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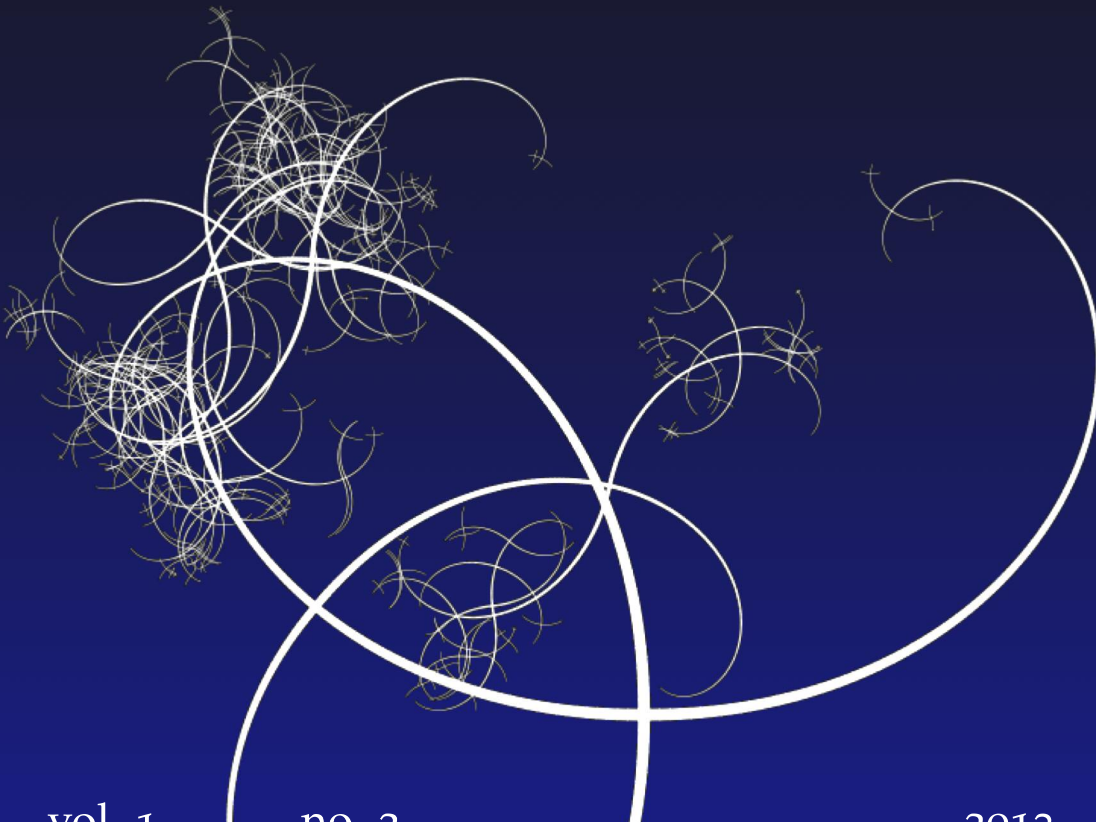

vol. 1 no. 2 2013

MATHEMATICS AND MECHANICS of Complex Systems

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MATHEMATICS AND MECHANICS OF COMPLEX SYSTEMS
Vol. 1, No. 2, 2013
dx.doi.org/10.2140/memocs.2013.1.233

A SUFFICIENT CONDITION FOR A DISCRETE SPECTRUM OF THE KIRCHHOFF PLATE WITH AN INFINITE PEAK

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Sufficient conditions for a discrete spectrum of the biharmonic equation in a two-dimensional peak-shaped domain are established. Different boundary conditions from Kirchhoff's plate theory are imposed on the boundary and the results depend both on the type of boundary conditions and the sharpness exponent of the peak.

1. Motivation

Elliptic boundary value problems on domains which have a Lipschitz boundary and a compact closure, in particular when they generate positive self-adjoint operators, have fully discrete spectra. However, if the domain loses the Lipschitz property or compactness, other situations may occur. It is well-known that for the Dirichlet case boundedness is sufficient but not necessary for having discrete spectrum. See the famous paper [Rellich 1948] or the more recent [Rozenbljum 1972; van den Berg 1984]. On the other hand, for the Neumann problem of the Laplace operator there exist numerous examples of bounded domains such that the spectrum gets a nonempty continuous component (see e.g. [Courant and Hilbert 1953; Maz'ya and Poborchii 2006; 1997; Simon 1992; Hempel et al. 1991]).

The literature on the spectra for the Laplace operator with various boundary conditions on special domains is focused on domains that have a cusp, a finite or infinite peak or horn [van den Berg 1984; Hempel et al. 1991; Jakšić et al. 1992; Davies and Simon 1992; Jakšić 1993; Ivrii 1999; Boyarchenko and Levendorskii 2000; van den Berg and Lianantonakis 2001; Kovařík 2011] or even a rolled horn [Simon 1992].

The criteria in [Adams and Fournier 2003] and [Evans and Harris 1987] for the embedding $H^1(\Omega) \subset L_2(\Omega)$ to be compact show that the Neumann–Laplace

This work is supported by the Russian Foundation for Basic Research (project 12-01-00348). Bakharev is also supported by the Chebyshev Laboratory (Department of Mathematics and Mechanics, Saint Petersburg State University) under the grant 11.G34.31.0026 of the Government of the Russian Federation and grant 6.38.64.2012 of Saint Petersburg State University.
MSC2010: 35P05, 47A10, 74K20.
Keywords: Kirchhoff plate, cusp, peak, discrete and continuous spectra.

