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ORTHOGONAL POLYNOMIALS AND RIESZ BASES APPLIED TO THE SOLUTION OF LOVE'S EQUATION

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In this paper we reinvestigate the structure of the solution of a well-known Love's problem, related to the electrostatic field generated by two circular coaxial conducting disks, in terms of orthogonal polynomial expansions, enlightening the role of the recently introduced class of the Lucas–Lehmer polynomials. Moreover we show that the solution can be expanded more conveniently with respect to a Riesz basis obtained starting from Chebyshev polynomials.

1. Introduction

In 1949, E. R. Love [1949] considered the electrostatic field generated by two identical circular coaxial conducting disks at equal, and at equal and opposite, potentials, the potential at infinity being taken equal to zero. He established a celebrated expression for the potential, involving the solution of an integral equation of well-known type, much simpler than that considered by other authors in previous works.

Love's integral equation is a Fredholm equation of the second kind. It has found applications in several applied physics fields such as polymer structures, aerodynamics, fracture mechanics, hydrodynamics, and elasticity engineering. Recently, a polynomial expansion scheme was proposed by M. Agida and A. S. Kumar [2010] as an analytical method for solving Love's integral equation in the case of a rational kernel. Their study is concerned with the calculation of the normalized field created conjointly by two similar plates of radius R, separated by a distance kR, where k is a positive real parameter, and at equal or opposite potential, with zero potential at infinity; the solution of this problem solves a Love's second kind integral equation; see also [Love 1990; Ren et al. 1999].

We propose two different approaches to this problem. In Section 2, starting from a classical technique, based on the expansion of the solution in orthogonal polynomials, we employ a class of polynomials introduced in [Vellucci and Bersani

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2016], in order to solve a modified version of the original Love's equation. In Section 3, we recall a work by M. Norgren and B. L. G. Jonsson [2009], and we show that their results are still valid, expanding the solution of Love's integral equation with respect to a nonharmonic Fourier cosine series, which is a particular case of a Riesz basis [Sun and Zhou 1999].

For literature related to the numerical solutions of singular integral equations of the deterministic type, we refer to the fundamental book by L. Fox and I. B. Parker [1968], where different analytical methods for the solution of random integral equations were investigated.

2. Chebyshev polynomials approach

2.1. ***Preliminaries.*** In the following, we introduce the mathematical tools to employ an analytical method for solving Love's integral equation in the case of a rational kernel. Afterwards, we recall a short summary on Love's original problem.

2.1.1. *Chebyshev and Lucas–Lehmer polynomials.* The Chebyshev polynomials of the first kind [Chebyshev 1858; Chebyshev 1875; Erdélyi et al. 1953; Gatteschi 1973; Rivlin 1990] satisfy the recurrence relation

$$\begin{cases} T_0(x) = 1, \\ T_1(x) = x, \\ T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x), \quad n \geq 2. \end{cases}$$

The polynomials $T_n(x)$ are orthogonal with respect to the weight function $1/\sqrt{1-x^2}$ defined on $x \in (-1, 1)$. In a previous paper [Vellucci and Bersani 2016], we studied a class of polynomials $L_n(x) = L_{n-1}^2(x) - 2$, created by means of the same iterative formula used to build the well-known Lucas–Lehmer sequence, employed in primality tests [Lucas 1878; Lehmer 1930; Ribenboim 1988; Bressoud 1989; Koshy 2001]. It is clearly crucial to choose the first term of the polynomial sequence. In [Vellucci and Bersani 2016], we showed that the Lucas–Lehmer polynomials are orthogonal in the interval $(-2, 2)$ with respect to the weight $w(x) = 1/\bigl(4\sqrt{4-x^2}\bigr)$ and such that their zeros belong to the interval $(-2, 2)$, that is to their orthogonality interval. From now on we will consider $L_0 = x$.

Let us first recall some important properties of these polynomials.

Proposition 1. *For each $n \geq 1$,*

$$L_n(x) = 2T_{2^{n-1}}\bigl(\tfrac{1}{2}x^2 - 1\bigr) \tag{1}$$

Proposition 2. *The polynomials $L_n(x)$ are orthogonal with respect to the weight function $1/\bigl(4\sqrt{4-x^2}\bigr)$ defined on $x \in (-2, 2)$.*

Corollary 3. *With $x = 2\cos\theta$, the polynomials $L_n(x)$ admit the representation*

$$L_n(2\cos\theta) = 2\cos(2^n\theta) \tag{2}$$

For $|x| \le 2$ we can show another formula for L_n:

$$L_n(x) = \Big(\tfrac{1}{2}x^2 - 1 - \sqrt{\big(\tfrac{1}{2}x^2-1\big)^2 - 1}\Big)^{2^{n-1}} + \Big(\tfrac{1}{2}x^2 - 1 + \sqrt{\big(\tfrac{1}{2}x^2-1\big)^2 - 1}\Big)^{2^{n-1}}. \tag{3}$$

