

NISSUNA UMANA INVESTIGAZIONE SI PUO DIMANDARE VERA SCIENZA
S'ESSA NON PASSA PER LE MATEMATICHE DIMOSTRAZIONI
LEONARDO DA VINCI

vol. 3

no. 2

2015

MATHEMATICS AND MECHANICS
of
Complex Systems

OLIVIER PANTZ AND KARIM TRABELSI

**DERIVATION OF NONLINEAR SHELL MODELS
COMBINING SHEAR AND FLEXURE:
APPLICATION TO BIOLOGICAL MEMBRANES**



DERIVATION OF NONLINEAR SHELL MODELS COMBINING SHEAR AND FLEXURE: APPLICATION TO BIOLOGICAL MEMBRANES

OLIVIER PANTZ AND KARIM TRABELSI

Biological membranes are often idealized as incompressible elastic surfaces whose strain energy only depends on their mean curvature and possibly on their shear. We show that this type of model can be derived using a formal asymptotic method by considering biological membranes to be thin, strongly anisotropic, elastic, locally homogeneous bodies.

1. Introduction

Shells, plates and membranes are solid deformable bodies having one characteristic dimension small by comparison with the other two dimensions. Their behavior is fully described by standard three-dimensional laws of continuum mechanics. Nevertheless, it is tempting, at least from the modeling viewpoint, to consider them as two-dimensional structures and to replace the genuine mechanical laws by two-dimensional reduced versions. This immediately raises two questions: (1) What is the correct model? and (2) How can it be mathematically justified? To this end, we consider the thickness ε of the plate/shell/membrane as a parameter and identify the limit behavior of the structure as ε goes to zero. According to the dependence of the elasticity moduli on the thickness of the shell, a full zoology of models may be derived. Membrane, isometric bending and von Kármán theories have been justified (amongst others), first formally (see Fox, Raoult and Simo [Fox et al. 1993]), then by means of Γ -convergence (see [Le Dret and Raoult 1995; 1996; Pantz 2003], Müller, Friesecke and James [Friesecke et al. 2002], Friesecke, James and Mora [Friesecke et al. 2003]; see also [Conti et al. 2006]). In those works, elasticity coefficients are assumed to scale like a power of the thickness ε of the plate or shell, that is, like $\varepsilon^{-\alpha}$. Membrane theory corresponds to the case $\alpha = 1$,

Communicated by Gilles A. Francfort.

The authors would like to thank the referees for their valuable comments, which helped to improve the manuscript.

This work was partly supported by the ANR Geometrya.

O. Pantz is a member of the DEFI team at INRIA Saclay Île-de-France.

MSC2010: 74K20, 74K25, 74B20.

Keywords: shell, nonlinear elasticity, Helfrich, red blood cell, vesicle.

isometric bending to the case $\alpha = 3$ and von Kármán to $\alpha = 4$. Intermediate values of α have also been considered, and an almost exhaustive hierarchy of models has thus been produced (see Müller, Friesecke and James [Friesecke et al. 2006]). Some cases remain to be treated; Conti and Maggi [2008], for instance, investigate the scaling of the energy corresponding to folds. The initial motivation for this article was the study of the mechanical behavior of red blood cells (RBCs), and our aim was to determine whether the classical RBC model could be derived by the above procedure.

Mature anucleate RBCs¹ are made of two mechanical structures: the cytoskeleton — a two-dimensional network of protein filaments that extends throughout the interior of the cell — and a lipid bilayer. Both are bound together by proteins linking the nodes of the mesh of the cytoskeleton to the lipid bilayer via transmembrane proteins. Lipid bilayers are self-assembled structures of phospholipids, which are small molecules containing a negatively charged phosphate group (called the head), and two highly hydrophobic fatty acid chains (called the tails). In an aqueous environment, phospholipids spontaneously form a double layer whose configuration isolates the hydrophobic tails from the surrounding water molecules. Modifying the area of such a lipid bilayer is energy-costly because it exposes some of the tails to the environment.

A bilayer that supports no other mechanical structure, is connected and has no boundary is called a vesicle. Vesicles are massively studied because they are easy to obtain experimentally. Moreover, they partially mimic the behavior of RBCs. Roughly speaking, they are RBCs without cytoskeleton (even if the RBC bilayer does embed a lot of different proteins responsible for different functions of the cell). They similarly resist bending. However, vesicles show no resistance to shear stress, while RBCs do, owing to their cytoskeletons.

A widely used model consists in considering that a lipid bilayer may be endowed with an elastic energy depending solely on the mean curvature of the vesicle. This energy is usually known as the Helfrich functional (named after Willmore in other contexts). It was introduced independently, as far as we know, by Canham [1970] and Helfrich [1973] some forty years ago. Evans [1974] has shown that the Helfrich functional can be derived by assuming a vesicle to be made of two interconnected elastic fluid membranes, each of them resistant to change of local area but not to bending itself. Jenkins [1977a] has extended the analysis of Helfrich to general two-dimensional liquid crystals [Singer and Nicolson 1972]. In particular, he derives the Euler–Lagrange equations satisfied by the equilibrium states, and examines the consequences of fluidity on the form of the strain energy (see also [Steigmann

¹Every mention to RBCs in this article will implicitly refer to anucleate mature RBCs without further notice.

1999]). As a means to take into account the various vesicle shapes observed, it is common to presume that the vesicle is endowed with a nonzero spontaneous curvature. The origin of this spontaneous curvature is usually attributed to different compositions of the outer and inner layers. Several refinements to this basic model have since been proposed as the so-called bilayer-couple model [Svetina and Žekš 1989], that consists in allowing the two lipid layers to slip on one another, and imposing that the total area of each layer remains constant (see also [Seifert et al. 1991] for a comparison between the two models). Miao, Seifert, Wortis and Döbereiner [Miao et al. 1994] proposed an intermediate model, called the area-difference elasticity model, where slight total area changes of each layer are allowed but still penalized.

As previously mentioned, the mechanical structure of the RBC is not only imputable to its bilayers. Their cytoskeleton endows them with resistance to shear stress. In most models, only the deformation of the RBC membrane is considered (that is, of the bilayer). To take into account the presence of the cytoskeleton an additional term is added to the total energy depending on the change of the metric of the membrane. Krishnaswamy [1996] proposed another model for which the deformations of the cytoskeleton and the fluid bilayer may differ.

The aforementioned models for vesicles and RBCs are backed up by numerous numerical studies that reproduce various shapes observed experimentally. Amongst others, Deuling and Helfrich [1976] (see also [Jenkins 1977b; Luke 1982; Luke and Kaplan 1979]) have computed axisymmetric vesicle shapes of minimum energy with respect to the values of the reduced volume and spontaneous curvature. Seifert, Berndl and Lipowsky [Seifert et al. 1991] have compared the axisymmetric solutions obtained using the spontaneous curvature model and the bilayer-couple model, whereas Agrawal and Steigmann [2009] have included contact conditions between the vesicle and a substrate. Full three-dimensional simulations have been performed by Feng and Klug [2006], Bonito, Nochetto and Pauletti [Bonito et al. 2010; 2011], Dziuk [2008] using a finite element method. Peng et al. [2013] use a dissipative particle dynamic approach and focus on the interaction between the lipid bilayer and the cytoskeleton. Du, Chun and Xiaoqiang perform numerical computations based on a phase field method [Du et al. 2006; 2004; Du and Zhang 2008]. Boundary integral methods have been used by Veerapaneni, Gueyffier and Zorin [Veerapaneni et al. 2009], Sohn, Tseng, Li, Voigt and Lowengrub [Sohn et al. 2010]. Another approach based on the immersed boundary method has been investigated by Kim and Lai [2010], Liu et al. [2004; 2006] and, together with a lattice Boltzmann approach, by Cowl and Fogelson [2010]. Finally, level set methods have also been implemented in this context by Salac and Miksis [2011], and Maitre, Milcent, Cottet, Raoult and Usson [Maitre et al. 2009] (see also [Doyeux et al. 2013]).

We prove in this article that the classical mechanical model of the RBCs can be recovered by means of a formal asymptotic analysis assuming that the RBC's membrane is made of a locally homogeneous, albeit strongly anisotropic, nonlinearly elastic material. The main difference with previous works on the justification of thin structures is that we assume different scalings for the elastic moduli in the tangential and normal directions to the midsection. Let us underline that our work cannot be considered as a justification of the classical RBC mechanical model. Indeed, the RBC is not a locally homogeneous elastic membrane, firstly because it is made of two different structures: the lipid bilayer (responsible for the resistance to bending) and a cytoskeleton (responsible for resistance to shear). Even the lipid bilayer could hardly be considered as made of a homogeneous material, the scale of the phospholipids it contains being of the same order as the thickness of the membrane. The cytoskeleton, being a two-dimensional spectrin network, is no more a homogeneous elastic body. Even if it is not overt at first glance, our work is strongly related to the justification, already mentioned, proposed in [Evans 1974].

We have chosen to consider a rather general setting (presented in Section 2) for which the modeling of the RBCs is obtained as a particular case (see Section 6). The asymptotic analysis is performed in Section 3. Assuming that the minimizers of the energy admit an asymptotic expansion with respect to the thickness (Section 3.2), they converge toward the solutions of a two-dimensional problem (see Section 3.3). The limit energy, computed in Section 3, contains membrane and flexural terms. In Section 4, we prove that, under invariance assumptions on the stored energy of the material, the flexural term depends only upon the second fundamental form, or even only upon the mean curvature of the shell. The isometric bending shell, RBC and vesicle models are obtained as particular applications in Section 6. The last section is devoted to some general remarks, in particular on the relaxation of the formal energy limit.

Finally, let us specify some notation. If M is a differentiable manifold, we denote by TM and T^*M its tangent and cotangent bundles. Moreover, $T^*(M; \mathbb{R}^3)$ will stand for the triple Whitney sum $T^*M \oplus T^*M \oplus T^*M$. The tangent spaces of a product of manifolds will be implicitly identified with the product of the tangent spaces, so that if M_1 and M_2 are differentiable manifolds and $M = M_1 \times M_2$, the bundle TM will be implicitly identified with $TM_1 \times TM_2$. If M is an open subset of \mathbb{R}^N , TM will be identified with $M \times \mathbb{R}^N$. The corresponding identifications will also be made for T^*M and $T^*(M; \mathbb{R}^3)$. The set of reals \mathbb{R} and its dual \mathbb{R}' will also be often implicitly identified. Sets will always be displayed with capital letters (for instance, the set of deformations ψ^ε will be denoted Ψ^ε). Sequences of terms of an asymptotic expansion are denoted using bold letters (for instance, $\boldsymbol{\psi} = (\psi_k)_{k \in \mathbb{N}}$ stands for the asymptotic expansion of ψ^ε). Accordingly, the sets of asymptotic expansions use both bold and capitalized letters (for instance, $\boldsymbol{\psi} \in \boldsymbol{\Psi}$). Moreover,

calligraphic letters will be exclusively used for fiber spaces. Two different reference configurations are used throughout our article; one is qualified to be abstract and the other geometrical. The same notations are used for both configurations, the only distinction being that a tilde is added over variables, sets and functionals defined on the geometric configuration (for instance, $\tilde{\psi}^\varepsilon$ is the deformation defined on the geometric configuration, whereas ψ^ε stands for the deformation over the abstract one). All of the notation introduced is recalled at the end of the article for convenience.

2. Elastic shells — three-dimensional modeling

We consider a thin nonlinearly elastic shell of midsurface S' and constant half-thickness $\varepsilon > 0$, and choose $S^\varepsilon = S' \times (-\varepsilon, \varepsilon)$ to be the reference configuration of this elastic body. We assume S' to be a regular two-dimensional orientable submanifold of \mathbb{R}^3 with or without boundary. In the following, S' is implicitly endowed with the metric induced by the Euclidean metric in \mathbb{R}^3 . Let ψ^ε be the deformation of the shell, that is, a map from S^ε into \mathbb{R}^3 . The differential $D\psi^\varepsilon(x^\varepsilon)$ of ψ^ε at x^ε is a linear map from $T_{x^\varepsilon}S^\varepsilon$ into $T_{\psi^\varepsilon(x^\varepsilon)}\mathbb{R}^3$. Since $T_{\psi^\varepsilon(x^\varepsilon)}\mathbb{R}^3$ is canonically isomorphic to \mathbb{R}^3 , $D\psi^\varepsilon(x^\varepsilon)$ is identified with an element of the Whitney sum $T^*S^\varepsilon \oplus T^*S^\varepsilon \oplus T^*S^\varepsilon$ denoted by $T^*(S^\varepsilon; \mathbb{R}^3)$. We denote by $J_\varepsilon(\psi^\varepsilon)$ the elastic energy of the shell under the deformation ψ^ε . We assume that the elastic energy is local and depends only on the first derivatives of the deformation. In other words, there exists a map W^ε from $T^*(S^\varepsilon; \mathbb{R}^3)$ into $\bar{\mathbb{R}}^+$ such that

$$J_\varepsilon(\psi^\varepsilon) := \int_{S^\varepsilon} W^\varepsilon(D\psi^\varepsilon) dx^\varepsilon,$$

where $dx^\varepsilon = dx' \wedge dx_3^\varepsilon$ and dx' is the two-dimensional Hausdorff measure restricted to S' , whereas $D\psi^\varepsilon(x^\varepsilon)$ stands for the differential of ψ^ε at $x^\varepsilon \in S^\varepsilon$. Note that this representation enables us to consider inhomogeneous shells. The shell is assumed to be subjected to volumic dead-body loads $f_\varepsilon \in L^2(S^\varepsilon)^3$, and we set

$$L_\varepsilon(\psi^\varepsilon) := \int_{S^\varepsilon} f_\varepsilon \cdot \psi^\varepsilon dx^\varepsilon.$$

The total energy of the system is accordingly given by

$$I_\varepsilon(\psi^\varepsilon) := J_\varepsilon(\psi^\varepsilon) - L_\varepsilon(\psi^\varepsilon).$$

Finally, boundary conditions may also be added. We set $\Gamma^\varepsilon = \gamma \times (-\varepsilon, \varepsilon)$, where $\gamma \subset \partial S'$ is the — possibly empty — part of the boundary where the shell is clamped, and we denote by ϕ^ε the imposed deformation on this set. Our aim is to determine the behavior of the minimizers φ^ε of I_ε over

$$\Psi^\varepsilon := \{\psi^\varepsilon \in W^{1,\infty}(S^\varepsilon)^3 : \psi^\varepsilon(x^\varepsilon) = \phi^\varepsilon(x^\varepsilon) \text{ for every } x^\varepsilon \in \Gamma^\varepsilon\}$$

as ε goes to zero. Note that the minimization problem of I_ε over Ψ^ε without any growth and polyconvex or quasiconvex assumptions on the stored energy function is generally not well posed. Here, we implicitly assume this problem to have a regular solution. Various assumptions have to be made regarding the dependence of the energy on the thickness for the needs of our analysis. These mainly concern the stored energy W^ε (see [Section 2.1](#)), but also the applied loads (see [Section 2.2](#)).

2.1. Dependence of the stored energy functions with respect to the thickness.

We set $S = S^1$, and assume the stored energy W^ε to be of the form

$$W^\varepsilon(F) = \varepsilon^{-1}(\varepsilon^{-2}W_2(F) + W_0(F)) \quad (1)$$

for every $F \in T^*(S^\varepsilon; \mathbb{R}^3)$ and $\varepsilon \leq 1$, where W_0 and W_2 are continuous nonnegative maps from $T^*(S; \mathbb{R}^3)$ into $\bar{\mathbb{R}}^+$. Note that we implicitly use the injection of

$$T^*(S^\varepsilon; \mathbb{R}^3) = T^*(S' \times (-\varepsilon, \varepsilon); \mathbb{R}^3) = T^*(S'; \mathbb{R}^3) \times T^*((-\varepsilon, \varepsilon); \mathbb{R}^3)$$

into

$$T^*(S; \mathbb{R}^3) = T^*(S' \times (-1, 1); \mathbb{R}^3) = T^*(S'; \mathbb{R}^3) \times T^*((-1, 1); \mathbb{R}^3)$$

in the definition (1) of W^ε . Standard analysis focuses on the case where only one element of this expansion is not zero. For instance, if $W_2 = 0$, we recover a nonlinear membrane model [[Le Dret and Raoult 1996](#)], and if $W_0 = 0$ we obtain the isometric bending one [[Friesecke et al. 2003](#)].

