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# DELAMINATED THIN ELASTIC INCLUSIONS INSIDE ELASTIC BODIES

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We propose a model for a two-dimensional elastic body with a thin elastic inclusion modeled by a beam equation. Moreover, we assume that a delamination of the inclusion may take place resulting in a crack. Nonlinear boundary conditions are imposed at the crack faces to prevent mutual penetration between the faces. Both variational and differential problem formulations are considered, and existence of solutions is established. Furthermore, we study the dependence of the solution on the rigidity of the embedded beam. It is proved that in the limit cases corresponding to infinite and zero rigidity, we obtain a rigid beam inclusion and cracks with nonpenetration conditions, respectively. Anisotropic behavior of the beam is also analyzed.

## 1. Introduction

The enforcement of elastic bodies using thin inclusions is a field of broad interest in solid and structural mechanics. The interplay between elastic fibers and matrix materials in general is important also in biological and medical problems involving tissues, muscles, tendon-couplings, etc. There are a number of different approaches in modeling such composites. The most classical approach assumes inextensible fibers; see, for example, [Saccomandi and Beatty 2002]. In this context, the modeling is often based directly on a finite elements. Another approach is based on a modeling of the matrix material as a supporting layer, like a Winkler support; see, for example, [Nassar and Hassen 1987]. Here, the fiber is represented by an Euler–Bernoulli beam. A very natural approach is based on asymptotic analysis [Argatov and Nazarov 1999]. Here, the embedded beams are taken with a small thickness parameter and the elastic layer is infinite. The limiting problem relates to a Winkler or Pasternak-type model. Finally, there are attempts to model hybrid partial differential equations coupling, say, the two-dimensional wave equation to a one-dimensional wave equation, using proper transmission conditions; see [Koch and Zuazua 2006].

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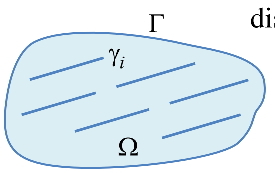
*Keywords:* thin inclusion, nonlinear boundary conditions, nonpenetration, crack, variational inequality.

In general, the terminology "thin inclusion" is used in cases where the dimension of the inclusion is less than that of the body. Among thin inclusions we can distinguish between rigid and elastic ones. Moreover, thin inclusions have a tendency to delaminate from the matrix material, thereby introducing cracks. A mathematical theory should be capable of consistently handling these different aspects. Therefore, in order to analyze composite materials one has to consider mathematical models of deformable bodies with elastic and rigid inclusions and cracks. In such a case, new types of boundary value problems and boundary conditions appear. Cracks also can be viewed as thin inclusions of zero rigidity. There are different approaches to modeling cracks in solids. The classical models are characterized by linear boundary conditions at the crack faces [Kozlov and Maz'ya 1991; Grisvard 1992; Nazarov and Plamenevsky 1994]. These linear models allow the opposite crack faces to penetrate each other which demonstrates a shortcoming of the model from a mechanical standpoint. For a discussion of singularities at the crack tip see, for example, [Kozlov and Maz'ya 1991; Nazarov and Plamenevsky 1994]. In recent years, a crack theory with nonpenetration conditions at the crack faces has been under active study. This theory is characterized by inequality-type boundary conditions which leads to free boundary value problems. The book [Khludnev and Kovtunenko 2000] contains results on crack models with the nonpenetration conditions for a wide class of constitutive laws. Elastic behavior of bodies with cracks and inequality-type boundary conditions is analyzed in [Khludnev 2010a]. In particular, the differentiability of energy functionals with respect to crack length is investigated. Finding the derivatives of the energy functionals is important from the standpoint of the Griffith rupture criterion; see [Kovtunenko 2003; Rudoy 2007; Frémiot et al. 2009; Khludnev et al. 2010]. The asymptotic behavior of the solution near crack tips was analyzed in [Khludnev and Kozlov 2008]. Existence theorems and qualitative properties of solutions in equilibrium problems for elastic bodies with thin and volume rigid inclusions can be found in [Khludnev et al. 2009; 2010b; Neustroeva 2009; Khludnev and Leugering 2010; 2011; Rudoy 2011; Rotanova 2011]. For behavior near rigid inclusion tips, see [Itou et al. 2012].

We propose a new model of a thin elastic inclusion inside of an elastic body. We consider a planar elastic body $\Omega$ with embedded elastic fibers $\gamma_i$, $i = 1, \ldots n$, as shown in the figure below. However, in this article we do not focus on the distribution of such fibers in such a domain but rather on the mathematical modeling and analysis of immersed fibers to begin with. We, therefore, without loss of generality, concentrate on a single fiber $\gamma$ embedded into $\Omega$ with boundary $\Gamma$.

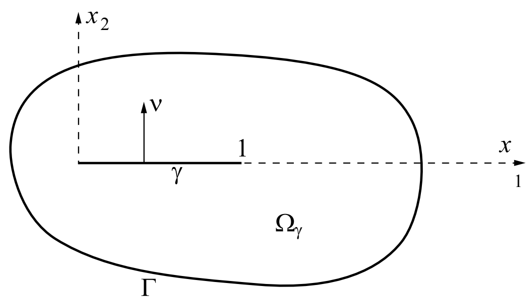



The mechanical behavior of the inclusion is modeled by the Kirchhoff–Love equations. The inclusion may be delaminated, providing therefore the presence of a crack. To exclude a mutual penetration between the crack faces, nonlinear boundary conditions of inequality type are considered along the cracks. Different problem formulations are proposed which are shown to be equivalent to each other. We prove the existence and uniqueness of solutions and analyze limit cases describing the passage to infinity and zero of the rigidity parameter of the inclusion. In particular, the models of rigid beam inclusions, semirigid beam inclusions, and crack models with the nonpenetration conditions are obtained in the limits.

The paper is organized as follows. In Section 2, we provide the problem formulation and handle the case where no delamination takes place. In Section 3, we derive the model for a one-sided delamination along the fiber. In Sections 4 and 5 we study the limiting model, as the rigidity of the fiber tends to infinity and zero, respectively. Sections 6 and 7 are concerned with two-sided delamination along the fiber and fibers that exhibit different stiffness properties with respect to longitudinal and vertical displacements. Oblique and kinking fibers as well as branching fibers can also be handled. Moreover, other beam models can be considered. However, this is subject to a forthcoming publication.

## 2. Problem formulation: the case without delamination

Denote by $\Omega \subset \mathbb{R}^2$ a bounded domain with Lipschitz boundary $\Gamma$ such that $\bar{\gamma} \subset \Omega$, $\gamma = (0, 1) \times \{0\}$. Denote by $\nu = (0, 1)$ a unit normal vector to $\gamma$, $\tau = (1, 0)$, and set $\Omega_\gamma = \Omega \setminus \bar{\gamma}$; see figure.




In what follows, the domain $\Omega_\gamma$ represents a region with an elastic material, and $\gamma$ is an elastic inclusion with specified properties. In particular, we consider $\gamma$ as a Kirchhoff–Love or Euler–Bernoulli beam incorporated in the elastic body. Let $A = \{a_{ijkl}\}$, $i, j, k, l = 1, 2$, be a given elasticity tensor with the usual properties of symmetry and positive definiteness,

$$a_{ijkl} = a_{jikl} = a_{klij}, \quad i, j, k, l = 1, 2, \quad a_{ijkl} \in L^\infty(\Omega),$$
$$a_{ijkl}\xi_{ij}\xi_{kl} \geq c_0|\xi|^2, \quad \forall\, \xi_{ji} = \xi_{ij}, \quad c_0 = \text{const.} > 0.$$

A summation convention over repeated indices is used: all functions with two lower indices are assumed to be symmetric in those indices.

