Journal of

Mechanics of Materials and Structures

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Volume 12, No. 5 December 2017

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JOURNAL OF MECHANICS OF MATERIALS AND STRUCTURES
Vol. 12, No. 5, 2017
dx.doi.org/10.2140/jomms.2017.12.711

TRANSIENT RESPONSE OF MULTILAYERED ORTHOTROPIC STRIPS WITH INTERFACIAL DIFFUSION AND SLIDING

XU WANG AND PETER SCHIAVONE

We use transfer matrix and state-space methods to derive exact solutions for the time-dependent and plane strain deformations of simply supported multilayered orthotropic elastic strips with simultaneous interfacial diffusion and rate-dependent sliding. Our analysis considers the corresponding interfacial tractions as the fundamental state variables. As a result, a homogeneous state-space equation can be obtained by enforcing the interfacial diffusion and sliding conditions. The relaxation times of the multilayered orthotropic strip and the evolution of the interfacial tractions can then be determined from the derived state-space equation. Once the transient interfacial tractions are known, all of the field variables at any time and at any position of the multilayered strip can be conveniently obtained.

1. Introduction

Interfacial diffusion and sliding are responsible for many phenomena at high temperatures [Sofronis and McMeeking 1994; Mori et al. 1997; He and Hu 2003] and are also closely related to room temperature plastic deformations in nanocrystalline materials [Wei et al. 2008]. The combined effect of interfacial diffusion and sliding in particulate, fibrous, laminated composites and in polycrystalline solids has been investigated in [Sofronis and McMeeking 1994; Kim and McMeeking 1995; Onaka et al. 1998; 1999; He and Hu 2003; Wang and Pan 2010; Wang et al. 2016; Wang and Wang 2016; Wei et al. 2008].

Most of the early discussions on laminated plates with rate-dependent imperfect interfaces are confined to short range diffusion-induced rate-dependent sliding (or viscous) interfaces in which the diffusion-induced long range mass transport at the interface is absent (see, for example, [He and Jiang 2003; Chen and Lee 2004]). Chen and Lee [2004] adopted power series expansions to approximate the variations of field variables with time. Very recently, Wang and Wang [2016] studied the time-dependent deformations of multilayered isotropic elastic strips with interfacial diffusion and sliding under cylindrical bending.

In this research, we endeavor to study the plane strain deformations of multilayered orthotropic elastic strips with simultaneous interfacial diffusion and sliding. First, we derive a general solution for displacements and stresses in a homogeneous orthotropic layer following Suo's method [1990]. Secondly, we use this general solution to obtain a transfer matrix relating the displacements and tractions on the upper interface of an orthotropic layer to those on its lower interface. Next, we derive a homogeneous state-space equation with interfacial tractions as state variables by imposing the interfacial diffusion and sliding conditions and by utilizing the transfer matrix method. It is noted that the construction of the state-space equation differs from that in [Wang and Wang 2016] in that here, the state vector is composed of the traction components on all of the existing interfaces whereas in that paper, the state vector is composed

Keywords: multilayered orthotropic strip, interfacial diffusion and sliding, transfer matrix, state-space equation, relaxation time.

of functions of time appearing in the expressions of displacements and stresses in all of the isotropic layers. Finally, the relaxation times and the transient elastic field in the multilayered strip can then be obtained by solving the corresponding state-space equation.

2. Analysis of a multilayered orthotropic strip

In a Cartesian coordinate system x_i $(i = 1, 2, 3)$, let u_i and σ_{ij} represent the displacements and stresses. As shown in Figure 1, we consider the plane strain deformations of a strip composed of N orthotropic elastic layers, labeled $1, 2, \dots, N$ from the bottom up. The $x_2 = 0$ plane coincides with the bottom surface of the strip and the x_2-axis is perpendicular to the strip. The strip of width l is simply supported at $x_1 = 0$ and $x_1 = l$. The thickness of layer j is h_j and the total thickness of the strip is $h = \sum_{j=1}^{N} h_j$. The subscript j or the superscript (j) will be used to denote the associated quantities in layer j. The strip is subjected only to a sinusoidal pressure loading $p = p_0 \sin kx_1$ with $k = \pi/l$ applied on its top surface. The boundary and interface conditions for the problem are specified as follows:

$$\sigma_{22}^{(N)} = -p_0 \sin kx_1, \quad \sigma_{12}^{(N)} = 0, \quad \text{at} \quad x_2 = h; \tag{1a}$$

$$\sigma_{22}^{(1)} = \sigma_{12}^{(1)} = 0 \quad \text{at} \quad x_2 = 0; \tag{1b}$$

$$\begin{gathered} \sigma_{22}^{(j+1)} = \sigma_{22}^{(j)}, \quad \sigma_{12}^{(j+1)} = \sigma_{12}^{(j)}, \quad \dot{u}_2^{(j)} - \dot{u}_2^{(j+1)} = D_j \frac{\partial^2 \sigma_{22}^{(j)}}{\partial x_1^2}, \\ \vartheta_j [\dot{u}_1^{(j+1)} - \dot{u}_1^{(j)}] = \sigma_{12}^{(j)}, \quad \text{at} \quad x_2 = \sum_{n=1}^{j} h_n, \quad j = 1, 2, \dots, N-1; \end{gathered} \tag{1c}$$

$$\sigma_{11}^{(1)} = \sigma_{11}^{(2)} = \cdots = \sigma_{11}^{(N)} = 0, \quad \text{and} \quad x_1 = 0 \quad \text{and} \quad x_1 = l; \tag{1d}$$

where the overdot denotes differentiation with respect to the time t, and D_j and ϑ_j are, respectively, the interface diffusion constant and viscosity for the interface between layer j and layer $j + 1$.