Behavior of strongly extended fibers. We assume that the stored energy W_2 is bounded from below by a positive constant for strongly extended fibers, namely, there exist $\delta, c > 0$ such that

$$W_2(F', F_3) \geq c$$

$$\text{for all } F' \in T^*(S'; \mathbb{R}^3), F_3 \in T^*((-1, 1); \mathbb{R}^3) \text{ such that } |F_3| \geq \delta. \quad (2)$$

Note that for every element F of a vector bundle endowed with a Riemann metric, the notation $|F|$ should be understood as the norm of the vectorial part of F . In particular, in (2), $|F_3| = |v|$ if $F_3 = (x_3, v) \in (-1, 1) \times \mathbb{R}^3 = T^*((-1, 1); \mathbb{R}^3)$.

Regularity and zero set of W_2 . We assume that W_2 is a nonnegative C^2 function and denote by \mathcal{M} the restriction of its zero set to the midsection, that is,

$$\mathcal{M} := \{F \in T^*(S'; \mathbb{R}^3) \times T_0^*((-1, 1); \mathbb{R}^3) : W_2(F) = 0\}.$$

Let \mathcal{M}' be the projection of \mathcal{M} onto $T^*(S'; \mathbb{R}^3)$, that is,

$$\mathcal{M}' := \{F' \in T^*(S'; \mathbb{R}^3) : \text{there exists } n_0 \in T_0^*((-1, 1); \mathbb{R}^3) \text{ with } (F', n_0) \in \mathcal{M}\}.$$

We assume that the projection of \mathcal{M} onto \mathcal{M}' is one-to-one. We denote by $n_0 : \mathcal{M}' \rightarrow T_0^*((-1, 1); \mathbb{R}^3)$ the function that maps every element F' of \mathcal{M}' to the corresponding element F_3 of $T_0^*((-1, 1); \mathbb{R}^3)$, so that

$$\mathcal{M} = \{(F', n_0(F')) \in T^*(S; \mathbb{R}^3) : F' \in \mathcal{M}'\}. \quad (3)$$

We recall that $S = S' \times (-1, 1)$ and that $T^*(S; \mathbb{R}^3)$ is identified with $T^*(S'; \mathbb{R}^3) \times T^*((-1, 1); \mathbb{R}^3)$. The vectorial part of $n_0(F') \in T_0^*((-1, 1); \mathbb{R}^3) = \{0\} \times \mathbb{R}^3$ will be denoted $n(F')$, so that $n_0(F') = (0, n(F'))$.

Local interpenetration. To avoid local interpenetration of matter, it is geometric to expect $D\psi^\varepsilon$ to be invertible. To this end, we require that $W_0(F) = \infty$ for every $F \in \mathcal{M}$ such that $\det F < 0$, and that

$$W_0(F) \rightarrow \infty \quad \text{if } F \in \mathcal{M} \text{ and } \det(F) \rightarrow 0. \quad (4)$$

2.2. Dependence of the applied loads on the thickness. The volumic loads are assumed to scale as the inverse of the thickness of the shell; more precisely, we assume that there exists $f : S \rightarrow \mathbb{R}^3$ such that, for every $\varepsilon \leq 1$,

$$f_\varepsilon(x) = \varepsilon^{-1} f(x) \quad \text{for every } x \in S^\varepsilon. \quad (5)$$

3. From 3D to 2D: a formal asymptotic analysis

3.1. Rescaling. We set $\psi(\varepsilon)(x', x_3) = \psi^\varepsilon(x', \varepsilon x_3)$, and define rescaled energies

$$J(\varepsilon)(\psi(\varepsilon)) := J_\varepsilon(\psi^\varepsilon) \quad \text{and} \quad I(\varepsilon)(\psi(\varepsilon)) := I_\varepsilon(\psi^\varepsilon).$$

The minimization problem of I_ε over Ψ^ε is then equivalent to the minimization problem of $I(\varepsilon)$ over

$$\Psi(\varepsilon) := \{\psi(\varepsilon) \in W^{1,\infty}(S)^3 : \psi(\varepsilon)(x) = \phi(\varepsilon)(x) \text{ for every } x \in \Gamma\},$$

where $\phi(\varepsilon)(x) = \phi^\varepsilon(x', \varepsilon x_3)$.

For every map $\psi^\varepsilon : S^\varepsilon \rightarrow \mathbb{R}^3$, we denote by $(D'\psi^\varepsilon, D_3\psi^\varepsilon)$ the decomposition of the differential ψ^ε along the sections of the cylinder S^ε and along its fibers, respectively. In other words, for every $x^\varepsilon = (x', x_3) \in S^\varepsilon$, $D'\psi^\varepsilon(x^\varepsilon)$ and $D_3\psi^\varepsilon(x^\varepsilon)$ stand for the elements of $T_{x'}^*(S'; \mathbb{R}^3)$ and $T_{x_3}^*((-1, 1); \mathbb{R}^3)$ such that $D\psi^\varepsilon(x^\varepsilon) = (D'\psi^\varepsilon(x^\varepsilon), D_3\psi^\varepsilon(x^\varepsilon))$.

For every deformation $\psi(\varepsilon)$ of S , we define its partial derivative $\partial_3\psi(\varepsilon)$ with respect to the normal direction as

$$\partial_3\psi(\varepsilon)(x', x_3) = \lim_{t \rightarrow 0} \frac{\psi(\varepsilon)(x', x_3 + t) - \psi(\varepsilon)(x', x_3)}{t}.$$

Performing a simple change of variable, we get

$$\begin{aligned} J(\varepsilon)(\psi(\varepsilon)) &= \varepsilon^{-1} \int_{S^\varepsilon} (\varepsilon^{-2} W_2 + W_0)(D' \psi^\varepsilon(x^\varepsilon), D_3 \psi^\varepsilon(x^\varepsilon)) dx^\varepsilon \\ &= \varepsilon^{-1} \int_{S^\varepsilon} (\varepsilon^{-2} W_2 + W_0)(D' \psi^\varepsilon(x^\varepsilon), (x_3^\varepsilon, \partial_3 \psi^\varepsilon(x^\varepsilon))) dx^\varepsilon \\ &= \int_S (\varepsilon^{-2} W_2 + W_0)(D' \psi(\varepsilon)(x), (\varepsilon x_3, \varepsilon^{-1} \partial_3 \psi(\varepsilon)(x))) dx. \end{aligned}$$

3.2. Ansatz. In order to perform our formal analysis, we assume that the minimizers $\varphi(\varepsilon)(x', x_3) = \varphi^\varepsilon(x', \varepsilon x_3)$ of the energy admit an asymptotic expansion

$$\varphi(\varepsilon)(x) = \sum_{k \geq 0} \varepsilon^k \phi_k(x) \quad \text{for every } x \in S, \quad (6)$$

with $(\phi_k) \in \ell^1(W^{1,\infty}(S)^3)$. Obviously, the same assumption has to be made on the applied Dirichlet boundary condition, and we let $\boldsymbol{\phi} = (\phi_k) \in \ell^1(W^{1,\infty}(S)^3)$ be the terms of the asymptotic expansion of the deformation $\phi(\varepsilon)(x) = \phi^\varepsilon(x', \varepsilon x_3)$ imposed on $\Gamma := \gamma \times (-1, 1)$, that is

$$\phi(\varepsilon)(x) = \sum_{k \geq 0} \varepsilon^k \phi_k(x) \quad \text{for every } x \in S. \quad (7)$$

The condition $\varphi^\varepsilon \in \Psi^\varepsilon$ reads as $\varphi_k(x) = \phi_k(x)$ for every $x \in \Gamma$. Consequently, we introduce the admissible set

$$\boldsymbol{\Psi} := \{\boldsymbol{\psi} = (\psi_k) \in \ell^1(W^{1,\infty}(S)^3) : \psi_k = \phi_k \text{ for every } x \in \Gamma\},$$

and the rescaled energies $\boldsymbol{J}(\varepsilon)$ and $\boldsymbol{I}(\varepsilon)$ from $\boldsymbol{\Psi}$ into $\bar{\mathbb{R}}$ defined by

$$\boldsymbol{J}(\varepsilon)(\boldsymbol{\psi}) := J(\varepsilon) \left(\sum_{k \geq 0} \varepsilon^k \psi_k \right) \quad \text{and} \quad \boldsymbol{I}(\varepsilon)(\boldsymbol{\psi}) := I(\varepsilon) \left(\sum_{k \geq 0} \varepsilon^k \psi_k \right). \quad (8)$$

3.3. Limit of the total energy. The first step of our analysis consists in computing the limit of $\boldsymbol{J}(\varepsilon)(\boldsymbol{\psi})$ as ε goes to zero for $\boldsymbol{\psi} \in \boldsymbol{\Psi}$. As we shall see in [Proposition 1](#), the limit of $\boldsymbol{J}(\varepsilon)$ contains two terms. Roughly speaking, one term measures the elastic energy due to the change of the metric of the midsection of the shell. It depends only on W_0 . The second term measures the elastic energy due to the variations of the orientation of its fibers. It depends on the second derivative of the stored energy function W_2 through a quadratic form $Q_{D' \psi_0}$.

In order to enhance the readability of the sequel, we introduce a practical notation. We recall that a section F of a vector bundle \mathcal{F} is a map from its base into \mathcal{F} such that $\pi_B(F)$ is the identity, where π_B stands for the projection of \mathcal{F} onto its

base B . Given such a section, we define the bundle map

$$\mathcal{F} \rightarrow T\mathcal{F}, \quad G \mapsto G_F = \frac{d}{dt}(F(\pi_B(G)) + tG)|_{t=0}. \quad (9)$$

Roughly speaking, G_F is the element G of $T_{F(\pi_B(G))}\mathcal{F}$. Similarly, for every $(x, v) \in \mathbb{R}^N \times \mathbb{R}^N$, we will sometimes denote $(x, v) \in T_x\mathbb{R}^N$ by v_x . For a section F' of \mathcal{M}' , for every $(G', s, v) \in T^*(S'; \mathbb{R}^3) \times \mathbb{R} \times (\mathbb{R}')^3$ we set

$$\mathcal{Q}_{F'}(G', s, v) := D^2W_2[G'_{F'}, s_0, v_{n(F')}]^2, \quad (10)$$

where $(G'_{F'}, s_0, v_{n(F')})$ is the element of $T_{(F', n_0(F'))}(T^*(S; \mathbb{R}^3))$ defined in (9) based on the decomposition of $T^*(S; \mathbb{R}^3) = T^*(S'; \mathbb{R}^3) \times (-1, 1) \times (\mathbb{R}')^3$, while D^2W_2 stands for the Hessian of W_2 . Namely, we have

$$(G'_{F'}, s_0, v_{n(F')}) = \frac{d\gamma}{dt}(0), \quad \text{where } \gamma(t) = (F'_{\pi_{S'}(G')} + tG', ts, n(F') + tv). \quad (11)$$

At first glance, the meaning of $D^2W_2[\dot{\gamma}(0)]^2$ is unclear, considering that the Hessian of a map defined on a manifold is not, in general, intrinsically defined. Nevertheless, it is well known that this is consistent on the set of critical points, which is precisely what is considered here. Indeed, $\gamma(0)$ is equal to the value of the section $(F', n_0(F'))$ at $\pi_{S'}(G)$. Yet F' is a section of \mathcal{M}' , hence $W_2(F', n_0(F)) = 0$, $W_2(\gamma(0)) = 0$ and $DW_2(\gamma(0)) = 0$. As a result, $D^2W_2[\dot{\gamma}(0)]^2$ is well defined, and, accordingly,

$$D^2W_2[\dot{\gamma}(0)]^2 = 2 \lim_{t \rightarrow 0} t^{-2} W_2(\gamma(t)). \quad (12)$$

Note that the right-hand side of (12) only depends on $\dot{\gamma}(0)$, so that the particular choice of the representative $\gamma(t)$ of $\dot{\gamma}(0)$ is irrelevant, as already mentioned.

We are now in a position to state the main result of this section.

Proposition 1. *Let Φ be the subset of the admissible set Ψ defined by*

$$\Phi := \{\psi \in \Psi : \partial_3\psi_0 = 0, D'\psi_0(x) \in \mathcal{M}', \text{ and } \partial_3\psi_1(x) = n(D'\psi_0(x)) \text{ for every } x \in S\}. \quad (13)$$

Let $\psi \in \Psi$. Then

$$\lim_{\varepsilon \rightarrow 0} \mathbf{I}(\varepsilon)(\psi) = \begin{cases} I_0(\psi_0, \frac{1}{2} \int_{-1}^1 \psi_1 dx_3, \partial_3\psi_2) & \text{if } \psi \in \Phi, \\ +\infty & \text{if } \psi \notin \Phi, \end{cases}$$

where

$$I_0(\psi_0, u, v) := J_0(\psi_0, u, v) - 2 \int_{S'} f_0 \cdot \psi_0 dx,$$

$$J_0(\psi_0, u, v) := \frac{1}{2} \int_S \mathcal{Q}_{D'\psi_0}(D'u + x_3 D'n, x_3, v) dx + 2 \int_{S'} W_0(D'\psi_0, n_0) dx',$$

where n and n_0 stand for $n(D'\psi_0)$ and $n_0(D'\psi_0)$, $Q_{D'\psi_0}$ is defined by (10), and $f_0(x') = f(x', 0)$.

Proof. We proceed in two steps. First, we prove that every sequence of deformations $\psi \in \Psi$ of finite elastic energy, that is, those that satisfy

$$\liminf_{\varepsilon \rightarrow 0} \mathbf{J}(\varepsilon)(\psi) < +\infty,$$

belongs to Φ . In particular, this implies that $\mathbf{J}(\varepsilon)(\psi)$ converges to infinity as ε goes to zero, for every ψ that is not in Φ . In a second step, we compute the limit of $\mathbf{J}(\varepsilon)(\psi)$ for every ψ in Φ .

Let $\psi \in \Psi$ be the asymptotic expansion of a deformation of finite elastic energy. From Fatou's lemma, we deduce

$$\begin{aligned} & \int_S \liminf_{\varepsilon \rightarrow 0} (\varepsilon^{-2} W_2) \left(\sum_{k \geq 0} \varepsilon^k (D'\psi_k, (\varepsilon x_3, \varepsilon^{-1} \partial_3 \psi_k)) \right) dx \\ & \leq \liminf_{\varepsilon \rightarrow 0} \int_S (\varepsilon^{-2} W_2 + W_0) \left(\sum_{k \geq 0} \varepsilon^k (D'\psi_k, (\varepsilon x_3, \varepsilon^{-1} \partial_3 \psi_k)) \right) dx \\ & = \liminf_{\varepsilon \rightarrow 0} \mathbf{J}(\varepsilon)(\psi) < \infty. \end{aligned}$$

Hence, we have

$$\liminf_{\varepsilon \rightarrow 0} W_2 \left(\sum_{k \geq 0} \varepsilon^k (D'\psi_k, (\varepsilon x_3, \varepsilon^{-1} \partial_3 \psi_k)) \right) = 0 \quad \text{a.e.}$$

From the assumption (2) made on the behavior of strongly extended fibers, it follows that, for almost every $x \in S$, $\sum_{k \geq 0} \varepsilon^{k-1} \partial_3 \psi_k$ remains bounded (up to a subsequence), that is, $\partial_3 \psi_0 = 0$. Now, since

$$\sum_{k \geq 0} \varepsilon^k (D'\psi_k, (\varepsilon x_3, \varepsilon^{-1} \partial_3 \psi_k)) \xrightarrow[\varepsilon \rightarrow 0]{L^\infty} (D'\psi_0, (0, \partial_3 \psi_1)), \quad (14)$$

and W_2 is assumed to be continuous, we have $W_2(D'\psi_0, (0, \partial_3 \psi_1)) = 0$ almost everywhere. From the hypothesis (3), we get $D'\psi_0 \in \mathcal{M}$ and $\partial_3 \psi_1 = n(D'\psi_0)$. As a conclusion, every sequence of deformations of finite elastic energy belongs to Φ , as announced.

