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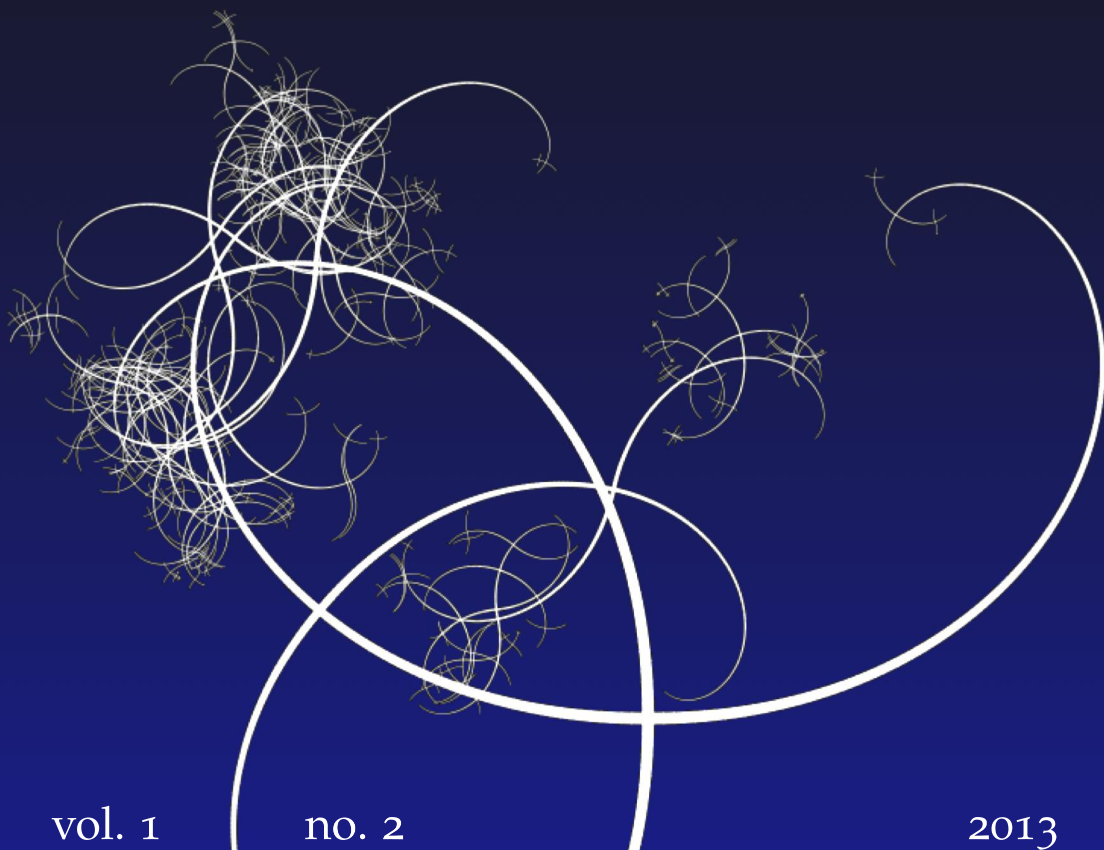

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WELL-POSEDNESS FOR DISLOCATION-BASED GRADIENT VISCOPLASTICITY, II: GENERAL NONASSOCIATIVE MONOTONE PLASTIC FLOWS

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In this work we extend the well-posedness for infinitesimal dislocation-based gradient viscoplasticity with linear kinematic hardening from the subdifferential case to general nonassociative monotone plastic flows. We assume an additive split of the displacement gradient into nonsymmetric elastic distortion and nonsymmetric plastic distortion. The thermodynamic potential is augmented with a term taking the dislocation density tensor Curl p into account. The constitutive equations in the models we study are assumed to be only of monotone type. Based on the generalized version of Korn's inequality for incompatible tensor fields (the nonsymmetric plastic distortion) due to Neff et al. the existence of solutions of quasistatic initial-boundary value problems under consideration is shown using a time-discretization technique and a monotone operator method.

1. Introduction

We study the existence of solutions of quasistatic initial-boundary value problems arising in gradient viscoplasticity. The models we study use rate-dependent constitutive equations with internal variables to describe the deformation behavior of metals at infinitesimally small strain.

Our focus is on a phenomenological model on the macroscale not including the case of single-crystal plasticity. From a mathematical point of view, the maze of equations, slip systems, and physical mechanisms in single-crystal plasticity is only obscuring the mathematical structure of the problem.

Our model has been first presented in [Neff et al. 2009a]. It is inspired by [2000]. Contrary to more classical strain gradient approaches, the model features a nonsymmetric plastic distortion field $p \in \mathbb{M}^3$ [Bardella 2010], a dislocation-based energy storage based solely on $|\text{Curl}\, p|$, and second gradients of the plastic distortion in the

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form of Curl Curl p acting as dislocation-based kinematical backstresses. We only consider energetic length-scale effects and not higher gradients in the dissipation.

The uniqueness of classical solutions in the subdifferential case (associated plasticity) for rate-independent and rate-dependent formulations is shown in [Neff 2008b]. The existence question for the rate-independent model in terms of a weak reformulation is addressed in [Neff et al. 2009a]. The rate-independent model with isotropic hardening is treated in [Ebobisse and Neff 2010]. The first numerical results for a simplified rate-independent irrotational formulation (no plastic spin, symmetric plastic distortion p) are presented in [Neff et al. 2009b]. In [Giacomini and Lussardi 2008; Reddy et al. 2008] well-posedness for a rate-independent model of [Gurtin and Anand 2005] is shown under the decisive assumption that the plastic distortion is symmetric (the irrotational case), in which case we may really speak of a strain gradient plasticity model, since the gradient acts on the plastic strain.

In order to appreciate the simplicity and elegance of our model we sketch some of its ingredients. First, as is usual in plasticity theory, we split the total displacement gradient into nonsymmetric elastic and plastic distortions:

$$\nabla u = e + p.$$

For invariance reasons, the elastic energy contribution may only depend on the elastic strains $\operatorname{sym} e = \operatorname{sym}(\nabla u - p)$. While p is nonsymmetric, a distinguishing feature of our model is that, similarly to classical approaches, only the symmetric part $\varepsilon_p := \operatorname{sym} p$ of the plastic distortion appears in the local Cauchy stress σ, while the higher-order stresses are nonsymmetric. The reason for this is that we assume that p has to obey the same transformation behavior as ∇u does, and thus the energy storage due to kinematical hardening should depend only on the plastic strains $\operatorname{sym} p$. For more on the basic invariance questions related to this issue dictating this type of behavior, see [Neff 2008a; Svendsen et al. 2009]. We assume as well plastic incompressibility: $\operatorname{tr} p = 0$.

The thermodynamic potential of our model can therefore be written as

$$\int_\Omega \Big(\underbrace{\mathbb{C}[x](\operatorname{sym}(\nabla u - p))(\operatorname{sym}(\nabla u - p))}_{\text{elastic energy}} + \underbrace{\frac{C_1}{2}|\operatorname{dev}\operatorname{sym} p|^2}_{\text{kinematical hardening}} + \underbrace{\frac{C_2}{2}|\operatorname{Curl} p|^2}_{\text{dislocation storage}} + \underbrace{u \cdot b}_{\text{external volume forces}} \Big)\, dx.$$

The positive definite elasticity tensor $\mathbb{C}$ is able to represent the elastic anisotropy of the material. The evolution equations for the plastic distortion p are taken in such a way that the stored energy is nonincreasing along trajectories of p at frozen displacement u; see [Neff et al. 2009a]. This ensures the validity of the second law of thermodynamics in the form of the reduced dissipation inequality.

For the reduced dissipation inequality we consider u fixed in time and consider the time derivative of the free energy (and taking into account that Curl is a self-adjoint operator provided that the appropriate boundary conditions are specified), we have

$$\begin{aligned}\frac{d}{dt}\int_\Omega W(\nabla u(t_0)-p(t),\, p(t), \operatorname{Curl} p(t))\,dx \\ &= \int_\Omega D_1 W\cdot(-\partial_t p) + D_2 W\cdot \partial_t p + D_3 W\cdot \operatorname{Curl}\partial_t p\,dx \\ &= -\int_\Omega (D_1 W - D_2 W - \operatorname{Curl} D_3 W)\cdot \partial_t p\,dx.\end{aligned}$$

Choosing $\partial_t p \in g(D_1 W - D_2 W - \operatorname{Curl} D_3 W)$ with a monotone function g we obtain the reduced dissipation inequality

$$\frac{d}{dt}\int_\Omega W(\nabla u(t_0)-p(t),\, p(t), \operatorname{Curl} p(t))\,dx \le 0.$$

Adapted to our situation, the plastic flow has the form

$$\partial_t p \in g(\sigma - C_1 \operatorname{dev}\operatorname{sym} p - C_2 \operatorname{Curl}\operatorname{Curl} p), \tag{1}$$

where $\sigma = \mathbb{C}[x]\operatorname{sym}(\nabla u - p)$ is the elastic symmetric Cauchy stress of the material and g is a multivalued monotone flow function which is not necessary the subdifferential of a convex plastic potential (associative plasticity). In this generality, our formulation comprises certain nonassociative plastic flows in which the yield condition and the flow direction are independent and governed by distinct functions. Moreover, the flow function g is supposed to induce a rate-dependent response as all materials are rate dependent.

Clearly, in the absence of energetic length-scale effects ($C_2 = 0$), the $\operatorname{Curl}\operatorname{Curl} p$ term is absent. In general we assume that g maps symmetric tensors to symmetric tensors. Thus, for $C_2 = 0$ the plastic distortion remains symmetric and the model reduces to a classical plasticity model. Therefore, the energetic length scale is solely responsible for the plastic spin in the model. The appearance of the $\operatorname{Curl}\operatorname{Curl} p$ term in the argument of g is clear: the argument of g consists of the Eshelby stress tensor Σ driving the plastic evolution, see [Neff et al. 2009a].

Regarding the boundary conditions necessary for the formulation of the higher-order theory we assume that the boundary is a perfect conductor, which means that the tangential component of p vanishes on $\partial\Omega$. In the context of dislocation dynamics these conditions express the requirement that there is no flux of the Burgers vector across a hard boundary. Gurtin and Needleman [2005] introduce the following different types of boundary conditions for the plastic distortion:[1]

$$\begin{aligned} \partial_t p \times n\big|_{\Gamma_{\text{hard}}} &= 0 \quad \text{“microhard” (perfect conductor),} \\ \partial_t p\big|_{\Gamma_{\text{hard}}} &= 0 \quad \text{“hard-slip”,} \\ \operatorname{Curl} p \times n\big|_{\Gamma_{\text{hard}}} &= 0 \quad \text{“microfree”.} \end{aligned} \tag{2}$$

We specify a sufficient condition for the microhard boundary condition, namely

$$p \times n\big|_{\Gamma_{\text{hard}}} = 0,$$

and assume $\Gamma_{\text{hard}} = \partial\Omega$. This is the correct boundary condition for tensor fields in $H(\operatorname{Curl})$ which admits tangential traces.