An equilibrium problem for the body $\Omega_\gamma$ and the elastic inclusion $\gamma$ (see, for example, [Bessoud et al. 2008]) is formulated as follows. For given external forces $f = (f_1, f_2) \in L^2(\Omega)^2$ acting on the body, we want to find a displacement field $u = (u_1, u_2)$, a stress tensor $\sigma = \{\sigma_{ij}\}$, $i, j = 1, 2$, and thin inclusion displacements $v$, $w$, defined in $\Omega$, $\Omega_\gamma$, and $\gamma$, respectively, such that

$$-\operatorname{div}\sigma = f \quad \text{in } \Omega_\gamma, \tag{2-1}$$

$$\sigma - A\varepsilon(u) = 0 \quad \text{in } \Omega, \tag{2-2}$$

$$EIv_{xxxx} = [\sigma_\nu] \quad \text{on } \gamma, \tag{2-3}$$

$$-ESw_{xx} = [\sigma_\tau] \quad \text{on } \gamma, \tag{2-4}$$

$$u = 0 \quad \text{on } \Gamma, \tag{2-5}$$

$$EIv_{xx} = EIv_{xxx} = ESw_x = 0 \quad \text{for } x = 0, 1, \tag{2-6}$$

$$v = u_\nu, \quad w = u_\tau \quad \text{on } \gamma. \tag{2-7}$$

Here $[h] = h^+ - h^-$ is a jump of a function $h$ on $\gamma$, where $h^\pm$ are the traces of $h$ on the faces of the beam $\gamma^\pm$. The signs $\pm$ correspond to the positive and negative directions of $\nu$; $v_x = dv/dx$, $x = x_1$, $(x_1, x_2) \in \Omega$; $\varepsilon(u) = \{\varepsilon_{ij}(u)\}$ is the strain tensor, $\varepsilon_{ij}(u) = \frac{1}{2}(u_{i,j} + u_{j,i})$, $i, j = 1, 2$; and $\sigma\nu = (\sigma_{1j}\nu_j, \sigma_{2j}\nu_j)$, $\sigma_\nu = \sigma_{ij}\nu_j\nu_i$, $\sigma_\tau = \sigma\nu \cdot \tau$, $u_\nu = u\nu$, $u_\tau = u\tau$. By $E$, $I$, and $S$ we denote the Young's modulus, the inertia of the cross section, and the area of cross section, respectively. Below, for the sake of simplicity, we put $EI = 1$ and $ES = 1$. The essence of the mathematical results obtained in this article does not change by this particular choice. When it comes to the asymptotic analysis for the stiffness of the beam, the role of the stiffness parameters will be taken into account. See Sections 4, 5, and 7.

Functions defined on $\gamma$ we identify with functions of the variable $x$.

Relations (2-1), (2-3), and (2-4) are the equilibrium equations for the elastic body and the inclusion, and (2-2) represents Hooke's law. According to (2-7), the vertical and tangential (along the axis $x_1$) displacements of the elastic body coincide with the inclusion displacements at $\gamma$.

Below we provide a variational formulation of the problem (2-1)–(2-7). To this end, we introduce the Sobolev space

$$V = \{(u, v, w) \in (H_0^1(\Omega))^2 \times H^2(\gamma) \times H^1(\gamma) \mid v = u_\nu,\ w = u_\tau \text{ on } \gamma\},$$

and the energy functional

$$\Pi(u, v, w) = \frac{1}{2}\int_\Omega \sigma(u)\varepsilon(u) - \int_\Omega fu + \frac{1}{2}\int_\gamma v_{xx}^2 + \frac{1}{2}\int_\gamma w_x^2.$$

Here $\sigma(u) = \sigma$ is defined by (2-2), that is, $\sigma(u) = A\varepsilon(u)$, and, for simplicity, we write $\sigma(u)\varepsilon(u) = \sigma_{ij}(u)\varepsilon_{ij}(u)$, $fu = f_iu_i$. We use standard notation for the spaces

$(H^1_0(\Omega))^2$, $H^2(\gamma)$, $H^1(\gamma)$. The functions $u$, $v$, and $w$ are independent, and the only relations are provided by the definition of $V$. In particular, $u_\nu$ and $u_\tau$ have more regularity as compared to that resulting from the inclusion $u \in (H^1_0(\Omega))^2$. We also use the notation $H^s(X)^2 := (H^s(X))^2$ for Sobolev spaces concerning functions in the plane.

Consider the minimization problem

$$\text{(P)} \qquad \text{Find } (u, v, w) \in V \text{ such that } \Pi(u, v, w) = \inf\nolimits_V \Pi.$$

**Theorem 2.1.** *Problem* (P) *admits a unique solution* $(u, v, w)$ *satisfying*

$$(u, v, w) \in V, \tag{2-8}$$

$$\int_\Omega \sigma(u)\varepsilon(\bar{u}) - \int_\Omega f\bar{u} + \int_\gamma v_{xx}\bar{v}_{xx} + \int_\gamma w_x\bar{w}_x = 0, \quad \forall (\bar{u}, \bar{v}, \bar{w}) \in V. \tag{2-9}$$

*Moreover*, *if the solution is smooth*, *then the strong representation*, (2-1)–(2-7), *and the weak representation*, (2-8) *and* (2-9), *are equivalent.*

*Proof.* In order to prove that the problem (2-8) and (2-9) admits a solution, it suffices to establish the coercivity of the functional $\Pi$ on the space $V$, since its weak lower semicontinuity is obvious. Due to Korn's inequality, we have

$$\Pi(u, v, w) \geq c_0\|u\|^2_{1,\Omega} - c_1\|u\|_{1,\Omega} + \frac{1}{2}\int_\gamma (v^2_{xx} + w^2_x) \pm \beta \int_\gamma (v^2 + w^2), \tag{2-10}$$

with positive constants $c_0$ and $c_1$ and a parameter $\beta > 0$, where $\|\cdot\|_{1,\Omega}$ is the norm in $H^1_0(\Omega)^2$ and $\|\cdot\|_{i,\gamma}$ is the norm in $H^i(\gamma)$, $i = 1, 2$. We have $v = u_\nu$ and $w = u_\tau$ at $\gamma$, hence, for small $\beta$, due to the trace inequality

$$\frac{c_0}{2}\|u\|^2_{1,\Omega} - \beta \int_\gamma (v^2 + w^2) \geq 0.$$

Thus, from (2-10) we obtain the desired limit:

$$\Pi(u, v, w) \geq \frac{c_0}{2}\|u\|^2_{1,\Omega} - c_1\|u\|_{1,\Omega} + \frac{1}{2}\int_\gamma (v^2_{xx} + w^2_x) + \beta \int_\gamma (v^2 + w^2) \to +\infty,$$
$$\|(u, v, w)\|_V \to \infty,$$

We now show the equivalence of (2-1)–(2-7) and (2-8) and (2-9) for smooth solutions. Let (2-1)–(2-7) be fulfilled. Take $(\bar{u}, \bar{v}, \bar{w}) \in V$ and multiply (2-1), (2-3), and (2-4) by $\bar{u}$, $\bar{v}$, and $\bar{w}$, respectively. Integrating over $\Omega_\gamma$ and $\gamma$, respectively, we get

$$\int_{\Omega_\gamma} (-\operatorname{div}\sigma - f)\bar{u} + \int_\gamma (v_{xxxx}\bar{v} - w_{xx}\bar{w}) - \int_\gamma ([\sigma_\nu]\bar{v} + [\sigma_\tau]\bar{w}) = 0.$$

Hence, by the boundary conditions (2-5) and (2-6),

$$\int_{\Omega_\gamma} (\sigma(u)\varepsilon(\bar{u}) - f\bar{u}) + \int_\gamma [\sigma\nu]\bar{u} + \int_\gamma (v_{xx}\bar{v}_{xx} + w_x\bar{w}_x) - \int_\gamma ([\sigma_\nu]\bar{v} + [\sigma_\tau]\bar{w}) = 0. \quad (2\text{-}11)$$