For the next step, let us consider an element $\psi \in \Phi$ and its associated energy

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \mathbf{J}(\varepsilon)(\psi) &= \lim_{\varepsilon \rightarrow 0} \varepsilon^{-2} \int_S W_2 \left(\sum_{k \geq 0} \varepsilon^k (D'\psi_k, (\varepsilon x_3, \varepsilon^{-1} \partial_3 \psi_k)) \right) dx \\ & \quad + \int_S W_0(D'\psi_0, (0, \partial_3 \psi_1)) dx. \end{aligned}$$

Since W_2 is a C^2 function and $W_2(D'\psi_0, (0, \partial_3\psi_1)) = 0$, using (12) with $\gamma(t) = (D'\psi_0 + tD'\psi_1, tx_3, \partial_3\psi_1 + t\partial_3\psi_2)$, Lebesgue's theorem implies

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \varepsilon^{-2} \int_S W_2 \left(\sum_{k \geq 0} \varepsilon^k (D'\psi_k, (\varepsilon x_3, \varepsilon^{-1} \partial_3 \psi_k)) \right) dx \\ = \frac{1}{2} \int_S D^2 W_2 [(D'\psi_1, D'\psi_0(x), x_{30}, \partial_3 \psi_2)_{\partial_3 \psi_1}]^2 dx \\ = \frac{1}{2} \int_S D^2 W_2 [(D'\psi_1, D'\psi_0(x), x_{30}, \partial_3 \psi_2)_{n(D'\psi_0)}]^2 dx \\ = \frac{1}{2} \int_S Q_{D'\psi_0}(D'\psi_1, x_3, \partial_3 \psi_2) dx. \end{aligned}$$

The limit of the elastic energy $J(\varepsilon)$ falls out from the fact that ψ_1 may be written as

$$\psi_1 = \frac{1}{2} \int_{-1}^1 \psi_1 dx_3 + x_3 n(D'\psi_0).$$

Finally, according to the definition (5) of f , we have

$$I(\varepsilon)(\psi) = J(\varepsilon)(\psi) - \int_S f(x', \varepsilon x_3) \cdot \left(\sum_{k \geq 0} \varepsilon^k \psi_k \right) dx.$$

The second term on the right-hand side converges to $2 \int_{S'} f \cdot \psi_0 dx$ as $\varepsilon \rightarrow 0$. \square

Since the limit energy is finite only for elements ψ in Φ , ϕ has to be equal to an element of Φ on the subset Γ of the boundary where clamping conditions are imposed.

Corollary 2. *If the minimizers $\varphi(\varepsilon)$ of the total rescaled energy $I(\varepsilon)$ over $\Psi(\varepsilon)$ admit an asymptotic expansion as in (6), and if their total energy $I(\varepsilon)(\varphi(\varepsilon))$ remains bounded, then $\phi_0(x', x_3)$ depends only on $x' \in \Gamma$. In addition, there exist $u_\gamma, n_\gamma \in W^{1,\infty}(\gamma)^3$ such that*

$$\phi_1(x', x_3) = u_\gamma(x') + x_3 n_\gamma(x') \quad \text{for } x \in \Gamma.$$

Note: since ϕ_0 depends only on x' , we shall write $\phi_0(x')$ instead of $\phi_0(x', x_3)$ henceforth.

3.4. Convergence of the minimizers.

Lemma 3. *If the minimizers $\varphi(\varepsilon)$ of the total rescaled energy $I(\varepsilon)$ over $\Psi(\varepsilon)$ admit an asymptotic expansion as in (6), and if their total energy $I(\varepsilon)(\varphi(\varepsilon))$ remains bounded, then*

$$I_0 \left(\varphi_0, \frac{1}{2} \int_{-1}^1 \varphi_1 dx_3, \partial_3 \varphi_2 \right) \leq \inf_{(\psi_0, u, v) \in \Phi_0} I_0(\psi_0, u, v),$$

where Φ_0 is the set of all

$$(\psi_0, u, v) \in W^{1,\infty}(S')^3 \times W^{1,\infty}(S')^3 \times W^{1,\infty}(S'; L^\infty(-1, 1)^3)$$

such that $D'\psi_0 \in \mathcal{M}$ a.e., $n(D'\psi_0) \in W^{1,\infty}(S')^3$, $\psi_0(x') = \phi_0(x')$, $u(x') = u_\gamma(x')$, $n(D'\psi_0(x')) = n_\gamma(x')$ for every $x' \in \gamma$, and $v(x) = \partial_3 \phi_2(x)$ for $x \in \Gamma$.

Proof. Let (φ_k) be the asymptotic expansion of a minimizer $\varphi(\varepsilon)$ of the total energy $I(\varepsilon)$. For every $(\psi_0, u, v) \in \Phi_0$, we set

$$\begin{aligned} \psi_1 &= u + x_3 n(D'\psi_0), \\ \psi_2(x) &= \phi_2(x', 0) + \int_0^{x_3} v(x', s) ds, \\ \psi_k &= \phi_k \quad \text{for all } k \geq 3. \end{aligned}$$

Using the Leibniz integral rule, we get that $D'\psi_2 = D'\phi_2 + \int_0^{x_3} D'v(x', s) ds$ and $\partial_3 \psi_2 = v$. It follows that ψ_2 belongs to $W^{1,\infty}(S')^3$. As u and $n(D'\psi_0)$ are assumed to be Lipschitzian, so is ψ_1 , and $\psi \in \ell^1(W^{1,\infty}(S')^3)$. From [Corollary 2](#), ϕ_0 depends only on x' and $\phi_1(x', x_3) = u_\gamma(x') + x_3 n_\gamma(x')$ on Γ . As $(\psi_0, u, v) \in \Phi_0$, we infer that $\psi_0 = \phi_0$ and $\psi_1 = \phi_1$ on Γ . Similarly, $\psi_2 = \phi_2$ on Γ . Thus, ψ belongs to Φ and $I(\varepsilon)(\varphi) \leq I(\varepsilon)(\psi)$. Letting ε goes to zero, we get from [Proposition 1](#) that

$$I_0\left(\varphi_0, \frac{1}{2} \int_{-1}^1 \varphi_1 dx_3, \partial_3 \phi_2\right) \leq I_0(\psi_0, u, v). \quad (15)$$

This completes the proof. \square

Lemma 4. *If the minimizers $\varphi(\varepsilon)$ of the total rescaled energy $I(\varepsilon)$ over $\Psi(\varepsilon)$ admit an asymptotic expansion as in (6), and if their total energy $I(\varepsilon)(\varphi(\varepsilon))$ remains bounded, then*

$$\left(\varphi_0, \frac{1}{2} \int_{-1}^1 \varphi_1 dx_3\right) = \arg \min_{(\psi_0, u) \in \Phi_1} I_1(\psi_0, u),$$

where Φ_1 is the set of all (ψ_0, u) in $W^{1,\infty}(S')^3 \times W^{1,\infty}(S')^3$ such that $D'\psi_0 \in \mathcal{M}$ a.e., $n(D'\psi_0) \in W^{1,\infty}(S')^3$, $\psi_0(x') = \phi_0(x')$, $u(x') = u_\gamma(x')$, and $n(D'\psi_0(x')) = n_\gamma(x')$ for $x' \in \gamma$, I_1 is given by

$$\begin{aligned} I_1(\psi_0, u) &:= \int_{S'} Q_{D'\psi_0}^0(D'u, 0) dx' + \frac{1}{3} \int_{S'} Q_{D'\psi_0}^0(D'n, 1) dx' \\ &\quad + 2 \int_{S'} W_0(D'\psi_0, n_0(D'\psi_0)) dx' - 2 \int_{S'} f_0 \cdot \psi_0 dx', \end{aligned}$$

$f_0(x') = f(x', 0)$ for every $x' \in S'$, and

$$Q_{F'}^0(G', x_3) = \inf_{v \in \mathbb{R}^3} Q_{F'}(G', x_3, v). \quad (16)$$

Proof. From [Lemma 3](#), we have

$$I_0\left(\varphi_0, \frac{1}{2} \int_{-1}^1 \varphi_1 dx_3, \partial_3 \varphi_2\right) \leq \inf_{(\psi_0, u, v) \in \Phi_0} I_0(\psi_0, u, v).$$

Moreover, from [Proposition 1](#), we have $\varphi \in \Phi$, which implies that

$$\left(\varphi_0, \frac{1}{2} \int_{-1}^1 \varphi_1 dx_3\right) \in \Phi_1.$$

Since $\Phi_0 = \Phi_1 \times V$ with

$$V := \{v \in W^{1,\infty}(S'; L^\infty(-1, 1)^3) : v(x) = \partial_3 \phi_2(x) \text{ for } x \in \Gamma\},$$

it follows that

$$I_1\left(\varphi_0, \frac{1}{2} \int_{-1}^1 \varphi_1 dx_3\right) = \inf_{(\psi_0, u) \in \Phi_1} \inf_{v \in V} I_0(\psi_0, u, v).$$

To complete the proof, we need to show that, for every $(\psi_0, u) \in \Phi_1$, we have

$$\inf_{v \in V} I_0(\psi_0, u, v) = I_1(\psi_0, u). \quad (17)$$

We recall that for every $(\psi_0, u) \in \Phi_1$ and every $v \in V$, we have

$$\begin{aligned} I_0(\psi_0, u, v) &= \frac{1}{2} \int_S \mathcal{Q}_{D'\psi_0}(D'u + x_3 D'n, x_3, v) dx \\ &\quad + 2 \int_{S'} W_0(D'\psi_0, n_0(D'\psi_0)) dx' - 2 \int_{S'} f_0 \cdot \psi_0 dx'. \end{aligned}$$

Next, the definition of $\mathcal{Q}_{D'\psi_0}^0$ entails that

$$\begin{aligned} I_0(\psi_0, u, v) &\geq \frac{1}{2} \int_S \mathcal{Q}_{D'\psi_0}^0((D'u, 0) + x_3(D'n, 1)) dx \\ &\quad + 2 \int_{S'} W_0(D'\psi_0, n_0(D'\psi_0)) dx' - 2 \int_{S'} f_0 \cdot \psi_0 dx'. \end{aligned}$$

Furthermore, for every $x' \in S'$ and every $F \in T_{(x', 0)}^*(S; \mathbb{R}^3)$, the quadratic form \mathcal{Q}_F^0 derives from a bilinear form. Hence,

$$\begin{aligned} &\int_{-1}^1 \mathcal{Q}_{D'\psi_0}^0((D'u, 0) + x_3(D'n, 1)) dx_3 \\ &= \int_{-1}^1 (\mathcal{Q}_{D'\psi_0}^0(D'u, 0) + x_3^2 \mathcal{Q}_{D'\psi_0}^0(D'n, 1)) dx_3 \\ &= 2\mathcal{Q}_{D'\psi_0}^0(D'u, 0) + \frac{2}{3}\mathcal{Q}_{D'\psi_0}^0(D'n, 1). \end{aligned}$$

Accordingly, we obtain that $I_0(\psi_0, u, v) \geq I_1(\psi_0, u)$, so that $\inf_{v \in V} I_0(\psi_0, u, v) \geq I_1(\psi_0, u)$. It remains to prove the reverse inequality to establish (17). For every $\delta \geq 0$, we have

$$I_0(\psi_0, u, v) \leq I_0(\psi_0, u, v) + \int_S \delta |v|^2 dx.$$

As a consequence,

$$\inf_{v \in V} I_0(\psi_0, u, v) \leq \inf_{v \in V} \left(I_0(\psi_0, u, v) + \int_S \delta |v|^2 dx \right).$$

From the assumptions made on W_2 , it follows that the quadratic form $Q_{D'\psi_0}$ is positive semidefinite almost everywhere. As a consequence, the map

$$v \mapsto Q_{D'\psi_0}(D'u + x_3 D'n, x_3, v) + \delta |v|^2$$

admits a unique minimizer for almost every $x \in S$. Let $v_\delta : S \rightarrow \mathbb{R}^3$ be the map such that $v_\delta = \arg \min_v Q_{D'\psi_0}(D'u + x_3 D'n, x_3, v) + \delta |v|^2$.

Since W_2 is assumed to be of class C^2 and $D'\psi_0$ is bounded, the norm of the quadratic form $Q_{D'\psi_0}$ is uniformly bounded. As a result, v_δ is measurable and belongs to $L^\infty(S)^3$. Also, there exists a sequence v_δ^k in V converging to v_δ in $L^2(S)^3$ as k goes to infinity, due to the density of V in $L^2(S)^3$. For every k , we have

$$\begin{aligned} \inf_{v \in V} I_0(\psi_0, u, v) &\leq \frac{1}{2} \int_S (Q_{D'\psi_0}(D'u + x_3 D'n, x_3, v_\delta^k) + \delta |v_\delta^k|^2) dx \\ &\quad + 2 \int_{S'} W_0(D'\psi_0, n_0(D'\psi_0)) dx' - 2 \int_{S'} f_0 \cdot \psi_0 dx'. \end{aligned}$$

Taking the limit with respect to k , we infer that

$$\begin{aligned} \inf_{v \in V} I_0(\psi_0, u, v) &\leq \frac{1}{2} \int_S (Q_{D'\psi_0}(D'u + x_3 D'n, x_3, v_\delta) + \delta |v_\delta|^2) dx \\ &\quad + 2 \int_{S'} W_0(D'\psi_0, n_0(D'\psi_0)) dx' - 2 \int_{S'} f_0 \cdot \psi_0 dx'. \end{aligned}$$

Note that $Q_{D'\psi_0}(D'u + x_3 D'n, x_3, v_\delta) + \delta |v_\delta|^2$ is a decreasing sequence (as δ goes to zero) of nonnegative functions. Therefore, its integral over S converges to its pointwise limit $Q_{D'\psi_0}^0(D'u + x_3 D'n, x_3)$, and the intended inequality follows:

$$\begin{aligned} \inf_{v \in V} I_0(\psi_0, u, v) &\leq \frac{1}{2} \int_S Q_{D'\psi_0}^0(D'u + x_3 D'n, x_3) dx \\ &\quad + 2 \int_{S'} W_0(D'\psi_0, n) dx' - 2 \int_{S'} f_0 \cdot \psi_0 dx' = I_1(\psi_0, u). \quad \square \end{aligned}$$

3.5. Boundary conditions. An interesting feature of the limit energy is that it depends on both ψ_0 and $u = \frac{1}{2} \int \psi_1 dx_3$. For general boundary conditions, it implies a coupling between both quantities through the term $\int_{S'} \mathcal{Q}_{D'\psi_0}^0(D'u, 0) dx'$ of $I_1(\psi_0, u)$. Hence, small perturbations scaling as the thickness of the shell may have an influence on the deformation ψ_0 of the midsection. In the literature, the boundary conditions are usually chosen to satisfy $u_\gamma = 0$, that is,

$$\phi_0(x) = \phi_0(x') \quad \text{and} \quad \phi_1(x) = x_3 n_\gamma(x') \quad \text{for every } (x', x_3) \in \Gamma^\varepsilon, \quad (18)$$

where n_γ is a unit vector. In this case, the minimization of $I_1(\psi_0, u)$ with respect to u is trivial and the limit energy can be expressed solely in terms of ψ_0 .

Proposition 5. *If the minimizers $\varphi(\varepsilon)$ of the total rescaled energy $I(\varepsilon)$ over $\Psi(\varepsilon)$ admit an asymptotic expansion as in (6), and if their total energy $I(\varepsilon)(\varphi(\varepsilon))$ remains bounded with $u_\gamma = 0$ on γ , then*

$$\varphi_0 = \arg \min_{\psi_0 \in \Psi_0} I_0(\psi_0),$$

where

$$I_0(\psi_0) := \frac{1}{3} \int_{S'} \mathcal{Q}_{D'\psi_0}^0(D'n, 1) dx' + 2 \int_{S'} W_0(D'\psi_0, n_0(D'\psi_0)) dx' - 2 \int_{S'} f_0 \cdot \psi_0 dx', \quad (19)$$

$f_0(x') = f(x', 0)$ for every $x' \in S'$, and

$$\Psi_0 := \{\psi_0 \in W^{1,\infty}(S')^3 : n = n(D'\psi_0) \in W^{1,\infty}(S')^3, D'\psi_0 \in \mathcal{M}', \\ \psi_0(x') = \phi_0(x'), \text{ and } n = n_\gamma(x') \text{ for every } x' \in \gamma\}.$$

4. Invariance and flexural energy

Under several assumptions on the stored energy function W_2 , the expression of the flexural part

$$I_{\text{flex}}(\psi_0) := \frac{1}{3} \int_{S'} \mathcal{Q}_{D'\psi_0}^0(D'n, 1) dx \quad (20)$$

of the total limit energy $I_0(\psi_0)$ may be reduced. More precisely, we shall consider the implications of homogeneity along the fibers, frame-indifference (left invariance under $\text{SO}(3)$), planar isotropy (right invariance under in-plane rotations), and finally right invariance of the stored energy under the special linear group of TS' .