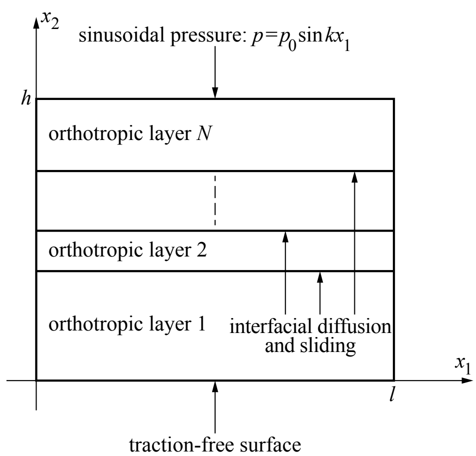

Figure 1. A multilayered orthotropic elastic strip with interfacial diffusion and rate-dependent sliding.

Let s_{ij} be the reduced elastic compliances in a certain layer. The general solution for displacements and stresses in this layer can be derived using the method proposed in [Suo 1990] as

$$\begin{bmatrix} \dfrac{u_1}{\cos kx_1} \\ \dfrac{u_2}{\sin kx_1} \end{bmatrix} =$$

$$\frac{s_{11}\lambda^2}{k}\begin{bmatrix} (n+m)^2-\delta & (n-m)^2-\delta & (n+m)^2-\delta & (n-m)^2-\delta \\ \dfrac{\lambda}{n+m}-\delta\lambda(n+m) & \dfrac{\lambda}{n-m}-\delta\lambda(n-m) & \delta\lambda(n+m)-\dfrac{\lambda}{n+m} & \delta\lambda(n-m)-\dfrac{\lambda}{n-m} \end{bmatrix}\begin{bmatrix} C_1e^{k\lambda(n+m)x_2} \\ C_2e^{k\lambda(n-m)x_2} \\ C_3e^{-k\lambda(n+m)x_2} \\ C_4e^{-k\lambda(n-m)x_2} \end{bmatrix},$$

$$\begin{bmatrix} \dfrac{\sigma_{12}}{\cos kx_1} \\ \dfrac{\sigma_{22}}{\sin kx_1} \end{bmatrix} = \begin{bmatrix} \lambda(n+m) & \lambda(n-m) & -\lambda(n+m) & -\lambda(n-m) \\ 1 & 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} C_1e^{k\lambda(n+m)x_2} \\ C_2e^{k\lambda(n-m)x_2} \\ C_3e^{-k\lambda(n+m)x_2} \\ C_4e^{-k\lambda(n-m)x_2} \end{bmatrix}, \tag{2}$$

where C_1, C_2, C_3, C_4 are coefficients to be determined, and λ, ρ, δ, n, m are dimensionless parameters defined by

$$\lambda = \left(\frac{s_{22}}{s_{11}}\right)^{1/4}, \quad \rho = \frac{2s_{12}+s_{66}}{2\sqrt{s_{11}s_{22}}}(\rho > -1), \quad \delta = \frac{s_{12}}{\sqrt{s_{12}s_{22}}} \quad (-1 \le \delta < \min\{\rho, 1\}),$$
$$n = \left[\tfrac{1}{2}(\rho+1)\right]^{1/2}, \quad m = \left[\tfrac{1}{2}(\rho-1)\right]^{1/2}. \tag{3}$$

The general solution in (2) is valid only for the nondegenerate case of $\rho \neq 1$. For the degenerate case $\rho = 1$ [Suo 1990], the displacements and stresses can be derived as

$$u_1 = [e^{kx_2^\lambda}(C_1 + x_2^\lambda C_2) + e^{-kx_2^\lambda}(C_3 + x_2^\lambda C_4)]\cos kx_1,$$
$$u_2 = \lambda\Big\{e^{kx_2^\lambda}\Big[C_1 + \Big(x_2^\lambda - \frac{3-4\tilde{\nu}}{k}\Big)C_2\Big] - e^{-kx_2^\lambda}\Big[C_3 + \Big(x_2^\lambda + \frac{3-4\tilde{\nu}}{k}\Big)C_4\Big]\Big\}\sin kx_1, \tag{4}$$

$$\sigma_{12} = 2\lambda\tilde{\mu}\big\{e^{kx_2^\lambda}[kC_1 + (kx_2^\lambda - 1 + 2\tilde{\nu})C_2] - e^{-kx_2}[kC_3 + (kx_2^\lambda + 1 - 2\tilde{\nu})C_4]\big\}\cos kx_1,$$
$$\sigma_{22} = 2\tilde{\mu}\big\{e^{kx_2^\lambda}[kC_1 + (kx_2^\lambda - 2 + 2\tilde{\nu})C_2] + e^{-kx_2^\lambda}[kC_3 + (kx_2^\lambda + 2 - 2\tilde{\nu})C_4]\big\}\sin kx_1, \tag{5}$$

where $x_2^\lambda = \lambda x_2$; C_1, C_2, C_3, C_4 are coefficients to be determined and

$$\tilde{\mu} = \frac{1}{s_{66}} = \frac{1}{2(\sqrt{s_{11}s_{22}} - s_{12})} = \frac{\lambda^2}{2s_{22}(1-\delta)},$$
$$\tilde{\nu} = 1 - \frac{\sqrt{s_{11}s_{22}}}{\sqrt{s_{11}s_{22}} - s_{12}} = 1 - \frac{2\sqrt{s_{11}s_{22}}}{s_{66}} = \frac{\delta}{\delta - 1} \le \frac{1}{2}. \tag{6}$$

Remark. Note that $\rho = 1$ is degenerate in the sense that we have $k\lambda(n+m) = k\lambda(n-m) = k\lambda$ and $-k\lambda(n+m) = -k\lambda(n-m) = -k\lambda$ in (2).