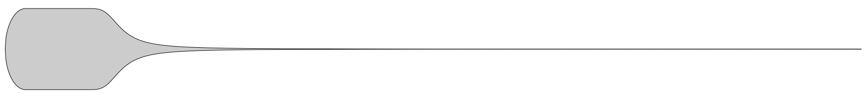
problem on a domain Ω with the infinite peak

$$\Pi_R = \{x = (x_1, x_2) \in \mathbb{R}^2 : x_1 > R,\ -H(x_1) < x_2 < H(x_1)\}, \tag{1}$$

where the function $H > 0$ is smooth and monotone decreasing, has discrete spectrum if and only if

$$\lim_{y\to+\infty} \int_y^{+\infty} \frac{H(\eta)}{H(y)}\, d\eta = 0 \quad \left(\iff \lim_{y\to+\infty} \frac{H(y+\epsilon)}{H(y)} = 0 \ \text{ for any } \epsilon > 0 \right). \tag{2}$$

The function H is assumed to have a first derivative that tends to zero and a bounded second derivative. It will be convenient to use the notation $\Upsilon(y) = (-H(y), H(y))$. Here is an image of such a domain:



The simplest boundary irregularity violating the Lipschitz condition is just the (finite) peak

$$\varpi_R = \{x : 0 < x_1 < R,\ -h(x_1) < x_2 < h(x_1)\}, \tag{3}$$

where $h(x_1) = h_0 x_1^{1+\alpha}$, $h_0 > 0$ and $\alpha > 0$. Nevertheless, the spectrum of the Neumann problem in the domain with this peak stays discrete. See Remark 5.1.

A criterion ([Nazarov 2009]) for having essential spectrum in the Neumann problem for elliptic systems of second order differential equations with a polynomial property is derived in [Nazarov 1999]. In particular it shows that the continuous spectrum of an elastic body with $\alpha \geq 1$ for the peak (3) is nonempty (see [Nazarov 2008; Bakharev and Nazarov 2009]). This phenomenon of generating wave processes in a finite volume, is known experimentally and used in the engineering practice to construct wave dampers, “black holes”, for elastic oscillations (see [Mironov 1988; Krylov and Tilman 2004], etc.).

In this paper we study the spectra of boundary value problems for the Kirchhoff model of a thin elastic plate described by the biharmonic operator Δ^2. The boundary conditions that we consider model the three mechanically most reasonable cases, namely where the lateral sides of the peak are supplied with one of the following three types of the boundary conditions: clamped edge (***Dirichlet***), traction-free edge (***Neumann***) and hinged edge (***Mixed***). In all these cases the spectrum of the problem in a bounded domain with the peak as in (3) is discrete. We derive sufficient conditions for the spectrum to be discrete for the boundary value problem on an unbounded domain with a peak as in (1).

If a sufficient number of Dirichlet conditions are imposed on the lateral sides of the peak (the cases ***D–N***, ***M–M***, ***D–M***, and ***D–D***; see formulas (5)–(7) and

(11), (12)), then the proof that the spectrum is discrete becomes rather simple (Theorem 5). Indeed, it suffices to apply the weighted Friedrich's inequality (13) and to take into account the decay of the quantity $H(y)$ as $y \to +\infty$.

Our main interest concerns the cases ***M–N*** and ***N–N***. By applying weighted inequalities of Hardy type (Lemmas 7 and 8) we obtain a sufficient condition for the case ***N–N*** to have discrete spectrum (Proposition 4). Indeed, as shown in [Adams and Fournier 2003], the second condition in (2) implies a criterion for the compact embedding $H^m(\Omega) \subset L_2(\Omega)$ for all m (we just need $m = 2$). One can also use that approach (Theorem 12) for the case ***M–N***. This approach differs from the one used in [Adams and Fournier 2003; Evans and Harris 1987]. The different argument allows to obtain a condition for having discrete spectrum if one of the peak's edges is traction-free and the other one is hinged (the case ***N–M***; see Theorem 13).

The obtained results essentially differ from each other: under the conditions (11) and also under (12) any decay of H is enough, the case ***M–N*** needs a power decay rate with the exponent $\alpha > 1$, while the case ***N–N*** needs a superexponential decay rate. See Remark 12.1 and 13.1.

2. The Kirchhoff plate model

Assumption 1. *Let Ω be a domain in the plane $\mathbb{R}^2$ with a smooth (of class C^∞) boundary Γ such that, for some $R > 0$ and some monotone decreasing function $H : [R, \infty) \to \mathbb{R}^+$ with $\lim_{t\to\infty} H(t) = 0$,*

(1) $\{(x_1, x_2) \in \Omega;\ x_1 > R\} = \{(x_1, x_2);\ x_1 > R$ *and* $|x_2| < H(x_1)\}$ *and*

(2) $\{(x_1, x_2) \in \Omega;\ x_1 < R\}$ *is bounded.*

We regard Ω as the projection of a thin isotropic homogeneous plate and apply the Kirchhoff theory (see [Mikhlin 1970, §30], [Nazarov 2002, Chapter 7], and so on). So we arrive at the fourth-order differential equation

$$\Delta^2 u(x) = \lambda u(x), \quad x \in \Omega, \tag{4}$$

which describes transverse oscillations of the plate. Here, $u(x)$ is the plate deflection, and λ a spectral parameter proportional to the square of the oscillation frequency.