We have $[\sigma\nu]\bar{u} = [\sigma_\nu]\bar{u}_\nu + [\sigma_\tau]\bar{u}_\tau$ on $\gamma$. Taking into account that $(\bar{u}, \bar{v}, \bar{w}) \in V$, from (2-11) the identity (2-9) follows. In so doing, we change the integration domain $\Omega_\gamma$ by $\Omega$, since $[u] = [\bar{u}] = 0$ on $\gamma$. Conversely, let (2-8) and (2-9) be fulfilled. We take test functions of the form $(\bar{u}, \bar{v}, \bar{w}) = (\varphi, 0, 0)$, $\varphi \in C_0^\infty(\Omega_\gamma)^2$. This gives the equilibrium equation (2-1). Next, from (2-9) it follows that

$$-\int_\gamma ([\sigma_\nu]\bar{u}_\nu + [\sigma_\tau]\bar{u}_\tau) + \int_\gamma (v_{xxxx}\bar{v} - w_{xx}\bar{w}) + w_x\bar{w}|_0^1 + v_{xx}\bar{v}_x|_0^1 - v_{xxx}\bar{v}|_0^1 = 0, \quad \forall\, (\bar{u}, \bar{v}, \bar{w}) \in V. \quad (2\text{-}12)$$

Choosing here $\bar{w} = 0$ and $\bar{v} = \bar{v}_x = 0$ at $x = 0, 1$, the relation follows:

$$-\int_\gamma [\sigma_\nu]\bar{u}_\nu - \int_\gamma [\sigma_\tau]\bar{u}_\tau + \int_\gamma (v_{xxxx}\bar{v} - w_{xx}\bar{w}) = 0.$$

Consequently, by the equalities $\bar{v} = \bar{u}_\nu$, and $\bar{w} = \bar{u}_\tau$ on $\gamma$, we obtain (2-3) and (2-4). In such a case, the identity (2-12) implies (2-6). Hence, the equivalence of (2-1)–(2-7) and (2-8) and (2-9) is proved. □

## 3. Delaminated elastic inclusion

Assume that a delamination of the elastic inclusion takes place at $\gamma^+$, thus we have a crack. In our model, inequality-type boundary conditions will be considered to prevent a mutual penetration between the crack faces. Displacements of the inclusion should coincide with the displacements of the elastic body at $\gamma^-$. The problem formulation is as follows. We have to find a displacement field $u = (u_1, u_2)$, a stress tensor $\sigma = \{\sigma_{ij}\}$, $i, j = 1, 2$, and thin inclusion displacements $v$ and $w$ defined in $\Omega_\gamma$, $\Omega_\gamma$, and $\gamma$, respectively, such that

$$-\operatorname{div}\sigma = f \quad \text{in } \Omega_\gamma, \quad (3\text{-}1)$$

$$\sigma - A\varepsilon(u) = 0 \quad \text{in } \Omega_\gamma, \quad (3\text{-}2)$$

$$v_{xxxx} = [\sigma_\nu] \quad \text{on } \gamma, \quad (3\text{-}3)$$

$$-w_{xx} = [\sigma_\tau] \quad \text{on } \gamma, \quad (3\text{-}4)$$

$$u = 0 \quad \text{on } \Gamma, \quad (3\text{-}5)$$

$$v_{xx} = v_{xxx} = w_x = 0 \quad \text{for } x = 0, 1, \quad (3\text{-}6)$$

$$[u_\nu] \ge 0, \quad v = u_\nu^-, \quad w = u_\tau^-, \quad \sigma_\nu^+[u_\nu] = 0 \quad \text{on } \gamma, \quad (3\text{-}7)$$

$$\sigma_\nu^+ \le 0, \quad \sigma_\tau^+ = 0 \quad \text{on } \gamma. \quad (3\text{-}8)$$

The first inequality in (3-7) provides a mutual nonpenetration between the crack faces. The second and the third relations of (3-7) show that the inclusion displacements coincide with the vertical and tangential displacements of the elastic body at $\gamma^-$.

First, we provide a variational formulation of the problem (3-1)–(3-8). We introduce the set of admissible displacements

$$K = \{(u, v, w) \in H^1_\Gamma(\Omega_\gamma)^2 \times H^2(\gamma) \times H^1(\gamma) \mid [u_\nu] \geq 0,\ v = u_\nu^-,\ w = u_\tau^- \text{ on } \gamma\}$$

and the energy functional

$$\Pi_1(u, v, w) = \frac{1}{2}\int_{\Omega_\gamma} \sigma(u)\varepsilon(u) - \int_{\Omega_\gamma} fu + \frac{1}{2}\int_\gamma v_{xx}^2 + \frac{1}{2}\int_\gamma w_x^2,$$

where the Sobolev space $H^1_\Gamma(\Omega_\gamma)$ is defined as

$$H^1_\Gamma(\Omega_\gamma) = \{v \in H^1(\Omega_\gamma) \mid v = 0 \text{ on } \Gamma\}.$$

**Theorem 3.1.** *There exists a unique solution of the problem*

$$\textit{Find } (u, v, w) \in K \textit{ such that } \Pi_1(u, v, w) = \inf\nolimits_K \Pi_1. \tag{3-9}$$

*This solution satisfies the variational inequality*

$$(u, v, w) \in K, \tag{3-10}$$

$$\int_{\Omega_\gamma} \sigma(u)\varepsilon(\bar{u} - u) - \int_{\Omega_\gamma} f(\bar{u} - u) + \int_\gamma v_{xx}(\bar{v}_{xx} - v_{xx}) + \int_\gamma w_x(\bar{w}_x - w_x) \geq 0, \quad \forall (\bar{u}, \bar{v}, \bar{w}) \in K. \tag{3-11}$$

*Moreover*, (3-1)–(3-8) *and* (3-10) *and* (3-11) *are equivalent for smooth solutions.*

*Proof.* The coercivity of the functional $\Pi_1$ can be proved as that in Section 2; hence, the problem (3-10) and (3-11) indeed has a solution. As for the equivalence of the representations for smooth solutions, assume that (3-1)–(3-8) hold. Take $(\bar{u}, \bar{v}, \bar{w}) \in K$ and multiply (3-1), (3-3), and (3-4) by $\bar{u} - u$, $\bar{v} - v$, and $\bar{w} - w$, respectively. Integrating over $\Omega_\gamma$ and $\gamma$, we have

$$\int_{\Omega_\gamma} (-\operatorname{div}\sigma - f)(\bar{u} - u) + \int_\gamma (v_{xxxx} - [\sigma_\nu])(\bar{v} - v) + \int_\gamma (-w_{xx} - [\sigma_\tau])(\bar{w} - w) = 0,$$

and hence

$$\int_{\Omega_\gamma} \sigma(u)\varepsilon(\bar{u} - u) - \int_{\Omega_\gamma} f(\bar{u} - u) + \int_\gamma [\sigma\nu(\bar{u} - u)] + \int_\gamma v_{xx}(\bar{v}_{xx} - v_{xx}) + \int_\gamma w_x(\bar{w}_x - w_x) - \int_\gamma [\sigma_\nu](\bar{v} - v) - \int_\gamma [\sigma_\tau](\bar{w} - w) = 0. \tag{3-12}$$

To prove the variational inequality (3-11), it suffices to state in (3-12) that

$$B \equiv \int_\gamma [\sigma\nu(\bar{u}-u)] - \int_\gamma [\sigma_\nu](\bar{v}-v) - \int_\gamma [\sigma_\tau](\bar{w}-w) \le 0.$$

This can be verified by (3-7) and (3-8). Hence the variational inequality (3-11) follows from (3-12), as required.