When $\rho = \lambda = 1$ for a transversely isotropic layer [Suo 1990], with isotropy being a special case, (4) and (5) simply reduce to those in [He and Jiang 2003]. It is interesting to note that the general solutions (2), (4) and (5) can be adapted to study the surface instability of orthotropic films due to surface van der

Waals forces [Wang and Li 2017]. From (2), (4) and (5) it follows that the displacements and tractions on the lower interface of layer j can be expressed in terms of those on its upper interface as

$$\begin{bmatrix} \tilde{\boldsymbol{u}}_0^{(j)} \\ \tilde{\boldsymbol{\sigma}}_0^{(j)} \end{bmatrix} = \boldsymbol{Q}(K_j, \delta_j, \rho_j) \begin{bmatrix} \tilde{\boldsymbol{u}}_1^{(j)} \\ \tilde{\boldsymbol{\sigma}}_1^{(j)} \end{bmatrix}, \tag{7}$$

where the subscripts 0 and 1 denote, respectively, the values on the lower and upper interfaces of layer j,

$$\tilde{\mathbf{u}}^{(j)} = \left[u_1^{(j)} \;\; u_2^{(j)}/\lambda_j\right]^T, \quad \tilde{\boldsymbol{\sigma}}^{(j)} = \left[\lambda_j s_{11}^{(j)} \sigma_{12}^{(j)}/k \;\; \lambda_j^2 s_{11}^{(j)} \sigma_{22}^{(j)}/k\right]^T, \quad K_j = k h_j \lambda_j; \tag{8}$$

$$\begin{aligned}
\mathbf{Q}(K,\delta,\rho) = {} & \frac{\cosh[K(n+m)]}{4nm} \\
& \times \begin{bmatrix} (n+m)^2-\delta & 0 & 0 & [\delta(n-m)^2-1][1-\delta(n+m)^2] \\ 0 & \delta-(n-m)^2 & [1-\delta(n-m)^2][1-\delta(n+m)^2] & 0 \\ 0 & -1 & (n+m)^2-\delta & 0 \\ 1 & 0 & 0 & \delta-(n-m)^2 \end{bmatrix} \\
& + \frac{\cosh[K(n-m)]}{4nm} \\
& \times \begin{bmatrix} \delta-(n-m)^2 & 0 & 0 & [1-\delta(n-m)^2][1-\delta(n+m)^2] \\ 0 & (n+m)^2-\delta & [\delta(n-m)^2-1][1-\delta(n+m)^2] & 0 \\ 0 & 1 & \delta-(n-m)^2 & 0 \\ -1 & 0 & 0 & (n+m)^2-\delta \end{bmatrix} \\
& + \frac{\sinh[K(n+m)]}{4nm(n+m)} \begin{bmatrix} 0 & (n+m)^2-\delta & -[(n+m)^2-\delta]^2 & 0 \\ \delta(n+m)^2-1 & 0 & 0 & [n-m-\delta(n+m)]^2 \\ -(n+m)^2 & 0 & 0 & 1-\delta(n+m)^2 \\ 0 & 1 & \delta-(n+m)^2 & 0 \end{bmatrix} \\
& + \frac{\sinh[K(n-m)]}{4nm(n+m)} \begin{bmatrix} 0 & \delta(n+m)^2-1 & [n-m-\delta(n+m)]^2 & 0 \\ (n+m)^2-\delta & 0 & 0 & -[(n+m)^2-\delta]^2 \\ 1 & 0 & 0 & \delta-(n+m)^2 \\ 0 & -(n+m)^2 & 1-\delta(n+m)^2 & 0 \end{bmatrix} \\
& \qquad \text{for } \rho \neq 1;
\end{aligned} \tag{9a}$$

and

$$\begin{aligned}
\mathbf{Q}(K,\delta,\rho) = {} & \cosh(K)\left[\mathbf{I} + \frac{K}{2}\begin{bmatrix} 0 & 1/(1-\tilde{\nu}) & -1/(1-\tilde{\nu})^2 & 0 \\ -1/(1-\tilde{\nu}) & 0 & 0 & 1/(1-\tilde{\nu})^2 \\ -1 & 0 & 0 & 1/(1-\tilde{\nu}) \\ 0 & 1 & -1/(1-\tilde{\nu}) & 0 \end{bmatrix}\right] \\
& + \frac{\sinh(K)}{2}\begin{bmatrix} K/(1-\tilde{\nu}) & (1-2\tilde{\nu})/(1-\tilde{\nu}) & -(3-4\tilde{\nu})/(1-\tilde{\nu})^2 & -K/(1-\tilde{\nu})^2 \\ (1-2\tilde{\nu})/(1-\tilde{\nu}) & -K/(1-\tilde{\nu}) & K/(1-\tilde{\nu})^2 & -(3-4\tilde{\nu})/(1-\tilde{\nu})^2 \\ -1 & -K & K/(1-\tilde{\nu}) & -(1-2\tilde{\nu})/(1-\tilde{\nu}) \\ K & -1 & -(1-2\tilde{\nu})/(1-\tilde{\nu}) & -K/(1-\tilde{\nu}) \end{bmatrix} \text{ for } \rho = 1.
\end{aligned} \tag{9b}$$

In writing (8), the factors $\cos kx_1$ and $\sin kx_1$ in the displacements and stresses have been excluded. Here $\mathbf{Q}(K, \delta, \rho)$ is a 4×4 transfer matrix. For convenience in the subsequent analysis, the transfer matrix is written in the following partitioned form

$$\mathbf{Q}(K_j, \delta_j, \rho_j) = \begin{bmatrix} \mathbf{Q}_1^{(j)} & \mathbf{Q}_2^{(j)} \\ \mathbf{Q}_3^{(j)} & \mathbf{Q}_4^{(j)} \end{bmatrix}, \tag{10}$$

where $\mathbf{Q}_1^{(j)}$, $\mathbf{Q}_2^{(j)}$, $\mathbf{Q}_3^{(j)}$ and $\mathbf{Q}_4^{(j)}$ are four 2×2 submatrices.