The following sets of boundary conditions have a clear physical interpretation (see [Mikhlin 1970, §30], [Gazzola et al. 2010, §1.1], and so on):

***(D)*:** Dirichlet for a clamped edge:

$$u(x) = \partial_n u(x) = 0, \quad x \in \Gamma_D. \tag{5}$$

(*N*): Neumann for a traction-free edge:

$$\begin{cases} \partial_n \Delta u(x) - (1-\nu)(\partial_s \varkappa(x) \partial_s u(x) - \partial_s^2 \partial_n u(x)) = 0, \\ \Delta u(x) - (1-\nu)(\partial_s^2 u(x) + \varkappa(x) \partial_n u(x)) = 0, \end{cases} \qquad x \in \Gamma_N. \tag{6}$$

(*M*): Mixed for a hinged edge:

$$u = \Delta u(x) - (1-\nu)\varkappa(x)\partial_n u(x) = 0, \quad x \in \Gamma_M. \tag{7}$$

Here, ∂_n and ∂_s stand for the normal and tangential derivatives, $\varkappa(x)$ is the signed curvature of the contour Γ at the point $x \in \Gamma$ positive for convex boundary parts, and $\nu \in [0, 1/2)$ is the Poisson ratio. Finally, Γ_D, Γ_N, and Γ_M are the unions of finite families of open curves and $\Gamma = \overline{\Gamma_D} \cup \overline{\Gamma_N} \cup \overline{\Gamma_M}$, two of which may be empty. In what follows it is convenient to use the notation $y = x_1$ and $z = x_2$.

The general properties of the spectra depend on which of the boundary conditions (5)–(7) are imposed on the upper (+) and lower (−) sides,

$$\Sigma_R^{\pm} = \{x : y > R,\ z = \pm H(y)\},$$

of the peak. Let us give a precise statement. We define a symmetric bilinear form on $H^2(\Omega)$ by

$$a(u,u) = \int_\Omega \left(\left| \frac{\partial^2 u}{\partial x_1^2} \right|^2 + \left| \frac{\partial^2 u}{i} \partial x_2^2 \right|^2 + 2(1-\nu) \left| \frac{\partial^2 u}{\partial x_1 \partial x_2} \right|^2 + 2\nu \frac{\partial^2 u}{\partial x_1^2} \frac{\partial^2 u}{\partial x_2^2} \right) dx \tag{8}$$

and $a(u, v) = \frac{1}{4} a(u+v, u+v) - \frac{1}{4} a(u-v, u-v)$. Then $\frac{1}{2} a(u,u)$ is the elastic energy stored in the plate. Since one directly verifies that

$$a(u,u) \geq (1-\nu) \sum_{j,k=1}^{2} \int_\Omega \left| \frac{\partial^2 u}{\partial x_j \partial x_k} \right|^2 dx, \tag{9}$$

the bilinear form $a(\cdot\,,\cdot)$ is nonnegative.

Definition 2. Let $\mathbb{H}$ be the subspace of functions $u \in H^2(\Omega)$, satisfying the conditions (5) on Γ_D and $u = 0$ on Γ_M in the sense of traces.

By [Birman and Solomyak 1980, §10.1] $\mathbb{H}$ is a Hilbert space with scalar product $a(\cdot\,,\cdot) + (\cdot\,,\cdot)_\Omega$, where $(\cdot\,,\cdot)_\Omega$ is the standard scalar product in the Lebesgue space $L_2(\Omega)$. Moreover, Theorem 2 of [Birman and Solomyak 1980, §10.1] implies that there exists a unique (unbounded) self-adjoint operator $A : D(A) \subset L_2(\Omega) \to L_2(\Omega)$ with $D(A) \subset D(A^{1/2}) = \mathbb{H}$ such that

$$(Au, v)_\Omega = a(u, v) \quad \text{for all } v \in \mathbb{H}. \tag{10}$$

Note that the eventually remaining boundary conditions in (5)–(7) appear in $D(A) \subset H^4(\Omega)$ as intrinsic natural boundary conditions from (9)–(8), see again [Mikhlin 1970, §30], [Gazzola et al. 2010, §1.1] etc.

Definition 3. By the spectrum for (4)–(7) we will mean $\sigma(A)$ with A defined in (10).

Since a is nonnegative, the spectrum $\sigma(A)$ belongs to $[0, \infty)$.

As a direct consequence of known results we may state the following.

Proposition 4. *If* (2) *holds, then the spectrum of the Equation* (4), *with either of the above boundary condition on the sides of the peak, is discrete.*

Proof. By [Birman and Solomyak 1980, §10.1 Theorem 5] the spectrum is discrete if and only if the embedding $\mathbb{H} \hookrightarrow L_2(\Omega)$ is compact. By [Adams and Fournier 2003; Evans and Harris 1987] one knows that $H^2(\Omega) \hookrightarrow L_2(\Omega)$ is compact whenever (2) holds true and $\mathbb{H} \subset H^2(\Omega)$. □

3. Simple cases: *D–N, M–M*, *D–M* and *D–D,*

Theorem 5. *Suppose Ω is as in Assumption 1 and suppose that the boundary conditions for problem* (4), *as given in* (5)-(7) *contain one of the cases* (11) *or* (12). *Then the spectrum is discrete.*