We combine this with a new inequality extending Korn's inequality to incompatible tensor fields, namely, for all $p \in H(\operatorname{Curl})$ such that $p \times n\big|_{\Gamma_{\text{hard}}} = 0$, we have

$$\underbrace{\|p\|_{L^2(\Omega)}}_{\text{plastic distortion}} \le C(\Omega)\Big(\underbrace{\|\operatorname{sym} p\|_{L^2(\Omega)}}_{\text{plastic strain}} + \underbrace{\|\operatorname{Curl} p\|_{L^2(\Omega)}}_{\text{dislocation density}}\Big). \tag{3}$$

Here, $\Gamma_{\text{hard}} \subset \partial\Omega$ with full two-dimensional surface measure and domain Ω needs to be *sliceable*, that is, cuttable into finitely many simply connected subdomains with Lipschitz boundaries. This inequality has been derived in [Neff et al. 2011; 2012a; 2012b] and is precisely motivated by the well-posedness question for our model [Neff et al. 2009a]. Inequality (3) expresses the fact that controlling the plastic strain sym p and the dislocation density Curl p in $L^2(\Omega)$ gives a control of the plastic distortion p in $L^2(\Omega)$ provided the correct boundary conditions are specified: namely the microhard boundary condition. Since in the sequel we assume that $\operatorname{tr}(p) = 0$ (plastic incompressibility) the quadratic terms in the thermodynamic potential provide a control of the right-hand side in (3).

It is worth noting that with g only monotone and not necessarily a subdifferential the powerful energetic solution concept [Giacomini and Lussardi 2008; Mainik and Mielke 2009; Kratochvíl et al. 2010] cannot be applied. In this contribution we face the combined challenge of a gradient plasticity model based on the dislocation density tensor Curl p involving the plastic spin, a general nonassociative monotone flow-rule, and a rate-dependent response.

Setting of the problem. Let $\Omega \subset \mathbb{R}^3$ be an open bounded set, the set of material points of the solid body, with a C^1-boundary. By T_e we denote a positive number (time of existence), which can be chosen arbitrarily large, and for $0 < t \le T_e$,

$$\Omega_t = \Omega \times (0, t).$$

[1]Here, $v \times n$ with $v \in \mathcal{M}^3$ and where $n \in \mathbb{R}^3$ denotes a row by column operation.

The sets $\mathcal{M}^3$ and $\mathcal{S}^3$ denote the sets of all 3×3 matrices and all symmetric 3×3 matrices, respectively. Let $\mathfrak{sl}(3)$ be the set of all traceless 3×3 matrices, that is,

$$\mathfrak{sl}(3) = \{v \in \mathcal{M}^3 \mid \operatorname{tr} v = 0\}.$$

Unknown in our small strain formulation are the displacement $u(x,t) \in \mathbb{R}^3$ of the material point x at time t and the nonsymmetric infinitesimal plastic distortion $p(x,t) \in \mathfrak{sl}(3)$.

The model equations of the problem are

$$-\operatorname{div}_x \sigma(x,t) = b(x,t), \tag{4}$$

$$\sigma(x,t) = \mathbb{C}[x]\big(\operatorname{sym}(\nabla_x u(x,t) - p(x,t))\big), \tag{5}$$

$$\partial_t p(x,t) \in g(x, \Sigma^{\text{lin}}(x,t)), \quad \Sigma^{\text{lin}} = \Sigma_e^{\text{lin}} + \Sigma_{\text{sh}}^{\text{lin}} + \Sigma_{\text{curl}}^{\text{lin}}, \tag{6}$$

$$\Sigma_{\text{e}}^{\text{lin}} = \sigma, \quad \Sigma_{\text{sh}}^{\text{lin}} = -C_1 \operatorname{dev}\operatorname{sym} p, \quad \Sigma_{\text{curl}}^{\text{lin}} = -C_2 \operatorname{Curl}\operatorname{Curl} p,$$

which must be satisfied in $\Omega \times [0, T_e)$. Here, $C_1, C_2 \geq 0$ are given material constants and Σ^{lin} is the infinitesimal Eshelby stress tensor driving the evolution of the plastic distortion p. The initial condition and Dirichlet boundary condition are

$$p(x,0) = p^{(0)}(x), \qquad x \in \Omega, \tag{7}$$

$$p(x,t) \times n(x) = 0, \qquad (x,t) \in \partial\Omega \times [0, T_e), \tag{8}$$

$$u(x,t) = 0, \qquad (x,t) \in \partial\Omega \times [0, T_e), \tag{9}$$

where n is a normal vector on the boundary $\partial\Omega$. For simplicity we consider only the homogeneous boundary condition. The elasticity tensor $\mathbb{C}[x] : \mathcal{S}^3 \to \mathcal{S}^3$ is a linear, symmetric, uniformly positive definite mapping. The mapping $x \mapsto \mathbb{C}[x] : \Omega \to \mathcal{S}^3$ is measurable. Classical linear kinematic hardening is included for $C_1 > 0$. Here, the nonlocal backstress contribution is given by the dislocation density motivated term $\Sigma_{\text{curl}}^{\text{lin}} = -C_2 \operatorname{Curl}\operatorname{Curl} p$ together with the corresponding microhard boundary conditions.

For the model we require that the nonlinear constitutive mapping $(v \mapsto g(\,\cdot\,, v)) : \mathcal{M}^3 \to 2^{\mathfrak{sl}(3)}$ is monotone[2], that is, it satisfies

$$0 \leq (v_1 - v_2) \cdot (v_1^* - v_2^*), \tag{10}$$

for all $v_i \in \mathcal{M}^3$, $v_i^* \in g(x, v_i)$, $i = 1, 2$, and for a.e. $x \in \Omega$. We also require that

$$0 \in g(x, 0), \quad \text{a.e. } x \in \Omega. \tag{11}$$

The mapping $x \mapsto g(x, \cdot\,) : \Omega \to 2^{\mathfrak{sl}(3)}$ is measurable (see Section 2 for the definition of the measurability of multivalued maps). Moreover, the function g has the

[2]Here $2^{\mathfrak{sl}(3)}$ denotes the power set of $\mathfrak{sl}(3)$.

following property:

$$g(x, v) \in \mathcal{S}^3 \quad \text{for any } v \in \mathcal{S}^3 \text{ and a.e. } x \in \Omega.$$

Given are the volume force $b(x,t) \in \mathbb{R}^3$ and the initial datum $p^{(0)}(x) \in \mathfrak{sl}(3)$.

Remark 1.1. It is well known that classical viscoplasticity (without gradient effects) gives rise to a well-posed problem. We extend this result to our formulation of rate-dependent gradient plasticity. The presence of the classical linear kinematic hardening in our model is related to $C_1 > 0$ whereas the presence of the nonlocal gradient term is always related to $C_2 > 0$.

In the recent work by the authors [Nesenenko and Neff 2012] the existence of solutions for the initial boundary problem (4)–(9) is studied under the assumption that the monotone function g is a subdifferential of a proper lower semicontinuous convex function $\phi : \mathcal{M}^3 \to \bar{\mathbb{R}}$ ($\bar{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$), that is, $g = \partial\phi$, and with the following different boundary condition:

$$\operatorname{Curl} p(x,t) \times n(x) = 0, \quad (x,t) \in \partial\Omega \times [0, T_e), \tag{12}$$

instead of (8). It is required there that the function ϕ satisfies the following two-sided estimate:

$$a_0|v|^q - b_0 \le \phi(v) \le a_1|v|^q + b_1, \tag{13}$$

for positive a_0 and a_1, some b_0 and b_1, and any $v \in \mathcal{M}^3$. Using methods of convex analysis we obtained in [Nesenenko and Neff 2012] the existence of weak solutions (see Definition 4.6) for the problem (4)–(7) + (12) + (9), with $g = \partial\phi$, under the restrictions on g given above. We note that the existence result derived in that paper is also valid for the new problem (4)–(9), that is, with the boundary condition (8) instead of (12), and of course the subdifferential structural assumption on g. In this work, assuming $\Omega \subset \mathbb{R}^3$ is a sliceable domain with a C^1-boundary and the homogeneous initial condition for p, that is, $p^{(0)}(x) = 0$ for $x \in \Omega$, we show the existence of strong solutions (see Definition 4.5) for the problem (4)–(9) with the monotone function g belonging to the class $\mathbb{M}(\Omega, \mathcal{M}^3, q, \alpha, m)$ defined in Section 4. The derivation of this result is based on the inequality (3), which is recently obtained in [Neff et al. 2011; 2012c] under the assumption that Ω is a sliceable domain, and on the monotonicity assumption for the function g. We note that in the case of the sliceable domain Ω the methods used in this work allow us to show the existence of strong solutions for (4)–(9) with $g = \partial\phi$, that is, the weak solutions for (4)–(9) with $g = \partial\phi$ derived in [Nesenenko and Neff 2012] are the strong solutions in the sense of Definition 4.5 in this case. However, we do not know how to extend our results on the existence of strong solutions to domains Ω which are not sliceable and to the nonhomogeneous initial condition. We note as well that the existence of strong solutions for the initial boundary problem formed by (4)–(7), (9), and (12), with $g \in \mathbb{M}(\Omega, \mathcal{M}^3, q, \alpha, m)$ or $g = \partial\phi$ with ϕ satisfying (13) for any domain Ω, is an open problem too.

Notation. Throughout we choose the numbers q and q^* satisfying the following conditions:

$$1 < q, q^* < \infty \quad \text{and} \quad 1/q + 1/q^* = 1,$$

where $|\cdot|$ denotes a norm in $\mathbb{R}^k$, $k \in \mathbb{N}$. Moreover, the following notations are used in this work. The space $W^{m,q}(\Omega, \mathbb{R}^k)$ with $q \in [1, \infty]$ consists of all functions in $L^q(\Omega, \mathbb{R}^k)$ with weak derivatives in $L^q(\Omega, \mathbb{R}^k)$ up to order m. If m is not integer, then $W^{m,q}(\Omega, \mathbb{R}^k)$ denotes the corresponding Sobolev–Slobodecki space. We set $H^m(\Omega, \mathbb{R}^k) = W^{m,2}(\Omega, \mathbb{R}^k)$. The norm in $W^{m,q}(\Omega, \mathbb{R}^k)$ is denoted by $\|\cdot\|_{m,q,\Omega}$ ($\|\cdot\|_q := \|\cdot\|_{0,q,\Omega}$). The operator Γ_0 defined by

$$\Gamma_0 : v \in W^{1,q}(\Omega, \mathbb{R}^k) \mapsto W^{1-1/q,q}(\partial\Omega, \mathbb{R}^k)$$

denotes the usual trace operator. The space $W_0^{m,q}(\Omega, \mathbb{R}^k)$ with $q \in [1, \infty]$ consists of all functions v in $W^{m,q}(\Omega, \mathbb{R}^k)$ with $\Gamma_0 v = 0$. One can define the bilinear form on the product space $L^q(\Omega, \mathcal{M}^3) \times L^{q^*}(\Omega, \mathcal{M}^3)$ by

$$(\xi, \zeta)_\Omega = \int_\Omega \xi(x) \cdot \zeta(x)\, dx.$$

The space

$$L^q_{\mathrm{Curl}}(\Omega, \mathcal{M}^3) = \{v \in L^q(\Omega, \mathcal{M}^3) \mid \mathrm{Curl}\, v \in L^q(\Omega, \mathcal{M}^3)\}$$

is a Banach space with respect to the norm

$$\|v\|_{q,\mathrm{Curl}} = \|v\|_q + \|\mathrm{Curl}\, v\|_q.$$

By $H(\mathrm{Curl})$ we denote the space of measurable functions in $L^2_{\mathrm{Curl}}(\Omega, \mathcal{M}^3)$, that is, $H(\mathrm{Curl}) = L^2_{\mathrm{Curl}}(\Omega, \mathcal{M}^3)$. The well-known result on the generalized trace operator can be easily adapted to functions with values in $\mathcal{M}^3$ (see [Sohr 2001, §II.1.2]). Then, according to this result, there is a bounded operator Γ_n on $L^q_{\mathrm{Curl}}(\Omega, \mathcal{M}^3)$:

$$\Gamma_n : v \in L^q_{\mathrm{Curl}}(\Omega, \mathcal{M}^3) \mapsto (W^{1-1/q^*,q^*}(\partial\Omega, \mathcal{M}^3))^*$$

with

$$\Gamma_n v = v \times n\big|_{\partial\Omega} \text{ if } v \in C^1(\bar{\Omega}, \mathcal{M}^3),$$

where X^* denotes the dual of a Banach space X. Next,

$$L^q_{\mathrm{Curl},0}(\Omega, \mathcal{M}^3) = \{w \in L^q_{\mathrm{Curl}}(\Omega, \mathcal{M}^3) \mid \Gamma_n(w) = 0\}.$$

We also define the space $Z^q_{\mathrm{Curl}}(\Omega, \mathcal{M}^3)$ by

$$Z^q_{\mathrm{Curl}}(\Omega, \mathcal{M}^3) = \{v \in L^q_{\mathrm{Curl},0}(\Omega, \mathcal{M}^3) \mid \mathrm{Curl}\,\mathrm{Curl}\, v \in L^q(\Omega, \mathcal{M}^3)\},$$

which is a Banach space with respect to the norm

$$\|v\|_{Z^q_{\mathrm{Curl}}} = \|v\|_{q,\mathrm{Curl}} + \|\mathrm{Curl}\,\mathrm{Curl}\,v\|_q.$$

For functions v defined on $\Omega \times [0, \infty)$ we denote by $v(t)$ the mapping $x \mapsto v(x, t)$, which is defined on Ω. The space $L^q(0, T_e; X)$ denotes the Banach space of all Bochner-measurable functions $u : [0, T_e) \to X$ such that $t \mapsto \|u(t)\|^q_X$ is integrable on $[0, T_e)$. Finally, we frequently use the spaces $W^{m,q}(0, T_e; X)$, which consist of Bochner-measurable functions having q-integrable weak derivatives up to order m.