It follows from (7) that the displacements on the two interfaces of layer j can be expressed in terms of the tractions on the two interfaces of the layer as follows:

$$\begin{aligned} \tilde{\boldsymbol{u}}_1^{(j)} &= [\boldsymbol{Q}_3^{(j)}]^{-1}\tilde{\boldsymbol{\sigma}}_0^{(j)} - [\boldsymbol{Q}_3^{(j)}]^{-1}\boldsymbol{Q}_4^{(j)}\tilde{\boldsymbol{\sigma}}_1^{(j)}, \\ \tilde{\boldsymbol{u}}_0^{(j)} &= \boldsymbol{Q}_1^{(j)}[\boldsymbol{Q}_3^{(j)}]^{-1}\tilde{\boldsymbol{\sigma}}_0^{(j)} + \big[\boldsymbol{Q}_2^{(j)} - \boldsymbol{Q}_1^{(j)}[\boldsymbol{Q}_3^{(j)}]^{-1}\boldsymbol{Q}_4^{(j)}\big]\tilde{\boldsymbol{\sigma}}_1^{(j)}. \end{aligned} \tag{11}$$

In view of the fact the tractions are *continuous* across all of the interfaces, we have from the above that

$$\tilde{\boldsymbol{u}}_1^{(j)} = \frac{1}{\alpha_{j-1}\beta_{j-1}}[\boldsymbol{Q}_3^{(j)}]^{-1}\boldsymbol{L}_{j-1}\tilde{\boldsymbol{\sigma}}_1^{(j-1)} - [\boldsymbol{Q}_3^{(j)}]^{-1}\boldsymbol{Q}_4^{(j)}\tilde{\boldsymbol{\sigma}}_1^{(j)}, \tag{12a}$$

$$\tilde{\boldsymbol{u}}_0^{(j+1)} = \frac{1}{\alpha_j\beta_j}\boldsymbol{Q}_1^{(j+1)}[\boldsymbol{Q}_3^{(j+1)}]^{-1}\boldsymbol{L}_j\tilde{\boldsymbol{\sigma}}_1^{(j)} + \big[\boldsymbol{Q}_2^{(j+1)} - \boldsymbol{Q}_1^{(j+1)}[\boldsymbol{Q}_3^{(j+1)}]^{-1}\boldsymbol{Q}_4^{(j+1)}\big]\tilde{\boldsymbol{\sigma}}_1^{(j+1)}, \tag{12b}$$

where

$$\alpha_j = \frac{\lambda_{j+1}}{\lambda_j}, \quad \beta_j = \sqrt{\frac{s_{11}^{(j)}s_{22}^{(j)}}{s_{11}^{(j+1)}s_{22}^{(j+1)}}}, \quad \boldsymbol{L}_j = \begin{bmatrix} 1 & 0 \\ 0 & \alpha_j \end{bmatrix}. \tag{13}$$

The interfacial diffusion and sliding conditions on the interface between layer j and layer $j+1$ in (1c) can be equivalently expressed as

$$\boldsymbol{L}_j\dot{\tilde{\boldsymbol{u}}}_0^{(j+1)} - \dot{\tilde{\boldsymbol{u}}}_1^{(j)} = \boldsymbol{\Lambda}_j\tilde{\boldsymbol{\sigma}}_1^{(j)}, \quad j = 1, 2, \dots, N-1, \tag{14}$$

where

$$\boldsymbol{\Lambda}_j = \begin{bmatrix} \dfrac{k}{\lambda_j s_{11}^{(j)}\vartheta_j} & 0 \\ 0 & \dfrac{k^3 D_j}{\lambda_j^3 s_{11}^{(j)}} \end{bmatrix}. \tag{15}$$

Substituting (12) into (14), we arrive at

$$\boldsymbol{R}_1^{(j)}\dot{\tilde{\boldsymbol{\sigma}}}_1^{(j-1)} + \boldsymbol{R}_2^{(j)}\dot{\tilde{\boldsymbol{\sigma}}}_1^{(j)} + \boldsymbol{R}_3^{(j)}\dot{\tilde{\boldsymbol{\sigma}}}_1^{(j+1)} = \boldsymbol{\Lambda}_j\tilde{\boldsymbol{\sigma}}_1^{(j)}, \quad j = 1, 2, \dots, N-1, \tag{16}$$

where

$$\begin{aligned} \boldsymbol{R}_1^{(j)} &= -\frac{1}{\alpha_{j-1}\beta_{j-1}}[\boldsymbol{Q}_3^{(j)}]^{-1}\boldsymbol{L}_{j-1}, \\ \boldsymbol{R}_2^{(j)} &= [\boldsymbol{Q}_3^{(j)}]^{-1}\boldsymbol{Q}_4^{(j)} + \frac{1}{\alpha_j\beta_j}\boldsymbol{L}_j\boldsymbol{Q}_1^{(j+1)}[\boldsymbol{Q}_3^{(j+1)}]^{-1}\boldsymbol{L}_j, \\ \boldsymbol{R}_3^{(j)} &= \boldsymbol{L}_j\big[\boldsymbol{Q}_2^{(j+1)} - \boldsymbol{Q}_1^{(j+1)}[\boldsymbol{Q}_3^{(j+1)}]^{-1}\boldsymbol{Q}_4^{(j+1)}\big]. \end{aligned} \tag{17}$$