Remark 5.1. By a similar reasoning, we may conclude that in the bounded domain ω, with the peak as in (3), Equation (4) has discrete spectrum for any set of conditions (5)–(7) on the arc $\partial\omega \setminus \mathbb{O}$. This fact follows from the inequality (see [Nazarov and Taskinen 2008]):

$$\left\| |x|^{-1} u; L_2(\omega) \right\|^2 \le c \left(\|\nabla u; L_2(\omega)\|^2 + \|u; L_2(\omega \setminus \varpi_R)\|^2 \right).$$

Proof. By assumption the boundary conditions provide at least one of the following two groups of relations:

$$u = 0 \qquad \text{on } \Sigma_R^+ \cup \Sigma_R^-; \tag{11}$$

$$u = \partial_n u = 0 \quad \text{on } \Sigma_R^+ \text{ or on } \Sigma_R^-. \tag{12}$$

In both cases (11) and (12) the following version of Friedrich's inequality is valid:

$$\int_{\Upsilon(y)} \left| \frac{\partial^2 u}{\partial z^2}(y, z) \right|^2 dz \ge \frac{c}{H(y)^4} \int_{\Upsilon(y)} |u(y, z)|^2 \, dz. \tag{13}$$

Therefore,

$$a(u, u) \ge c \int_\Omega H(y)^{-4} \, |u(x)|^2 \, dx. \tag{14}$$

The embedding operator $\gamma : \mathbb{H} \to L_2(\Omega)$ can be represented as the sum $\gamma_0 + \gamma_\rho$, where $\rho \geq R$ is large and positive, $\gamma_0 = \gamma - \gamma_\rho$, and γ_ρ contains the operator of multiplication by the characteristic function of Π_ρ. The operator γ_0 is compact, and the norm of γ_ρ, in view of (14), does not exceed $c \max\{H(x_1)^{-2};\ x_1 \geq \rho\}$. Since the function H decays, this quantity goes to zero when $\rho \to +\infty$, i.e., the operator γ can be approximated by compact operators in the operator norm. Thus γ is compact and the result is proved. □

4. Auxiliary inequalities

First of all we prove some one-dimensional weighted inequalities, two of which are of Hardy type involving a weight function h as follows.

Assumption 6. *Let h be a positive weight function of class C^2 on $[0, +\infty)$ such that*

- $\displaystyle\int_0^\infty h(s)\,ds < \infty$ *and*
- *for some large T, we have $h'(t) < 0$ and $h''(t) > 0$ for $t \in (T, \infty)$.*

Throughout this section h is supposed to satisfy this assumption.

Lemma 7. *If U is differentiable for $y \geq R$ and $U(R) = 0$, then*

$$\int_R^{+\infty} h(y)\,|U(y)|^2\,dy \leq \int_R^{+\infty} F_h(y)\,\left|\partial_y U(y)\right|^2\,dy,$$

where

$$F_h(y) = \frac{4}{h(y)}\left(\int_y^{+\infty} h(\tau)\,d\tau\right)^2.$$

Proof. Using the Cauchy–Bunyakovsky–Schwarz inequality, we have

$$\begin{aligned}
&\int_R^{+\infty} h(y)\,|U(y)|^2\,dy \\
&\quad = 2\int_R^{+\infty} h(y)\int_R^{y} \partial_t U(t)U(t)\,dt\,dy \\
&\quad \leq 2\int_R^{+\infty}\int_t^{+\infty} h(y)\left|\partial_t U(t)U(t)\right|dy\,dt \\
&\quad \leq 2\left(\int_R^{+\infty} h(t)|U(t)|^2\,dt\right)^{1/2}\left(\int_R^{+\infty} h(t)^{-1}\left(\int_t^{+\infty} h(y)\,dy\right)^2 |\partial_t U(t)|^2\,dt\right)^{1/2},
\end{aligned}$$

and the result follows through division by a common factor. □

Lemma 8. *If U is differentiable for $y \geq R$ and $U(R) = 0$, then*

$$\int_R^{+\infty} h(y) \left|\partial_y U(y)\right|^2 \, dy \geq \int_R^{+\infty} G_h(y) \, |U(y)|^2 \, dy,$$

where

$$G_{h,R}(y) = \frac{1}{4h(y)} \left(\int_R^y h(\tau)^{-1} \, d\tau \right)^{-2}.$$

Proof. For functions v with $v(0) = 0$ the Hardy inequality tells us that

$$\int_0^{+\infty} t^{-2} \, |v(t)|^2 \, dt \leq 4 \int_0^{+\infty} |\partial_t v(t)|^2 \, dt.$$

We make the change $t \mapsto y \in [R, +\infty)$ where $t = \int_R^y h(\tau)^{-1} \, d\tau$, and set $U(y) = v(t)$. Then $\partial_t v(t) = h(y) \partial_y U(y)$ leads to the desired estimate. $\square$

Corollary 9. *If the function U is twice differentiable for $y \geq R$ and $U(R) = U'(R) = 0$, then*

$$\int_R^{+\infty} h(t) \left|\partial_t^2 U(t)\right|^2 \, dt \geq W_h(R) \int_R^{+\infty} h(t) \, |U(t)|^2 \, dt \tag{15}$$

is valid with

$$\begin{aligned} W_h(R) &:= \inf_{t \in [R, +\infty)} \frac{G_{h,R}(t)}{F_h(t)} \\ &= \inf_{t \in [R, +\infty)} \frac{1}{16} \left(\int_R^t h(\tau)^{-1} \, d\tau \right)^{-2} \left(\int_t^{\infty} h(\tau) \, d\tau \right)^{-2}. \end{aligned} \tag{16}$$