We change the sign inside the radical, factoring out the imaginary unit:

$$L_n(x) = \Big(\tfrac{1}{2}x^2 - 1 - \imath\sqrt{1-\big(\tfrac{1}{2}x^2-1\big)^2}\Big)^{2^{n-1}} + \Big(\tfrac{1}{2}x^2 - 1 + \imath\sqrt{1-\big(\tfrac{1}{2}x^2-1\big)^2}\Big)^{2^{n-1}}. \tag{4}$$

We then calculate the powers of the complex conjugate numbers L_n^+ and L_n^-, depending on the variable x. Let

$$\begin{aligned} L_n(x) &= \Big(\tfrac{1}{2}x^2 - 1 + \sqrt{\big(\tfrac{1}{2}x^2-1\big)^2 - 1}\Big)^{2^{n-1}} + \Big(\tfrac{1}{2}x^2 - 1 - \sqrt{\big(\tfrac{1}{2}x^2-1\big)^2 - 1}\Big)^{2^{n-1}} \\ &= L_n^+(x) + L_n^-(x). \end{aligned} \tag{5}$$

The absolute value of both complex numbers is unitary, since

$$\begin{aligned} |L_n^-| &= \Big|\tfrac{1}{2}x^2 - 1 - \imath\sqrt{1-\big(\tfrac{1}{2}x^2-1\big)^2}\Big|^{2^{n-1}}, \\ |L_n^+| &= \Big|\tfrac{1}{2}x^2 - 1 + \imath\sqrt{1-\big(\tfrac{1}{2}x^2-1\big)^2}\Big|^{2^{n-1}}, \end{aligned}$$

and

$$|L_n^+| = |L_n^-| = \sqrt{\big(\tfrac{1}{2}x^2-1\big)^2 + 1 - \big(\tfrac{1}{2}x^2-1\big)^2} = 1. \tag{6}$$

Moreover, since $L_1(\pm\sqrt{2}) = 0$, $L_2(\pm\sqrt{2}) = -2$, and $L_n(\pm\sqrt{2}) = 2$ for all $n \ge 3$, the argument of $L_n(\pm\sqrt{2})$ is 0 for every $n \ge 3$. In the other cases, since we can write $x = 2\cos\vartheta$ when $|x| \le 2$, it follows that $\frac{1}{2}x^2 - 1 = \cos 2\vartheta$. Thus for $|x| \ne \sqrt{2}$, we can also put

$$\vartheta(x) = \tfrac{1}{2}\arctan\frac{\sqrt{1-\big(\tfrac{1}{2}x^2-1\big)^2}}{\tfrac{1}{2}x^2-1} + b\pi \tag{7}$$

where b is a binary digit. Finally, using (2), we obtain $L_n(x) = 2\cos(2^n\vartheta(x))$.

By further setting

$$\theta(x) = \tfrac{1}{2}\arctan\frac{\sqrt{1-\big(\tfrac{1}{2}x^2-1\big)^2}}{\tfrac{1}{2}x^2-1}, \tag{8}$$

we can write

$$L_n(x) = 2\cos(2^n\theta(x) + 2^n b\pi) = 2\cos(2^n\theta(x)). \tag{9}$$

On the other hand, the Chebyshev polynomials of the first kind can be defined as the unique polynomials satisfying

$$T_n(t) = \cos(n \arccos t)$$

or, in other words, as the unique polynomials satisfying

$$T_n(\cos(\vartheta)) = \cos(n\vartheta)$$

for $n = 0, 1, 2, 3, \ldots$. Therefore, by Proposition 1,

$$L_n(x) = 2T_{2^{n-1}}\big(\tfrac{1}{2}x^2 - 1\big) = 2\cos\big(2^{n-1} \arccos\big(\tfrac{1}{2}x^2 - 1\big)\big).$$

2.1.2. *Love's problem.* Two leading cases of the problem are considered here: to specify the field generated by two identical circular coaxial conducting disks (a) at equal potentials and (b) at equal and opposite potentials, the potential at infinity being taken as zero. The results established by Love are as follows: the upper sign referring to the case of equally charged disks and the lower to that of oppositely charged disks. For Theorem 4 we refer to [Love 1949, Figures 1 and 2].

Theorem 4 [Love 1949]. *In the two leading cases described above, the potential at any point (p, ζ, ζ'), specified by its distance $r = \rho a$ from the axis of the disks and its axial distances $z = \zeta a$ and $z' = \zeta' a$ from their planes, is*

$$\frac{V_0}{\pi}\int_{-1}^{1}\left(\frac{1}{\sqrt{\rho^2 + (\zeta + it)^2}} \pm \frac{1}{\sqrt{\rho^2 + (\zeta' + it)^2}}\right) f(t)\, dt, \tag{10}$$

where V_0 is potential of the disks, a is the radius of the disks, each square root has positive real part, and $f(t)$ is the solution of the integral equation

$$f(x) \pm \frac{1}{\pi}\int_{-1}^{1} \frac{k}{k^2 + (x-t)^2} f(t)\, dt = 1, \quad |x| \le 1 \tag{11}$$

where k is the spacing parameter.