By considering the fact that the pressure prescribed on the top surface of the multilayered strip is static and that the bottom surface of the strip is traction-free (i.e., $\dot{\tilde{\boldsymbol{\sigma}}}_1^{(0)} = \dot{\tilde{\boldsymbol{\sigma}}}_0^{(1)} = \dot{\tilde{\boldsymbol{\sigma}}}_1^{(N)} = 0$), (16) can be recast into the following standard homogeneous state-space equation:

$$\boldsymbol{A}\dot{\boldsymbol{\xi}} = \boldsymbol{B}\boldsymbol{\xi}, \tag{18}$$

where

$$\boldsymbol{\xi} = \begin{bmatrix} \tilde{\boldsymbol{\sigma}}_1^{(1)} \\ \tilde{\boldsymbol{\sigma}}_1^{(2)} \\ \vdots \\ \tilde{\boldsymbol{\sigma}}_1^{(N-1)} \end{bmatrix}, \tag{19}$$

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{R}_2^{(1)} & \boldsymbol{R}_3^{(1)} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \boldsymbol{R}_1^{(2)} & \boldsymbol{R}_2^{(2)} & \boldsymbol{R}_3^{(2)} & \mathbf{0} & \cdots & \mathbf{0} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \cdots & \mathbf{0} & \boldsymbol{R}_1^{(N-2)} & \boldsymbol{R}_2^{(N-2)} & \boldsymbol{R}_3^{(N-2)} \\ \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \boldsymbol{R}_1^{(N-1)} & \boldsymbol{R}_2^{(N-1)} \end{bmatrix}, \tag{20}$$

$$\boldsymbol{B} = \mathrm{diag}[\boldsymbol{\Lambda}_1 \;\; \boldsymbol{\Lambda}_2 \;\; \cdots \;\; \boldsymbol{\Lambda}_{N-1}]. \tag{21}$$

It is observed that

(i) the state vector $\boldsymbol{\xi}$ is composed of the $2(N-1)$ traction components on all of the existing $N-1$ interfaces;

(ii) the dimensionless matrix $\boldsymbol{A}$ can be completely determined by the following dimensionless parameters of the N layers: δ_j, ρ_j, K_j $(j = 1, 2, \ldots, N)$ and α_j, β_j $(j = 1, 2, \ldots, N-1)$; and

(iii) the matrix $\boldsymbol{B}$ having the dimension of 1/time depends on the diffusion and sliding properties of the $N-1$ interfaces and is independent of the thicknesses of all the layers.

The general solution to the homogeneous state-space equation in (18) is simply given by

$$\boldsymbol{\xi}(t) = \exp(\boldsymbol{A}^{-1}\boldsymbol{B}t)\boldsymbol{\xi}(0), \quad t \geq 0, \tag{22}$$

where $\exp(\boldsymbol{A}^{-1}\boldsymbol{B}t)$ is the state transition matrix, and the initial state $\boldsymbol{\xi}(0)$ can be simply determined by assuming that all of the interfaces are initially perfectly bonded. Indeed, by assuming that all of the interfaces are initially perfectly bonded, we can derive the relationship

$$\begin{bmatrix} \tilde{\boldsymbol{u}}_0^{(1)} \\ \tilde{\boldsymbol{\sigma}}_0^{(1)} \end{bmatrix} = \begin{bmatrix} \boldsymbol{M}_{11} & \boldsymbol{M}_{12} \\ \boldsymbol{M}_{21} & \boldsymbol{M}_{22} \end{bmatrix} \begin{bmatrix} \tilde{\boldsymbol{u}}_1^{(N)} \\ \tilde{\boldsymbol{\sigma}}_1^{(N)} \end{bmatrix} \quad \text{at} \quad t = 0, \tag{23}$$

where the four 2×2 sub-matrices $\boldsymbol{M}_{11}$, $\boldsymbol{M}_{12}$, $\boldsymbol{M}_{21}$ and $\boldsymbol{M}_{22}$ are given by

$$\begin{bmatrix} \boldsymbol{M}_{11} & \boldsymbol{M}_{12} \\ \boldsymbol{M}_{21} & \boldsymbol{M}_{22} \end{bmatrix} = \boldsymbol{Q}(K_1, \delta_1, \rho_1) \times \boldsymbol{T}(\alpha_1, \beta_1) \times \boldsymbol{Q}(K_2, \delta_2, \rho_2) \times \boldsymbol{T}(\alpha_2, \beta_2) \times \cdots \\ \times \boldsymbol{T}(\alpha_{N-1}, \beta_{N-1}) \times \boldsymbol{Q}(K_N, \delta_N, \rho_N), \tag{24}$$

with

$$\boldsymbol{T}(\alpha, \beta) = \mathrm{diag}[1 \;\; \alpha \;\; \alpha\beta \;\; \beta]. \tag{25}$$

In view of the fact that $\tilde{\boldsymbol{\sigma}}_0^{(1)} = \mathbf{0}$ and $\tilde{\boldsymbol{\sigma}}_1^{(N)} = -(p_0\sqrt{s_{11}^{(N)} s_{22}^{(N)}}/k)[0 \;\; 1]^T$ at any time, we find from (23) that

$$\tilde{\boldsymbol{u}}_1^{(N)} = \frac{p_0\sqrt{s_{11}^{(N)} s_{22}^{(N)}}}{k} \boldsymbol{M}_{21}^{-1} \boldsymbol{M}_{22} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \text{at} \quad t = 0. \tag{26}$$

Consequently, the initial displacements and tractions on the upper interface of layer j at can be arrived at by using the recursion relation

$$\begin{bmatrix} \tilde{\boldsymbol{u}}_1^{(j)} \\ \tilde{\boldsymbol{\sigma}}_1^{(j)} \end{bmatrix} = \boldsymbol{T}(\alpha_j, \beta_j) \times \boldsymbol{Q}(K_{j+1}, \delta_{j+1}, \rho_{j+1}) \begin{bmatrix} \tilde{\boldsymbol{u}}_1^{(j+1)} \\ \tilde{\boldsymbol{\sigma}}_1^{(j+1)} \end{bmatrix}, \quad j = N-1, N-2, \dots, 1 \quad \text{at} \quad t = 0. \tag{27}$$