2. Maximal monotone operators

In this section we recall some basics about monotone and maximal monotone operators. For more details see [Barbu 1976; Pascali and Sburlan 1978; Hu and Papageorgiou 1997], for example.

Let V be a reflexive Banach space with the norm $\|\cdot\|$, and let V^* be its dual space with the norm $\|\cdot\|_*$. The brackets $\langle \cdot, \cdot \rangle$ denote the dual pairing between V and V^*. Under V we shall always mean a reflexive Banach space throughout this section. For a multivalued mapping $A : V \to 2^{V^*}$ we define the *effective domain* of A as

$$D(A) = \{v \in V \mid Av \neq \varnothing\}$$

and the *graph* of A as

$$\mathrm{Gr}\,A = \{[v, v^*] \in V \times V^* \mid v \in D(A),\ v^* \in Av\}.$$

Definition 2.1. A mapping $A : V \to 2^{V^*}$ is called *monotone* if the inequality

$$\langle v^* - u^*, v - u \rangle \geq 0$$

holds for all $[v, v^*], [u, u^*] \in \mathrm{Gr}\,A$. A monotone mapping $A : V \to 2^{V^*}$ is called *maximal monotone* if the inequality

$$\langle v^* - u^*, v - u \rangle \geq 0 \quad \text{for all } [u, u^*] \in \mathrm{Gr}\,A$$

implies $[v, v^*] \in \mathrm{Gr}\,A$.

A mapping $A : V \to 2^{V^*}$ is called *generalized pseudomonotone* if the set Av is closed, convex, and bounded for all $v \in D(A)$ and, for every pair of sequences $\{v_n\}$ and $\{v^*_n\}$ such that $v^*_n \in Av_n$, $v_n \rightharpoonup v_0$, $v^*_n \rightharpoonup v^*_0 \in V^*$ and

$$\limsup_{n\to\infty} \langle v^*_n, v_n - v_0 \rangle \leq 0,$$

we have $[v_0, v^*_0] \in \mathrm{Gr}\,A$ and $\langle v^*_n, v_n \rangle \to \langle v^*_0, v_0 \rangle$.

A mapping $A : V \to 2^{V^*}$ is called *strongly coercive* if either $D(A)$ is bounded or $D(A)$ is unbounded and the condition

$$\frac{\langle v^*, v - w\rangle}{\|v\|} \to +\infty \quad \text{as } \|v\| \to \infty, \quad [v, v^*] \in \operatorname{Gr} A,$$

is satisfied for each $w \in D(A)$.

It is well known [Pascali and Sburlan 1978, p. 105] that if A is a maximal monotone operator, then for any $v \in D(A)$ the image Av is a closed convex subset of V^* and the graph $\operatorname{Gr} A$ is demiclosed.[3] A maximal monotone operator is also generalized pseudomonotone; see [Barbu 1976; Pascali and Sburlan 1978; Hu and Papageorgiou 1997].

Remark 2.2. We recall that the subdifferential of a lower semicontinuous and convex function is maximal monotone; see, for example, [Phelps 1993, Theorem 2.25].

Definition 2.3. The *duality mapping* $J : V \to 2^{V^*}$ is defined by

$$J(v) = \{v^* \in V^* \mid \langle v^*, v\rangle = \|v\|^2 = \|v^*\|_*^2\}$$

for all $v \in V$.

Without loss of generality (due to Asplund's theorem) we can assume that both V and V^* are strictly convex, that is, that the unit ball in the corresponding space is strictly convex. By virtue of [Barbu 1976, Theorem II.1.2], the equation

$$J(v_\lambda - v) + \lambda A v_\lambda \ni 0$$

has a solution $v_\lambda \in D(A)$ for every $v \in V$ and $\lambda > 0$ if A is maximal monotone. The solution is unique; see [Barbu 1976, p. 41].

Definition 2.4. Setting

$$v_\lambda = j_\lambda^A v \quad \text{and} \quad A_\lambda v = -\lambda^{-1} J(v_\lambda - v)$$

we define two single-valued operators: the *Yosida approximation* $A_\lambda : V \to V^*$ and the *resolvent* $j_\lambda^A : V \to D(A)$ with $D(A_\lambda) = D(j_\lambda^A) = V$.

By this definition, one immediately sees that $A_\lambda v \in A(j_\lambda^A v)$. For the main properties of the Yosida approximation we refer to [Barbu 1976; Pascali and Sburlan 1978; Hu and Papageorgiou 1997] and mention only that both are continuous operators and that A_λ is bounded and maximal monotone.

Next, the maximality of the sum of two maximal monotone operators is given by the following result.

[3] A set $A \in V \times V^*$ is demiclosed if, whenever v_n converges strongly to v_0 in V and v_n^* converges weakly to v_0^* in V^* (or v_n converges weakly to v_0 in V and v_n^* converges strongly to v_0^* in V^*) and $[v_n, v_n^*] \in \operatorname{Gr} A$, we have $[v, v^*] \in \operatorname{Gr} A$.

Theorem 2.5. *Let V be a reflexive Banach space, and let A and B be maximal. Suppose that the condition*

$$D(A) \cap \operatorname{int} D(B) \neq \varnothing$$

is fulfilled. Then the sum $A + B$ is a maximal monotone operator.

Proof. See [Pascali and Sburlan 1978, Theorem III.3.6] or [Barbu 1976, Theorem II.1.7]. □

For deeper results on the maximality of the sum of two maximal monotone operators we refer the reader to the book [Simons 1998]. The next surjectivity result plays an important role in the existence theory for monotone operators.

Theorem 2.6. *If V is a (strictly convex) reflexive Banach space and $A : V \to 2^{V^*}$ is maximal monotone and coercive, then A is surjective.*

Proof. See [Pascali and Sburlan 1978, Theorem III.2.10]. □

Measurability of multivalued mappings. In this subsection we present briefly some facts about measurable multivalued mappings. We assume that V, and hence V^*, is separable and denote the set of maximal monotone operators from V to V^* by $\mathfrak{M}(V \times V^*)$. Further, let $(S, \Sigma(S), \mu)$ be a σ-finite μ-complete measurable space.

Definition 2.7. A function $A : S \to \mathfrak{M}(V \times V^*)$ is measurable if, for every open set $U \in V \times V^*$, the set

$$\{x \in S \mid A(x) \cap U \neq \varnothing\}$$

is measurable in S. Here "open set" could be replaced by "closed set", "Borel set", "open ball", or "closed ball", with an equivalent result.

The next result states that the notion of measurability for maximal monotone mappings can be equivalently defined in terms of the measurability for appropriate single-valued mappings.

Proposition 2.8. *Let $A : S \to \mathfrak{M}(V \times V^*)$, let $\lambda > 0$ and let E be dense in V. The following are equivalent:*

(a) *A is measurable,*

(b) *for every $v \in E$, $x \mapsto j_\lambda^{A(x)} v$ is measurable, and*

(c) *$v \in E$, $x \mapsto A_\lambda(x) v$ is measurable.*

Proof. See [Damlamian et al. 2007, Proposition 2.11]. □

For further reading on measurable multivalued mappings we refer the reader to [Castaing and Valadier 1977; Hu and Papageorgiou 1997; Pankov 1997].

Canonical extensions of maximal monotone operators. Given a mapping

$$A : S \to \mathfrak{M}(V \times V^*),$$

one can define a monotone graph from $L^q(S, V)$ to $L^{q^*}(S, V^*)$, where $1/q + 1/q^* = 1$, as follows.

Definition 2.9. Let $A : S \to \mathfrak{M}(V \times V^*)$, the canonical extension of A from $L^q(S, V)$ to $L^{q^*}(S, V^*)$, where $1/q + 1/q^* = 1$, is defined by:

$$\mathrm{Gr}\,\mathcal{A} = \{[v, v^*] \in L^p(S, V) \times L^q(S, V^*) \mid [v(x), v^*(x)] \in \mathrm{Gr}\,A(x) \text{ for a.e. } x \in S\}.$$

Monotonicity of $\mathcal{A}$ as defined in Definition 2.9 is obvious, while its maximality follows from the next proposition.

Proposition 2.10. *Let $A : S \to \mathfrak{M}(V \times V^*)$ be measurable. If $\mathrm{Gr}\,\mathcal{A} \neq \varnothing$, then $\mathcal{A}$ is maximal monotone.*

Proof. See [Damlamian et al. 2007, Proposition 2.13]. □

We have to point out here that the maximality of $A(x)$ for almost every $x \in S$ does not imply the maximality of $\mathcal{A}$ as the latter can be empty [Damlamian et al. 2007]: $S = (0, 1)$ and $\mathrm{Gr}\,A(x) = \{[v, v^*] \in \mathbb{R}^m \times \mathbb{R}^m \mid v^* = x^{-1/q}\}$.

3. Some properties of the Curl Curl operator

In this section we present some results concerning the Curl Curl operator, which are relevant to further investigations. For the Curl Curl operator with a slightly different domain of definition similar results are obtained in [Nesenenko and Neff 2012, §4]. Here we adopt the results of that papers to our purposes.