Conversely, let (3-10) and (3-11) be fulfilled. First, it is easy to derive the equilibrium equation (3-1) from (3-10) and (3-11). We next substitute the test functions $(\bar{u}, \bar{v}, \bar{w}) = (u, v, w) \pm (\varphi, \omega, \psi)$ in (3-11), with $[\varphi_\nu] = 0$, $\varphi_\nu^- = \omega$, $\varphi_\tau^- = \psi$, on $\gamma$. This gives

$$\int_{\Omega_\gamma} \sigma(u)\varepsilon(\varphi) - \int_{\Omega_\gamma} f\varphi + \int_\gamma v_{xx}\omega_{xx} + \int_\gamma w_x\psi_x = 0.$$

Hence,

$$-\int_\gamma [\sigma\nu\cdot\varphi] + \int_\gamma v_{xxxx}\omega - \int_\gamma w_{xx}\psi - v_{xxx}\omega|_0^1 + v_{xx}\omega_x|_0^1 + w_x\psi|_0^1 = 0. \quad (3\text{-}13)$$

Assuming $\omega = \omega_x = \psi = 0$ as $x = 0, 1$, from (3-13) one gets

$$-\int_\gamma ([\sigma_\nu]\varphi_\nu + [\sigma_\tau\varphi_\tau]) + \int_\gamma (v_{xxxx}\omega - w_{xx}\psi) = 0. \quad (3\text{-}14)$$

Due to the arbitrariness of $\varphi_\tau^+$, we obtain $\sigma_\tau^+ = 0$ on $\gamma$. Since $\omega = \varphi_\nu$ and $\psi = \varphi_\tau^-$ on $\gamma$ we obtain the equations (3-3) and (3-4). Now, taking into account (3-3) and (3-4), it follows from (3-13) that boundary conditions (3-6) are fulfilled. Let us prove the last relation of (3-7) and the inequality in (3-8). To this end, we take in (3-11) test functions of the form $(\bar{u}, \bar{v}, \bar{w}) = (u, v, w) + (\varphi, 0, 0)$, with $\varphi_\nu^+ \ge 0$ on $\gamma$, $\varphi_\nu^- = 0$, and $\varphi_\tau^- = 0$ on $\gamma$. This provides

$$\int_{\Omega_\gamma} \sigma(u)\varepsilon(\varphi) - \int_{\Omega_\gamma} f\varphi \ge 0,$$

and thus

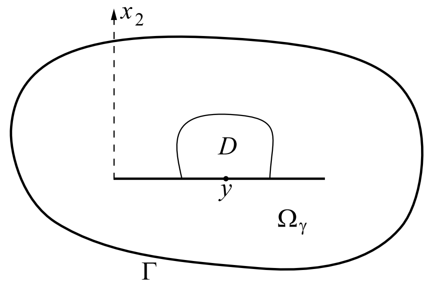
$$-\int_\gamma \sigma^+\nu\cdot\varphi^+ \ge 0.$$

This relation implies

$$\int_\gamma \sigma_\nu^+\varphi_\nu^+ \le 0.$$

Since $\varphi_\nu^+$ is an arbitrary nonnegative function, we conclude that $\sigma_\nu^+ \le 0$ on $\gamma$.

Next, assume that at any point $y \in \gamma$ we have $[u_\nu(y)] > 0$. It necessarily gives $\sigma_\nu^+(y) = 0$, since in such a case a function $(\bar{u}, \bar{v}, \bar{w}) = (u, v, w) \pm (\lambda\varphi, 0, 0)$ can be substituted in (3-11) with a smooth function $\varphi$, $\operatorname{supp}\varphi \subset \bar{D}$, $\lambda$ a small parameter, and $D$ a small neighborhood:




This provides the relation

$$\int_{\Omega_\gamma} \sigma(u)\varepsilon(\varphi) - \int_{\Omega_\gamma} f\varphi = 0,$$

hence the statement follows. On the other hand, if $\sigma_\nu^+(y) < 0$ we derive $[u_\nu(y)] = 0$, and, consequently, the last relation of (3-7) is proved. The proof of the equivalency of (3-1)–(3-8) and (3-10) and (3-11) is complete. □

## 4. Convergence as the rigidity tends to infinity

In fact, a solution of the problem (3-1)–(3-8) should depend on the rigidity parameter of the thin inclusion. In the model (3-1)–(3-8), this parameter was taken to be equal to 1. In this section we introduce the parameter into the model and analyze its passage to infinity. To this end, we define the energy functional

$$\Pi_\delta(u, v, w) = \frac{1}{2}\int_{\Omega_\gamma} \sigma(u)\varepsilon(u) - \int_{\Omega_\gamma} fu + \frac{\delta}{2}\int_\gamma v_{xx}^2 + \frac{\delta}{2}\int_\gamma w_x^2, \quad \delta > 0.$$

**Theorem 4.1.** *There exists a unique solution to the problem*

*Find* $(u^\delta, v^\delta, w^\delta) \in K$ *such that* $\Pi_\delta(u^\delta, v^\delta, w^\delta) = \inf_K \Pi_\delta$

*that satisfies the variational inequality*

$$(u^\delta, v^\delta, w^\delta) \in K, \tag{4-1}$$

$$\int_{\Omega_\gamma} \sigma(u^\delta)\varepsilon(\bar{u}-u^\delta) - \int_{\Omega_\gamma} f(\bar{u}-u^\delta) + \delta\int_\gamma v_{xx}^\delta(\bar{v}_{xx}-v_{xx}^\delta) + \delta\int_\gamma w_x^\delta(\bar{w}_x - w_x^\delta) \ge 0,$$
$$\forall\,(\bar{u}, \bar{v}, \bar{w}) \in K. \tag{4-2}$$

*Proof.* The proof is analogous to that of Theorem 2.1 and is omitted. □

Our aim in this section is to pass to the limit in (4-1) and (4-2) as $\delta \to +\infty$. To this end, we introduce the notation for vertical rigid displacements $R_s(\gamma)$ and for admissible displacements $K_r$:

$$R_s(\gamma) := \big\{ l(x) \mid l(x) = c_0 + c_1 x,\ x \in \gamma;\ c_0, c_1 \in \mathbb{R} \big\},$$
$$K_r := \big\{ u \in H^1_\Gamma(\Omega_\gamma)^2 \mid [u_\nu] \ge 0,\ u^-_\nu|_\gamma \in R_s(\gamma),\ u^-_\tau|_\gamma \in \mathbb{R} \big\}.$$

**Theorem 4.2.** *Let* $(u^\delta, v^\delta, w^\delta) \in K$. *Then we can pass to the limit as* $\delta \to +\infty$ *and obtain a unique element* $(u, v, w) \in K_r$ *such that* $(u, v, w)$ *satisfies*

$$u^\delta \to u \quad \text{weakly in } H^1_\Gamma(\Omega_\gamma)^2, \tag{4-3}$$
$$v^\delta \to v \quad \text{weakly in } H^2(\gamma), \qquad v_{xx} = 0 \quad \text{on } \gamma, \tag{4-4}$$
$$w^\delta \to w \quad \text{weakly in } H^1(\gamma), \qquad w_x = 0 \quad \text{on } \gamma. \tag{4-5}$$

*In particular,* $v(x) = c_0 + c_1 x$, $w(x) = q_0$, $q_0 = const.$, $x \in (0, 1)$. *Moreover,* $(u, v, w)$ *satisfies the limiting problem*

$$u \in K_r, \tag{4-6}$$
$$\int_{\Omega_\gamma} \sigma(u)\varepsilon(\bar{u} - u) - \int_{\Omega_\gamma} f(\bar{u} - u) \ge 0, \quad \forall\, \bar{u} \in K_r. \tag{4-7}$$

*Proof.* From (4-2) it follows that

$$\int_{\Omega_\gamma} \sigma(u^\delta)\varepsilon(u^\delta) - \int_{\Omega_\gamma} f u^\delta \pm \beta \int_\gamma ((w^\delta)^2 + (v^\delta)^2) + \delta \int_\gamma (v^\delta_{xx})^2 + \delta \int_\gamma (w^\delta_x)^2 = 0. \tag{4-8}$$