The initial state $\boldsymbol{\xi}(0)$ can then be extracted from the above expression. It is seen from (22) that as time t approaches infinity, all of the interfacial tractions will be relaxed to zero. The evolution of the displacements on the upper interfaces of the N layers can then be obtained from (12a). Consequently, the field variables at any position within layer j and at any time can be determined from (7) with h_j reinterpreted as the distance to the upper interface of layer j. In addition, the relaxation times of the multilayered strip can be obtained by solving the following generalized eigenvalue problem:

$$(\boldsymbol{A} + \eta \boldsymbol{B})\boldsymbol{v} = \mathbf{0}, \tag{28}$$

where η is the eigenvalue and $\boldsymbol{v}$ the associated eigenvector. Equation (28) follows immediately from (18) by assuming the solution to be of the form $\boldsymbol{\xi} = \boldsymbol{v}e^{-t/\eta}$. The $2(N-1)$ positive real eigenvalues $\eta_1, \eta_2, \dots, \eta_{2(N-1)}$ are just the $2(N-1)$ relaxation times of the laminated plate.

3. Illustrative examples and discussion

In this section, we demonstrate the exact solution derived in the previous section via

(i) a bilayered strip,

(ii) a trilayered strip, and

(iii) a 1001-layered strip.

Remember that the factors $\cos kx_1$ and $\sin kx_1$ have been excluded in the illustrations of displacements and stresses.

3.1. ***A bilayered strip.*** We first consider a bilayered orthotropic strip. For a bilayered strip with $N = 2$, both $\boldsymbol{A}$ and $\boldsymbol{B}$ are 2×2 matrices and are given by

$$\begin{aligned} \boldsymbol{A} &= \boldsymbol{R}_2^{(1)}[\boldsymbol{Q}_3^{(1)}]^{-1}\boldsymbol{Q}_4^{(1)} + \frac{1}{\alpha_1\beta_1}\boldsymbol{L}_1\boldsymbol{Q}_1^{(2)}[\boldsymbol{Q}_3^{(2)}]^{-1}\boldsymbol{L}_1, \\ \boldsymbol{B} &= \boldsymbol{\Lambda}_1. \end{aligned} \tag{29}$$

In this case, analytical expressions of the two relaxation times of the bilayered strip are given by

$$\eta_{1,2} = -(A_{11}\chi + A_{22}\gamma) \pm \sqrt{(A_{11}\chi - A_{22}\gamma)^2 + 4A_{12}A_{21}\chi\gamma} > 0, \tag{30}$$

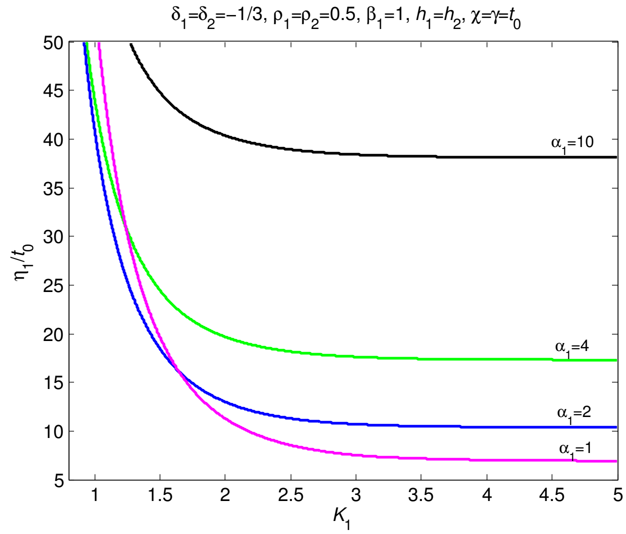

Figure 2. The larger relaxation time η_1 of a bilayered strip as a function of K_1 for different values of α_1 with $\delta_1 = \delta_2 = -\frac{1}{3}$, $\rho_1 = \rho_2 = 0.5$, $\beta_1 = 1$, $h_1 = h_2$, $\chi = \gamma = t_0$.

where

$$\chi = \frac{\lambda_1 s_{11}^{(1)} \vartheta_1}{2k}, \quad \gamma = \frac{\lambda_1^3 s_{11}^{(1)}}{2k^3 D_1}. \tag{31}$$

We illustrate in Figures 2 and 3 the two relaxation times η_1 and η_2 as functions of K_1 for different values of α_1 with $\delta_1 = \delta_2 = -\frac{1}{3}$, $\rho_1 = \rho_2 = 0.5$, $\beta_1 = 1$, $h_1 = h_2$, $\chi = \gamma = t_0$. The elastic constants for layer 2 are obtained after an in-plane coordinate rotation of the angle $\pi/2$ for layer 1. It is seen from

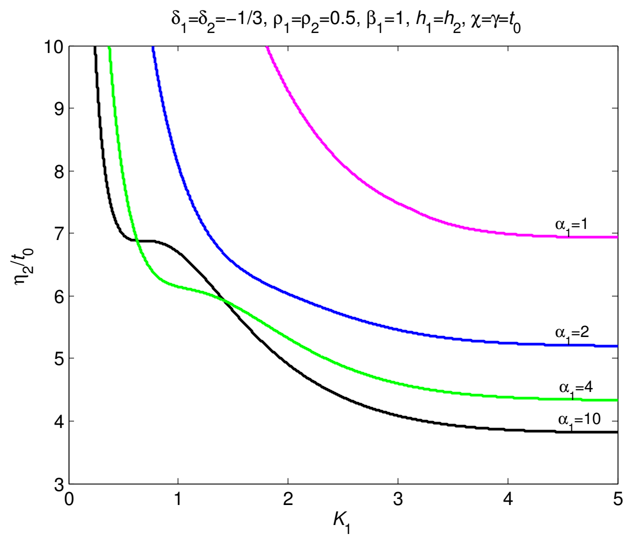

Figure 3. The smaller relaxation time η_2 of a bilayered strip as a function of K_1 for different values of α_1 with $\delta_1 = \delta_2 = -\frac{1}{3}$, $\rho_1 = \rho_2 = 0.5$, $\beta_1 = 1$, $h_1 = h_2$, $\chi = \gamma = t_0$.