4.1. Homogeneity along the fibers. We say that the shell is homogeneous along the fibers if, for every

$$(F', x_3, v) \in T^*(S; \mathbb{R}^3) = T^*(S'; \mathbb{R}^3) \times (-1, 1) \times (\mathbb{R}')^3,$$

we have $W_2(F', x_3, v) = W_2(F', 0, v)$. In this case, for every

$$(G', s, v) \in T^*(S'; \mathbb{R}^3) \times \mathbb{R} \times (\mathbb{R}')^3$$

and every section F' of \mathcal{M}' , we have $D^2 W_2[G'_{F'}, s_0, v_{n(F')}]^2 = D^2 W_2[G'_{F'}, 0, v_{n(F')}]^2$. It follows that $Q_{F'}^0(G', s)$ is independent of s , and hence we denote it by $Q_{F'}^0(G')$, so that

$$I_{\text{flex}}(\psi_0) = \frac{1}{3} \int_{S'} Q_{D'\psi_0}^0(D'n) dx.$$

4.2. Frame-indifference. The principle of frame-indifference states that the space is invariant under rotation, which translates in our case to the following condition on the stored energy function W^ε :

$$W^\varepsilon(F) = W^\varepsilon(RF) \quad \text{for every rotation } R \in \text{SO}(3).$$

This is assumed in the sequel. Accordingly, W_2 satisfies the same property, i.e., $W_2(F) = W_2(RF)$ for every $R \in \text{SO}(3)$.

In the following, we denote by $\mathcal{E}_{S'}$ the set of symmetric bilinear forms on TS' , that is, the fiber bundle with base space S' whose fiber $(\mathcal{E}_{S'})_{x'}$ at $x' \in S'$ is the set of symmetric bilinear forms on $T_{x'}S'$. The fiber bundle \mathcal{E}_S is defined in a similar way, and \mathcal{E}'_S stands for its restriction to S' . Moreover, if $F \in T_x^*(S; \mathbb{R}^3)$, $F^T F$ stands for the element of $(\mathcal{E}_{S'})_x$ that maps every element (u, v) of $(T_x S)^2$ to the scalar product between Fu and Fv . A similar notation is used to define $(F')^T F' \in \mathcal{E}_{S'}$ for every $F' \in T^*(S'; \mathbb{R}^3)$.

Lemma 6. *If the stored energy W_2 is frame-indifferent, then for every $F' \in \mathcal{M}'$ of maximum rank and every $R \in \text{SO}(3)$, we have*

$$RF' \in \mathcal{M}' \quad \text{and} \quad n(RF') = Rn(F').$$

Moreover, there exists a bundle map $\tau' : \mathcal{M}' \rightarrow TS'$ and a map $\tau_3 : \mathcal{M}' \rightarrow \mathbb{R}$ such that, for every $F' \in \mathcal{M}'$ of maximal rank,

$$n(F') = F'\tau'(F') + n_{F'}\tau_3(F'),$$

where both $\tau'(F')$ and $\tau_3(F')$ depend only on $C' = (F')^T F'$, and $n_{F'} \in \mathbb{R}^3$ is defined by

$$n_{F'} \cdot w = \det(F', w) \quad \text{for every } w \in \mathbb{R}^3. \quad (21)$$

Lastly, $C = (F', n_0(F'))^T (F', n_0(F'))$ depends only on C' .

Proof. The first part of the proposition is obvious. Next, since F' is of maximum rank, $(F', n_{F'})$ is invertible, so we can set $(\tau'(F'), \tau_3(F')) = (F', n_{F'})^{-1} n(F')$.

Moreover, we can check that

$$\begin{aligned}
 (\tau'(RF'), \tau_3(RF')) &= (RF', n_{RF'})^{-1}n(RF) \\
 &= (RF', R(n_{F'}))^{-1}Rn(F) = (R(F', n_{F'}))^{-1}Rn(F) \\
 &= (F', n_{F'})^{-1}n(F') = (\tau'(F'), \tau_3(F')),
 \end{aligned}$$

whence both τ' and τ_3 only depend on $(F')^T F'$. Finally, it is readily verified that both $n(F')^T n(F')$ and $n(F')^T F'$ are invariant under rotations of F' . As a result, $C = (F', n_0(F'))^T (F', n_0(F'))$ depends only on $(F')^T F'$ as well. \square

Since $T_{(x', x_3)}S = T_{x'}S' \times T_{x_3}(-1, 1) = T_{x'}S' \times \mathbb{R}$, every element $C \in (\mathcal{E}_S)_x$ can be decomposed uniquely into $(C', C_3, C_{33}) \in (\mathcal{E}_{S'})_{x'} \times T_{x'}^*S' \times \mathbb{R}$ such that, for every (u', u_3) and every (v', v_3) in $T_{x'}S' \times \mathbb{R} = T_xS$,

$$C((u', u_3), (v', v_3)) = C'(u', v') + u_3 C_3(v') + v_3 C_3(u') + C_{33}u_3 v_3.$$

In addition, we write this decomposition as

$$C = \begin{pmatrix} C' & C_3^T \\ C_3 & C_{33} \end{pmatrix}.$$

Let us introduce the fiber bundle \mathcal{P} with base space S' whose fiber at $x' \in S'$ is the set of polynomials of degree less than or equal to two on $(\mathcal{E}_{S'})_{x'}$.

Proposition 7. *If the stored energy function W_2 is frame-indifferent, then there exists a bundle map $P : C' \mapsto P_{C'}$ over S' from $\mathcal{E}_{S'}$ into \mathcal{P} such that, for every deformation ψ_0 of finite limit energy $I_0(\psi_0)$ and every $G' \in T^*(S'; \mathbb{R}^3)$, we have*

$$Q_{D'\psi_0}^0(G', 1) = P_{C'}(D'\psi_0^T G' + G'^T D'\psi_0),$$

where $C' = D'\psi_0^T D'\psi_0$ for short. Moreover, if W_2 is homogeneous along the fibers, then $P_{C'}$ is homogeneous of degree two.

Proof. Let $\mathcal{M}^+ = \{F \in \mathcal{M} : \det F > 0\}$. Since W_2 is assumed to be frame-indifferent, there exists a map $W_S : \mathcal{E}_S \rightarrow \mathbb{R}$ such that, for every F in a neighborhood of \mathcal{M}^+ ,

$$W_2(F) = W_S \circ m(F), \tag{22}$$

where $m : T^*(S; \mathbb{R}^3) \rightarrow \mathcal{E}_S$ is the bundle map defined by $m(G) = G^T G$. Let F' be a section of \mathcal{M}' , $s \in \mathbb{R}$ and $G' \in T^*(S'; \mathbb{R}^3)$ such that $(F', n_0(F')) \in \mathcal{M}^+$ a.e. Then definition (10) combined with (22) gives

$$Q_{F'}^0(G', s) = \inf_{v \in \mathbb{R}^3} D^2 W_2[G'_{F'}, s_0, v_{n(F')}]^2 = \inf_{v \in \mathbb{R}^3} D^2 W_S[Dm(G'_{F'}, s_0, v_{n(F')})]^2.$$

Since $\mathcal{E}_S = (-1, 1) \times \mathcal{E}'_S$, we can identify $T\mathcal{E}_S$ with $T(-1, 1) \times T\mathcal{E}'_S$. Doing so, we obtain that

$$Dm(G'_{F'}, s_0, v_{n(F')}) = \frac{d}{dt}(ts, C + tE)(t=0) = (s_0, E_C),$$

where

$$E = \begin{pmatrix} (F')^T G' + (G')^T F' & (F')^T v + (G')^T n \\ v^T F' + n^T G' & n^T v + v^T n \end{pmatrix},$$

$n(F')$ is denoted by n for short, and $C = m(F)$. Since $(F', n(F'))(x')$ is assumed to be invertible for every $x' \in S'$, setting $w' = F'^T v + G'^T n$ and $w_3 = n^T v + v^T n$, we get

$$Q_{F'}^0(G', 1) = P_{C'}((F')^T G' + (G')^T F'), \quad (23)$$

where $P_{C'}$ is the section of \mathcal{P} defined by

$$P_{C'}(M) = \inf_{w \in \mathbb{R}^3} D^2 W_S \left[1_0, \begin{pmatrix} M & w' \\ (w')^T & w_3 \end{pmatrix}_C \right]^2 \quad (24)$$

for every $M \in \mathcal{E}_{S'}$, and $C = (F', n_0(F'))^T (F', n_0(F'))$, which depends only on C' , according to [Lemma 6](#). Finally, if ψ_0 is a deformation satisfying $I_0(\psi_0) < \infty$, owing to the noninterpenetration assumptions made, we know that $D\psi_0 \in \mathcal{M}^+$ a.e. As a result, $Q_{D'\psi_0}^0(D'n, 1) = P_{C'}(M)$ a.e. on S' , with $M = F^T D'n + D'n^T F'$, as claimed. Moreover, if the shell is homogeneous along its fibers, (24) reduces to

$$P_{C'}(M) = \inf_{w \in \mathbb{R}^3} D^2 W_S \left[0, \begin{pmatrix} M & w' \\ (w')^T & w_3 \end{pmatrix}_C \right]^2, \quad (25)$$

which is homogeneous of degree two with respect to M . \square

Remark. Note that $D'\psi_0^T D'n + D'n^T D'\psi_0$ is not, in general, the second fundamental form of the deformation, except in the case where $n(D'\psi_0)$ is the normal vector to $D'\psi_0$.

4.3. Planar isotropy. We say that the material is isotropic along the midsection of the shell if, for every planar rotation R in the set $\text{SO}(T_{x'}S')$ of rotations of $T_{x'}S'$ and every $F = (F', F_3) \in T_{x'}^*(S^\varepsilon; \mathbb{R}^3)$, we have

$$W^\varepsilon(F', F_3) = W^\varepsilon(F' R, F_3). \quad (26)$$

As a consequence, for deformations of finite energy, the fibers of the shell remain normal to its section.

Lemma 8. *Assume that the shell is isotropic along its midsection. Then there exists a map $\tau_3 : \mathcal{M}' \rightarrow \mathbb{R}$ such that*

$$n(F') = n_{F'} \tau_3(F')$$

for every $F' \in \mathcal{M}'$ of maximal rank, where $n_{F'}$ is defined by (21). Moreover, $\tau_3(F')$ depends only on the metric $C' = F'^T F'$.

Proof. Let F' be an element of \mathcal{M}' of maximal rank. By definition, we have

$$n_0(F') = \arg \min_{F_3 \in T_0^*((-1,1); \mathbb{R}^3)} W_2(F', F_3).$$

For every rotation $R \in \text{SO}(TS')$, the isotropy property yields

$$\begin{aligned} n_0(F'R) &= \arg \min_{F_3 \in T_0^*((-1,1); \mathbb{R}^3)} W_2(F'R, F_3) \\ &= \arg \min_{F_3 \in T_0^*((-1,1); \mathbb{R}^3)} W_2(F', F_3) = n_0(F'). \end{aligned}$$

In particular, this entails that $n(-F') = n(F')$. What is more, by frame-indifference, we have from Lemma 6 that

$$n(F') = F' \tau'(F') + n_{F'} \tau_3(F'),$$

where τ' is a bundle map from \mathcal{M}' into TS' and τ_3 is a map from \mathcal{M}' into \mathbb{R} , both of them depending only on the metric $C' = F'^T F'$. Thus,

$$\begin{aligned} F' \tau'(F') + n_{F'} \tau_3(F') &= n(F') = n(-F') = -F' \tau'(-F') + n_{-F'} \tau_3(-F') \\ &= -F' \tau'(F') + n_{F'} \tau_3(F'). \end{aligned}$$

Consequently, $F' \tau'(F') = 0$ and $n(F') = n_{F'} \tau_3(F')$, as claimed. \square

Proposition 9. *Assume that the shell is isotropic along its midsection. Then the flexural energy $I_{\text{flex}}(\psi_0)$ depends only on the metric and the second fundamental form of the deformed surface. Namely, we have*

$$I_{\text{flex}}(\psi_0) = \frac{1}{3} \int_S P_{C'}(|n(D'\psi_0)| b_{D'\psi_0}) dx',$$

where $b_{D'\psi_0}$ is the second fundamental form of ψ_0 , i.e.,

$$b_{F'} = D'N^T F' + F'^T D'N, \quad (27)$$

where $N = n_{F'}/|n_{F'}|$, and $P_{C'}$ is defined by Proposition 7.

Proof. Since $n(D'\psi_0)$ (denoted n for short) is normal collinear to $n_{D'\psi_0}$ and thus to the normal N to the deformed surface, we get

$$D'n^T D'\psi_0 + D'\psi_0^T D'n = (n \cdot N)(D'N^T D'\psi_0 + D'\psi_0^T D'N). \quad \square$$

4.4. Right invariance under the special linear group. We denote by $\mathrm{SL}(TS')$ the special linear group over TS' , that is the fiber bundle over S' whose fiber at x' is the linear diffeomorphisms of $T_{x'}S'$ with determinant equal to one. In this section, we consider the case where the energy W_2 is right invariant under the special linear group, that is, $W_2(F', F_3) = W_2(F'U, F_3)$ for every $x' \in S'$, $U \in \mathrm{SL}(T_{x'}S')$ and $(F', F_3) \in T_{x'}^*(S'; \mathbb{R}^3) \times T^*((-1, 1); \mathbb{R}^3)$.

Proposition 10. *Assume that W_2 is right invariant under $\mathrm{SL}(TS')$. Then the flexural energy $I_{\mathrm{flex}}(\psi_0)$ depends only on the metric and on the mean curvature*

$$H = \mathrm{Tr}(C'^{-1/2} b_{D'\psi} C'^{-1/2})$$

of the deformation. More precisely, we have

$$I_{\mathrm{flex}}(\psi_0) = \frac{1}{3} \int_{S'} K_{x', \det(C')} (H) dx', \quad (28)$$

where $K : S' \times \mathbb{R} \rightarrow \Pi_2$; here Π_2 is the set of polynomials of degree at most two. Moreover, if the shell is homogeneous along its fibers, then

$$I_{\mathrm{flex}}(\psi_0) = \frac{1}{3} \int_{S'} \kappa_{x', \det(C')} |H|^2 dx', \quad (29)$$

where κ is a map from $S' \times \mathbb{R}^+$ into \mathbb{R}^+ .