For small $\beta > 0$, due to $v^\delta = u^{\delta -}_\nu$ and $w^\delta = u^{\delta -}_\tau$ on $\gamma$, the following relation holds:

$$\frac{1}{2} \int_{\Omega_\gamma} \sigma(u^\delta)\varepsilon(u^\delta) - \beta \int_\gamma ((w^\delta)^2 + (v^\delta)^2) \ge 0.$$

Consequently, from (4-8) one gets, as $\delta \ge \beta$,

$$c_0 \|u^\delta\|^2_{1,\Omega_\gamma} + \beta \|v^\delta\|^2_{2,\gamma} + \beta \|w^\delta\|^2_{1,\gamma} \le c_1 \|u^\delta\|_{1,\Omega_\gamma}, \quad c_0 > 0.$$

Hence, uniformly in $\delta \ge \delta_0$,

$$\|u^\delta\|^2_{1,\Omega_\gamma} + \|v^\delta\|^2_{2,\gamma} + \|w^\delta\|^2_{1,\gamma} \le c. \tag{4-9}$$

On the other hand, the relation (4-8) implies for $\delta \ge \delta_0$,

$$\delta \int_\gamma (v^\delta_{xx})^2 + \delta \int_\gamma (w^\delta_x)^2 \le c. \tag{4-10}$$

Thus, we can pass to the limit on a subsequence and obtain (4-3)–(4-5). Let us choose $(\bar{u}, l, q) \in K$ as a test function in (4-2), $l \in R_s(\gamma)$, $q \in \mathbb{R}$. Notice that

$\bar{u} \in K_r$. Then, from (4-2) it follows that

$$\int_{\Omega_\gamma} \sigma(u^\delta)\varepsilon(\bar{u} - u^\delta) - \int_{\Omega_\gamma} f(\bar{u} - u^\delta) \geq \delta \int_\gamma (v^\delta_{xx})^2 + \delta \int_\gamma (w^\delta_x)^2. \tag{4-11}$$

Again, passing to the limit, as $\delta \to \infty$, according to (4-3)–(4-5) we obtain the variational inequality (4-6) and (4-7) just as in [Khludnev 2010a; 2010b; Khludnev and Leugering 2010], with $u|_\gamma = \rho$, where, for any $x \in \gamma$, we have

$$\rho(x) = b(x_2, -x_1) + (a_1, a_2), \quad \text{with } a_1, a_2, b \in \mathbb{R}.$$

Hence $u$ is an infinitesimal rigid displacement at $\gamma$. The convergence of the entire sequence and the uniqueness follows as usual. □

**Remark.** The inclusion $\gamma$ in the limit problem (4-6) and (4-7) can be interpreted as a rigid beam inclusion. Solvability of this problem can be also proved independently by minimizing the functional

$$\pi(v) = \frac{1}{2}\int_{\Omega_\gamma} \sigma(v)\varepsilon(v) - \int_{\Omega_\gamma} f v$$

over the set $K_r$.

We are now going to establish two strong formulations of (4-6) and (4-7), which, in turn, are equivalent to (4-6) and (4-7) if the solutions are smooth.

**Theorem 4.3.** *We consider two problems*:

(i) *Find a displacement field* $u = (u_1, u_2)$, *a stress tensor* $\sigma = \{\sigma_{ij}\}$, $i, j = 1, 2$, *and thin inclusion displacements* $l_0 \in R_s(\gamma)$, *and* $q_0 \in \mathbb{R}$ *defined in* $\Omega_\gamma$, $\Omega_\gamma$, *and* $\gamma$, *respectively*, *such that*

$$-\operatorname{div}\sigma = f \quad \text{in } \Omega_\gamma, \tag{4-12}$$

$$\sigma - A\varepsilon(u) = 0 \quad \text{in } \Omega_\gamma, \tag{4-13}$$

$$u = 0 \quad \text{on } \Gamma, \tag{4-14}$$

$$[u_\nu] \geq 0, \quad l_0 = u_\nu^-, \quad q_0 = u_\tau^- \quad \text{on } \gamma, \tag{4-15}$$

$$\textstyle\int_\gamma [\sigma\nu \cdot u] = 0, \tag{4-16}$$

$$-\textstyle\int_\gamma [\sigma\nu \cdot \bar{u}] \geq 0, \quad \forall\, \bar{u} \in K_r. \tag{4-17}$$

(ii) *Find a displacement field* $u = (u_1, u_2)$, *a stress tensor* $\sigma = \{\sigma_{ij}\}$, $i, j = 1, 2$, *and thin inclusion displacements* $l_0 \in R_s(\gamma)$ *and* $q_0 \in \mathbb{R}$ *defined in* $\Omega_\gamma$, $\Omega_\gamma$, *and $\gamma$, respectively, such that*

$$-\operatorname{div}\sigma = f \quad \text{in } \Omega_\gamma, \tag{4-18}$$

$$\sigma - A\varepsilon(u) = 0 \quad \text{in } \Omega_\gamma, \tag{4-19}$$

$$u = 0 \quad \text{on } \Gamma, \tag{4-20}$$

$$[u_\nu] \ge 0, \quad l_0 = u_\nu^-, \quad q_0 = u_\tau^- \quad \text{on } \gamma, \tag{4-21}$$

$$\sigma_\tau^+ = 0, \quad \sigma_\nu^+ \le 0, \quad \sigma_\nu^+[u_\nu] = 0 \quad \text{on } \gamma, \tag{4-22}$$

$$\textstyle\int_\gamma \sigma_\tau^- = 0, \quad \int_\gamma [\sigma_\nu] l = 0, \quad \forall\, l \in R_s(\gamma). \tag{4-23}$$

(*The conditions in* (4-23) *guarantee that the principal vector of forces and the principal vector of moments acting at $\gamma$ are equal to zero.*)

*Then, if the solution to problem* (4-6) *and* (4-7) *of Theorem 4.1 is smooth enough, the two problems are equivalent.*

*Proof.* We first prove that (4-6) and (4-7) and (4-12)–(4-17) are equivalent for smooth solutions. Assume that (4-6) and (4-7) hold. We take test functions $\bar u$ in (4-7) such that $\bar u = u \pm \varphi$, $\varphi \in C_0^\infty(\Omega_\gamma)^2$. This provides the equilibrium equation (4-12). From (4-7) it follows

$$\int_{\Omega_\gamma} \sigma(u)\varepsilon(u) - \int_{\Omega_\gamma} f u = 0. \tag{4-24}$$

Integrating by parts in (4-24) we get (4-16). By (4-24), the variational inequality (4-7) can be rewritten as

$$\int_{\Omega_\gamma} \sigma(u)\varepsilon(\bar u) - \int_{\Omega_\gamma} f\bar u \ge 0, \quad \forall\, \bar u \in K_r,$$

thus (4-17) follows. Conversely, let (4-12)–(4-17) be fulfilled. We take $\bar u \in K_r$ and multiply (4-12) by $\bar u - u$. Integrating over $\Omega_\gamma$ we get

$$\int_{\Omega_\gamma} (-\operatorname{div}\sigma - f)(\bar u - u) = 0.$$

Hence

$$\int_\gamma [\sigma\nu(\bar u - u)] + \int_{\Omega_\gamma} \sigma(u)\varepsilon(\bar u - u) - \int_{\Omega_\gamma} f(\bar u - u) = 0.$$

In order to obtain the variational inequality (4-7), it suffices to prove

$$-\int_\gamma [\sigma\nu(\bar u - u)] \ge 0. \tag{4-25}$$

But the inequality (4-25) follows from (4-16) and (4-17). Thus, the equivalence of (4-6) and (4-7) and (4-12)–(4-17) is established.

