1D case, we choose the kernel function $C(x, y)$ to be the *canonical* kernel function $\chi_\delta(x, y)$, whose definition is given as follows. For $(x, y) \in \Omega$,

$$\chi_\delta(x, y) := \begin{cases} 1 & \text{if } (x, y) \in (-\delta, \delta) \times (-\delta, \delta), \\ 0 & \text{otherwise.} \end{cases}$$

The agreement of the operators $\mathcal{L}$ and $\mathcal{L}_{\text{orig}}$ in the 1D bulk shown in Lemma 2.1 carries over to the 2D bulk whose definition is given by

$$\text{bulk} = \big\{(x, y) \in \Omega : (x, y) \in (-1+\delta, 1-\delta) \times (-1+\delta, 1-\delta)\big\}.$$

Inspired by the projections that give the even and odd parts of a univariate function given in (1-7), we define the following operators that act on a bivariate function.

$$P_{e,x'}, P_{o,x'}, P_{e,y'}, P_{o,y'} : L^2(\Omega) \to L^2(\Omega),$$

whose definitions are

$$P_{e,x'} u(x', y') := \tfrac{1}{2}(u(x', y') + u(-x', y')), \qquad P_{o,x'} u(x', y') := \tfrac{1}{2}(u(x', y') - u(-x', y')), \quad (6\text{-}1)$$

$$P_{e,y'} u(x', y') := \tfrac{1}{2}(u(x', y') + u(x', -y')), \qquad P_{o,y'} u(x', y') := \tfrac{1}{2}(u(x', y') - u(x', -y')). \quad (6\text{-}2)$$

Each operator is an orthogonal projection and possesses the following decomposition property:

$$P_{e,x'} + P_{o,x'} = I_{x'}, \quad P_{e,y'} + P_{o,y'} = I_{y'}.$$

One can easily check that all four orthogonal projections in (6-1) and (6-2) commute with each other. We define the following new operators obtained from the products of these projections:

$$
\begin{aligned}
P_{e,x'}P_{e,y'}u(x',y') &:= \tfrac{1}{4}\big\{[u(x',y')+u(x',-y')]+[u(-x',y')+u(-x',-y')]\big\},\\
P_{e,x'}P_{o,y'}u(x',y') &:= \tfrac{1}{4}\big\{[u(x',y')-u(x',-y')]+[u(-x',y')-u(-x',-y')]\big\},\\
P_{o,x'}P_{o,y'}u(x',y') &:= \tfrac{1}{4}\big\{[u(x',y')-u(x',-y')]-[u(-x',y')-u(-x',-y')]\big\},\\
P_{o,x'}P_{e,y'}u(x',y') &:= \tfrac{1}{4}\big\{[u(x',y')+u(x',-y')]-[u(-x',y')+u(-x',-y')]\big\}.
\end{aligned}
$$

These operators are also orthogonal projections and satisfy the following decomposition property:

$$P_{e,x'}P_{e,y'}+P_{e,x'}P_{o,y'}+P_{o,x'}P_{e,y'}+P_{o,x'}P_{o,y'}=I_{x',y'}.$$

They will be used in the definition of the operators $\mathcal{M}_{\mathtt{D}}$ and $\mathcal{M}_{\mathtt{N}}$.

Theorem 6.1 (Main Theorem in 2D). *Let* $\Omega := [-1,1]\times[-1,1]$ *and the kernel function be separable in the form*

$$C(x,y)=X(x)Y(y), \tag{6-3}$$

where X *and* Y *are even functions. Then, the operators* $\mathcal{M}_{\mathtt{D}}$ *and* $\mathcal{M}_{\mathtt{N}}$ *defined by*

$$
\begin{aligned}
&(\mathcal{M}_{\mathtt{D}}-c)u(x,y) :=\\
&\quad -\iint_{\Omega}\big[\widehat{X}_{\mathtt{a}}(x'-x)P_{e,x'}+\widehat{X}_{\mathtt{p}}(x'-x)P_{o,x'}\big]\big[\widehat{Y}_{\mathtt{a}}(y'-y)P_{e,y'}+\widehat{Y}_{\mathtt{p}}(y'-y)P_{o,y'}\big]u(x',y')\,\mathrm{d}x'\,\mathrm{d}y',\\
&(\mathcal{M}_{\mathtt{N}}-c)u(x,y) :=\\
&\quad -\iint_{\Omega}\big[\widehat{X}_{\mathtt{p}}(x'-x)P_{e,x'}+\widehat{X}_{\mathtt{a}}(x'-x)P_{o,x'}\big]\big[\widehat{Y}_{\mathtt{p}}(y'-y)P_{e,y'}+\widehat{Y}_{\mathtt{a}}(y'-y)P_{o,y'}\big]u(x',y')\,\mathrm{d}x'\,\mathrm{d}y',
\end{aligned}
$$

agree with $\mathcal{L}u(x,y)$ *in the bulk, i.e., for* $(x,y)\in(-1+\delta,1-\delta)\times(-1+\delta,1-\delta)$. *Furthermore, the operators* $\mathcal{M}_{\mathtt{D}}$ *and* $\mathcal{M}_{\mathtt{N}}$ *enforce pure Dirichlet and pure Neumann BC, respectively*:

$$
\begin{aligned}
(\mathcal{M}_{\mathtt{D}}-c)u(x,\pm1) &= (\mathcal{M}_{\mathtt{D}}-c)u(\pm1,y)=0,\\
\frac{\partial}{\partial n}[(\mathcal{M}_{\mathtt{N}}-c)u](x,\pm1) &= \frac{\partial}{\partial n}[(\mathcal{M}_{\mathtt{N}}-c)u](\pm1,y)=0,
\end{aligned}
$$

where n *denotes the outward unit normal vector.*

Remark 6.2. Although we assume a separable kernel function $C(x,y)=X(x)Y(y)$ as in (6-3), note that we do not impose a separability assumption on the solution $u(x,y)$.

Remark 6.3. In Theorem 6.1, the function u is scalar valued, which corresponds to the solution of a nonlocal diffusion problem. In higher dimensional PD problems, the function u is vector valued. The extension of our construction to such problems is the subject of ongoing work.

7. Conclusion

We presented novel governing operators $\mathcal{M}_{\mathtt{D}}$ and $\mathcal{M}_{\mathtt{N}}$ in the theory of PD constructed by the guiding principle that they agree with the original PD operator $\mathcal{L}_{\text{orig}}$ in the bulk, and, at the same time, enforce local Dirichlet or Neumann BC. We also presented the operators $\mathcal{M}_{\mathtt{a}}$ and $\mathcal{M}_{\mathtt{p}}$ that enforce local antiperiodic and periodic BC. In [Aksoylu et al. 2017d], we give an overview of local BC in general nonlocal problems. We believe that our contribution is an important step towards extending the applicability of PD to problems that require local BC such as contact, shear, and traction. For future research, we plan to investigate if our approach of enforcing local BC can be used to eliminate surface effects. Finally, we presented the extension of the 1D governing operators to 2D on rectangular domains. The generalization to 3D box domains is straightforward. The construction of the operators for general geometries remains an open problem and constitutes the subject of ongoing work.

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BURAK AKSOYLU: burak@wayne.edu
Department of Mathematics, Wayne State University, 656 W. Kirby, Detroit, MI 48202, United States

FATIH CELIKER: celiker@wayne.edu
Department of Mathematics, Wayne State University, 656 W. Kirby, Detroit, MI 48202, United States

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