Proof. Let \mathbb{O} be the fiber bundle over S' whose fibers are the maps from $T_{x'}S'$ into itself of zero trace. For every $O \in \mathbb{O}$ and $x' = \pi_{S'}(O)$, there exists a regular map $U : (0, 1) \rightarrow \mathrm{SL}(T_{x'}S')$ such that $\dot{U}(0) = O$ and $U(0) = \mathrm{Id}$. Let F be a section of \mathcal{M} , $(G', s, v) \in T_{x'}^*(S'; \mathbb{R}^3) \times \mathbb{R} \times (\mathbb{R}^3)^3$, and let $\gamma(t) = (\gamma'(t), \gamma_3(t))$ be a curve in $T^*(S; \mathbb{R}^3)$ such that $\dot{\gamma}(0) = (G'_{F'}, s_0, v_{n(F')})$, as in (11). From the right invariance under the special linear group, for every $U_0 \in \mathrm{SL}(T_{x'}S')$, we have

$$W_2(\gamma(t)) = W_2(\gamma'(t)U_0U(t), \gamma_3(t)).$$

As a consequence,

$$D^2 W_2[\dot{\gamma}(0)]^2 = D^2 W_2 \left[\frac{d}{dt} (\gamma' U_0 U, \gamma_3) |_{t=0} \right]^2. \quad (30)$$

Then a simple computation yields

$$\frac{d}{dt} (\gamma' U_0 U, \gamma_3) |_{t=0} = ((G' U_0 + F' U_0 O)_{F' U_0}, s_0, v_{n(F')}),$$

which, owing to (30) and (10), leads to $Q_{F'}^0(G', s) = Q_{F'U_0}^0(G'U_0 + F'U_0O, s)$. From (23), recalling that $C' = F'^T F'$, we get

$$\begin{aligned} P_{C'}(F'^T G' + G'^T F') &= Q_{F'}^0(G', 1) = Q_{F'U_0}^0(G'U_0 + F'U_0O, 1) \\ &= P_{U_0^T F'^T F' U_0}((F'U_0)^T (G'U_0 + F'U_0O) + (G'U_0 + F'U_0O)^T (F'U_0)) \\ &= P_{U_0^T C' U_0}(U_0^T (F'^T G' + G'^T F')U_0 + U_0^T C' U_0 O + O^T U_0^T C' U_0) \\ &= P_{C'_0}((C'_0)^{1/2} [C'_0^{-1/2} U_0^T (F'^T G' + G'^T F') U_0 C'_0^{-1/2} \\ &\quad + C'_0^{1/2} O C'_0^{-1/2} + C'_0^{-1/2} O^T C'_0^{1/2}] C'_0^{1/2}), \end{aligned}$$

where $C' = F'^T F'$ and $C'_0 = U_0^T C' U_0$. Since the map

$$O \mapsto C'_0^{1/2} O C'_0^{-1/2} + C'_0^{-1/2} O^T C'_0^{1/2}$$

is a diffeomorphism over the set of symmetric trace-free matrices, the above expression leads to

$$P_{C'}(F'^T G' + G'^T F') = P_{C'_0} \left(\frac{1}{2} \text{Tr}(C'_0^{-1/2} U_0^T (F'^T G' + G'^T F') U_0 C'_0^{-1/2}) C'_0 \right).$$

In addition,

$$\text{Tr}(C'_0^{-1/2} U_0^T (F'^T G' + G'^T F') U_0 C'_0^{-1/2}) = \text{Tr}(C'^{-1/2} (F'^T G' + G'^T F') C'^{-1/2}),$$

so that we may write

$$P_{C'}(F'^T G' + G'^T F') = P_{C'_0} \left(\frac{1}{2} \text{Tr}(C'^{-1/2} (F'^T G' + G'^T F') C'^{-1/2}) C'_0 \right).$$

Since C' is symmetric and nonnegative, there exists a rotation $R \in \text{SO}(T_{x'} S')$ and nonnegative reals λ_1, λ_2 such that

$$C' = R^T \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} R.$$

Let us choose $U_0 \in \text{SL}(T_{x'} S')$ such that

$$U_0 = (\det C')^{1/4} R^T \begin{pmatrix} \lambda_1^{-1/2} & 0 \\ 0 & \lambda_2^{-1/2} \end{pmatrix}.$$

Hence, $C'_0 = U_0^T C' U_0 = (\det C')^{1/2} \text{Id}$, so that

$$\begin{aligned} P_{C'}(F'^T G' + G'^T F') &= \\ &P_{(\det C')^{1/2} \text{Id}} \left(\frac{1}{2} \text{Tr}(C'^{-1/2} (F'^T G' + G'^T F') C'^{-1/2}) (\det C')^{1/2} \text{Id} \right). \quad (31) \end{aligned}$$

Using the definition of I_{flex} and $D'\psi_0^T D'n + D'n^T D'\psi_0 = |n(D'\psi_0)|b_{D'\psi_0}$, we infer that

$$\begin{aligned} I_{\text{flex}}(\psi_0) &= \frac{1}{2} \int_{S'} P_{(\det C')^{1/2} \text{Id}} \left(\frac{1}{2} |n(D'\psi_0)| \text{Tr}(C'^{-1/2} b_{D'\psi} C'^{-1/2}) (\det C')^{1/2} \text{Id} \right) dx' \\ &= \frac{1}{2} \int_{S'} P_{(\det C')^{1/2} \text{Id}} \left(\frac{1}{2} |n(D'\psi_0)| H(\det C')^{1/2} \text{Id} \right) dx'. \end{aligned}$$

Finally, the right invariance of W_2 with respect to $\text{SL}(TS')$ implies that $|n(F')|$ depends only on $\det C'$. Setting

$$K_{x', \det C'}(H) = P_{(\det C')^{1/2} \text{Id}} \left(\frac{1}{2} |n(F')| H(\det C')^{1/2} \text{Id} \right),$$

we get (28). Moreover, if the shell is homogeneous along its fibers, then $P_{(\det C')^{1/2} \text{Id}}$ is homogeneous of degree two and, accordingly, $K_{x', \det C'}(H)$ is a monomial. \square

5. Geometric configuration

Classically, the energy of an elastic body is not written in terms of the deformation ψ^ε of S^ε , but in terms of the deformation $\tilde{\psi}^\varepsilon$ of the *geometric* configuration $\tilde{S}^\varepsilon := g_\varepsilon(S^\varepsilon)$, where

$$\begin{aligned} g_\varepsilon : S' \times (-\varepsilon, \varepsilon) &\rightarrow \mathbb{R}^3 \\ (x', x_3) &\mapsto x' + x_3 n'(x'), \end{aligned}$$

$n' : S' \rightarrow \mathbb{R}^3$ being the normal to S' . We set $\tilde{S} := \tilde{S}^{\varepsilon_0}$, for a small enough ε_0 such that g_{ε_0} is one-to-one. In the following, we will always assume that $\varepsilon \leq \varepsilon_0$. We intend to recast our results in this geometric configuration. This is easily achieved by a mere change of variables. To begin with, we have to recast our initial three-dimensional problem in the geometric configuration.

5.1. Recast of the problem. We denote by $\tilde{J}_\varepsilon(\tilde{\psi}^\varepsilon)$ the elastic energy of a deformation $\tilde{\psi}^\varepsilon$ of \tilde{S}^ε , and assume that it has the form

$$\tilde{J}_\varepsilon(\tilde{\psi}^\varepsilon) := \int_{\tilde{S}^\varepsilon} \tilde{W}^\varepsilon(\tilde{x}, \nabla \tilde{\psi}^\varepsilon(\tilde{x})) d\tilde{x},$$

where \tilde{W}^ε stands for the stored energy function of the solid. Furthermore, we assume the shell to be subjected to dead-body loads \tilde{f}_ε , so that the total energy of the system is given by

$$\tilde{I}_\varepsilon(\tilde{\psi}^\varepsilon) := \tilde{J}_\varepsilon(\tilde{\psi}^\varepsilon) - \int_{\tilde{S}^\varepsilon} \tilde{f}_\varepsilon \cdot \tilde{\psi}^\varepsilon d\tilde{x}. \quad (32)$$

Finally, clamping boundary conditions are added on a part of the boundary: $g_\varepsilon(\Gamma^\varepsilon) = g_\varepsilon(\gamma \times (-\varepsilon, \varepsilon))$, where $\gamma \subset \partial S'$.

Our aim is to determine the behavior of the minimizers $\tilde{\varphi}^\varepsilon$ of \tilde{I}_ε over

$$\tilde{\Psi}^\varepsilon := \{\tilde{\psi}^\varepsilon \in W^{1,\infty}(\tilde{S}^\varepsilon)^3 : \tilde{\psi}^\varepsilon \circ g_\varepsilon(x^\varepsilon) = \phi^\varepsilon(x^\varepsilon), x^\varepsilon \in \Gamma^\varepsilon\}$$

as ε goes to zero, under the assumptions on the stored energy and the applied loads made hereunder.

In order to apply our results, several assumptions, similar to the ones we made on W^ε and ϕ^ε , have to be imposed on \tilde{W}^ε and on the minimization sequences $\tilde{\varphi}^\varepsilon$.

Dependence of the stored energy on the thickness. We assume that there exist continuous nonnegative maps \tilde{W}_2 and \tilde{W}_0 such that, for every $(\tilde{x}, F) \in \tilde{S} \times \mathbb{R}^{3 \times 3}$, we have

$$\tilde{W}^\varepsilon(\tilde{x}, F) = \varepsilon^{-1}(\varepsilon^{-2}\tilde{W}_2(\tilde{x}, F) + \tilde{W}_0(\tilde{x}, F)).$$

Behavior of strongly extended fibers. We assume that the stored energy \tilde{W}_2 is bounded from below by a positive constant for strongly extended fibers, that is, there exist $\delta, c > 0$ such that

$$\begin{aligned} \tilde{W}_2(\tilde{x}, F' \circ \pi'_{x'} + F_3 \otimes n'(x')) &\geq c \\ \text{for all } \tilde{x} \in \tilde{S}, F' \in T_{x'}^*(S'; \mathbb{R}^3), F_3 \in \mathbb{R}^3 \text{ such that } |F_3| &\geq \delta, \end{aligned} \quad (33)$$

where x' is the projection of \tilde{x} onto S' and $\pi'_{x'}$ is the projection of \mathbb{R}^3 onto $T_{x'}(S'; \mathbb{R}^3)$.

Regularity and zero set of \tilde{W}_2 . We assume that \tilde{W}_2 is a nonnegative C^2 function, and denote by $\tilde{\mathcal{M}}$ the restriction of its zero set to $S' \times \mathbb{R}^{3 \times 3}$, that is,

$$\tilde{\mathcal{M}} := \{(x', F) \in S' \times \mathbb{R}^{3 \times 3} : \tilde{W}_2(x', F) = 0\}.$$

Let \mathcal{M}' be the projection of $\tilde{\mathcal{M}}$ onto $T^*(S'; \mathbb{R}^3)$, that is,

$$\mathcal{M}' := \bigcup_{x' \in S'} \{F' \in T_{x'}^*(S'; \mathbb{R}^3) : \text{there exists } n \in \mathbb{R}^3 \text{ with } (x', F' \circ \pi'_{x'} + n \otimes n'(x')) \in \tilde{\mathcal{M}}\}.$$

Once again, we assume that the projection of $\tilde{\mathcal{M}}$ onto \mathcal{M}' is one-to-one, that is, there exists a map $n : \mathcal{M}' \rightarrow \mathbb{R}^3$ such that

$$\tilde{\mathcal{M}} = \{(x', F' \circ \pi'_{x'} + n(F') \otimes n'(x')) \in S' \times \mathbb{R}^{3 \times 3} : F' \in \mathcal{M}'\}. \quad (34)$$

Interpenetration. To avoid interpenetration of matter, it is geometric to expect $D\tilde{\psi}^\varepsilon$ to be invertible. To this end, we require that $\tilde{W}_0(x', F) = \infty$ for every $(x', F) \in \tilde{\mathcal{M}}$ such that $\det F < 0$, and that

$$\tilde{W}_0(x', F) \rightarrow \infty \quad \text{if } (x', F) \in \tilde{\mathcal{M}} \text{ and } \det F \rightarrow 0.$$

Applied loads. The volumic loads are assumed to scale as the inverse of the thickness of the shell; more precisely, we assume that there exists $\tilde{f} : \tilde{S} \rightarrow \mathbb{R}^3$ such that

$$\tilde{f}_\varepsilon(\tilde{x}) = \varepsilon^{-1} \tilde{f}(\tilde{x}) \quad \text{for every } \tilde{x} \in \tilde{S}.$$

Ansatz. We assume that the minimizers of the energy admit an asymptotic expansion in the form

$$\tilde{\varphi}^\varepsilon(x' + \varepsilon x_3 n') = \sum_{k \geq 0} \varepsilon^k \varphi_k(x', x_3). \quad (35)$$

5.2. Change of variable. In order to apply our result, we first have to rewrite the energy in terms of the associated deformation $\psi^\varepsilon = \tilde{\psi}^\varepsilon \circ g_\varepsilon$ of S^ε . We have

$$\tilde{J}(\tilde{\psi}^\varepsilon) = J(\psi^\varepsilon) = \int_{S^\varepsilon} W^\varepsilon(D\psi^\varepsilon) dx, \quad (36)$$

with $W^\varepsilon = \varepsilon^{-1}(\varepsilon^{-2}W_2 + W_0)$, and for every $F \in T_{x'}^*(S; \mathbb{R}^3)$,

$$W_k(F) = \tilde{W}_k(F \circ (Dg_\varepsilon(x))^{-1}) \det(Dg_\varepsilon(x)), \quad k = 0, 2. \quad (37)$$

Note that W_2 and W_0 are independent of ε since $Dg_\varepsilon = (\text{Id}', n') + x_3(D'n', 0)$ (which is denoted by Dg hereafter). In addition, these energies satisfy the assumptions made in [Section 2.1](#). Finally, the minimizers $\varphi^\varepsilon = \tilde{\varphi}^\varepsilon \circ g_\varepsilon$ admit the same asymptotic expansion as $\tilde{\varphi}^\varepsilon$. Thus, all of the results of [Sections 3](#) and [4](#) apply and may be expressed in terms of \tilde{W}_0 and \tilde{W}_2 up to a change of variable. Moreover, the definitions of \mathcal{M}' and of the map $n : \mathcal{M}' \rightarrow \mathbb{R}^3$ are independent of the chosen approach.

Lemma 11. *If the function W_2 is defined by (37), and F' is a section of \mathcal{M}' , then, for every $G' \in T_{x'}^*(S'; \mathbb{R}^3)$ and $s \in \mathbb{R}$, we have*

$$Q_{F'}^0(G', s) = \tilde{Q}_{F'}^0(G' - sF'D'n', s),$$

where

$$\tilde{Q}_{F'}^0(G', s) := \inf_{v \in \mathbb{R}^3} D^2 \tilde{W}_2(x', F)[sn', G'\pi_{x'}' + v \otimes n']^2,$$

where $\pi_{x'}'$ is the projection of \mathbb{R}^3 onto $T_{x'}S'$ and $F = F'(x') \circ \pi_{x'}' + n(F'(x')) \otimes n'(x')$.

Proof. Let F' be a section of \mathcal{M}' , x' be an element of S' and $G' \in T_{x'}^*(S'; \mathbb{R}^3)$. From the definition of $Q_{F'}^0$, we have

$$Q_{F'}^0(G', s) = \inf_{v \in \mathbb{R}^3} D^2 W_2[G'_{F'}, s_0, v_{n(F')}]^2,$$

where $(G'_{F'}, s_0, v_{n(F')}) = \dot{\gamma}(0)$, and

$$\gamma(t) = (F'(x') + tG', ts, n(F'(x')) + vt)$$

is an element of

$$T^*(S'; \mathbb{R}^3) \times (-1, 1) \times (\mathbb{R}^3)^3 = T^*(S'; \mathbb{R}^3) \times T^*((-1, 1); \mathbb{R}^3) = T^*(S; \mathbb{R}^3).$$

From (37), we deduce

$$Q_{F'}^0(G', s) = \det(Dg(x', 0)) \inf_{v \in \mathbb{R}^3} D^2 \tilde{W}_2[\dot{\tilde{\gamma}}(0)]^2 = \inf_{v \in \mathbb{R}^3} D^2 \tilde{W}_2[\dot{\tilde{\gamma}}(0)]^2, \quad (38)$$

where $\tilde{\gamma}(t) = \gamma(t) \circ Dg(x', ts)^{-1}$. On the other hand, $Dg = (\text{Id}', n') + x_3(D'n', 0)$, and hence

$$\begin{aligned} (Dg)^{-1} &= ((\text{Id}', n') + x_3(D'n', 0))^{-1} \\ &= ((\text{Id}', n')(\text{Id} + x_3(\text{Id}', n')^{-1}(D'n', 0)))^{-1} \\ &= (\text{Id} - x_3(\text{Id}', n')^{-1}(D'n', 0))(\text{Id}', n')^{-1} + o(x_3). \end{aligned}$$

Since $(\text{Id}' + n' \otimes e_3)^{-1} = \begin{pmatrix} \pi' \\ n'/T \end{pmatrix}$, the above identity reads

$$\begin{aligned} (Dg)^{-1} &= \begin{pmatrix} \pi' \\ n'/T \end{pmatrix} - x_3 \begin{pmatrix} \pi' \\ n'/T \end{pmatrix} (D'n', 0) \begin{pmatrix} \pi' \\ n'/T \end{pmatrix}^{-1} + o(x_3) \\ &= \begin{pmatrix} \pi' \\ n'/T \end{pmatrix} - x_3 \begin{pmatrix} \pi' D'n' \pi' \\ 0 \end{pmatrix} + o(x_3). \end{aligned}$$

It follows that $\tilde{\gamma}(t) = (x(t), F(t)) + o(t)$ with $x(t) = x' + tsn'$, and, using the notation F' for $F'(x')$ for short,

$$\begin{aligned} F(t) &= ((F', n(F')) + t(G', v)) \begin{pmatrix} \pi' \\ n'/T \end{pmatrix} - ts \begin{pmatrix} \pi' D'n' \pi' \\ 0 \end{pmatrix} \\ &= (F', n(F')) \begin{pmatrix} \pi' \\ n'/T \end{pmatrix} + t(G' \pi' + v \otimes n') - ts F' D'n' \pi'. \end{aligned}$$

Consequently,

$$\begin{aligned} \dot{\tilde{\gamma}}(0) &= \left[\begin{pmatrix} x', (F', n(F')) \begin{pmatrix} \pi' \\ n'/T \end{pmatrix} \end{pmatrix}, (sn', G' \pi' + v \otimes n' - s(F' D'n' \pi')) \right] \\ &= \left[(x', F' \pi' + n(F') \otimes n'), (sn', (G' - sF' D'n') \pi' + v \otimes n') \right] \\ &= \left[(x', F), (sn', (G' - sF' D'n') \pi' + v \otimes n') \right] \end{aligned}$$

The conclusion follows from (38). \square

From now on, we limit our analysis to the case where standard boundary conditions (18) are applied. From Proposition 5, we immediately infer the next result.