Lemma 3.1 (self-adjointness of Curl Curl). *Let $\Omega \subset \mathbb{R}^3$ be an open bounded set with a Lipschitz boundary and $A : L^2(\Omega, \mathcal{M}^3) \to L^2(\Omega, \mathcal{M}^3)$ be the linear operator defined by*

$$Av = \mathrm{Curl}\,\mathrm{Curl}\,v$$

with $\mathrm{dom}(A) = Z^2_{\mathrm{Curl}}(\Omega, \mathcal{M}^3)$. *The operator A is self-adjoint and nonnegative.*

Proof. Consider the closed linear operator $S : L^2(\Omega, \mathcal{M}^3) \to L^2(\Omega, \mathcal{M}^3)$ defined by

$$Sv = \mathrm{Curl}\,v, \quad v \in \mathrm{dom}(S) = L^2_{\mathrm{Curl},0}(\Omega, \mathcal{M}^3).$$

It is easily seen that its adjoint is given by

$$S^*v = \mathrm{Curl}\,v, \quad v \in \mathrm{dom}(S^*) = L^2_{\mathrm{Curl}}(\Omega, \mathcal{M}^3).$$

Then, by [Kato 1966, Theorem V.3.24], the operator S^*S with

$$\mathrm{dom}(S^*S) = \{v \in \mathrm{dom}(S) \mid Sv \in \mathrm{dom}(S^*)\},$$

which is exactly the operator A, is self-adjoint in $L^2(\Omega, \mathcal{M}^3)$. The nonnegativity of A follows from its representation by the operator S, that is, $A = S^*S$, and the identity

$$(Av, u)_\Omega = (S^*Sv, u)_\Omega = (Sv, Su)_\Omega,$$

which holds for all $v \in \mathrm{dom}(A)$ and $u \in \mathrm{dom}(S)$. □

Corollary 3.2. *The operator* $A : L^2(\Omega, \mathcal{M}^3) \to L^2(\Omega, \mathcal{M}^3)$ *defined in Lemma 3.1 is maximal monotone.*

Proof. According to [Brézis 1970, Theorem 1], a linear monotone operator A is maximal monotone if it is a densely defined closed operator whose adjoint A^* is monotone. The statement of the corollary follows then directly from Lemma 3.1 and the mentioned result of Brézis. □

Boundary value problems. Let $\Omega \subset \mathbb{R}^3$ be an open bounded set with a Lipschitz boundary. For every $v \in L^2(\Omega, \mathcal{M}^3)$ we define a functional Ψ on $L^2(\Omega, \mathcal{M}^3)$ by

$$\Psi(v) = \begin{cases} \dfrac{1}{2} \displaystyle\int_\Omega |\mathrm{Curl}\, v(x)|^2 \, dx, & v \in L^2_{\mathrm{Curl},0}(\Omega, \mathcal{M}^3), \\ +\infty, & \text{otherwise.} \end{cases}$$

It is easy to check that Ψ is proper, convex, and lower semicontinuous. The next lemma gives a precise description of the subdifferential $\partial\Psi$.

Lemma 3.3. *We have that* $\partial\Psi = \mathrm{Curl}\,\mathrm{Curl}$ *with*

$$\mathrm{dom}(\partial\Psi) = Z^2_{\mathrm{Curl}}(\Omega, \mathcal{M}^3).$$

Proof. Let $A : L^2(\Omega, \mathcal{M}^3) \to L^2(\Omega, \mathcal{M}^3)$ be the linear operator defined by

$$Av = \mathrm{Curl}\,\mathrm{Curl}\, v$$

and $\mathrm{dom}(A) = Z^2_{\mathrm{Curl}}(\Omega, \mathcal{M}^3)$. Due to Lemma 3.1, the identity

$$\int_\Omega \mathrm{Curl}\,\mathrm{Curl}\, v(x) \cdot w(x) \, dx = \int_\Omega \mathrm{Curl}\, v(x) \cdot \mathrm{Curl}\, w(x) \, dx \tag{14}$$

holds for any $v, w \in Z^2_{\mathrm{Curl}}(\Omega, \mathcal{M}^3)$. Therefore, using (14) we obtain

$$\int_\Omega \mathrm{Curl}\,\mathrm{Curl}\, v \cdot (w - v) \, dx = \int_\Omega \mathrm{Curl}\, v \cdot (\mathrm{Curl}\, w - \mathrm{Curl}\, v) \, dx \le \Psi(w) - \Psi(v)$$

for every $v, w \in \mathrm{dom}(A)$. This shows that $A \subset \partial\Psi$. Since A is maximal monotone (see Corollary 3.2) we conclude that $A = \partial\Psi$. □

4. Existence of strong solutions

In this section we prove the main existence result for (4)–(9). To show the existence of weak solutions a time-discretization method is used in this work. In the first step, we prove the existence of the solutions of the time-discretized problem in appropriate Hilbert spaces based on the Helmholtz projection in $L^2(\Omega, \mathcal{S}^3)$ (Appendix A) and monotone operator methods (Section 2). In order to be able to apply the monotone operator method to the time-discretized problem we regularize it by a linear positive definite term. In the second step, we derive the uniform a priori estimates for the solutions of the time-discretized problem using the polynomial growth of the function g (see Definition 4.1) and then we pass to the weak limit in the equivalent formulation of the time-discretized problem employing the weak lower semicontinuity of lower semicontinuous convex functions and the maximal monotonicity of g.

Main result. First, we define the class of maximal monotone functions we deal with in this work.

Definition 4.1. For $m \in L^1(\Omega, \mathbb{R})$, $\alpha \in \mathbb{R}_+$, and $q > 1$, $\mathbb{M}(\Omega, \mathbb{R}^k, q, \alpha, m)$ is the set of multivalued functions $h : \Omega \times \mathbb{R}^k \to 2^{\mathbb{R}^k}$ with the following properties:

- $v \mapsto h(x, v)$ is maximal monotone for almost all $x \in \Omega$,
- the mapping $x \mapsto j_\lambda(x, v) : \Omega \to \mathbb{R}^k$ is measurable for all $\lambda > 0$, where $j_\lambda(x, v)$ is the inverse of $v \mapsto v + \lambda h(x, v)$,
- for a.e. $x \in \Omega$ and every $v^* \in h(x, v)$
$$\alpha\left(\frac{|v|^q}{q} + \frac{|v^*|^{q^*}}{q^*}\right) \le (v, v^*) + m(x), \tag{15}$$
where $1/q + 1/q^* = 1$.

Remark 4.2. The condition (15) is equivalent to the following two inequalities:
$$|v^*|^{q^*} \le m_1(x) + \alpha_1 |v|^q, \quad (v, v^*) \ge m_2(x) + \alpha_2 |v|^q, \tag{16}$$
for a.e. $x \in \Omega$ and every $v^* \in h(x, v)$ and with suitable functions $m_1, m_2 \in L^1(\Omega, \mathbb{R})$ and numbers $\alpha_1, \alpha_2 \in \mathbb{R}_+$.

Remark 4.3. Viscoplasticity is typically included in the former conditions by choosing the function g to be in Norton–Hoff form, that is,
$$g(\Sigma) = [|\Sigma| - \sigma_{\mathrm{y}}]_+^r \frac{\Sigma}{|\Sigma|}, \quad \Sigma \in \mathcal{M}^3, \tag{17}$$
where σ_{y} is the flow stress and r is some parameter together with $[x]_+ := \max(x, 0)$. If $g : \mathcal{M}^3 \mapsto \mathcal{S}^3$ then the flow is called irrotational (no plastic spin).

In case of a nonassociative flow rule, g is not a subdifferential but may, for example, be written as

$$g(\Sigma) = \mathcal{F}_1(\Sigma)\,\partial\mathcal{F}_2(\Sigma),$$

where $\mathcal{F}_1$ describes the yield function and $\mathcal{F}_2$ the flow direction.

The main properties of the class $\mathbb{M}(\Omega, \mathbb{R}^k, q, \alpha, m)$ are collected in the following proposition.

Proposition 4.4. *Let $\mathcal{H}$ be a canonical extension of a function $h : \mathbb{R}^k \to 2^{\mathbb{R}^k}$, which belongs to $\mathbb{M}(\Omega, \mathbb{R}^k, q, \alpha, m)$. Then $\mathcal{H}$ is maximal monotone and surjective, and $D(\mathcal{H}) = L^p(\Omega, \mathbb{R}^k)$.*

Proof. See [Damlamian et al. 2007, Corollary 2.15]. □

Next, we define two notions of solution for the initial boundary value problem (4)–(9). Both notions are introduced without assuming the homogeneity of the initial condition (7).

Definition 4.5 (strong solution). A function (u, σ, p) such that

$$(u, \sigma) \in H^1(0, T_e; H_0^1(\Omega, \mathbb{R}^3) \times L^2(\Omega, \mathcal{S}^3)), \quad \Sigma^{\text{lin}} \in L^q(\Omega_{T_e}, \mathcal{M}^3),$$
$$p \in H^1(0, T_e; L^2_{\text{Curl}}(\Omega, \mathcal{M}^3)) \cap L^2(0, T_e; Z^2_{\text{Curl}}(\Omega, \mathcal{M}^3)),$$

is called a *strong solution* of the initial boundary value problem (4)–(9) if, for every $t \in [0, T_e]$, the function $(u(t), \sigma(t))$ is a weak solution of the boundary value problem (73) with $\hat{\varepsilon}_p = \text{sym}\, p(t)$, and the condition $\hat{b} = b(t)$, the evolution inclusion (6) and the initial condition (7) are satisfied pointwise.

For the reader's convenience we give here also the definition of a weak solution for the problem (4)–(9) in the case when the monotone function g is a subdifferential of a proper lower semicontinuous convex function ϕ, that is, $g = \partial\phi$.

Definition 4.6 (weak solution). A function (u, σ, p) such that

$$(u, \sigma) \in W^{1,q^*}(0, T_e; W_0^{1,q^*}(\Omega, \mathbb{R}^3) \times L^{q^*}(\Omega, \mathcal{S}^3)), \quad \Sigma^{\text{lin}} \in L^q(\Omega_{T_e}, \mathcal{M}^3),$$
$$p \in W^{1,q^*}(0, T_e; L^{q^*}(\Omega, \mathcal{M}^3)) \cap L^{q^*}(0, T_e; Z^{q^*}_{\text{Curl}}(\Omega, \mathcal{M}^3)),$$

with

$$(\sigma, \text{dev}\,\text{sym}\, p, \text{Curl}\, p) \in L^\infty(0, T_e; L^2(\Omega, \mathcal{S}^3 \times \mathcal{M}^3 \times \mathcal{M}^3)),$$

is called a *weak solution* of the initial boundary value problem (4)–(9) if for every $t \in [0, T_e]$ the function $(u(t), \sigma(t))$ is a weak solution of the boundary value problem (73) with $\hat{\varepsilon}_p = \text{sym}\, p(t)$ and $\hat{b} = b(t)$, the initial condition (7) is satisfied, and the inequality[4]

$$\frac{1}{2}\int_\Omega \mathbb{C}^{-1}[x]\sigma(x,t)\cdot\sigma(x,t)\,dx + C_1\|\operatorname{dev}\operatorname{sym} p(t)\|_2^2 + C_2\|\operatorname{Curl} p(t)\|_2^2$$
$$+\int_0^t\int_\Omega \big(\phi^*(x,\partial_s p(x,s)) + \phi(x,\Sigma^{\mathrm{lin}}(x,s))\big)\,dxds \le \int_0^t (b(s),\partial_s u(s))_\Omega\,ds$$
$$+\frac{1}{2}\int_\Omega \mathbb{C}^{-1}[x]\sigma^{(0)}(x)\cdot\sigma^{(0)}(x)\,dx + C_1\|\operatorname{dev}\operatorname{sym} p^{(0)}\|_2^2 + C_2\|\operatorname{Curl} p^{(0)}\|_2^2$$

holds for all $t\in(0,T_e)$, with the function $\sigma^{(0)}\in L^2(\Omega,\mathcal{S}^3)$ determined by (73) for $\hat\varepsilon_p = \operatorname{sym} p^{(0)}$ and $\hat b = b(0)$.

In our previous paper [Nesenenko and Neff 2012] it is shown that under some additional regularity the weak solutions of the problem (4)–(9) with $g=\partial\phi$ become strong solutions of (4)–(9) in the sense of Definition 4.5.

Next, we state the main result of this work.

Theorem 4.7. *Suppose that $1<q^*\le 2\le q<\infty$. Assume that $\Omega\subset\mathbb{R}^3$ is a sliceable domain with a C^1-boundary and $\mathbb{C}\in L^\infty(\Omega,\mathcal{S}^3)$. Let the functions $b\in W^{1,q}(0,T_e;L^q(\Omega,\mathbb{R}^3))$ be given. Assume that $g\in\mathbb{M}(\Omega,\mathcal{M}^3,q,\alpha,m)$ and that for a.e. $x\in\Omega$ the relations*

$$p^{(0)}(x)=0 \quad \text{and} \quad 0\in g(x,\sigma^{(0)}(x)) \tag{18}$$

hold, where the function $\sigma^{(0)}\in L^2(\Omega,\mathcal{S}^3)$ is determined by (73) *for $\hat\varepsilon_p=0$ and $\hat b=b(0)$.*

Then there exists a solution (u,σ,p) of the initial boundary value problem (4)–(9).