Lemma 10. *We have*

$$\inf_{t \in [R, +\infty)} \frac{G_{h^3,R}(t)}{h(t) h'(t)^2} \geq 1. \tag{17}$$

Suppose moreover that

$$\lim_{t \to +\infty} \partial_t (\log h(t)) = -\infty. \tag{18}$$

Then

$$W_h(R) \to +\infty \quad \text{and} \quad W_{h^3}(R) \to +\infty \quad \text{for } R \to \infty.$$

Proof. Since $-h'(\tau) \geq -h'(t) > 0$ for $\tau < t$, we find

$$\begin{aligned} \frac{G_{h^3,R}(t)}{h(t) h'(t)^2} &= \frac{1}{4h'(t)^2 h(t)^4} \left(\int_R^t h(\tau)^{-3} d\tau \right)^{-2} \geq \frac{1}{4h(t)^4} \left(- \int_R^t h'(\tau) h(\tau)^{-3} d\tau \right)^{-2} \\ &= \frac{1}{4h(t)^4} \left(\frac{1}{2h(t)^2} - \frac{1}{2h(R)^2} \right)^{-2} \geq 1. \end{aligned}$$

Since (18) equals $h'(t)/h(t) \to -\infty$ for $t \to \infty$, we find that for $t \to \infty$ both

$$h(t)\left(\int_t^\infty h(\tau)\,d\tau\right)^{-1} \to \infty \quad \text{and} \quad \frac{1}{h(t)}\left(\int_R^t \frac{1}{h(\tau)}d\tau\right)^{-1} \to \infty.$$

Hence

$$\frac{G_{h,R}(t)}{F_h(t)} \to \infty \quad \text{as } t \to \infty,$$

and since the quotient also goes to infinity for $t \downarrow R$, it has a minimum in some $t_R \in (R, \infty)$. Calculating $\bigl(G_{h,R}(t)/F_h(t)\bigr)' = 0$ we find

$$\frac{1}{h(t)}\int_t^\infty h(\tau)\,d\tau - h(t)\int_R^t h(\tau)^{-1}\,d\tau = 0.$$

Hence

$$\inf_{t\in[R,\infty)}\left(\int_R^t h(\tau)^{-1}\,d\tau \int_t^\infty h(\tau)\,d\tau\right)^{-1} = h(t_R)^2\left(\int_{t_R}^\infty h(\tau)\,d\tau\right)^{-2},$$

which goes to infinity for $R \to \infty$ since $t_R > R$. The claim for $W_h(R)$ follows. The same argument holds true for $W_{h^3}(R)$. □

5. Estimates for a traction-free boundary

We assume that the Neumann boundary conditions (6) are imposed at the both sides of the peak (1). Let us describe for $\rho \to +\infty$ the behavior of the multiplier $K(\rho)$ in the inequality

$$K(\rho)\int_{\Pi_\rho} |u(y,z)|^2\,dy \le \|u;\, H^2(\Omega)\|^2, \quad u \in H^2(\Omega). \tag{19}$$

If $K(\rho)$ increases unboundedly as $\rho \to +\infty$ then, as above, Theorem 10.1.5 of [Birman and Solomyak 1980] ensures that the spectrum of the equation (4)–(7) stays discrete even in the case both sides of the peak are supplied with the traction-free boundary conditions (N) and, moreover, for any other boundary conditions from the list (5)–(7).

Proposition 11. *Suppose that Assumption 6 is satisfied for $h = H$ and that*

$$\lim_{t\to\infty} \partial_t\bigl(\log H(t)\bigr) = -\infty.$$

Then for ρ sufficiently large (19) *holds true with*

$$K(\rho) = c\min\{H^{-4}(\rho),\, W_H(\rho),\, W_{H^3}(\rho)\}. \tag{20}$$

Proof. It is sufficient to check the inequality (19) for smooth functions which vanish for $y < \rho$. We use the representation

$$u(x) = u(y, z) = u_0(y) + zu_1(y) + u^\perp(y, z)$$

where, for $y > \rho$, the component $u^\perp$ is subject to the orthogonality conditions

$$\begin{aligned} &\int_{\Upsilon(y)} u^\perp(y, z)\, dz = 0, \\ &\int_{\Upsilon(y)} \partial_z u^\perp(y, z)\, dz = u^\perp(y, H(y)) - u^\perp(y, -H(y)) = 0. \end{aligned} \tag{21}$$

Let us process the integrals on the right-hand side of

$$\int_{\Pi_\rho} |\nabla_x^2 u(x)|^2\, dx = I_1 + 4I_2 + I_3, \tag{22}$$

where

$$I_1 := \int_{\Pi_\rho} |\partial_z^2 u(x)|^2 dx, \quad I_2 := \int_{\Pi_\rho} |\partial_y \partial_z u(x)|^2 dx, \quad I_3 := \int_{\Pi_\rho} |\partial_y^2 u(x)|^2 dx.$$