We now turn to the second problem and demonstrate that (4-6)–(4-7) is equivalent to (4-18)–(4-23) for smooth solutions. Let (4-6) and (4-7) be fulfilled. As before, we check that the equilibrium equation (4-18) follows from (4-7). Next, we choose test functions $\bar{u}=u\pm\tilde{u}$, $[\tilde{u}_\nu]=0$, $\tilde{u}_\nu^-|_\gamma \in R_s(\gamma)$, $\tilde{u}_\tau^-|_\gamma \in \mathbb{R}$, and $\tilde{u}\in H^1_\Gamma(\Omega_\gamma)^2$. This gives

$$\int_{\Omega_\gamma} \sigma(u)\varepsilon(\tilde{u}) - \int_{\Omega_\gamma} f\tilde{u} = 0,$$

and, hence,

$$-\int_\gamma [\sigma_\nu]\tilde{u}_\nu - \int_\gamma [\sigma_\tau \tilde{u}_\tau] = 0. \tag{4-26}$$

Since $\tilde{u}_\tau^+$ is arbitrary on $\gamma$, we derive the first relation of (4-22). By $\tilde{u}_\tau^- \in \mathbb{R}$ on $\gamma$, from (4-26) we also obtain the first and the second relations of (4-23). Now we choose test functions in (4-7) as $\bar{u} = u + \tilde{u}$, $\tilde{u} \in H^1_\Gamma(\Omega_\gamma)^2$, and $\tilde{u}_\nu^+ \geq 0$ on $\gamma$, $\operatorname{supp} \tilde{u} \subset \bar{D}$; see figure on page 9. This gives

$$\int_{\Omega_\gamma} \sigma(u)\varepsilon(\tilde{u}) - \int_{\Omega_\gamma} f\tilde{u} \geq 0.$$

Consequently,

$$\int_\gamma (\sigma_\nu^+\tilde{u}_\nu^+ + \sigma_\tau^+\tilde{u}_\tau^+) \leq 0. \tag{4-27}$$

By the choice of $\tilde{u}$, from (4-27) the second relation of (4-22) follows.

In order to derive (4-18)–(4-23) from (4-6) and (4-7), it remains to check the last condition of (4-22). To this end, assume that at a given point $y \in \gamma$ we have $[u_\nu(y)] > 0$. Take test functions in (4-7) of the form $\bar{u} = u \pm \lambda\varphi$, $\operatorname{supp}\varphi \subset \bar{D}$, where $\lambda$ is a small parameter, $D$ is a small neighborhood, and $\varphi$ is a smooth function; see again figure on page 9. We get

$$\int_{\Omega_\gamma} \sigma(u)\varepsilon(\varphi) - \int_{\Omega_\gamma} f\varphi = 0;$$

thus

$$\int_\gamma \sigma_\nu^+\varphi_\nu^+ = 0,$$

and $\sigma_\nu^+(y) = 0$, that is, $\sigma_\nu^+(y)[u_\nu(y)] = 0$. On the other hand, assuming that $\sigma_\nu^+(y) < 0$, we easily derive $[u_\nu(y)] = 0$, and the last relation of (4-22) follows. Thus, from (4-6) and (4-7) we have derived all relations (4-18)–(4-23). To complete the proof of equivalence of (4-6) and (4-7) and (4-18)–(4-23), assume the converse, that is, let (4-18)–(4-23) be fulfilled. We take $\bar{u} \in K_r$ and multiply (4-18) by $\bar{u} - u$.

Integrating over $\Omega_\gamma$, one gets

$$\int_{\Omega_\gamma} (-\operatorname{div}\sigma - f)(\bar{u} - u) = 0,$$

and, consequently,

$$\int_{\Omega_\gamma} \sigma(u)\varepsilon(\bar{u} - u) - \int_{\Omega_\gamma} f(\bar{u} - u) = -\int_\gamma [\sigma\nu(\bar{u} - u)]. \tag{4-28}$$

To derive the variational inequality (4-7) from (4-28), it suffices to prove

$$-\int_\gamma [\sigma\nu(\bar{u} - u)] \geq 0. \tag{4-29}$$

We have, by (4-22) and by $\bar{u} \in K_r$, that

$$-\int_\gamma \sigma_\nu^+([\bar{u}_\nu] - [u_\nu]) \geq 0. \tag{4-30}$$

In view of (4-23) and the first relation of (4-22), the inequality (4-30) can be rewritten as

$$-\int_\gamma [\sigma_\nu(\bar{u}_\nu - u_\nu)] - \int_\gamma [\sigma_\tau(\bar{u}_\tau - u_\tau)] \geq 0. \tag{4-31}$$

From (4-31), (4-29) follows. We already mentioned that from (4-28) and (4-29) the variational inequality (4-7) follows. Thus, equivalence of (4-6) and (4-7) and (4-18)–(4-23) is completely proved. □

## 5. Convergence as the rigidity tends to zero

In this section we analyze the case where the rigidity parameter $\delta$ for the inclusion convergence to zero. Again, consider the problem (4-1) and (4-2). Our aim is to pass to the limit in (4-1) and (4-2) as $\delta \to 0$. To this end, we define the set of admissible displacements

$$K_0 = \{u \in H^1_\Gamma(\Omega_\gamma)^2 \mid [u_\nu] \geq 0 \text{ on } \gamma\}.$$

**Theorem 5.1.** *Let* $(u^\delta, v^\delta, w^\delta) \in K$ *be the unique solution of* (4-1) *and* (4-2). *Then, as* $\delta \to 0$, *we find a unique element* $w \in K_0$ *such that*

$$u^\delta \to u \quad \textit{weakly in } H^1_\Gamma(\Omega_\gamma)^2, \tag{5-1}$$

$$\sqrt{\delta}v^\delta \to \tilde{v} \quad \textit{weakly in } H^2(\gamma), \tag{5-2}$$

$$\sqrt{\delta}w^\delta \to \tilde{w} \quad \textit{weakly in } H^1(\gamma). \tag{5-3}$$

*Moreover*, $(u, v, w)$ *satisfies the variational inequality*

$$u \in K_0, \tag{5-4}$$

$$\int_{\Omega_\gamma} \sigma(u)\varepsilon(\bar{u} - u) - \int_{\Omega_\gamma} f(\bar{u} - u) \geq 0, \quad \forall\, \bar{u} \in K_0. \tag{5-5}$$

*Proof.* First note that (4-2) implies

$$\int_{\Omega_\gamma} \sigma(u^\delta)\varepsilon(u^\delta) - \int_{\Omega_\gamma} f u^\delta + \delta \int_\gamma (v^\delta_{xx})^2 + \delta \int_\gamma (w^\delta_x)^2 = 0. \tag{5-6}$$

Hence, we have a uniform-in-$\delta$ estimate

$$\|u^\delta\|^2_{1,\Omega_\gamma} \leq c. \tag{5-7}$$

On the other hand, the relation (5-6) implies, for all $\delta$,

$$\delta \int_\gamma (v^\delta_{xx})^2 + \delta \int_\gamma (w^\delta_x)^2 \leq c. \tag{5-8}$$

By (5-7),

$$\int_\gamma (v^\delta)^2 = \int_\gamma (u^{\delta-}_\nu)^2 \leq c, \quad \int_\gamma (w^\delta)^2 = \int_\gamma (u^{\delta-}_\tau)^2 \leq c, \tag{5-9}$$

hence, in view of (5-8),

$$\delta \|v^\delta\|^2_{2,\gamma} + \delta \|w^\delta\|^2_{1,\gamma} \leq c.$$

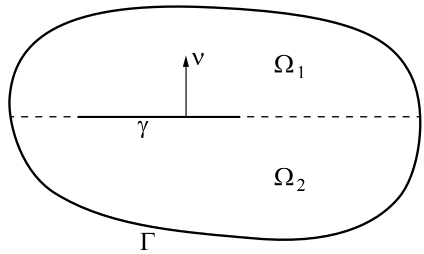
By (5-1)–(5-3), a passage to the limit in (4-1) and (4-2) is possible. We choose $\bar{u} \in K_0$ such that $\bar{u}_\nu$ and $\bar{u}_\tau$ are smooth at $\gamma^-$, and define the functions $\bar{v} = \bar{u}^-_\nu$ and $\bar{w} = \bar{u}^-_\tau$ on $\gamma$. Then $(\bar{u}, \bar{v}, \bar{w}) \in K$, and a substitution of this test function in (4-2) implies

$$\int_{\Omega_\gamma} \sigma(u^\delta)\varepsilon(\bar{u}) - \int_{\Omega_\gamma} f(\bar{u} - u^\delta) \geq \int_{\Omega_\gamma} \sigma(u^\delta)\varepsilon(u^\delta) + \delta \int_\gamma (v^\delta_{xx})^2 - \delta \int_\gamma v^\delta_{xx}\bar{v}_{xx} + \delta \int_\gamma (w^\delta_x)^2 - \delta \int_\gamma w^\delta_x \bar{w}_x.$$