In order to deal with the measurable elasticity tensor $\mathbb{C}$, we reformulate the problem (4)–(9) as follows: Let the function $(\hat v,\hat\sigma)\in W^{1,q}(0,T_e,W_0^{1,q}(\Omega,\mathbb{R}^3)\times L^q(\Omega,\mathcal{S}^3))$ be a solution of the linear elasticity problem formed by

$$-\operatorname{div}_x\hat\sigma(x,t) = b(x,t), \qquad x\in\Omega, \tag{19}$$
$$\hat\sigma(x,t) = \hat{\mathbb{C}}(\operatorname{sym}(\nabla_x\hat v(x,t))), \quad x\in\Omega, \tag{20}$$
$$\hat v(x,t) = 0, \qquad x\in\partial\Omega, \tag{21}$$

where $\hat{\mathbb{C}}:\mathcal{S}^3\to\mathcal{S}^3$ is any positive definite linear mapping independent of (x,t). Such a function $(\hat v,\hat\sigma)$ exists (see Appendix A). Then the solution (u,σ,p) of the initial boundary value problem (4)–(9) has the form

$$(u,\sigma,p) = (\tilde v+\hat v,\tilde\sigma+\hat\sigma,p),$$

where the function $(\tilde v,\tilde\sigma,p)$ solves the problem

[4]Here ϕ^* is the Legendre–Fenchel conjugate of ϕ.

$$-\operatorname{div}_x \tilde{\sigma}(x,t) = 0, \tag{22}$$

$$\tilde{\sigma}(x,t) = \mathbb{C}[x]\big(\operatorname{sym}(\nabla_x \tilde{v}(x,t) - p(x,t))\big) + (\mathbb{C}[x] - \hat{\mathbb{C}})\big(\operatorname{sym}(\nabla_x \hat{v}(x,t))\big), \tag{23}$$

$$\partial_t p(x,t) \in g\big(x, \Sigma^{\text{lin}}(x,t)\big), \qquad \Sigma^{\text{lin}} = \Sigma_e^{\text{lin}} + \Sigma_{\text{sh}}^{\text{lin}} + \Sigma_{\text{curl}}^{\text{lin}} \tag{24}$$

$$\big(\Sigma_{\text{e}}^{\text{lin}} = \tilde{\sigma} + \hat{\sigma}, \quad \Sigma_{\text{sh}}^{\text{lin}} = -C_1 \operatorname{dev}\operatorname{sym} p, \quad \Sigma_{\text{curl}}^{\text{lin}} = -C_2 \operatorname{Curl}\operatorname{Curl} p\big),$$

$$p(x,0) = 0, \qquad x \in \Omega, \tag{25}$$

$$p(x,t) \times n(x) = 0, \quad (x,t) \in \partial\Omega \times [0, T_e), \tag{26}$$

$$\tilde{v}(x,t) = 0, \quad (x,t) \in \partial\Omega \times [0, T_e). \tag{27}$$

Here, the function $(\hat{v}, \hat{\sigma})$ given as the solution of (19) is considered as known. Next, we show that the problem (22)–(27) has a solution. This will prove the existence of solutions for (4)–(9).

Proof. We will show the existence of solutions using Rothe's method (a time-discretization method, see [Roubíček 2005] for details). In order to introduce a time-discretized problem, let us fix any $m \in \mathbb{N}$ and set

$$h := \frac{T_e}{2^m}, \quad p_m^0 := 0, \quad \hat{\sigma}_m^n := \frac{1}{h}\int_{(n-1)h}^{nh} \hat{\sigma}(s)\,ds \in L^q(\Omega, \mathbb{R}^3), \quad n = 1, \dots, 2^m.$$

Then we are looking for functions $u_m^n \in H^1(\Omega, \mathbb{R}^3)$, $\sigma_m^n \in L^2(\Omega, \mathcal{S}^3)$, and $p_m^n \in Z^2_{\text{Curl}}(\Omega, \mathcal{M}^3)$ with $p_m^n(x) \in \mathfrak{sl}(3)$ for a.e. $x \in \Omega$ and

$$\Sigma_{n,m}^{\text{lin}} := \sigma_m^n + \hat{\sigma}_m^n - C_1 \operatorname{dev}\operatorname{sym} p_m^n - \frac{1}{m} p_m^n - C_2 \operatorname{Curl}\operatorname{Curl} p_m^n \in L^q(\Omega, \mathcal{M}^3)$$

solving the problem given by

$$-\operatorname{div}_x \sigma_m^n(x) = 0, \tag{28}$$

$$\sigma_m^n(x) = \mathbb{C}[x]\big(\operatorname{sym}(\nabla_x u_m^n(x) - p_m^n(x))\big) + (\mathbb{C}[x] - \hat{\mathbb{C}})(\hat{\mathbb{C}})^{-1}\hat{\sigma}_m^n(x), \tag{29}$$

$$\frac{p_m^n(x) - p_m^{n-1}(x)}{h} \in g(\Sigma_{n,m}^{\text{lin}}(x)), \tag{30}$$

together with the boundary conditions

$$p_m^n(x) \times n(x) = 0, \quad x \in \partial\Omega, \tag{31}$$

$$u_m^n(x) = 0, \quad x \in \partial\Omega. \tag{32}$$

Next, we adopt the reduction technique proposed in [Alber and Chełmiński 2004] to the equations above. Let $(u_m^n, \sigma_m^n, p_m^n)$ be a solution of the boundary value problem (28)–(32). Equations (28), (29), and (32) form a boundary value problem for the solution (u_m^n, σ_m^n) of the problem of linear elasticity. Due to the linearity of this problem we can write these components of the solution in the form

$$(u_m^n, \sigma_m^n) = (U_m^n, \Sigma_m^n) + (w_m^n, \tau_m^n),$$

with the solution (w_m^n, τ_m^n) of the Dirichlet boundary value problem (73) for the data $\hat{b} = 0$, $\hat{\varepsilon}_p = (\mathbb{C} - \hat{\mathbb{C}})(\hat{\mathbb{C}})^{-1}\hat{\sigma}_m^n$, and with the solution (U_m^n, Σ_m^n) of the problem (73) for the data $\hat{b} = 0$, $\hat{\varepsilon}_p = \operatorname{sym}(p_m^n)$. We thus obtain

$$\operatorname{sym}(\nabla_x u_m^n) - \operatorname{sym}(p_m^n) = (P_2 - I)\operatorname{sym}(p_m^n) + \operatorname{sym}(\nabla_x w_m^n),$$

where the operator P_2 is defined in Definition A.8. We insert this equation into (29) and get that (30) can be rewritten in the form

$$\frac{p_m^n - p_m^{n-1}}{h} \in \mathcal{G}(-M_m p_m^n - C_2 \operatorname{Curl}\operatorname{Curl} p_m^n + (\hat{\sigma}_m^n + \tau_m^n)), \tag{33}$$

$$p_m^n(x) \times n(x) = 0, \quad x \in \partial\Omega, \tag{34}$$

where

$$M_m := (\mathbb{C}Q_2 + L)\operatorname{sym} + \frac{1}{m} I : L^2(\Omega, \mathcal{M}^3) \to L^2(\Omega, \mathcal{M}^3),$$

with the Helmholtz projection Q_2 and the operator L defined by (76). Here $\mathcal{G}$ denotes the canonical extension of g. Next, the problem (33) reads

$$\Psi(p_m^n) \ni \hat{\sigma}_m^n + \tau_m^n, \tag{35}$$

where

$$\Psi(v) = \mathcal{G}^{-1}\left(\frac{v - p_m^{n-1}}{h}\right) + M_m(v) + \partial\Phi(v).$$

Here, the functional $\Phi : L^2(\Omega, \mathcal{M}^3) \to \bar{\mathbb{R}}$ is given by

$$\Phi(v) := \begin{cases} \frac{1}{2}\int_\Omega |\operatorname{Curl} v(x)|^2 \, dx, & v \in L^2_{\operatorname{Curl},0}(\Omega, \mathcal{M}^3), \\ +\infty, & \text{otherwise.} \end{cases}$$

That Φ is a proper convex lower semicontinuous functional and $\operatorname{Curl}\operatorname{Curl} = \partial\Phi$ is proved in Section 3. Since M_m is bounded, self-adjoint, and positive definite (see Corollary A.10 and the definition of M_m), it is maximal monotone by [Barbu 1976, Theorem II.1.3]. The last thing which we have to verify is whether the operator

$$\Psi = \mathcal{G}^{-1} + M_m + \partial\Phi$$

is maximal monotone. Since $g \in \mathbb{M}(\Omega, \mathcal{M}^3, q, \alpha, m)$, using the boundedness of M_m we conclude that the domains of $\mathcal{G}^{-1}$ and M_m are equal to the whole space

$L^2(\Omega, \mathcal{M}^3)$. Therefore, Theorem 2.5 guarantees that the sum $\mathcal{G}^{-1} + M_m + \partial\Phi$ is maximal monotone with

$$\mathrm{dom}(\Psi) = \mathrm{dom}(\partial\Phi) := Z^2_{\mathrm{Curl}}(\Omega, \mathcal{M}^3).$$

Since M_m is coercive in $L^2(\Omega, \mathcal{M}^3)$, which obviously yields the coercivity of Ψ, the operator Ψ is surjective by Theorem 2.6. Thus, we conclude that (35), as well as the problem (33), has the solutions with the required regularity, that is,

$$p^n_m \in Z^2_{\mathrm{Curl}}(\Omega, \mathcal{M}^3).$$

By the constructions this implies that the boundary value problem (28)–(32) is solvable as well (for more details we refer the reader to [Alber and Chełmiński 2004]). Moreover, $p^n_m(x) \in \mathfrak{sl}(3)$ for a.e. $x \in \Omega$.

Rothe approximation functions. For any family $\{\xi^n_m\}_{n=0,\dots,2m}$ of functions in a reflexive Banach space X, we define the *piecewise affine interpolant* $\xi_m \in C([0, T_e], X)$ by

$$\xi_m(t) := \Big(\frac{t}{h} - (n-1)\Big)\xi^n_m + \Big(n - \frac{t}{h}\Big)\xi^{n-1}_m \quad \text{for } (n-1)h \le t \le nh, \tag{36}$$

and the *piecewise constant interpolant* $\bar{\xi}_m \in L^\infty(0, T_e; X)$ by

$$\bar{\xi}_m(t) := \xi^n_m \quad \text{for } (n-1)h < t \le nh,\ n = 1, \dots, 2^m, \text{ and } \bar{\xi}_m(0) := \xi^0_m. \tag{37}$$

For further analysis we recall the following property of $\bar{\xi}_m$ and ξ_m:

$$\|\xi_m\|_{L^q(0,T_e;X)} \le \|\bar{\xi}_m\|_{L^q(-h,T_e;X)} \le \big(h\|\xi^0_m\|^q_X + \|\bar{\xi}_m\|^q_{L^q(0,T_e;X)}\big)^{1/q}, \tag{38}$$

where $\bar{\xi}_m$ is formally extended to $t \le 0$ by ξ^0_m and $1 \le q \le \infty$; see [Roubíček 2005].