Since $I_1 = \int_\rho^{+\infty} \int_{\Upsilon(y)} |\partial_z^2 u^\perp(y, z)|^2\, dz\, dy$ and since, by the orthogonality conditions in (21), inequality (13) holds here also, we find that

$$I_1 \geq c \int_{\Pi_\rho} H(y)^{-4} \left|u^\perp(x)\right|^2\, dx. \tag{23}$$

For the last term in (22) we have

$$I_3 = \int_{\Pi_\rho} \left|\partial_y^2 u_0(y) + z\partial_y^2 u_1(y) + \partial_y^2 u^\perp(y, z)\right|^2 dx \geq J_1 + J_2 + 2J_3 + 2J_4, \tag{24}$$

where

$$\begin{aligned} J_1 &= g \int_{\Pi_\rho} |\partial_y^2 u_0(y)|^2 dx, & J_3 &= g \int_{\Pi_\rho} \partial_y^2 u_0(y) \partial_y^2 u^\perp(y, z)\, dx, \\ J_2 &= g \int_{\Pi_\rho} |z\partial_y^2 u_1(y)|^2 dx, & J_4 &= g \int_{\Pi_\rho} z\partial_y^2 u_1(y) \partial_y^2 u^\perp(y, z)\, dx. \end{aligned}$$

We readily notice that according to the inequality (15) the estimates

$$J_1 \geq W_H(\rho) \int_\rho^{+\infty} 2H(y)\, |u_0(y)|^2\, dy = W_H(\rho) \int_{\Pi_\rho} |u_0(y)|^2\, dx \tag{25}$$

and

$$J_2 = \frac{2}{3}\int_\rho^{+\infty} H^3(y)|\partial_y^2 u_1(y)|^2\,dy \geq \frac{2}{3}\,W_{H^3}(\rho)\int_\rho^{+\infty} H^3(y)|u_1(y)|^2\,dy$$
$$= W_{H^3}(\rho)\int_{\Pi_\rho} |z u_1(y)|^2\,dx \qquad (26)$$

are fulfilled. For our purpose we need $W_H(\rho)\to+\infty$ and $W_{H^3}(\rho)\to+\infty$ for $\rho\to\infty$ and this we will assume.

Besides, by the Cauchy–Bunyakovsky–Schwarz inequality, we have

$$|J_3| \leq J_1^{1/2}\biggl(\int_\rho^{+\infty}\frac{1}{2H(y)}\biggl(\int_{\Upsilon(y)}\partial_y^2 u^\perp(y,z)\,dz\biggr)^2 dy\biggr)^{1/2}.$$

We now deal with the inner integral in z in the last expression. To this end, we take into account the orthogonality conditions (21) and the trace inequality. We differentiate the first equality in (21) twice with respect to y and obtain

$$\sum_\pm\bigl(2\partial_y u^\perp(y,\pm H(y))\partial_y H(y)+u^\perp(y,\pm H(y))\partial_y^2 H(y)\pm\partial_z u^\perp(y,\pm H(y))(\partial_y H(y))^2\bigr)$$
$$+\int_{\Upsilon(y)}\partial_y^2 u^\perp(y,z)\,dz = 0.$$

Thus,

$$\biggl(\int_{\Upsilon(y)}\partial_y^2 u^\perp(y,z)\,dz\biggr)^2$$
$$\leq c\sum_\pm\Bigl(\bigl|\partial_z u^\perp(y,\pm H(y))\bigr|^2|\partial_y H(y)|^4+\bigl|u^\perp(y,\pm H(y))\bigr|^2|\partial_y^2 H(y)|^2$$
$$+\bigl|\partial_y u^\perp(y,\pm H(y))\bigr|^2|\partial_y H(y)|^2\Bigr).$$

For the first two terms between the brackets we use the trace inequality

$$\bigl|\partial_z u^\perp(y,\pm H(y))\bigr|^2|H(y)|^2+\bigl|u^\perp(y,\pm H(y))\bigr|^2\leq c\,|H(y)|^3\int_{\Upsilon(y)}\bigl|\partial_z^2 u^\perp(y,z)\bigr|^2 dz.$$

For the third term, we write down the chain of inequalities

$$\bigl|\partial_y u^\perp(y,\pm H(y))\bigr|^2$$
$$\leq cH(y)\int_{\Upsilon(y)}\bigl|\partial_y\partial_z u^\perp(y,z)\bigr|^2\,dz+\,c|H(y)|^{-2}\biggl(\int_{\Upsilon(y)}\partial_y u^\perp(y,z)\,dz\biggr)^2$$
$$\leq cH(y)\int_{\Upsilon(y)}\bigl|\partial_y\partial_z u^\perp(y,z)\bigr|^2\,dz$$
$$+\,2c\biggl(\frac{\partial_y H(y)}{H(y)}\biggr)^2\Bigl(\bigl|u^\perp(y,H(y))\bigr|^2+\bigl|u^\perp(y,-H(y))\bigr|^2\Bigr).$$

As a result, we find that

$$\left|\int_{\Upsilon(y)} \partial_y^2 u^\perp(y,z)\,dz\right|^2 \le c\big(|\partial_y H(y)|^4 H(y) + |\partial_y^2 H(y)|^2 |H(y)|^3\big) \int_{\Upsilon(y)} \left|\partial_z^2 u^\perp(y,z)\right|^2 dz + c\,|\partial_y H(y)|^2\,|H(y)| \int_{\Upsilon(y)} \left|\partial_y \partial_z u^\perp(y,z)\right|^2 dz.$$