Taking the lower limit as $\delta \to 0$ in both parts of this inequality, we derive

$$\int_{\Omega_\gamma} \sigma(u)\varepsilon(\bar{u} - u) - \int_{\Omega_\gamma} f(\bar{u} - u) \geq 0. \tag{5-10}$$

Inequality (5-10) holds for all functions $\bar{u} \in K_0$ such that $\bar{u}_\nu$ and $\bar{u}_\tau$ are quite smooth at $\gamma^-$. We state that it will be valid for all $\bar{u} \in K_0$. Indeed, let $\bar{u} \in K_0$ be any fixed function. We divide the domain $\Omega_\gamma$ into two subdomains $\Omega_1$ and $\Omega_2$, as shown:




Consider the restriction $\bar{u}|_{\Omega_2} \in H^1(\Omega_2)^2$, and extend this function to $\Omega$ as a function from $H_0^1(\Omega)^2$. Denote this extension by $v$. Then we put $\tilde{u} = \bar{u} - v$. It is clear that $[\tilde{u}_\nu] \geq 0$ on $\gamma$, and $\tilde{u} = 0$ in $\Omega_2$, thus $\tilde{u}_\nu = 0$ and $\tilde{u}_\tau = 0$ at $\gamma^-$. Next we choose a sequence $v^n \in C_0^\infty(\Omega)^2$ such that

$$v^n \to v \quad \text{strongly in } H_0^1(\Omega)^2.$$

In this case

$$\tilde{u} + v^n \to \bar{u} \quad \text{strongly in } H_\Gamma^1(\Omega_\gamma)^2.$$

On the other hand, $\tilde{u} + v^n \in K_0$, and $\tilde{u}_\nu + v_\nu^n$ and $\tilde{u}_\tau + v_\tau^n$ are smooth functions at $\gamma^-$. Hence, the limit function $u$ from (5-1) satisfies the variational inequality (5-4), as stated. □

**Remark.** We have proved that the limit problem for (4-1) and (4-2) as $\delta \to 0$ coincides with the well-known boundary value problem describing the equilibrium of the elastic body with the crack $\gamma$. This model provides a mutual nonpenetration between the crack faces, hence it is suitable from the mechanical standpoint. The strong formulation of the problem (5-4) and (5-5) is as follows. We have to find functions $u = (u_1, u_2)$ and $\sigma = \{\sigma_{ij}\}$, $i, j = 1, 2$, defined in $\Omega_\gamma$, such that

$$-\operatorname{div} \sigma = f \quad \text{in } \Omega_\gamma, \tag{5-11}$$

$$\sigma - A\varepsilon(u) = 0 \quad \text{in } \Omega_\gamma, \tag{5-12}$$

$$u = 0 \quad \text{on } \Gamma, \tag{5-13}$$

$$[u_\nu] \geq 0, \quad \sigma_\nu^\pm \leq 0, \quad [\sigma_\nu] = 0, \quad \sigma_\tau^\pm = 0, \quad \sigma_\nu[u_\nu] = 0 \quad \text{on } \gamma. \tag{5-14}$$

Many results concerning this model can be found in [Khludnev and Kovtunenko 2000; Khludnev 2010a].

## 6. Two-sided delamination of the inclusion

In this section we analyze the case when a delamination takes place at both sides of the elastic inclusion $\gamma$. First, we remark that a delamination of the elastic inclusion can be considered at $\gamma_0^+$, where $\gamma_0$ is a part of $\gamma$. In particular, set $\gamma_0 = \left(0, \frac{1}{2}\right) \times \{0\}$.

Suppose that there is no delamination at $\gamma \setminus \gamma_0$. In this case a differential formulation of the equilibrium problem is as follows.

**Theorem 6.1.** *We consider the following problem*: *Find a displacement field* $u = (u_1, u_2)$, *a stress tensor* $\sigma = \{\sigma_{ij}\}$, $i, j = 1, 2$, *and thin inclusion displacements* $v$ *and* $w$ *defined in* $\Omega_\gamma$, $\Omega_\gamma$, *and* $\gamma$, *respectively*, *such that*

$$-\operatorname{div} \sigma = f \quad \text{in } \Omega_\gamma, \tag{6-1}$$

$$\sigma - A\varepsilon(u) = 0 \quad \text{in } \Omega_\gamma, \tag{6-2}$$

$$v_{xxxx} = [\sigma_\nu] \quad \text{on } \gamma, \tag{6-3}$$

$$-w_{xx} = [\sigma_\tau] \quad \text{on } \gamma, \tag{6-4}$$

$$u = 0 \quad \text{on } \Gamma, \tag{6-5}$$

$$v_{xx} = v_{xxx} = 0, \quad w_x = 0 \quad \text{for } x = 0, 1, \tag{6-6}$$

$$v = u_\nu, \quad w = u_\tau \quad \text{on } \gamma \setminus \gamma_0, \tag{6-7}$$

$$[u_\nu] \geq 0, \quad v = u_\nu^-, \quad w = u_\tau^- \quad \text{on } \gamma_0, \tag{6-8}$$

$$\sigma_\nu^+ \leq 0, \quad \sigma_\tau^+ = 0, \quad \sigma_\nu^+[u_\nu] = 0 \quad \text{on } \gamma_0. \tag{6-9}$$

*The problem* (6-1)–(6-9) *admits a variational formulation.*

*Proof.* The arguments are similar to those of the proofs above. The details are omitted. □

Moreover, we can consider a different type of delamination along $\gamma$. Denote $\gamma_1 = \left(0, \frac{2}{3}\right) \times \{0\}$ and $\gamma_2 = \left(\frac{1}{3}, 1\right) \times \{0\}$, and assume that delamination takes place at $\gamma_1^+$ and $\gamma_2^-$. In this case the part $\left(\frac{1}{3}, \frac{2}{3}\right) \times \{0\}$ of the inclusion is delaminated at both sides. We introduce the energy functional

$$\Pi_1(u, v, w) = \frac{1}{2} \int_{\Omega_\gamma} \sigma(u)\varepsilon(u) - \int_{\Omega_\gamma} fu + \frac{1}{2} \int_\gamma v_{xx}^2 + \frac{1}{2} \int_\gamma w_x^2,$$

and the set of admissible displacements

$$\begin{aligned} K_1 = \big\{&(u, v, w) \in H_\Gamma^1(\Omega_\gamma)^2 \times H^2(\gamma) \times H^1(\gamma) \mid \\ &[u_\nu] \geq 0 \text{ on } \gamma \setminus (\gamma_1 \cap \gamma_2);\ v = u_\nu^-, w u_\tau^- \text{ on } \gamma \setminus \gamma_2; \\ &v = u_\nu^+, w = u_\tau^+ \text{ on } \gamma \setminus \gamma_1;\ u_\nu^+ - v \geq 0, v - u_\nu^- \geq 0 \text{ on } \gamma_1 \cap \gamma_2\big\}. \end{aligned}$$

**Theorem 6.2.** *There exists a unique solution to the problem*:

$$\textit{Find } (u, v, w) \in K_1 \textit{ such that } \Pi_1(u, v, w) = \inf_{K_1} \Pi_1.$$