A priori estimates. Multiplying (28) by $\dfrac{u^n_m - u^{n-1}_m}{h}$ and integrating over Ω we get

$$\big(\sigma^n_m, \mathrm{sym}(\nabla_x(u^n_m - u^{n-1}_m)/h)\big)_\Omega = 0.$$

Equations (29) and (30) imply that for a.e. $x \in \Omega$

$$\begin{aligned}
&\sigma^n_m \cdot \big(\mathrm{sym}(\nabla_x(u^n_m - u^{n-1}_m)/h) - \mathbb{C}^{-1}[x](\sigma^n_m - \sigma^{n-1}_m)/h\big)\\
&\quad + \sigma^n_m \cdot \big(\mathbb{C}^{-1}[x](\mathbb{C}[x] - \hat{\mathbb{C}})(\hat{\mathbb{C}})^{-1}(\hat{\sigma}^n_m - \hat{\sigma}^{n-1}_m)/h\big)\\
&\quad - \frac{p^n_m - p^{n-1}_m}{h} \cdot \Big(C_1 \,\mathrm{dev}\,\mathrm{sym}\, p^n_m + \frac{1}{m} p^n_m + C_2 \,\mathrm{Curl}\,\mathrm{Curl}\, p^n_m\Big) + \frac{p^n_m - p^{n-1}_m}{h} \cdot \hat{\sigma}^n_m\\
&\qquad = g^{-1}\Big(\frac{p^n_m - p^{n-1}_m}{h}\Big) \cdot \Big(\frac{p^n_m - p^{n-1}_m}{h}\Big).
\end{aligned}$$

After integrating the last identity over Ω, the above computations imply

$$\frac{1}{h}(\mathbb{C}^{-1}\sigma_m^n, \sigma_m^n - \sigma_m^{n-1})_\Omega + C_1\frac{1}{h}(\operatorname{dev}\operatorname{sym}(p_m^n - p_m^{n-1}), \operatorname{dev}\operatorname{sym} p_m^n)_\Omega$$
$$+ \frac{1}{m}\frac{1}{h}(p_m^n - p_m^{n-1}, p_m^n)_\Omega + C_2\frac{1}{h}(\operatorname{Curl}(p_m^n - p_m^{n-1}), \operatorname{Curl} p_m^n)_\Omega$$
$$+ \frac{\alpha}{q}\|\Sigma_{n,m}^{\mathrm{lin}}\|_q^q + \frac{\alpha}{q^*}\left\|\frac{p_m^n - p_m^{n-1}}{h}\right\|_{q^*}^{q^*}$$
$$\le \int_\Omega m(x)\,dx + \frac{1}{h}(\sigma_m^n, \bar{\mathbb{C}}(\hat{\sigma}_m^n - \hat{\sigma}_m^{n-1}))_\Omega + \frac{1}{h}(\hat{\sigma}_m^n, p_m^n - p_m^{n-1})_\Omega,$$

where $\bar{\mathbb{C}} := \mathbb{C}^{-1}(\mathbb{C} - \hat{\mathbb{C}})(\hat{\mathbb{C}})^{-1}$. Multiplying by h and summing the obtained relation for $n = 1, \dots, l$ for any fixed $l \in [1, 2^m]$ we derive the following inequality (here $\mathbb{B} := \mathbb{C}^{-1}$):

$$\frac{1}{2}\Big(\|\mathbb{B}^{1/2}\sigma_m^l\|_2^2 + C_1\|\operatorname{dev}\operatorname{sym} p_m^l\|_2^2 + \frac{1}{m}\|p_m^l\|_2^2 + C_2\|\operatorname{Curl} p_m^l\|_2^2\Big)$$
$$+ \frac{h\alpha}{q}\sum_{n=1}^{l}\|\Sigma_{n,m}^{\mathrm{lin}}\|_q^q + \frac{h\alpha}{q^*}\sum_{n=1}^{l}\left\|\frac{p_m^n - p_m^{n-1}}{h}\right\|_{q^*}^{q^*} \le C^{(0)} + \int_\Omega m(x)\,dx$$
$$+ h\sum_{n=1}^{l}\left(\sigma_m^n, \bar{\mathbb{C}}\frac{\hat{\sigma}_m^n - \hat{\sigma}_m^{n-1}}{h}\right)_\Omega + h\sum_{n=1}^{l}\left(\hat{\sigma}_m^n, \frac{p_m^n - p_m^{n-1}}{h}\right)_\Omega, \quad (39)$$

where[5]

$$2C^{(0)} := \|\mathbb{B}^{1/2}\sigma^{(0)}\|_2^2.$$

Since $\hat{\sigma}_m^n \in L^q(\Omega, \mathcal{S}^3)$, using Young's inequality with $\epsilon > 0$ we get that

$$\left(\hat{\sigma}_m^n, \frac{p_m^n - p_m^{n-1}}{h}\right)_\Omega$$
$$\le \|\hat{\sigma}_m^n\|_q\left\|\frac{p_m^n - p_m^{n-1}}{h}\right\|_{q^*} \le C_\epsilon\|\hat{\sigma}_m^n\|_q^q + \epsilon\left\|\frac{p_m^n - p_m^{n-1}}{h}\right\|_{q^*}^{q^*}, \quad (40)$$

where C_ϵ is a positive constant appearing in the Young inequality. Analogically, we obtain

$$\left(\sigma_m^n, \bar{\mathbb{C}}\frac{\hat{\sigma}_m^n - \hat{\sigma}_m^{n-1}}{h}\right)_\Omega \le \epsilon\|\sigma_m^n\|_2^2 + C_\epsilon\left\|\frac{\hat{\sigma}_m^n - \hat{\sigma}_m^{n-1}}{h}\right\|_2^2, \quad (41)$$

[5]Here we use the inequality

$$\sum_{n=1}^{l}(\phi_m^n - \phi_m^{n-1}, \phi_m^n)_\Omega = \tfrac{1}{2}\sum_{n=1}^{l}(\|\phi_m^n\|_2^2 - \|\phi_m^{n-1}\|_2^2) + \tfrac{1}{2}\sum_{n=1}^{l}\|\phi_m^n - \phi_m^{n-1}\|_2^2 \ge \tfrac{1}{2}\|\phi_m^l\|_2^2 - \tfrac{1}{2}\|\phi_m^0\|_2^2,$$

valid for any family of functions $\phi_m^0, \phi_m^1, \dots, \phi_m^m$.

with some other constant C_ϵ. Combining inequalities (39), (40), and (41), and choosing an appropriate value for $\epsilon > 0$, we obtain the estimate

$$\frac{1}{2}\Big(\|\mathbb{B}^{1/2}\sigma_m^l\|_2^2 + C_1\|\operatorname{dev}\operatorname{sym} p_m^l\|_2^2 + \frac{1}{m}\|p_m^l\|_2^2 + C_2\|\operatorname{Curl} p_m^l\|_2^2\Big) + h\hat{C}_\epsilon\sum_{n=1}^{l}\bigg(\|\Sigma_{n,m}^{\mathrm{lin}}\|_q^q + \bigg\|\frac{p_m^n - p_m^{n-1}}{h}\bigg\|_{q^*}^{q^*}\bigg)$$
$$\le C^{(0)} + \int_\Omega m(x)\,dx + h\epsilon\sum_{n=1}^{l}\|\sigma_m^n\|_2^2 + h\tilde{C}_\epsilon\sum_{n=1}^{l}\bigg(\|\hat{\sigma}_m^n\|_q^q + \bigg\|\frac{\hat{\sigma}_m^n - \hat{\sigma}_m^{n-1}}{h}\bigg\|_2^2\bigg), \tag{42}$$

where $\tilde{C}$, $\tilde{C}_\epsilon$, and $\hat{C}_\epsilon$ are some positive constants. Now, taking [Roubíček 2005, Remark 8.15] and the definition of Rothe's approximation functions into account we rewrite (42) as follows:

$$\|\mathbb{B}^{1/2}\bar{\sigma}_m(t)\|_2^2 + C_1\|\operatorname{dev}\operatorname{sym}\bar{p}_m(t)\|_2^2 + \frac{1}{m}\|\bar{p}_m(t)\|_2^2 + C_2\|\operatorname{Curl}\bar{p}_m(t)\|_2^2 + 2\hat{C}_\epsilon\int_0^{T_e}\int_\Omega\big(|\partial_t p_m(x,t)|^{q^*} + |\overline{\Sigma}_m^{\mathrm{lin}}(x,t)|^q\big)\,dx\,dt$$
$$\le 2C^{(0)} + \|m\|_{1,\Omega} + \epsilon\|\sigma_m\|_{2,\Omega\times(0,T_e)}^2 + 2\tilde{C}_\epsilon\|\hat{\sigma}\|_{W^{1,q}(0,T_e;L^q(\Omega,\mathcal{S}^3))}^q. \tag{43}$$

From (43) we get immediately that

$$\bar{C}_\epsilon\|\sigma_m\|_{2,\Omega\times(0,t)}^2 + C_1\|\operatorname{dev}\operatorname{sym}\bar{p}_m(t)\|_2^2 + \frac{1}{m}\|\bar{p}_m(t)\|_2^2 + C_2\|\operatorname{Curl}\bar{p}_m(t)\|_2^2 + 2\hat{C}_\epsilon\big(\|\partial_t p_m\|_{q^*,\Omega\times(0,T_e)}^{q^*} + \|\overline{\Sigma}_m^{\mathrm{lin}}\|_{q,\Omega\times(0,T_e)}^q\big)$$
$$\le 2C^{(0)} + \|m\|_{1,\Omega} + 2\tilde{C}_\epsilon\|\hat{\sigma}\|_{W^{1,q}(0,T_e;L^q(\Omega,\mathcal{S}^3))}^q, \tag{44}$$

where $\bar{C}_\epsilon$ is some other constant depending on ϵ. Altogether, from estimate (44) we get that

$$\{p_m\}_m \text{ is uniformly bounded in } W^{1,q^*}(0,T_e;L^{q^*}(\Omega,\mathcal{M}^3)), \tag{45}$$
$$\{\operatorname{dev}\operatorname{sym}\bar{p}_m\}_m \text{ is uniformly bounded in } L^\infty(0,T_e;L^2(\Omega,\mathcal{M}^3)), \tag{46}$$
$$\{\sigma_m\}_m, \text{ is uniformly bounded in } L^2(0,T_e;L^2(\Omega,\mathcal{S}^3)), \tag{47}$$
$$\{\operatorname{Curl}\bar{p}_m\}_m \text{ is uniformly bounded in } L^\infty(0,T_e;L^2(\Omega,\mathcal{M}^3)), \tag{48}$$
$$\{\overline{\Sigma}_m^{\mathrm{lin}}\}_m \text{ is uniformly bounded in } L^q(0,T_e;L^q(\Omega,\mathcal{M}^3)), \tag{49}$$
$$\Big\{\frac{1}{\sqrt{m}}\bar{p}_m\Big\}_m \text{ is uniformly bounded in } L^\infty(0,T_e;L^2(\Omega,\mathcal{M}^3)). \tag{50}$$

In particular, the uniform boundedness of the sequences in (45)–(50) yields

$$\{u_m\}_m \text{ is uniformly bounded in } W^{1,q^*}(0, T_e; W_0^{1,q^*}(\Omega, \mathbb{R}^3)), \tag{51}$$

$$\{\operatorname{Curl}\operatorname{Curl} \bar{p}_m\}_m \text{ is uniformly bounded in } L^2(0, T_e; L^2(\Omega, \mathcal{M}^3)). \tag{52}$$

In view of (38), the estimates (45)–(52) further imply that the sequences $\{\sigma_m\}_m$, $\{\operatorname{dev}\operatorname{sym} p_m\}_m$, $\{\operatorname{Curl} p_m\}_m$, $\{p_m/\sqrt{m}\}_m$, $\{\Sigma_m^{\text{lin}}\}_m$, and $\{\operatorname{Curl}\operatorname{Curl} p_m\}_m$ are also uniformly bounded in the corresponding spaces. As a result, we have

$$\{p_m\}_m \text{ is uniformly bounded in } L^{q^*}(0, T_e; Z_{\operatorname{Curl}}^{q^*}(\Omega, \mathcal{M}^3)). \tag{53}$$

Furthermore, due to (3), (46), (48), and (52) we obtain that

$$\{\bar{p}_m\}_m \text{ and } \{p_m\}_m \text{ are uniformly bounded in } L^2(0, T_e; Z_{\operatorname{Curl}}^2(\Omega, \mathcal{M}^3)). \tag{54}$$

Moreover, (36) and (37) yield $\{p_m(x,t), \bar{p}_m(x,t)\}_m \in \mathfrak{sl}(3)$ for a.e. $(x,t) \in \Omega_{T_e}$.