Proposition 12. *Assume that the standard boundary conditions (18) apply to the shell. Let $\tilde{\varphi}^\varepsilon$ be the minimizer of the total energy $\tilde{I}_\varepsilon(\tilde{\varphi}^\varepsilon)$ over the space of admissible deformations. If $\tilde{\varphi}^\varepsilon$ admits an asymptotic expansion as in (35), and if the total energy $\tilde{I}_\varepsilon(\tilde{\varphi}^\varepsilon)$ remains bounded, then*

$$\varphi_0 = \arg \min_{\psi_0 \in \Psi_0} \tilde{I}_0(\psi_0),$$

where

$$\begin{aligned} \tilde{I}_0(\psi_0) &= \frac{1}{3} \int_{S'} \tilde{Q}_{D'\psi_0}^0 (D'n - D'\psi_0 D'n', 1) dx' \\ &\quad + 2 \int_{S'} \tilde{W}_0(x', (D'\psi_0, n)) dx' - 2 \int_{S'} \tilde{f}_0 \cdot \psi_0 dx', \end{aligned}$$

$\tilde{f}_0(x') = \tilde{f}(x', 0)$ for every $x' \in S'$, $n = n(D'\psi_0)$, and

$$\begin{aligned} \Psi_0 &:= \{\psi_0 \in W^{1,\infty}(S')^3 : D'\psi_0 \in \mathcal{M}', n = n(D'\psi_0) \in W^{1,\infty}(S')^3 \\ &\quad \text{such that } \psi_0(x') = \phi_0(x') \text{ and } n(x') = n_\gamma(x') \text{ for every } x' \in \gamma\}. \end{aligned}$$

Note that more general Dirichlet conditions could have been considered in the same fashion as in [Lemma 4](#).

5.3. Homogeneity along the fibers. We say that the shell is homogeneous along its fibers in the geometric configuration if, for every $x' \in S'$, $s \in (-1, 1)$ and $F \in \mathbb{R}^{3 \times 3}$, we have $\tilde{W}_2(x' + sn', F) = \tilde{W}_2(x', F)$.

Proposition 13. *If the shell is homogeneous along its fibers in the geometric configuration, then $\tilde{Q}_{F'}^0(G', s)$ is independent of s , and is denoted by $\tilde{Q}_{F'}^0(G')$.*

5.4. Frame-indifference. In the following, we assume the stored energy to be frame-indifferent, that is, $\tilde{W}^\varepsilon(\tilde{x}, RF) = \tilde{W}^\varepsilon(\tilde{x}, F)$ for every $(\tilde{x}, F) \in \tilde{S}^\varepsilon \times \mathbb{R}^{3 \times 3}$ (with $\varepsilon > 0$ small enough), and every rotation $R \in \text{SO}(3)$. Note that this is equivalent to the frame-indifference of W^ε .

Proposition 14. *If the stored energy function \tilde{W}_2 is frame-indifferent, then there exists a bundle map $\tilde{P} : C' \mapsto \tilde{P}_{C'}$ over S' from $\mathcal{E}_{S'}$ into \mathcal{P} such that, for every deformation ψ_0 of finite energy $\tilde{I}_0(\psi_0)$ and for every $G' \in T^*(S'; \mathbb{R}^3)$, we have*

$$\tilde{Q}_{D'\psi_0}^0(G', 1) = \tilde{P}_{C'}(D'\psi_0^T G' + G'^T D'\psi_0),$$

with $C' = D'\psi_0^T D'\psi_0$ and $n = n(D'\psi_0)$. Moreover, if the shell is homogeneous along its fibers in the geometric configuration, then $\tilde{P}_{C'}$ is homogeneous of degree two.

Proof. The proof is similar to the one devised for the abstract configuration. Once again, there exists a map \tilde{W}_S such that, at least in a neighborhood of $\tilde{\mathcal{M}}^+ = \{F \in \tilde{\mathcal{M}} : \det F > 0\}$, we may write $\tilde{W}_2(x, F) = \tilde{W}_S(x, F^T F)$. After some computations, we derive the claimed result with

$$\begin{aligned} \tilde{P}_{C'}(M) &= \\ &\quad \inf_{\substack{w' \in T_{x'} S' \\ w_3 \in \mathbb{R}}} D^2 \tilde{W}_S(x', \tilde{C}) [n', \pi'^T M \pi' + n' w'^T \pi' + \pi'^T w' n'^T + n' w_3 n'^T]^2, \quad (39) \end{aligned}$$

where $\tilde{C} = (D'\psi_0\pi' + n(F) \otimes n')^T (D'\psi_0\pi' + n(F) \otimes n')$. Moreover, frame-indifference implies also that \tilde{C} depends only on C' . Finally, if the shell is homogeneous along its fibers in the geometric configuration, we have

$$\tilde{P}_{C'}(M) = \inf_{\substack{w' \in T_{x'}^* S' \\ w_3 \in \mathbb{R}}} \frac{\partial^2 \tilde{W}_S}{\partial \tilde{C}^2}(x', \tilde{C}) [\pi'^T M \pi' + n' w'^T \pi' + \pi'^T w' n'^T + n' w_3 n'^T]^2. \quad (40)$$

This completes the proof. \square

5.5. Planar isotropy. We say the shell is isotropic along its midsection if, for every $x' \in S'$ and $(F', F_3) \in T_{x'}^*(S'; \mathbb{R}^3) \times \mathbb{R}^3$, we have

$$\tilde{W}^\varepsilon(x', F' R \pi'_{x'} + F_3 \otimes n') = \tilde{W}^\varepsilon(x', F' \pi'_{x'} + F_3 \otimes n')$$

for every $R \in \text{SO}(T_{x'} S')$. This is equivalent to the definition used in the abstract configuration. We investigate the consequences of planar isotropy on the flexural part of the energy

$$\tilde{I}_{\text{flex}}(\psi_0) := \frac{1}{3} \int_{S'} \tilde{Q}_{D'\psi_0}^0(D'n - D'\psi_0 D'n', 1) dx'.$$

Proposition 15. *If the shell is isotropic along its midsection, then*

$$\tilde{I}_{\text{flex}}(\psi_0) = \frac{1}{3} \int_{S'} \tilde{P}_{C'}(|n(D'\psi_0)| b_{D'\psi_0} - (C' D'n' + (D'n')^T C')) dx',$$

where $b_{D'\psi_0}$ is the second fundamental form of the deformed surface, given by (27).

Proof. The proof is similar to Proposition 9. \square

5.6. Right invariance under the special linear group. We say that the stored energy \tilde{W}_2 is invariant under the special linear group if, for every $\tilde{x} = x' + x_3 n'$ and $(F', F_3) \in T_{x'}^*(S'; \mathbb{R}^3) \times \mathbb{R}^3$, we have

$$\tilde{W}_2(\tilde{x}, F' U \pi'_{x'} + F_3 \otimes n') = \tilde{W}_2(\tilde{x}, F' \pi'_{x'} + F_3 \otimes n')$$

for every $U \in \text{SL}(T_{x'} S')$. Note that this definition is equivalent to the one given in the abstract configuration.

Proposition 16. *If \tilde{W}_2 is right invariant under the special linear group, then*

$$\tilde{I}_{\text{flex}}(\psi_0) = \frac{1}{3} \int_{S'} \tilde{K}_{x', \det(C')} (|n(D'\psi_0)| H - H_0) dx', \quad (41)$$

where H and H_0 are the mean curvatures of the deformed shell $\psi_0(S')$ and undeformed shell S' , respectively. Moreover, if the shell is homogeneous along its fibers,

then

$$\tilde{I}_{\text{flex}}(\psi) = \frac{1}{3} \int_{S'} \tilde{\kappa}_{x', \det(C')} (|n(D'\psi_0)|H - H_0)^2 dx'.$$

Proof. For all sections F' of $T^*(S'; \mathbb{R}^3)$ and $G' \in T_{x'}^*(S'; \mathbb{R}^3)$, we have

$$\tilde{P}_{C'}(F'^T(G' - F'D'n') + (G' - F'D'n')^T F') = P_{C'}(F'^T G' + G'^T F').$$

From [Proposition 10](#) and [\(31\)](#), we deduce that

$$\tilde{P}_{C'}(F'^T(G' - F'D'n') + (G' - F'D'n')^T F') = \tilde{P}_{(\det C')^{1/2} \text{Id}}\left(\frac{1}{2}\alpha(\det C')^{1/2} \text{Id}\right),$$

with

$$\alpha = \text{Tr}(C'^{-1/2}(F'^T(G' - F'D'n') + (G' - F'D'n')^T F')C'^{-1/2}).$$

We thus obtain [\(41\)](#) with

$$\tilde{K}_{x', \det(C')}(H) = \tilde{P}_{\det(C') \text{Id}}\left(\frac{1}{2}H(\det C')^{1/2} \text{Id}\right). \quad (42)$$

If the shell is homogeneous along its fibers, then the $\tilde{P}_{(\det C') \text{Id}}$ is homogeneous of degree two, whence the conclusion in this case. \square

6. Examples

We are now in position to apply our formal convergence result to derive different models for isometric bending shells, vesicles and RBCs. Note that, in our setting, we do not derive the nonlinear membrane shell model (see [\[Le Dret and Raoult 1996\]](#)) since W_2 cannot be chosen to be equal to zero. Throughout this section, we assume that W^ε and \tilde{W}^ε satisfy the assumptions [\(2\)](#), [\(3\)](#) and [\(33\)](#), [\(34\)](#), respectively, that the Dirichlet boundary conditions on Γ^ε are given by [\(7\)](#) and [\(18\)](#) and that the minimizers φ^ε and $\tilde{\varphi}^\varepsilon$ of the total energy admit asymptotic expansions as in [\(6\)](#) and [\(35\)](#), respectively, while their total energies $I_\varepsilon(\varphi^\varepsilon)$ and $\tilde{I}_\varepsilon(\tilde{\varphi}^\varepsilon)$ remain bounded. Moreover, the stored energies are assumed to be frame-indifferent.

6.1. Isometric bending shells. In this section, we recover the isometric bending shell model by choosing $\tilde{W}_0 = 0$ and the set $\tilde{\mathcal{M}}$ of the zeros of \tilde{W}_2 restricted to the midsection to be equal to

$$\tilde{\mathcal{M}}_{\text{iso}} := S' \times \text{SO}(3). \quad (43)$$

The sequence of minimizers of the energy converges toward the minimizer of an energy whose elastic part depends only on the difference between the second fundamental form of the deformed shell and that of its reference configuration.

Proposition 17. *If $\tilde{W}^\varepsilon = \varepsilon^{-3}\tilde{W}_2$ and if \tilde{W}_2 is such that $\tilde{\mathcal{M}} = \tilde{\mathcal{M}}_{\text{iso}}$, given by [\(43\)](#), then*

$$\varphi_0 = \arg \min_{\psi_0 \in \Psi_0} \tilde{I}_0(\psi_0),$$

where

$$\begin{aligned} \tilde{I}_0(\psi_0) = \frac{1}{3} \int_{S'} \tilde{P}_{x', \text{Id}}((D'\psi_0)^T D'n + D'n^T D'\psi_0 - (D'n' + D'n'^T)) dx' \\ - 2 \int_{S'} f_0 \cdot \psi_0 dx', \quad (44) \end{aligned}$$

$f_0(x') = f(x', 0)$ for every $x' \in S'$, $n = n(D'\psi_0)$, $\tilde{P}_{x', \text{Id}}$ is a polynomial of degree at most two given by (39), and

$$\begin{aligned} \Psi_0 = \{\psi_0 \in W^{1,\infty}(S')^3 : (D'\psi_0)^T D'\psi_0 = \text{Id}, n = n(D'\psi_0) \in W^{1,\infty}(S')^3, \\ \psi_0(x') = \phi_0(x') \text{ and } n(x') = n_\gamma(x') \text{ for every } x' \in \gamma\}. \end{aligned}$$

Moreover, if the shell is homogeneous along its fibers, then $\tilde{P}_{x', \text{Id}}$ is homogeneous of degree two.

Proof. It is a straightforward application of Proposition 12 and Proposition 14. \square

Example. Let us give a practical example. For instance, one can choose the Saint Venant–Kirchhoff nonlinearly elastic stored energy function

$$\tilde{W}_2(F) = \mu \text{Tr}((F^T F - \text{Id})^2) + \frac{\lambda}{2} \text{Tr}(F^T F - \text{Id})^2.$$

A simple computation leads to the energy

$$\tilde{I}_0(\psi_0) = \frac{1}{3} \int_{S'} 2\mu \text{Tr}((b - b_{\text{ref}})^2) + \frac{\lambda\mu}{2\mu + \lambda} \text{Tr}(b - b_{\text{ref}})^2 - f_0 \cdot \psi_0 dx',$$

where $b = (D'\psi_0)^T D'n + (D'n)^T D'\psi_0$ is the second fundamental form of the deformed shell and $b_{\text{ref}} = (D'n')^T + D'n'$ is the second fundamental form of the undeformed shell.

6.2. Vesicles. In this section we derive Helfrich functionals, with or without spontaneous curvature, from three-dimensional elasticity. The main difference with the isometric case lies in the fact that we assume the energy to be right invariant under the special linear group $\text{SL}(TS')$. Note that this readily implies that it may not be chosen to be isotropic. The Helfrich functional without spontaneous curvature is derived using the abstract configuration S , while the one with spontaneous curvature is obtained using the geometric configuration \tilde{S} of the shell.

6.2.1. Without spontaneous curvature. In this section we consider the case where the zero set of W_2 restricted to the midsection is given by

$$\begin{aligned} \mathcal{M}_H := \{(F', F_3) \in T^*(S'; \mathbb{R}^3) \times T_0^*((-1, 1); \mathbb{R}^3) : \\ \det(F) = 1, F_3 \cdot v = \det(F', v) \text{ for every } v \in T_0^*((-1, 1); \mathbb{R}^3)\}. \quad (45) \end{aligned}$$

From Propositions 5 and 10, we obtain that the minimizers of the energy formally converge to the Helfrich functional with no spontaneous curvature.

Proposition 18. *Suppose that $W^\varepsilon = \varepsilon^{-3}W_2$, W_2 is right invariant under $SL(TS')$, and that $\mathcal{M} = \mathcal{M}_H$ as given by (45). If the shell is homogeneous along its fibers in the abstract configuration, then*

$$\varphi_0 = \arg \min_{\psi_0 \in \Psi_0} I_0(\psi_0),$$

where

$$I_0(\psi_0) = \frac{1}{3} \int_{S'} \kappa |H|^2 dx' - 2 \int_{S'} f_0 \cdot \psi_0 dx', \quad (46)$$

$f_0(x') = f(x', 0)$ for every $x' \in S'$, H is the mean curvature of the deformed shell $\psi_0(S')$, $\kappa(x') = P_{\text{Id}}(\frac{1}{2} \text{Id})$, where P_{Id} is given by (25), and

$$\begin{aligned} \Psi_0 = \{ \psi_0 \in W^{1,\infty}(S')^3 : n \in W^{1,\infty}(S')^3 \text{ and } \det((D'\psi_0)^T D'\psi_0) = 1, \\ \psi_0(x') = \phi_0(x') \text{ and } n(x') = n_\gamma(x') \text{ for } x' \in \gamma \}, \end{aligned}$$

where n is the normal to the deformed surface $\psi_0(S')$.