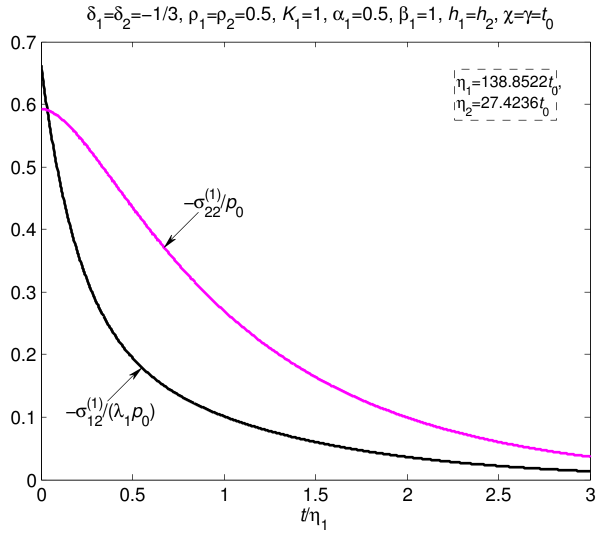

Figure 4. The evolution of the normal and shear tractions on the interface with $\delta_1 = \delta_2 = -\frac{1}{3}$, $\rho_1 = \rho_2 = 0.5$, $K_1 = 1$, $\alpha_1 = 0.5$, $\beta_1 = 1$, $h_1 = h_2$, $\chi = \gamma = t_0$.

the two figures that as the thickness of the strip increases (or equivalently K_1 increases), the relaxation times decrease. Figure 4 shows the evolution of the normal and shear tractions on the interface with $\delta_1 = \delta_2 = -\frac{1}{3}$, $\rho_1 = \rho_2 = 0.5$, $K_1 = 1$, $\alpha_1 = 0.5$, $\beta_1 = 1$, $h_1 = h_2$, $\chi = \gamma = t_0$. It is observed from Figure 4 that

(i) the shear traction decays faster than the normal traction;

(ii) the interfacial tractions evolve on the time scale of the larger relaxation time $\eta_1 = 138.8522t_0$ and their magnitudes are minimal when $t > 3\eta_1$.

When $h_1, h_2 \to \infty$, the two relaxation times of a bilayered strip are given explicitly by

$$\begin{aligned}\eta_{1,2} = {} & 2\chi(n_1 + \alpha_1^{-1}\beta_1^{-1}n_2) + 2\gamma(n_1 + \alpha_1\beta_1^{-1}n_2) \\ & \pm 2\sqrt{[\chi(n_1 + \alpha_1^{-1}\beta_1^{-1}n_2) - \gamma(n_1 + \alpha_1\beta_1^{-1}n_2)]^2 + \chi\gamma[1 + \delta_1 - \beta_1^{-1}(1 + \delta_2)]^2},\end{aligned} \tag{32}$$

where χ and γ have been defined by (31). The above analytical result is derived using analytic continuation and the positive definite Hermitian matrix for an orthotropic material in [Suo 1990, (9.6)]. If the two half-planes of the bimaterial are elastically isotropic, we have

$$\alpha_1 = \lambda_1 = \lambda_2 = n_1 = n_2 = 1, \quad \beta_1 = \frac{\mu_2(1-\nu_1)}{\mu_1(1-\nu_2)}; \quad s_{11}^{(j)} = \frac{1-\nu_j}{2\mu_j}, \quad \delta_j = \frac{\nu_j}{\nu_j - 1}, \quad j = 1, 2, \tag{33}$$

where μ and ν are the shear modulus and Poisson's ratio. In this case, (32) simply reduces to [Wang and Wang 2016, (38)].

Furthermore, we note that the relaxation times in Figures 2 and 3 for $K_1 = 5$ are very close to the result in (32).

3.2. ***A trilayered strip.*** Next we consider a trilayered orthotropic strip. For a trilayered strip with $N = 3$, both $\boldsymbol{A}$ and $\boldsymbol{B}$ are 4×4 matrices and are given by

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{R}_2^{(1)} & \boldsymbol{R}_3^{(1)} \\ \boldsymbol{R}_1^{(2)} & \boldsymbol{R}_2^{(2)} \end{bmatrix}, \qquad \boldsymbol{B} = \mathrm{diag}[\boldsymbol{\Lambda}_1 \;\; \boldsymbol{\Lambda}_2]. \tag{34}$$

It is convenient to numerically determine the four relaxation times in the trilayered strip. For example, if the parameters of the trilayered strip are chosen as

$$\begin{aligned} &\delta_1 = \delta_2 = \delta_3 = -\tfrac{1}{3}, \quad \rho_1 = \rho_2 = \rho_3 = 0.5, \quad h_1 = h_2 = h_3, \quad K_1 = 1, \\ &\alpha_1 = \frac{1}{\alpha_2} = 0.5, \quad \beta_1 = \beta_2 = 1, \quad \vartheta_1 = \vartheta_2, \quad D_1 = D_2, \quad \chi = \gamma = t_0, \end{aligned} \tag{35}$$

where χ and γ have been defined by (31); the four relaxation times of the trilayered strip are determined as

$$\eta_1 = 258.9914 t_0, \quad \eta_2 = 38.3432 t_0, \quad \eta_3 = 21.5497 t_0, \quad \eta_4 = 13.6673 t_0. \tag{36}$$