The final inequality for the integral J_3 takes the form

$$|J_3| \le c_1(\rho) J_1^{1/2} I_1^{1/2} + c_2(\rho) J_1^{1/2} K_1^{1/2} \tag{27}$$

where $K_1 = \left\|\partial_{yz}^2 u^\perp;\, L_2(\Pi_\rho)\right\|^2$ and

$$c_1(\rho) = c \sup_{y\in[\rho,+\infty)} \big(|\partial_y H(y)|^2 + |\partial_y^2 H(y)|\,|H(y)|\big),$$
$$c_2(\rho) = c \sup_{y\in[\rho,+\infty)} |\partial_y H(y)|.$$

Both suprema tend to 0 for $\rho \to +\infty$. A similar argument shows that

$$|J_4| \le c_1(\rho) J_2^{1/2} I_1^{1/2} + c_2(\rho) J_2^{1/2} K_1^{1/2}. \tag{28}$$

It remains to process the second term in (22), that is,

$$\begin{aligned} I_2 &= \int_{\Pi_\rho} \left|z\partial_y u_1(y) + \partial_y \partial_z u^\perp(y,z)\right|^2 dx \\ &= \underbrace{\int_{\Pi_\rho} \left|\partial_y \partial_z u^\perp(y,z)\right|^2 dx}_{=\,K_1} + \underbrace{\int_{\Pi_\rho} \left|\partial_y u_1(y)\right|^2 dx}_{=:\,K_2} + 2 \underbrace{\int_{\Pi_\rho} \partial_y u_1(y) \partial_y \partial_z u^\perp(y,z)\,dx}_{=:\,K_3}. \end{aligned}$$

So it follows that

$$K_1 = I_2 - K_2 - 2K_3 \le I_2 + 2\,|K_3|\,. \tag{29}$$

We continue by estimating the integral K_3:

$$\begin{aligned} |K_3| &= \left|\int_\rho^{+\infty} \partial_y u_1(y) \int_{\Upsilon(y)} \partial_y \partial_z u^\perp(y,z)\,dz\,dy\right| \\ &\le \left(\int_\rho^{+\infty} G_{H^3,\rho}(y) |\partial_y u_1(y)|^2\,dy\right)^{\frac12} \left(\int_\rho^{+\infty} G_{H^3,\rho}(y)^{-1} \left|\int_{\Upsilon(y)} \partial_y \partial_z u^\perp(y,z)\,dz\right|^2 dy\right)^{\frac12} \\ &\le c J_2^{1/2} \left(\int_\rho^{+\infty} G_{H^3,\rho}(y)^{-1} \left|\int_{\Upsilon(y)} \partial_y \partial_z u^\perp(y,z)\,dz\right|^2 dy\right)^{1/2}. \end{aligned}$$

Differentiating the second formula (21) with respect to y yields

$$\int_{\Upsilon(y)} \partial_y \partial_z u^{\perp}(y,z)\,dz + \sum_{\pm} \partial_z u^{\perp}(y, \pm H(y))\partial_y H(y) = 0.$$

By the trace inequality we find that

$$\begin{aligned}\left|\int_{\Upsilon(y)} \partial_y \partial_z u^{\perp}(y,z)\,dz\right|^2 &= \Big(\sum_{\pm} \partial_z u^{\perp}(y, \pm H(y))\Big)^2 |\partial_y H(y)|^2 \\ &\le c|H(y)||\partial_y H(y)|^2 \int_{\Upsilon(y)} |\partial_z^2 u^{\perp}(y,z)|^2\,dz.\end{aligned}$$

Thus, from the relation (18), which implies (17), we get

$$|K_3| \le c \sup_{y\in[\rho,+\infty)} \big\{|G_{H^3,\rho}(y)|^{-1/2} |\partial_y H(y)||H(y)|^{1/2}\big\} J_2^{1/2} I_1^{1/2} \le c J_2^{1/2} I_1^{1/2}. \tag{30}$$

We find by combining (29) and (30) that

$$K_1 \le I_2 + c J_2^{1/2} I_1^{1/2}$$

and so (27) and (28) yield, respectively,

$$|J_3| \le c_1(\rho) J_1^{1/2} I_1^{1/2} + c_2(\rho) J_1^{1/2} \big(I_2 + c J_2^{1/2} I_1^{1/2}\big)^{1/2}, \tag{31}$$

$$|J_4| \le c_1(\rho) J_2^{1/2} I_1^{1/2} + c_2(\rho) J_2^{1/2} \big(I_2 + c J_2^{1/2} I_1^{1/2}\big)^{1/2}. \tag{32}$$

Using first (22) and (24), next (31) and (32) for ρ large enough, and finally (23), (25) and (26) we conclude that indeed

$$\begin{aligned}\left\|\nabla_x^2 u;\, L_2(\Pi_\rho)\right\|^2 &= I_1 + 4I_2 + I_3 \\ &\ge I_1 + 4I_2 + J_1 + J_2 + 2J_3 + 2J_4 \\ &\ge \tfrac{1}{2}(I_1 + 4I_2 + J_1 + J_2) \ge \tfrac{1}{2}(I_1 + J_1 + J_2) \\ &\ge c \min\{H^{-4}(\rho),\, W_H(\rho),\, W_{H^3}(\rho)\} \left\|u;\, L_2(\Pi_\rho)\right\|^2,\end{aligned}$$

whenever ρ is large enough. □

6. Traction-free boundaries: *N–N*

Theorem 12. *Suppose that H satisfies Assumption 6 with $h = H$ and that*

$$\lim_{t\to\infty} \partial_t \left(\log H(t)\right) = -\infty. \tag{33}$$

Then the embedding $H^2(\Omega) \hookrightarrow L_2(\Omega)$ is compact and the spectrum of the problem (4) *with the Neumann boundary conditions* (6) *on both sides of the peak is discrete.*