Example. Proposition 18 can be applied with

$$W_2(F) = W_S(C) = \alpha(\det(C) - 1)^2 + \beta|Ce_3 - e_3|^2, \quad (47)$$

where $C = F^T F$ and α and β are positive real constants. A simple computation leads to

$$D^2W_S \left[0, \begin{pmatrix} \frac{1}{2} \text{Id} & w' \\ w' & w_3 \end{pmatrix}_{\text{Id}} \right]^2 = 2\alpha(1 + w_3)^2 + 2\beta|(w', w_3)|^2.$$

Then, from the expression (25) of P_{Id} , we get

$$\kappa = P_{\text{Id}}(\frac{1}{2} \text{Id}) = \inf_w 2\alpha(1 + w_3)^2 + 2\beta|(w', w_3)|^2 = 2(\alpha^{-1} + \beta^{-1})^{-1}.$$

Hence, the limit energy in this case is

$$I_0(\psi_0) = \frac{2}{3} \int_{S'} (\alpha^{-1} + \beta^{-1})^{-1} |H|^2 dx' - 2 \int_{S'} f_0 \cdot \psi_0 dx'.$$

6.2.2. With spontaneous curvature. Here we derive from three-dimensional elasticity a model of shells whose limit energy is the Helfrich functional with nonzero spontaneous curvature. Basically, such a model is obtained by using the same assumptions as in the previous case but cast in the geometric configuration, with a set of zeros restricted to the midsection for \tilde{W}_2 given by

$$\tilde{\mathcal{M}}_H := \{(x', F) \in S' \times \mathbb{R}^{3 \times 3} : \det(F) = 1 \text{ and } (\text{Cof } F - F)n' = 0\}. \quad (48)$$

The following proposition is a direct application of Propositions 12 and 16.

Proposition 19. *Suppose that $\tilde{W}^\varepsilon = \varepsilon^{-3} \tilde{W}_2$, \tilde{W}_2 is right invariant under $\text{SL}(TS')$, and such that $\tilde{M} = \tilde{M}_H$ as given by (48). If the shell is homogeneous along its fibers in the geometric configuration, then*

$$\varphi_0 = \arg \min_{\psi_0 \in \Psi_0} \tilde{I}_0(\psi_0),$$

where

$$\tilde{I}_0(\psi_0) = \frac{1}{3} \int_{S'} \tilde{\kappa} |H - H_0|^2 dx' - 2 \int_{S'} f_0 \cdot \psi_0 dx', \quad (49)$$

$f_0(x') = \tilde{f}(x', 0)$ for every $x' \in S'$, where H is the mean curvature of the deformed shell $\psi_0(S')$ and H_0 is the mean curvature of S' , with

$$\tilde{\kappa}(x') = \tilde{P}_{\text{Id}}\left(\frac{1}{2} \text{Id}\right),$$

where \tilde{P}_{Id} is given by (40) and

$$\begin{aligned} \Psi_0 = \{ \psi_0 \in W^{1,\infty}(S')^3 : n \in W^{1,\infty}(S')^3 \text{ and } \det(D'\psi_0^T D'\psi_0) = 1, \\ \psi_0(x') = \phi_0(x') \text{ and } n(x') = n_\gamma(x') \text{ for every } x' \in \gamma \}, \quad (50) \end{aligned}$$

where n is the normal to the deformed surface $\psi_0(S')$.

Example. The stored energy function

$$\tilde{W}_2(x', F) = \tilde{W}_{Sx'} \tilde{C} = \alpha(\det(F^T F) - 1)^2 + \beta |F^T F n' - n'|^2$$

satisfies the assumptions of [Proposition 19](#), and we have

$$\tilde{\kappa} = \inf_{\substack{w' \in T_{x'} S' \\ w_3 \in \mathbb{R}}} \frac{\partial \tilde{W}_S}{\partial \tilde{C}^2}(x', \text{Id}) \left[\frac{1}{2} \pi'^T \pi' + n' w'^T \pi' + \pi'^T w' n'^T + n' w_3 n'^T \right]^2.$$

Furthermore, we have

$$\frac{\partial \tilde{W}_S}{\partial \tilde{C}^2}(x', \text{Id}) [\delta C]^2 = 2(\alpha \text{Tr}(\delta C)^2 + \beta |\delta C n'|^2),$$

so that

$$\tilde{\kappa} = \inf_{\substack{w' \in T_{x'} S' \\ w_3 \in \mathbb{R}}} 2(\alpha(1 + w_3)^2 + \beta(|w'|^2 + w_3^2)) = 2(\alpha^{-1} + \beta^{-1})^{-1}.$$

For such a choice of \tilde{W}_2 , and under the assumptions made in [Proposition 19](#), the sequence of minimizers $\tilde{\varphi}^\varepsilon$ formally converges toward a minimizer of

$$\tilde{I}_0(\psi_0) = \frac{2}{3} \int_{S'} (\alpha^{-1} + \beta^{-1})^{-1} |H - H_0|^2 dx' - 2 \int_{S'} \tilde{f}_0 \cdot \psi_0 dx'$$

over Ψ_0 given by (50). Note that this is the set of deformations that preserve the local area of the shell supplemented with boundary conditions on a subset of the boundary.

6.3. Red blood cells. The mechanical behavior of a red blood cell (RBC) is driven by the nature of its membrane, which is mainly made of a lipid bilayer. Note that in addition to the lipid bilayer, RBCs are also composed of a protein skeleton. This skeleton ensures a small resistance of the RBCs to shear stress. Such a model may be obtained as the limit of the three-dimensional elasticity. In this section, we derive a model of the mechanical behavior of RBCs as the limit of genuine three-dimensional elasticity. To this end, we consider a stored energy W^ε whose asymptotic assumption reads as

$$W^\varepsilon = \varepsilon^{-1}(\varepsilon^{-2}W_2 + W_0),$$

where W_2 satisfies the same assumption as in the study of vesicles without spontaneous curvature (see Section 6.2.1), namely, its zero set restricted to the midsection is given by (45). We get that the sequence φ^ε of minimizers formally converges to

$$\varphi_0 = \arg \min_{\psi_0 \in \Psi_0} I_0(\psi_0),$$

where

$$I_0(\psi_0) = \frac{1}{3} \int_{S'} k|H|^2 dx' + 2 \int_{S'} W_0(D'\psi_0, n) dx' - 2 \int_{S'} f_0 \cdot \psi_0 dx',$$

$f_0(x') = f(x', 0)$, n is the normal to the deformed shell $\psi_0(S')$ and Ψ_0 is the set of deformations that preserve the local area of the shell and satisfy the boundary conditions

$$\psi_0(x') = \phi_0(x') \quad \text{and} \quad n(x') = n_\gamma(x') \quad \text{for every } x' \in \gamma.$$

Example. As an example, we can choose the nonlinearly elastic Saint Venant–Kirchhoff stored energy function

$$W_0(F) = \mu \operatorname{Tr}((C - \operatorname{Id})^2) + \frac{\lambda}{2} \operatorname{Tr}(C - \operatorname{Id})^2 \quad \text{with } C = F^T F,$$

and $W_2(F)$ as in (47). This leads to a limit energy

$$I_0(\psi_0) = \frac{2}{3} \int_{S'} (\alpha^{-1} + \beta^{-1})^{-1} |H|^2 dx' + 2 \int_{S'} \left(\mu \operatorname{Tr}((D'\psi_0^T D'\psi_0 - \operatorname{Id}')^2) + \frac{\lambda}{2} \operatorname{Tr}(D'\psi_0^T D'\psi_0 - \operatorname{Id}')^2 \right) dx' - 2 \int_{S'} f_0 \cdot \psi_0 dx'.$$

7. Conclusion

In this article we prove, using a formal approach, that new nonlinearly elastic shell models may be derived assuming the shell to be highly anisotropic. Notably, it enables us to derive some models used in the study of vesicles and RBCs. Part of the results presented in this article have since been proved by a Γ -convergence approach in an Eulerian setting for the justification of the modeling of vesicles by Merlet [2013a; 2013b]. Finally, let us recall and emphasize the fact that the computation of the limit energy should include a relaxation step that is not taken into account in our formal framework. The only interesting case being the one where the flexural term $Q_{F'}^0(G, s)$ is not fully degenerate, that is not independent of G . In such a case, a relaxation of the membrane term of the limit energy is expected to take place. The correct limit energy in Proposition 5 should read

$$I_0'(\psi_0) = \frac{1}{3} \int_{S'} Q_{D'\psi_0}^0(D'n, 1) dx' + 2 \int_{S'} \mathcal{Q}'W_0(D'\psi_0, n) dx' - 2 \int_{S'} f_0 \cdot \psi_0 dx',$$

where $\mathcal{Q}'W_0$ is the in-plane quasiconvexification of W_0 , defined for every element F' of $T^*(S'; \mathbb{R}^3)$ by

$$\mathcal{Q}'W_0(F', n) = \inf_{\varphi \in C_0^\infty(\omega; T_{x'}S')} |\omega|^{-1} \int_{\omega} W_0(F'(\text{Id}' + D'\varphi), n) dy',$$

where $x' = \pi_{S'}(F')$ and ω is a bounded regular open set of $T_{x'}S'$.

Notation

- \mathbb{R} , set of reals
- \mathbb{R}' , dual set of reals
- \mathbb{N} , set of nonnegative integers
- S' , midsurface of the shell in the reference configuration
- ε , thickness of the shell
- $S^\varepsilon := S' \times (-\varepsilon, \varepsilon)$, abstract reference configuration of the shell
- $S := S^1$, rescaled abstract reference configuration of the shell
- \tilde{S}^ε , geometric reference configuration
- \tilde{S} , geometric reference configuration of maximum thickness
- x^ε , element of S^ε
- \tilde{x} , element of \tilde{S}
- x' , element of S'
- TM , tangent space to M

- $T_x M$, tangent fiber to M at x
- T^*M , cotangent space to M
- $T_x^* M$, cotangent space to M at x
- $T(M; \mathbb{R}^3) := TM \oplus TM \oplus TM$, Whitney triple sum of TM
- $T_x(M; \mathbb{R}^3) := T_x M \oplus T_x M \oplus T_x M$, Whitney triple sum of $T_x M$
- $T^*(M; \mathbb{R}^3) := T^*M \oplus T^*M \oplus T^*M$, Whitney triple sum of T^*M
- $T_x^*(M; \mathbb{R}^3) := T_x^* M \oplus T_x^* M \oplus T_x^* M$, Whitney triple sum of $T_x^* M$
- $\pi_B : \mathcal{F} \rightarrow B$, projection of a fiber bundle \mathcal{F} onto its base B
- $\pi'_{x'} : \mathbb{R}^3 \rightarrow T_{x'} S'$, projection onto $T_{x'} S'$
- ψ^ε , deformation of S^ε
- $\psi(\varepsilon) : S \rightarrow \mathbb{R}^3$, rescaled map of the deformation ψ^ε
- $\boldsymbol{\psi} = (\psi_k)$, expansion of the deformation $\psi(\varepsilon) = \sum_k \varepsilon^k \psi_k$ with respect to the thickness
- $\boldsymbol{\varphi} = (\varphi_k)$, expansion of the minimizers $\varphi(\varepsilon) = \sum_k \varepsilon^k \varphi_k$ with respect to the thickness
- $D\psi^\varepsilon$, differential of $\psi^\varepsilon : S^\varepsilon \rightarrow \mathbb{R}^3$
- $D\psi^\varepsilon(x^\varepsilon)$, differential of $\psi^\varepsilon : S^\varepsilon \rightarrow \mathbb{R}^3$ at x^ε
- $(D'\psi^\varepsilon, D_3\psi^\varepsilon)$, decomposition of $D\psi^\varepsilon \in T^*(S^\varepsilon; \mathbb{R}^3)$ with respect to the product $T^*(S'; \mathbb{R}^3) \times T^*((-\varepsilon, \varepsilon); \mathbb{R}^3)$, where $\psi^\varepsilon : S^\varepsilon \rightarrow \mathbb{R}^3$
- $(D'\psi^\varepsilon(x), D_3\psi^\varepsilon(x))$, decomposition of $D\psi^\varepsilon(x) \in T_x^*(S^\varepsilon; \mathbb{R}^3)$ with respect to the product $T_x^*(S'; \mathbb{R}^3) \times T_{x_3}^*((-\varepsilon, \varepsilon); \mathbb{R}^3)$, where $\psi^\varepsilon : S^\varepsilon \rightarrow \mathbb{R}^3$
- ∂_3 , partial differentiation along the fibers
- $J_\varepsilon(\psi^\varepsilon)$, elastic energy of a deformation $\psi^\varepsilon : S^\varepsilon \rightarrow \mathbb{R}^3$
- $I_\varepsilon(\psi^\varepsilon)$, total energy of a deformation $\psi^\varepsilon : S^\varepsilon \rightarrow \mathbb{R}^3$
- $L_\varepsilon(\psi^\varepsilon)$, work of the external loads
- $J(\varepsilon)(\psi(\varepsilon))$, rescaled elastic energy of the deformation ψ^ε
- $I(\varepsilon)(\psi(\varepsilon))$, rescaled total energy of the deformation ψ^ε
- $\boldsymbol{J}(\varepsilon)(\boldsymbol{\psi})$, elastic energy of the deformation of asymptotic expansion $\boldsymbol{\psi}$
- $\boldsymbol{I}(\varepsilon)(\boldsymbol{\psi})$, total energy of the deformation of asymptotic expansion $\boldsymbol{\psi}$
- $I_0(\boldsymbol{\psi}_0)$, limit of the total energy (for standard boundary conditions)
- f_ε , external loads
- $W^\varepsilon : T^*(S^\varepsilon; \mathbb{R}^3) \rightarrow \bar{\mathbb{R}}^+$, stored energy
- Ψ^ε , set of admissible deformations of S^ε

- Ψ , set of admissible asymptotic expansions for the deformations of S^ε
- $W_k : T^*(S; \mathbb{R}^3) \rightarrow \mathbb{R}^+$, k -th term of the asymptotic expansion of the stored energy
- $\mathcal{M} \subset T^*(S; \mathbb{R}^3)$, the restriction of the zero set of W_2 to the midsection S'
- $\mathcal{M}' \subset T^*(S'; \mathbb{R}^3)$, projection of \mathcal{M} on $T^*(S'; \mathbb{R}^3)$
- $n_0 : \mathcal{M}' \rightarrow T_0^*((-1, 1); \mathbb{R}^3)$, orientation of the normal fiber in the deformed configuration
- $n : \mathcal{M}' \rightarrow \mathbb{R}^3$, orientation of the normal fiber in the deformed configuration (only the vectorial part)
- n' , normal to the midsection S' of the reference configuration
- D^2W_2 , second derivative of W_2 on \mathcal{M}
- $a[\cdot]^2 = a(\cdot, \cdot)$, where a is a bilinear form
- $Q_F : T^*(S'; \mathbb{R}^3) \times \mathbb{R}^3 \rightarrow \mathbb{R}$, quadratic form associated to the flexural limit energy, where F is a section of \mathcal{M}'
- $Q_F^0 := \inf_v Q_F(\cdot, v) : T^*(S'; \mathbb{R}^3) \rightarrow \mathbb{R}$
- $I_{\text{flex}}(\psi)$, flexural limit energy of a deformation ψ
- dx' , 2-dimensional Hausdorff measure restricted to S'
- $\text{Tr}(A)$, the trace of the matrix A
- $\tilde{\psi}^\varepsilon, \tilde{W}^\varepsilon, \tilde{W}_k, \tilde{\mathcal{M}}, \tilde{L}_\varepsilon, \tilde{f}_\varepsilon, \dots$, variables with tildes are defined on the geometric configuration
- $\text{SO}(TM)$, rotations of TM , that is, the fiber bundle made of all rotations of $T_x M$ (with $x \in M$), where M is a Riemannian manifold
- $\text{SO}(n)$, rotations of \mathbb{R}^n
- $\text{SL}(TM)$, special group of TM , that is, the fiber bundle made of all linear diffeomorphisms of $T_x M$ with determinant one (with $x \in M$), where M is a Riemannian manifold
- \mathcal{E}_M , set of symmetric bilinear forms on the tangent space of M
- \mathcal{O} , fiber bundle over S' whose fibers are the traceless maps from $T_{x'} S'$ into itself
- \mathcal{P} , set of polynomials of degree at most two on $\mathcal{E}_{S'}$

References

- [Agrawal and Steigmann 2009] A. Agrawal and D. J. Steigmann, “Boundary-value problems in the theory of lipid membranes”, *Contin. Mech. Thermodyn.* **21**:1 (2009), 57–82.
- [Bonito et al. 2010] A. Bonito, R. H. Nochetto, and M. S. Pauletti, “Parametric FEM for geometric biomembranes”, *J. Comput. Phys.* **229**:9 (2010), 3171–3188.