Additional regularity of discrete solutions. In order to get the additional a priori estimates, we extend the function b to $t < 0$ by setting $b(t) = b(0)$. The extended function b is in the space $W^{1,p}(-2h, T_e; W^{-1,p}(\Omega, \mathbb{R}^3))$. Then, we set $b_m^0 = b_m^{-1} := b(0)$. Let us further set

$$p_m^{-1} := p_m^0 - h\mathcal{G}(\Sigma_{0,m}^{\text{lin}}).$$

The assumption (18) implies that $p_m^{-1} = 0$. Next, we define functions $(u_m^{-1}, \sigma_m^{-1})$ and (u_m^0, σ_m^0) as solutions of the linear elasticity problem (73) to the data $\hat{b} = b_m^{-1}$, $\hat{\gamma} = 0$, and $\hat{\varepsilon}_p = 0$ and $\hat{b} = b_m^0$, $\hat{\gamma} = 0$, and $\hat{\varepsilon}_p = 0$, respectively. Obviously, the following estimate holds:

$$\left\{ \left\| \frac{u_m^0 - u_m^{-1}}{h} \right\|_2, \left\| \frac{\sigma_m^0 - \sigma_m^{-1}}{h} \right\|_2 \right\} \le C, \tag{55}$$

where C is some positive constant independent of m. Taking now the incremental ratio of (30) for $n = 1, \dots, 2^m$, we obtain[6]

$$\operatorname{rt} p_m^n - \operatorname{rt} p_m^{n-1} = \mathcal{G}(\Sigma_{n,m}^{\text{lin}}) - \mathcal{G}\big(\Sigma_{(n-1),m}^{\text{lin}}\big).$$

Let us now multiply the last identity by $-\big(\Sigma_{n,m}^{\text{lin}} - \Sigma_{(n-1),m}^{\text{lin}}\big)/h$. Then, using the monotonicity of $\mathcal{G}$ we obtain

$$\begin{aligned}\frac{1}{m}&(\operatorname{rt} p_m^n - \operatorname{rt} p_m^{n-1}, \operatorname{rt} p_m^n)_\Omega \\ &+ \big(\operatorname{rt} p_m^n - \operatorname{rt} p_m^{n-1}, \operatorname{dev}\operatorname{sym}(\operatorname{rt} p_m^n)\big)_\Omega + \big(\operatorname{rt} p_m^n - \operatorname{rt} p_m^{n-1}, \operatorname{Curl}\operatorname{Curl}(\operatorname{rt} p_m^n)\big)_\Omega \\ &\le (\operatorname{rt} p_m^n - \operatorname{rt} p_m^{n-1}, \operatorname{rt} \sigma_m^n)_\Omega + (\operatorname{rt} p_m^n - \operatorname{rt} p_m^{n-1}, \operatorname{rt} \hat{\sigma}_m^n)_\Omega.\end{aligned}$$

[6]For simplicity we use the notation $\operatorname{rt}\phi_m^n := (\phi_m^n - \phi_m^{n-1})/h$, where $\phi_m^0, \phi_m^1, \dots, \phi_m^m$ is any family of functions.

With (28) and (29) the previous inequality can be rewritten as follows:

$$\frac{1}{m}(\operatorname{rt} p_m^n - \operatorname{rt} p_m^{n-1}, \operatorname{rt} p_m^n)_\Omega + \big(\operatorname{rt} p_m^n - \operatorname{rt} p_m^{n-1}, \operatorname{dev}\operatorname{sym}(\operatorname{rt} p_m^n)\big)_\Omega + \big(\operatorname{rt} p_m^n - \operatorname{rt} p_m^{n-1}, \operatorname{Curl}\operatorname{Curl}(\operatorname{rt} p_m^n)\big)_\Omega + (\operatorname{rt}\sigma_m^n - \operatorname{rt}\sigma_m^{n-1}, \mathbb{C}^{-1}\operatorname{rt}\sigma_m^n)_\Omega \le (\operatorname{rt}\hat\sigma_m^n - \operatorname{rt}\hat\sigma_m^{n-1}, \bar{\mathbb{C}}\operatorname{rt}\sigma_m^n)_\Omega + (\operatorname{rt} p_m^n - \operatorname{rt} p_m^{n-1}, \operatorname{rt}\hat\sigma_m^n)_\Omega.$$

As in the proof of (39), multiplying the last inequality by h and summing with respect to n from 1 to l for any fixed $l \in [1, 2^m]$ we get the estimate

$$\frac{h}{m}\|\operatorname{rt} p_m^l\|_2^2 + h\|\operatorname{dev}\operatorname{sym}\operatorname{rt} p_m^l\|_2^2 + h\|\mathbb{B}^{1/2}\operatorname{rt}\sigma_m^l\|_2^2 + h\|\operatorname{Curl}\operatorname{rt} p_m^l\|_2^2 \le 2hC^{(0)} + 2h\sum_{n=1}^{l}(\operatorname{rt}\hat\sigma_m^n, \operatorname{rt} p_m^n - \operatorname{rt} p_m^{n-1})_\Omega + 2h\sum_{n=1}^{l}(\operatorname{rt}\hat\sigma_m^n - \operatorname{rt}\hat\sigma_m^{n-1}, \bar{\mathbb{C}}\operatorname{rt}\sigma_m^n)_\Omega, \tag{56}$$

where now $C^{(0)}$ is defined by

$$2C^{(0)} := \|\mathbb{B}^{1/2}\operatorname{rt}\sigma_m^0\|_2^2.$$

We note that (55) yields the uniform boundness of $C^{(0)}$ with respect to m. Summing now (56) for $l = 1, \dots, 2^m$ we derive the inequality

$$\frac{1}{m}\|\partial_t p_m\|_{2,\Omega_{T_e}}^2 + \|\operatorname{dev}\operatorname{sym}(\partial_t p_m)\|_{2,\Omega_{T_e}}^2 + \|\operatorname{Curl}(\partial_t p_m)\|_{2,\Omega_{T_e}}^2 + C\|\partial_t\sigma_m\|_{2,\Omega_{T_e}}^2 \le C\|\partial_t\hat\sigma_m\|_{2,\Omega_{T_e}}(\|\partial_t\sigma_m\|_{2,\Omega_{T_e}} + \|\partial_t p_m\|_{2,\Omega_{T_e}}). \tag{57}$$

Using now inequality (3), the condition $\partial_t p_m(x, t) \in \mathfrak{sl}(3)$ for a.e. $(x, t) \in \Omega_{T_e}$, and Young's inequality with $\epsilon > 0$ in (57), we obtain that

$$\frac{1}{m}\|\partial_t p_m\|_{2,\Omega_{T_e}}^2 + C_\epsilon\|\partial_t p_m\|_{2,\Omega_{T_e}}^2 + C_\epsilon\|\partial_t\sigma_m\|_{2,\Omega_{T_e}}^2 \le C\|\partial_t\hat\sigma_m\|_{2,\Omega_{T_e}}^2. \tag{58}$$

Since $\hat\sigma_m$ is uniformly bounded in $W^{1,q}(\Omega_{T_e}, \mathcal{S}^3)$, estimates (57) and (58) imply

$$\{\operatorname{dev}\operatorname{sym}\partial_t p_m\}_m \text{ is uniformly bounded in } L^2(0, T_e; L^2(\Omega, \mathcal{M}^3)), \tag{59}$$

$$\{\partial_t\sigma_m\}_m \text{ is uniformly bounded in } L^2(0, T_e; L^2(\Omega, \mathcal{M}^3)), \tag{60}$$

$$\{\operatorname{Curl}\partial_t p_m\}_m \text{ is uniformly bounded in } L^2(0, T_e; L^2(\Omega, \mathcal{M}^3)), \tag{61}$$

$$\left\{\frac{1}{\sqrt{m}}\partial_t p_m\right\}_m \text{ is uniformly bounded in } L^2(0, T_e; L^2(\Omega, \mathcal{M}^3)), \tag{62}$$

$$\{p_m\}_m \text{ is uniformly bounded in } H^1(0, T_e; L^2_{\operatorname{Curl}}(\Omega, \mathcal{M}^3)). \tag{63}$$

Existence of solutions. By estimates (45)–(54) and (59)–(63) and at the expense of extracting a subsequence, we have that the sequences in (45)–(54) and (59)–(63) converge with respect to weak and weak-star topologies in corresponding spaces, respectively. Next, we claim that weak limits of $\{\bar{p}_m\}_m$ and $\{p_m\}_m$ coincide. Indeed, using (45) this can be shown as follows:

$$\begin{aligned}\|p_m-\bar{p}_m\|^2_{\Omega_{T_e}} &= \sum_{n=1}^{m}\int_{(n-1)h}^{nh}\Big\|(p_m^n-p_m^{n-1})\frac{t-nh}{h}\Big\|_2^2\,dt\\ &= \frac{h^{2+1}}{2+1}\sum_{n=1}^{m}\Big\|\frac{p_m^n-p_m^{n-1}}{h}\Big\|_2^2=\frac{h^2}{2+1}\Big\|\frac{dp_m}{dt}\Big\|^2_{2,\Omega_{T_e}},\end{aligned}$$

which implies that $\bar{p}_m-p_m$ converges strongly to 0 in $L^2(\Omega_{T_e},\mathcal{M}^3)$. The proof of the fact that the difference $\bar{\sigma}_m-\sigma_m$ converges weakly to 0 in $L^2(\Omega_{T_e},\mathcal{S}^3)$ can be performed as on p. 210 of [Roubíček 2005]. For the reader's convenience we reproduce the reasoning used there. Let us choose some appropriate number $d\in\mathbb{N}$ and then fix any integer $n_0\in[1,2^d]$. Let $h_0=T_e/2^{n_0}$. Consider functions $I_{[h_0(n_0-1),h_0n_0]}v$ with $v\in L^2(\Omega,\mathcal{S}^3)$, where I_K denotes the indicator function of a set K. We note that, according to Proposition 1.36 of the same reference, the linear combinations of all such functions are dense in $L^2(\Omega_{T_e},\mathcal{S}^3)$. Then for any $h\le h_0$[7]we have

$$\begin{aligned}&(\sigma_m-\bar{\sigma}_m,\,I_{[h_0(n_0-1),h_0n_0]}v)_{\Omega_{T_e}}\\ &=\int_{h_0(n_0-1)}^{h_0n_0}(\sigma_m(t)-\bar{\sigma}_m(t),\,v)_\Omega\,dt=\sum_{n=\frac{h_0}{h}(n_0-1)+1}^{h_0n_0/h}\int_{(n-1)h}^{nh}\Big((\sigma_m^n-\sigma_m^{n-1})\frac{t-nh}{h},\,v\Big)_\Omega dt\\ &=-\frac{h}{2}(\sigma_m^{h_0n_0/h}-\sigma_m^{h_0(n_0-1)/h},\,v)_\Omega=-\frac{h}{2}(\bar{\sigma}_m(h_0n_0)-\bar{\sigma}_m(h_0(n_0-1)),\,v)_\Omega.\end{aligned}$$