In this example, the top and bottom layers have identical elastic constants, whilst the elastic constants for the middle layer are obtained after an in-plane coordinate rotation of the angle $\pi/2$ for the top or bottom layer. We illustrate in Figure 5 the evolution of the normal and shear tractions on the two interfaces. It is observed from Figure 5 that

(1) the tractions on the lower interface and the shear stress on the upper interface will eventually change sign with time whilst the normal traction on the upper interface will always maintain its sign;

(2) on each interface, the shear traction decays faster than the normal traction;

(3) at any instant, the magnitude of the normal traction on the upper interface is always higher than that on the lower interface.

3.3. ***A 1001-layered strip.*** Finally, we consider a strip composed of 1001 identical orthotropic layers of equal thickness. The diffusional and sliding properties on all of the interfaces are identical (i.e., $\vartheta_1 = \vartheta_2 = \cdots = \vartheta_{1000}$ and $D_1 = D_2 = \cdots = D_{1000}$). Our results indicate that the relaxation times are independent of the specific value of δ. We further set $K_1 = 1$ and $\chi = \gamma = t_0$, where χ and γ have been defined by (31). The variation of all the $2(N-1) = 2000$ relaxation times η_j as a decreasing function of j for different values of ρ is illustrated in Figure 6. It is observed from Figure 6 that

(i) the relaxation times are increasing functions of ρ, which implies that a decrement in ρ will expedite the relaxation process;

(ii) the curve for $\rho \to -1$ forms the lower bound of the relaxation times;

(iii) there is a large jump between η_{1001} and η_{1002}, there is a small jump between η_{999} and η_{1000}.

4. Conclusions

Based on the rigorous theory of elasticity, we derive an exact solution to the plane strain problem associated with a multilayered orthotropic elastic strip with simultaneous interfacial diffusion and rate-dependent sliding. By using the transfer matrix method and by enforcing the diffusion and sliding

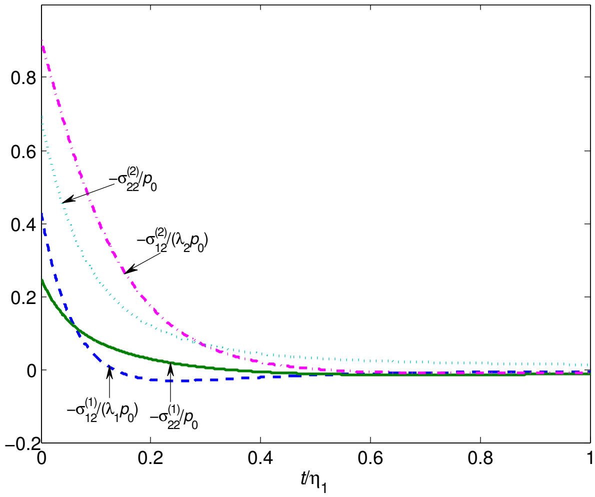

Figure 5. The evolution of the normal and shear tractions on the two interfaces in the trilayered strip described by (35).

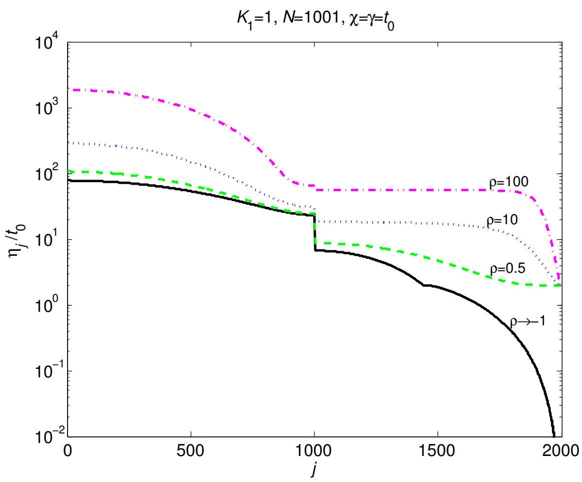

Figure 6. Variation of all the 2000 relaxation times η_j as a decreasing function of j for different values of the anisotropic parameter ρ with $K_1 = 1$ and $\chi = \gamma = t_0$ in a 1001-layered orthotropic strip.

conditions on all of the existing interfaces, we obtain a homogeneous state-space equation with state variables identified as the traction components on the interfaces. Once the initial values of the interfacial tractions are known by assuming that initially all interfaces are perfectly bonded, the evolution of the state variables can be uniquely determined. It is seen that the transfer matrix $\boldsymbol{Q}(K, \delta, \rho)$ and the state transition matrix $\exp(\boldsymbol{A}^{-1}\boldsymbol{B}t)$ are the key components of our solution. Using Matlab, numerical results for relaxation times and evolution of interfacial tractions for bilayered, trilayered and 1001-layered orthotropic strips are presented to demonstrate the exact solution. The solution strategy adopted in this research can also be employed to study the case in which the orthotropic elastic coefficients of each layer vary continuously along the thickness direction.

Acknowledgements

This work is supported by the National Natural Science Foundation of China (Grant No. 11272121) and through a Discovery Grant from the Natural Sciences and Engineering Research Council of Canada (RGPIN 2017-03716115112).

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Received 30 May 2017. Revised 27 Jul 2017. Accepted 5 Aug 2017.

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Journal of Mechanics of Materials and Structures

Volume 12, No. 5 December 2017

1559-3959(2017)12:5;1-0