- [Bonito et al. 2011] A. Bonito, R. H. Nochetto, and M. S. Pauletti, “Dynamics of biomembranes: effect of the bulk fluid”, *Math. Model. Nat. Phenom.* **6**:5 (2011), 25–43.
- [Canham 1970] P. Canham, “The minimum energy of bending as a possible explanation of the bi-concave shape of the human red blood cell”, *Journal of Theoretical Biology* **26**:1 (1970), 61–81.
- [Conti and Maggi 2008] S. Conti and F. Maggi, “Confining thin elastic sheets and folding paper”, *Arch. Ration. Mech. Anal.* **187**:1 (2008), 1–48.
- [Conti et al. 2006] S. Conti, F. Maggi, and S. Müller, “Rigorous derivation of Föppl’s theory for clamped elastic membranes leads to relaxation”, *SIAM J. Math. Anal.* **38**:2 (2006), 657–680.
- [Crowl and Fogelson 2010] L. M. Crowl and A. L. Fogelson, “Computational model of whole blood exhibiting lateral platelet motion induced by red blood cells”, *Int. J. Numer. Methods Biomed. Eng.* **26**:3-4 (2010), 471–487.
- [Deuling and Helfrich 1976] H. Deuling and W. Helfrich, “The curvature elasticity of fluid membranes : A catalogue of vesicle shapes”, *J. Phys. France* **37**:11 (1976), 1335–1345.
- [Doyeux et al. 2013] V. Doyeux, Y. Guyot, V. Chabannes, C. Prud’homme, and M. Ismail, “Simulation of two-fluid flows using a finite element/level set method. Application to bubbles and vesicle dynamics”, *J. Comput. Appl. Math.* **246** (2013), 251–259.
- [Du and Zhang 2008] Q. Du and J. Zhang, “Adaptive finite element method for a phase field bending elasticity model of vesicle membrane deformations”, *SIAM J. Sci. Comput.* **30**:3 (2008), 1634–1657.
- [Du et al. 2004] Q. Du, C. Liu, and X. Wang, “A phase field approach in the numerical study of the elastic bending energy for vesicle membranes”, *J. Comput. Phys.* **198**:2 (2004), 450–468.
- [Du et al. 2006] Q. Du, C. Liu, and X. Wang, “Simulating the deformation of vesicle membranes under elastic bending energy in three dimensions”, *J. Comput. Phys.* **212**:2 (2006), 757–777.
- [Dziuk 2008] G. Dziuk, “Computational parametric Willmore flow”, *Numer. Math.* **111**:1 (2008), 55–80.
- [Evans 1974] E. Evans, “Bending resistance and chemically induced moments in membrane bilayers”, *Biophysical Journal* **14**:12 (1974), 923 – 931.
- [Feng and Klug 2006] F. Feng and W. S. Klug, “Finite element modeling of lipid bilayer membranes”, *J. Comput. Phys.* **220**:1 (2006), 394–408.
- [Fox et al. 1993] D. D. Fox, A. Raoult, and J. C. Simo, “A justification of nonlinear properly invariant plate theories”, *Arch. Rational Mech. Anal.* **124**:2 (1993), 157–199.
- [Friesecke et al. 2002] G. Friesecke, R. D. James, and S. Müller, “A theorem on geometric rigidity and the derivation of nonlinear plate theory from three-dimensional elasticity”, *Comm. Pure Appl. Math.* **55**:11 (2002), 1461–1506.
- [Friesecke et al. 2003] G. Friesecke, R. D. James, M. G. Mora, and S. Müller, “Derivation of nonlinear bending theory for shells from three-dimensional nonlinear elasticity by Gamma-convergence”, *C. R. Math. Acad. Sci. Paris* **336**:8 (2003), 697–702.
- [Friesecke et al. 2006] G. Friesecke, R. D. James, and S. Müller, “A hierarchy of plate models derived from nonlinear elasticity by gamma-convergence”, *Arch. Ration. Mech. Anal.* **180**:2 (2006), 183–236.
- [Helfrich 1973] W. Helfrich, “Elastic properties of lipid bilayers: theory and possible experiments”, *Naturforsch C* **28**:11 (1973), 693–703.
- [Jenkins 1977a] J. T. Jenkins, “The equations of mechanical equilibrium of a model membrane”, *SIAM J. Appl. Math.* **32**:4 (1977), 755–764.
- [Jenkins 1977b] J. T. Jenkins, “Static equilibrium configurations of a model red blood cell”, *Journal of Mathematical Biology* **4** (1977), 149–169.

- [Kim and Lai 2010] Y. Kim and M.-C. Lai, “Simulating the dynamics of inextensible vesicles by the penalty immersed boundary method”, *J. Comput. Phys.* **229**:12 (2010), 4840–4853.
- [Krishnaswamy 1996] S. Krishnaswamy, “A cosserat-type model for the red blood cell wall”, *International Journal of Engineering Science* **34**:8 (1996), 873–899.
- [Le Dret and Raoult 1995] H. Le Dret and A. Raoult, “The nonlinear membrane model as variational limit of nonlinear three-dimensional elasticity”, *J. Math. Pures Appl.* (9) **74**:6 (1995), 549–578.
- [Le Dret and Raoult 1996] H. Le Dret and A. Raoult, “The membrane shell model in nonlinear elasticity: a variational asymptotic derivation”, *J. Nonlinear Sci.* **6**:1 (1996), 59–84.
- [Liu and Liu 2006] Y. Liu and W. K. Liu, “Rheology of red blood cell aggregation by computer simulation”, *J. Comput. Phys.* **220**:1 (2006), 139–154.
- [Liu et al. 2004] Y. Liu, L. Zhang, X. Wang, and W. K. Liu, “Coupling of Navier–Stokes equations with protein molecular dynamics and its application to hemodynamics”, *Internat. J. Numer. Methods Fluids* **46**:12 (2004), 1237–1252.
- [Luke 1982] J. C. Luke, “A method for the calculation of vesicle shapes”, *SIAM J. Appl. Math.* **42**:2 (1982), 333–345.
- [Luke and Kaplan 1979] J. C. Luke and J. I. Kaplan, “On the theoretical shapes of bilipid vesicles under conditions of increasing membrane area”, *Biophysical Journal* **25**:1 (1979), 107–111.
- [Maitre et al. 2009] E. Maitre, T. Milcent, G.-H. Cottet, A. Raoult, and Y. Usson, “Applications of level set methods in computational biophysics”, *Math. Comput. Modelling* **49**:11-12 (2009), 2161–2169.
- [Merlet 2013a] B. Merlet, “A highly anisotropic nonlinear elasticity model for vesicles I. Eulerian formulation, rigidity estimates and vanishing energy limit”, 2013, http://hal.archives-ouvertes.fr/hal-00848547/PDF/Merlet_VesiclesPartI.pdf. 23 pages, preprint.
- [Merlet 2013b] B. Merlet, “A highly anisotropic nonlinear elasticity model for vesicles. II. Derivation of the thin bilayer bending theory”, 2013, http://hal.archives-ouvertes.fr/hal-00848552/PDF/Merlet_VesiclesPartII.pdf. 57 pages, preprint.
- [Miao et al. 1994] L. Miao, U. Seifert, M. Wortis, and H.-G. Döbereiner, “Budding transitions of fluid-bilayer vesicles: The effect of area-difference elasticity”, *Phys. Rev. E* **49**:6 (1994), 5389–5407.
- [Pantz 2003] O. Pantz, “On the justification of the nonlinear inextensional plate model”, *Arch. Ration. Mech. Anal.* **167**:3 (2003), 179–209.
- [Peng et al. 2013] Z. Peng, X. Li, I. V. Pivkin, M. Dao, G. E. Karniadakis, and S. Suresh, “Lipid bilayer and cytoskeletal interactions in a red blood cell”, *Proc. Nat. Acad. Sci.* (2013).
- [Salac and Miksis 2011] D. Salac and M. Miksis, “A level set projection model of lipid vesicles in general flows”, *J. Comput. Phys.* **230**:22 (2011), 8192–8215.
- [Seifert et al. 1991] U. Seifert, K. Berndl, and R. Lipowsky, “Shape transformations of vesicles: Phase diagram for spontaneous-curvature and bilayer-coupling models”, *Phys. Rev. A* **44** (1991), 1182–1202.
- [Singer and Nicolson 1972] S. J. Singer and G. L. Nicolson, “The fluid mosaic model of the structure of cell membranes”, *Science* **175**:4023 (1972), 720–731.
- [Sohn et al. 2010] J. S. Sohn, Y.-H. Tseng, S. Li, A. Voigt, and J. S. Lowengrub, “Dynamics of multicomponent vesicles in a viscous fluid”, *J. Comput. Phys.* **229**:1 (2010), 119–144.
- [Steigmann 1999] D. J. Steigmann, “Fluid films with curvature elasticity”, *Archive for Rational Mechanics and Analysis* **150** (1999), 127–152.

[Svetina and Žekš 1989] S. Svetina and B. Žekš, “Membrane bending energy and shape determination of phospholipid vesicles and red blood cells”, *European Biophysics Journal* **17** (1989), 101–111.

[Veerapaneni et al. 2009] S. K. Veerapaneni, D. Gueyffier, D. Zorin, and G. Biros, “A boundary integral method for simulating the dynamics of inextensible vesicles suspended in a viscous fluid in 2D”, *J. Comput. Phys.* **228**:7 (2009), 2334–2353.

Received 24 Jul 2012. Revised 6 Jun 2014. Accepted 9 Jun 2014.

OLIVIER PANTZ: olivier.pantz@polytechnique.org

*Centre de Mathématiques Appliquées, École Polytechnique, Route de Saclay,
91128 Palaiseau CEDEX, France*

KARIM TRABELSI: karim.trabelsi@polytechnique.edu

*Direction de la Recherche et de l’Innovation, Institut Polytechnique des Sciences Avancées,
5-9 rue Maurice Grandcoing, 94200 Ivry-sur-Seine, France*



MATHEMATICS AND MECHANICS OF COMPLEX SYSTEMS

msp.org/memocs

EDITORIAL BOARD

ANTONIO CARCATERRA	Università di Roma "La Sapienza", Italia
ERIC A. CARLEN	Rutgers University, USA
FRANCESCO DELL'ISOLA	(CO-CHAIR) Università di Roma "La Sapienza", Italia
RAFFAELE ESPOSITO	(TREASURER) Università dell'Aquila, Italia
ALBERT FANNJIANG	University of California at Davis, USA
GILLES A. FRANCFORT	(CO-CHAIR) Université Paris-Nord, France
PIERANGELO MARCATI	Università dell'Aquila, Italy
JEAN-JACQUES MARIGO	École Polytechnique, France
PETER A. MARKOWICH	DAMTP Cambridge, UK, and University of Vienna, Austria
MARTIN OSTOJA-STARZEWSKI	(CHAIR MANAGING EDITOR) Univ. of Illinois at Urbana-Champaign, USA
PIERRE SEPPECHER	Université du Sud Toulon-Var, France
DAVID J. STEIGMANN	University of California at Berkeley, USA
PAUL STEINMANN	Universität Erlangen-Nürnberg, Germany
PIERRE M. SUQUET	LMA CNRS Marseille, France

MANAGING EDITORS

MICOL AMAR	Università di Roma "La Sapienza", Italia
CORRADO LATTANZIO	Università dell'Aquila, Italy
ANGELA MADEO	Université de Lyon-INSA (Institut National des Sciences Appliquées), France
MARTIN OSTOJA-STARZEWSKI	(CHAIR MANAGING EDITOR) Univ. of Illinois at Urbana-Champaign, USA


ADVISORY BOARD

ADNAN AKAY	Carnegie Mellon University, USA, and Bilkent University, Turkey
HOLM ALTENBACH	Otto-von-Guericke-Universität Magdeburg, Germany
MICOL AMAR	Università di Roma "La Sapienza", Italia
HARM ASKES	University of Sheffield, UK
TEODOR ATANACKOVIĆ	University of Novi Sad, Serbia
VICTOR BERDICHEVSKY	Wayne State University, USA
GUY BOUCHITTÉ	Université du Sud Toulon-Var, France
ANDREA BRAIDES	Università di Roma Tor Vergata, Italia
ROBERTO CAMASSA	University of North Carolina at Chapel Hill, USA
MAURO CARFORE	Università di Pavia, Italia
ERIC DARVE	Stanford University, USA
FELIX DARVE	Institut Polytechnique de Grenoble, France
ANNA DE MASI	Università dell'Aquila, Italia
GIANPIETRO DEL PIERO	Università di Ferrara and International Research Center MEMOCS, Italia
EMMANUELE DI BENEDETTO	Vanderbilt University, USA
BERNOLD FIEDLER	Freie Universität Berlin, Germany
IRENE M. GAMBA	University of Texas at Austin, USA
SERGEY GAVRILYUK	Université Aix-Marseille, France
TIMOTHY J. HEALEY	Cornell University, USA
DOMINIQUE JEULIN	École des Mines, France
ROGER E. KHAYAT	University of Western Ontario, Canada
CORRADO LATTANZIO	Università dell'Aquila, Italy
ROBERT P. LIPTON	Louisiana State University, USA
ANGELO LUONGO	Università dell'Aquila, Italia
ANGELA MADEO	Université de Lyon-INSA (Institut National des Sciences Appliquées), France
JUAN J. MANFREDI	University of Pittsburgh, USA
CARLO MARCHIORO	Università di Roma "La Sapienza", Italia
GÉRARD A. MAUGIN	Université Paris VI, France
ROBERTO NATALINI	Istituto per le Applicazioni del Calcolo "M. Picone", Italy
PATRIZIO NEFF	Universität Duisburg-Essen, Germany
ANDREY PIATNITSKI	Narvik University College, Norway, Russia
ERRICO PRESUTTI	Università di Roma Tor Vergata, Italy
MARIO PULVIRENTI	Università di Roma "La Sapienza", Italia
LUCIO RUSSO	Università di Roma "Tor Vergata", Italia
MIGUEL A. F. SANJUAN	Universidad Rey Juan Carlos, Madrid, Spain
PATRICK SELVADURAI	McGill University, Canada
ALEXANDER P. SEYRANIAN	Moscow State Lomonosov University, Russia
MIROSLAV ŠILHAVÝ	Academy of Sciences of the Czech Republic
GUIDO SWEERS	Universität zu Köln, Germany
ANTOINETTE TORDESILLAS	University of Melbourne, Australia
LEV TRUSKINOVSKY	École Polytechnique, France
JUAN J. L. VELÁZQUEZ	Bonn University, Germany
VINCENZO VESPRI	Università di Firenze, Italia
ANGELO VULPIANI	Università di Roma La Sapienza, Italia

MEMOCS (ISSN 2325-3444 electronic, 2326-7186 printed) is a journal of the International Research Center for the Mathematics and Mechanics of Complex Systems at the Università dell'Aquila, Italy.

Cover image: "Tangle" by © John Horigan; produced using the *Context Free* program (contextfreeart.org).

PUBLISHED BY

 **mathematical sciences publishers**
nonprofit scientific publishing

<http://msp.org/>

© 2015 Mathematical Sciences Publishers

Derivation of nonlinear shell models combining shear and flexure: application to biological membranes Olivier Pantz and Karim Trabelsi	101
Canonical duality theory and triality for solving general global optimization problems in complex systems Daniel Morales-Silva and David Y. Gao	139
Neutrality of eccentrically coated elastic inclusions Xu Wang and Peter Schiavone	163
Relative Cauchy evolution for the vector potential on globally hyperbolic spacetimes Marco Benini	177

MEMOCS is a journal of the International Research Center for the Mathematics and Mechanics of Complex Systems at the Università dell'Aquila, Italy.