Employing (47) we get that $\bar{\sigma}_m-\sigma_m$ converges weakly to 0 in $L^2(\Omega_{T_e},\mathcal{S}^3)$. Next, by (50) the sequence $\{p_m/m\}_m$ converges strongly to 0 in $L^2(\Omega_{T_e},\mathcal{M}^3)$. Summarizing all observations made above we may conclude that the limit functions denoted by $\tilde{v}$, $\tilde{\sigma}$, p, and Σ^{lin} have the following properties:

$$\begin{gathered}(\tilde{v},\tilde{\sigma})\in H^1(0,T_e;\,H_0^1(\Omega,\mathbb{R}^3)\times L^2(\Omega,\mathcal{S}^3)),\qquad \Sigma^{\text{lin}}\in L^q(\Omega_{T_e},\mathcal{M}^3),\\ p\in H^1(0,T_e;\,L^2_{\text{Curl}}(\Omega,\mathcal{M}^3))\cap L^2(0,T_e;\,Z^2_{\text{Curl}}(\Omega,\mathcal{M}^3)).\end{gathered}$$

Moreover, $p(x,t)\in\mathfrak{sl}(3)$ holds for a.e. $(x,t)\in\Omega_{T_e}$. Before passing to the weak limit, we note that the Rothe approximation functions satisfy the equations

$$-\operatorname{div}_x\bar{\sigma}_m(x,t)=\bar{b}_m(x,t),\tag{64}$$

$$\sigma_m(x,t)=\mathbb{C}\big(\operatorname{sym}(\nabla_x u_m(x,t)-p_m(x,t))\big)+(\mathbb{C}[x]-\hat{\mathbb{C}})(\hat{\mathbb{C}})^{-1}\hat{\sigma}_m(x),\tag{65}$$

$$\partial_t p_m(x,t)\in g(\overline{\Sigma}^{\text{lin}}_m(x,t)),\tag{66}$$

[7] We recall that h is chosen to be equal to $T_e/2^m$ for some $m\in\mathbb{N}$.

together with the initial and boundary conditions

$$p_m(x,0)=0, \quad x\in\Omega, \tag{67}$$

$$p_m(x,t)\times n(x)=0, \quad x\in\partial\Omega, \tag{68}$$

$$u_m(x,t)=0, \quad x\in\partial\Omega. \tag{69}$$

Passing to the weak limit in (64), (65), and (69) we obtain that the limit functions $\tilde{v}$, $\tilde{\sigma}$, p, and Σ^{lin} satisfy (22) and (27). To show that the limit functions satisfy also (24) we proceed as follows: As above, the system (64)–(69) can be rewritten as

$$\begin{aligned}\int_0^{T_e}\int_\Omega \big(g^{-1}(\partial_t p_m(x,t))\cdot\partial_t p_m(x,t)\big)\,dx\,dt \\ = -\Big(\frac{d\sigma_m}{dt},\mathbb{C}^{-1}\bar{\sigma}_m\Big)_{\Omega_{T_e}} - C_1\Big(\frac{dp_m}{dt},\operatorname{dev}\operatorname{sym}\bar{p}_m\Big)_{\Omega_{T_e}} - \frac{1}{m}\Big(\frac{dp_m}{dt},\bar{p}_m\Big)_{\Omega_{T_e}} \\ - C_2\Big(\frac{dp_m}{dt},\operatorname{Curl}\operatorname{Curl}\bar{p}_m\Big)_{\Omega_{T_e}} + (\hat{\sigma}_m,\partial_t p_m)_{\Omega_{T_e}} + (\bar{\mathbb{C}}\bar{\sigma}_m,\partial_t\hat{\sigma}_m)_{\Omega_{T_e}}.\end{aligned} \tag{70}$$

Due to (59)–(63) and Lemma B.11 we can pass to the weak limit inferior in (70) to get the following inequality:

$$\begin{aligned}\limsup_{m\to\infty}\int_0^{T_e}\int_\Omega \big(g^{-1}(\partial_t p_m(x,t))\cdot\partial_t p_m(x,t)\big)\,dx\,dt \\ \le (\partial_t p,\tilde{\sigma}+\hat{\sigma}-\operatorname{dev}\operatorname{sym}p-\operatorname{Curl}\operatorname{Curl}p)_{\Omega_{T_e}}.\end{aligned} \tag{71}$$

Let $\mathcal{G}$ denote the canonical extension of g. Then (71) reads as follows:

$$\limsup_{m\to\infty}(\mathcal{G}^{-1}(\partial_t p_m),\partial_t p_m)_{\Omega_{T_e}} \le (\partial_t p,\tilde{\sigma}+\hat{\sigma}-\operatorname{dev}\operatorname{sym}p-\operatorname{Curl}\operatorname{Curl}p)_{\Omega_{T_e}}. \tag{72}$$

Since $\mathcal{G}^{-1}$ is pseudomonotone, inequality (72) yields that, for a.e. $(x,t)\in\Omega_{T_e}$,

$$\partial_t p(x,s)\in g(\tilde{\sigma}(x,t)+\hat{\sigma}(x,t)-\operatorname{dev}\operatorname{sym}p(x,t)-\operatorname{Curl}\operatorname{Curl}p(x,t)).$$

Therefore, we conclude that the limit functions $\tilde{v}$, $\tilde{\sigma}$, p, and Σ^{lin} satisfy (22)–(27) and the existence of strong solutions is herewith established.

This completes the proof of Theorem 4.7. □

Appendix A: Helmholtz's projection

In this section we present some results concerning projection operators to spaces of tensor fields, which are symmetric gradients, and to spaces of tensor fields with vanishing divergence. For details the reader is referred to [Alber and Chełmiński 2007].

In linear elasticity theory it is well known (see [Giusti 2003, Theorem 10.15]) that a Dirichlet boundary value problem formed by the equations

$$-\operatorname{div}_x \sigma(x) = \hat{b}(x), \qquad x \in \Omega, \tag{73}$$

$$\sigma(x) = \mathbb{C}[x](\operatorname{sym}(\nabla_x u(x)) - \hat{\varepsilon}_p(x)), \quad x \in \Omega, \tag{74}$$

$$u(x) = 0, \qquad x \in \partial\Omega, \tag{75}$$

for given $\hat{b} \in W^{-1,q}(\Omega, \mathbb{R}^3)$ and $\hat{\varepsilon}_p \in L^q(\Omega, \mathcal{S}^3)$ has a unique weak solution $(u, \sigma) \in W_0^{1,q}(\Omega, \mathbb{R}^3) \times L^q(\Omega, \mathcal{S}^3)$ provided the open set Ω has a C^1-boundary and $\mathbb{C} \in C(\overline{\Omega}, \mathcal{S}^3)$. Here the number q satisfies $1 < q < \infty$. For $\hat{b} = 0$ the solution of (73) satisfies the inequality

$$\|\operatorname{sym}(\nabla_x u)\|_q \le C \|\hat{\varepsilon}_p\|_q$$

with some positive constant C.

Definition A.8. For every $\hat{\varepsilon}_p \in L^q(\Omega, \mathcal{S}^3)$ we define a linear operator $P_q : L^q(\Omega, \mathcal{S}^3) \to L^q(\Omega, \mathcal{S}^3)$ by

$$P_q \hat{\varepsilon}_p := \operatorname{sym}(\nabla_x u),$$

where $u \in W_0^{1,q}(\Omega, \mathbb{R}^3)$ is the unique weak solution of (73) to the given function $\hat{\varepsilon}_p$ and $\hat{b} = 0$.

Next, a subset $\mathcal{G}^q$ of $L^q(\Omega, \mathcal{S}^3)$ is defined by

$$\mathcal{G}^q = \{\operatorname{sym}(\nabla_x u) \mid u \in W_0^{1,q}(\Omega, \mathbb{R}^3)\}.$$

The main properties of P_q are stated in the following lemma.

Lemma A.9. *For every $1 < q < \infty$ the operator P_p is a bounded projector onto the subset $\mathcal{G}^q$ of $L^q(\Omega, \mathcal{S}^3)$. The projector $(P_q)^*$ adjoint with respect to the bilinear form $[\xi, \zeta]_\Omega := (\xi, \zeta)_\Omega$ on $L^q(\Omega, \mathcal{S}^3) \times L^{q^*}(\Omega, \mathcal{S}^3)$ satisfies*

$$(P_q)^* = P_{q^*}, \quad \textit{where } \frac{1}{q^*} + \frac{1}{q} = 1.$$

Due to Lemma A.9 the projection operator

$$Q_q = (I - P_q) : L^q(\Omega, \mathcal{S}^3) \to L^q(\Omega, \mathcal{S}^3)$$

is well defined and generalizes the classical Helmholtz projection.

Let $L : \mathcal{S}^3 \to \mathcal{S}^3$ be the linear, positive semidefinite mapping given by

$$Lv = C_1 \operatorname{dev} v. \tag{76}$$

The next result is needed for the subsequent analysis.

Corollary A.10. *Let*

$$(\mathbb{C}P_q + L)^*$$

be the operator adjoint to $\mathbb{C}P_q + L : L^q(\Omega, \mathcal{S}^3) \to L^q(\Omega, \mathcal{S}^3)$ *with respect to the bilinear form* $(\xi, \zeta)_\Omega$ *on the product space* $L^q(\Omega, \mathcal{S}^3) \times L^{q^*}(\Omega, \mathcal{S}^3)$. *Then* $(\mathbb{C}P_q + L)^* = \mathbb{C}P_{q^*} + L$. *Moreover, the operator* $\mathbb{C}Q_2 + L$ *is nonnegative and self-adjoint.*

For the proof of this result the reader is referred to [Alber and Chełmiński 2004].

Appendix B

In this appendix we prove the following lemma (see [Roubíček 2005]).

Lemma B.11. *Let* X *be a reflexive Banach space embedded continuously and densely into a Hilbert space* H, *let the functions* ϕ_m *and* $\bar{\phi}_m$ *be defined by* (36) *and* (37) *for any family of functions* $\phi_m^0, \phi_m^1, \dots, \phi_m^m$, *respectively, and let* ϕ *be a weak limit of* ϕ_m. *Then the following inequality*:

$$\limsup_{m\to\infty}\left\langle \frac{d\phi_m}{dt}, \bar{\phi}_m\right\rangle_{L^q(X^*),L^p(X)} \geq \left\langle \frac{d\phi}{dt}, \phi\right\rangle_{L^q(X^*),L^p(X)}$$

holds, where $\langle\,\cdot\,,\,\cdot\,\rangle_{L^q(X^*),L^p(X)}$ *denotes the dual pairing between* $L^p(X)$ *and* $L^q(X^*)$.

Proof. The last inequality results from the next line by taking lim sup from both sides and using the lower semicontinuity of the norm

$$\begin{aligned}\left\langle \frac{d\phi_m}{dt}, \bar{\phi}_m\right\rangle_{L^q(X^*),L^p(X)} &= \sum_{n=1}^m \int_{h(n-1)}^{hn} \left\langle \frac{\phi_m^n - \phi_m^{n-1}}{h}, \phi_m^n\right\rangle_{X^*,X} dt = \sum_{n=1}^m \langle \phi_m^n - \phi_m^{n-1}, \phi_m^n\rangle_{X^*,X} \\ &= \sum_{n=1}^m \frac{1}{2}\|\phi_m^n\|_H^2 - \frac{1}{2}\|\phi_m^{n-1}\|_H^2 + \frac{1}{2}\|\phi_m^n - \phi_m^{n-1}\|_H^2 \geq \frac{1}{2}\|\phi_m^m\|_H^2 - \frac{1}{2}\|\phi_m^0\|_H^2.\end{aligned}$$

The proof is completed by generalized integration by parts. □

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