*This solution satisfies the variational inequality*

$$(u, v, w) \in K_1, \tag{6-10}$$

$$\int_{\Omega_\gamma} \sigma(u)\varepsilon(\bar{u}-u) - \int_{\Omega_\gamma} f(\bar{u}-u) + \int_\gamma v_{xx}(\bar{v}_{xx}-v_{xx}) + \int_\gamma w_x(\bar{w}_x - w_x) \ge 0, \quad \forall\,(\bar{u},\bar{v},\bar{w}) \in K_1. \tag{6-11}$$

*Moreover, if* $(u, v, w) \in K_1$ *is a smooth solution of* (6-10) *and* (6-11) *then it solves the following strong problem and vice versa*:

*Find a displacement field* $u = (u_1, u_2)$, *a stress tensor* $\sigma = \{\sigma_{ij}\}$, $i, j = 1, 2$, *and thin inclusion displacements* $v$ *and* $w$ *defined in* $\Omega_\gamma$, $\Omega_\gamma$, *and* $\gamma$, *respectively*, *such that*

$$-\operatorname{div}\sigma = f \quad \text{in } \Omega_\gamma, \tag{6-12}$$

$$\sigma - A\varepsilon(u) = 0 \quad \text{in } \Omega_\gamma, \tag{6-13}$$

$$v_{xxxx} = [\sigma_\nu] \quad \text{on } \gamma, \tag{6-14}$$

$$-w_{xx} = [\sigma_\nu] \quad \text{on } \gamma, \tag{6-15}$$

$$u = 0 \quad \text{on } \Gamma, \tag{6-16}$$

$$v_{xx} = v_{xxx} = 0, \quad w_x = 0 \quad \text{for } x = 0, 1, \tag{6-17}$$

$$v = u_\nu^-, \quad w = u_\tau^-, \quad \sigma_\nu^+ \le 0, \quad \sigma_\nu^+[u_\nu] = 0 \quad \text{on } \gamma \setminus \gamma_2, \tag{6-18}$$

$$[u_\nu] \ge 0, \quad \sigma_\tau^+ = 0 \quad \text{on } \gamma \setminus \gamma_2, \tag{6-19}$$

$$v = u_\nu^+, \quad w = u_\tau^+, \quad \sigma_\nu^- \le 0, \quad \sigma_\nu^-[u_\nu] = 0 \quad \text{on } \gamma \setminus \gamma_1, \tag{6-20}$$

$$[u_\nu] \ge 0, \quad \sigma_\tau^- = 0 \quad \text{on } \gamma \setminus \gamma_1, \tag{6-21}$$

$$u_\nu^+ - v \ge 0, \quad \sigma_\tau^+ = 0, \quad \sigma_\nu^+ \le 0, \quad \sigma_\nu^+(u_\nu^+ - v) = 0 \quad \text{on } \gamma_1 \cap \gamma_2, \tag{6-22}$$

$$v - u_\nu^- \ge 0, \quad \sigma_\tau^- = 0, \quad \sigma_\nu^- \le 0, \quad \sigma_\nu^-(v - u_\nu^-) = 0 \quad \text{on } \gamma_1 \cap \gamma_2. \tag{6-23}$$

*Proof.* We omit the proof, as it uses the same techniques as above. □

## 7. Anisotropic thin elastic inclusion

For the sake of completeness, we consider a case when the rigidity parameters of the elastic inclusion are different in the $x_1$ and $x_2$ directions. In this section we consider passages to limits for this situation. Assume that the rigidity parameter along the axis $x_2$ is fixed, and we change the rigidity parameter along the axis $x_1$. For a given parameter $\delta > 0$, the problem formulation is as follows:

*Find* $(u^\delta, v^\delta, w^\delta)$ *such that*

$$(u^\delta, v^\delta, w^\delta) \in K, \tag{7-1}$$

$$\int_{\Omega_\gamma} \sigma(u^\delta)\varepsilon(\bar{u}-u^\delta) - \int_{\Omega_\gamma} f(\bar{u}-u^\delta) + \int_\gamma v^\delta_{xx}(\bar{v}_{xx}-v^\delta_{xx}) + \delta\int_\gamma w^\delta_x(\bar{w}_x - w^\delta_x) \ge 0, \quad \forall\,(\bar{u},\bar{v},\bar{w}) \in K. \tag{7-2}$$

Our aim is to pass to the limit in (7-1) and (7-2) as $\delta \to +\infty$ and $\delta \to 0$. We omit a justification of the limiting procedures, and just formulate the limit problems. Observe that this justification recalls those of Sections 4 and 5.

**7.1.** ***Passage to the limit as $\delta \to +\infty$.*** The formulation of the limiting problem is as follows. We have to find a displacement field $u = (u_1, u_2)$, a stress tensor $\sigma = \{\sigma_{ij}\}$, $i, j = 1, 2$, and thin inclusion displacements $q_0 \in \mathbb{R}$ and $v$ defined in $\Omega_\gamma$, $\Omega_\gamma$, and $\gamma$, respectively, such that

$$-\operatorname{div}\sigma = f \quad \text{in } \Omega_\gamma, \tag{7-3}$$

$$\sigma - A\varepsilon(u) = 0 \quad \text{in } \Omega_\gamma, \tag{7-4}$$

$$v_{xxxx} = [\sigma_\nu] \quad \text{on } \gamma, \tag{7-5}$$

$$u = 0 \quad \text{on } \Gamma, \tag{7-6}$$

$$v_{xx} = v_{xxx} = 0 \quad \text{for } x = 0, 1, \tag{7-7}$$

$$[u_\nu] \geq 0, \quad v = u_\nu^-, \quad q_0 = v_\tau^- \quad \text{on } \gamma, \tag{7-8}$$

$$\sigma_\nu^+ \leq 0, \quad \sigma_\nu^+[u_\nu] = 0, \quad \sigma_\tau^+ = 0 \quad \text{on } \gamma, \tag{7-9}$$

$$\int_\gamma \sigma_\tau^- = 0. \tag{7-10}$$

We remark that the inclusion $\gamma$ in the limit problem (7-3)–(7-10) can be interpreted as a semirigid beam inclusion. It is possible to give a variational formulation of the problem (7-3)–(7-10).

**7.2.** ***Passage to the limit as $\delta \to 0$.*** In this case the formulation of the limiting problem is the following. We have to find a displacement field $u = (u_1, u_2)$, a stress tensor $\sigma = \{\sigma_{ij}\}$, $i, j = 1, 2$, and a thin inclusion displacement $v$ defined in $\Omega_\gamma$, $\Omega_\gamma$, and $\gamma$, respectively, such that

$$-\operatorname{div}\sigma = f \quad \text{in } \Omega_\gamma, \tag{7-11}$$

$$\sigma - A\varepsilon(u) = 0 \quad \text{in } \Omega_\gamma, \tag{7-12}$$

$$v_{xxxx} = [\sigma_\nu] \quad \text{on } \gamma, \tag{7-13}$$

$$u = 0 \quad \text{on } \Gamma, \tag{7-14}$$

$$v_{xx} = v_{xxx} = 0 \quad \text{for } x = 0, 1, \tag{7-15}$$

$$[u_\nu] \geq 0, \quad v = u_\nu^- \quad \text{on } \gamma, \tag{7-16}$$

$$\sigma_\nu^+ \leq 0, \quad \sigma_\tau^\pm = 0, \quad \sigma_\nu^+[u_\nu] = 0 \quad \text{on } \gamma. \tag{7-17}$$

Note that the thin inclusion $\gamma$ in the limit problem (7-11)–(7-17) describes only vertical displacements of the beam, and tangential displacements of the beam coincide with the tangential displacements of the elastic body at $\gamma^-$. We omit a variational formulation of the problem (7-11)–(7-17) since this model was analyzed in [Khludnev and Negri 2012].

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