Proof. By Proposition 11, $K(\rho)$ can be estimated as in (20). Assumption 6 implies that Lemma 10 holds true and hence $K(\rho) \to +\infty$ for $\rho \to +\infty$. One concludes as in the proof of Theorem 5 through an approximation by compact operators. □

Remark 12.1. Note that the functions $H(y) = y^{-\alpha}$ and $H(y) = \exp(-\alpha y)$, $\alpha > 0$, do not satisfy the requirement in (2) or (33). The functions $H(y) = \exp(-y^{1+\alpha})$ with $\alpha > 0$ however do.

7. An incomplete Dirichlet condition: *M–N*

In this section the boundary conditions only contain a single stable condition

$$u(x) = 0, \quad x \in \Sigma_\rho^+ \ \text{(or } x \in \Sigma_\rho^-\text{)}. \tag{34}$$

Theorem 13. *Suppose that* $\lim_{y\to\infty} H(y)^{-3} G_H(y) = +\infty$. *Then the problem in* (4) *with the boundary condition as in* (34) *has discrete spectrum.*

Remark 13.1. The functions $H(y) = y^{-1-\alpha}$ with $\alpha > 0$ meet the condition in Theorem 13.

Proof. By (34), Friedrich's inequality holds on the section $\Upsilon(y)$ and, consequently,

$$\left\| \mathscr{A}\, \partial_z u;\, L_2(\Pi_\rho) \right\|^2 \ge c \left\| \mathscr{A}\, H^{-2} u;\, L_2(\Pi_\rho) \right\|^2$$

for every positive weight function $y \mapsto \mathscr{A}(y)$. The function $v = \partial_z u$ can be represented as the sum $v(x) = v_0(y) + v^\perp(x)$ where, for $y > \rho$, the component $v^\perp$ satisfies the first condition in (21). Therefore,

$$\begin{aligned}
&\int_{\Pi_\rho} |\nabla_x^2 u(x)|^2\, dx \\
&\ge \int_{\Pi_\rho} |\nabla_x v(x)|^2\, dx \\
&\ge \int_\rho^{+\infty} 2H(y)|\partial_y v_0(y)|^2\, dy + \int_{\Pi_\rho} |\partial_z v^\perp(y,z)|^2\, dx + 2\int_{\Pi_\rho} \partial_y v_0(y) \partial_y v^\perp(y,z)\, dx \\
&=: I_4 + I_5 + 2I_6.
\end{aligned}$$

Setting $Z_H(y) = H(y)^{-1} G_H(y)$, we get

$$\begin{aligned}
I_4 \ge \int_\rho^{+\infty} 2G_H(y)|v_0(y)|^2 dy &\ge \int_\rho^{+\infty} 2Z_H(y)H(y)|v_0(y)|^2 dy \\
&= \|Z_H v_0;\, L_2(\Pi_\rho)\|^2.
\end{aligned}$$

Friedrich's inequality implies

$$I_5 = \|\partial_z v^\perp;\, L_2(\Pi_\rho)\| \ge c\|H^{-2} v^\perp;\, L_2(\Pi_\rho)\|^2.$$

Furthermore,

$$I_6 = \int_{\Pi_\rho} \partial_y v_0(y) \partial_y v^\perp(y,z)\, dx \le I_5^{1/2} \left(\int_\rho^{+\infty} \frac{1}{2H(y)} \left| \int_{\Upsilon(y)} \partial_y v^\perp(y,z)\, dz \right|^2 \right)^{1/2}$$
$$\le I_5^{1/2} \left(\int_\rho^{+\infty} \frac{1}{2H(y)} \left| v^\perp(y, H(y)) \partial_y H(y) - v^\perp(y, -H(y)) \partial_y H(y) \right|^2 \right)^{1/2}.$$

Thus $I_6 \le c |\partial_y H(\rho)| I_5^{1/2} I_6^{1/2}$ holds and hence

$$\left\| \nabla_x^2 u;\, L_2(\Pi_\rho) \right\|^2 \ge c \left\| \min\{H^{-4};\, H^{-3} G_H\} u;\, L_2(\Pi_\rho) \right\|^2.$$

Compactness and hence the discrete spectrum follow from the assumption on H. □

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Received 7 Mar 2012. Revised 21 Jul 2012. Accepted 28 Aug 2012.

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MATHEMATICS AND MECHANICS OF COMPLEX SYSTEMS

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MEMOCS (ISSN 2325-3444 electronic, 2326-7186 printed) is a journal of the International Research Center for the Mathematics and Mechanics of Complex Systems at the Università dell'Aquila, Italy.

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MATHEMATICS AND MECHANICS OF COMPLEX SYSTEMS
vol. 1 no. 2 2013

MEMOCS is a journal of the International Research Center for the Mathematics and Mechanics of Complex Systems at the Università dell'Aquila, Italy.

2326-7186(2013)1:2;1-H