Theorem 5 [Love 1949]. *For every positive k, (11) has a continuous solution, and no other solution: it is real and even, and is specifiable by the Neumann series*

$$f(x) = 1 + \sum_{n=1}^{\infty} (\mp 1)^n \int_{-1}^{1} K_n(x, t)\, dt, \tag{12}$$

where the iterated kernels $K_n(x, t)$, for $n \in \mathbb{N}$, $n > 1$, are given by

$$K_1(x, t) = \frac{1}{\pi}\frac{k}{k^2 + (x-t)^2} \quad \textit{and} \quad K_n(x, t) = \int_{-1}^{1} K_{n-1}(x, s) K_1(s, t)\, ds.$$

Theorem 6 [Love 1949]. *The capacitance of each disk in the two cases is*

$$\frac{a}{\pi}\int_{-1}^{1} f(t)\,dt,$$

and the components of the field at all points not on the disks are given by the appropriate formal differentiations of (10).

2.2. ***The classical approach to the problem in terms of orthogonal polynomials.*** For the solution of the problems we will refer to [Fox and Parker 1968]. When the upper and lower disks are at potentials V_0 and $\pm V_0$, the potential V at any point whose spheroidal coordinates are (μ, η) with respect to the upper disk and (μ', η') with respect to the lower one is expressed in terms of Legendre functions. The upper disk, specified in cylindrical polar coordinates (r, θ, z) by $r \le a$ and $z = 0$, is taken as "focal disk" $\eta = 0$ of spheroidal coordinates (μ, η); in actual study these are such that $-2 \le \mu \le 2$, $\eta \ge 0$.

Then (10) can be rewritten in the form

$$\frac{V_0}{2\pi}\int_{-2}^{2}\left(\frac{1}{\sqrt{\rho^2+(\zeta+it/2)^2}} \pm \frac{1}{\sqrt{\rho^2+(\zeta'+it/2)^2}}\right) f(t/2)\,dt, \tag{13}$$

where each square root has positive real part, and $f(t)$ is the solution of the integral equation

$$f(x) \pm \frac{1}{2\pi}\int_{-2}^{2}\frac{k}{k^2+(x-t/2)^2}\, f(t/2)\,dt = 1, \quad |x| \le 2. \tag{14}$$

By the linear transformation $t = 2y$, both equations can be reduced to Love's original form. In (14) we put $k = 1$ and consider positive sign, so

$$f(x) + \frac{1}{2\pi}\int_{-2}^{2}\frac{1}{1+(x-t/2)^2}\, f(t/2)\,dt = 1, \quad |x| \le 2. \tag{15}$$

We make the substitution $x \mapsto \frac{1}{2}x^2 - 1$ in (15), yielding

$$f\left(\tfrac{1}{2}x^2-1\right) + \frac{1}{2\pi}\int_{-2}^{2}\frac{1}{1+\left(\frac{1}{2}(x^2-t)-1\right)^2}\, f\left(\tfrac{1}{2}t\right)dt = 1.$$

We can find a Chebyshev series solution as follows: write

$$f(x) = \sum_{r=0}^{\infty} a_r T_r(x),$$

substitute it into (15), interchange the order of integration and summation in the first term. Then we arrive at the equation

$$\sum_{r=0}^{\infty} a_r T_r\left(\tfrac{1}{2}x^2-1\right)+\frac{1}{2\pi}\sum_{s=0}^{\infty} a_s \int_{-2}^{2} \frac{T_s\left(\frac{1}{2}t\right)}{1+\left(\frac{1}{2}(x^2-t)-1\right)^2}\,dt=1 \tag{16}$$

for $|x|\le 2$. If we can now determine the expansion

$$\frac{1}{2}\int_{-2}^{2} \frac{T_s\left(\frac{1}{2}t\right)}{1+\left(\frac{1}{2}(x^2-t)-1\right)^2}\,dt=\sum_{r=0}^{\infty} b_{sr} T_r\left(\tfrac{1}{2}x^2-1\right),$$

we can equate the corresponding coefficients of each $T_r(x)$ on both sides of (15), which is legitimate since the Chebyshev polynomials form a complete set of independent functions, to produce an infinite set of algebraic equations for the required coefficients a_r, given by

$$a_r+\sum_{s=0}^{\infty} a_s b_{sr}=0, \quad r=1,2,\ldots \tag{17}$$

and, for $r=0$,

$$a_0+\sum_{s=0}^{\infty} a_s b_{s,0}=1.$$

The a_r will decrease rapidly for sufficiently large r, so that in a convenient method of solving (17) we select the first $n+1$ rows and columns, perform Gaussian elimination and back-substitution for the last few coefficients — a_n, a_{n-1}, a_{n-2}, say — decide by inspection whether convergence is sufficiently rapid for the required precision with this selected value of n, and if necessary add some extra rows and columns with only a small additional amount of work.

Let's go back to (16). Let

$$J=\{1,2,4,\ldots\}=\{2^{r-1}\mid r\in\mathbb{N}\},$$

and rewrite (16) in this way:

$$\sum_{r=0}^{\infty} a_r T_r\left(\tfrac{1}{2}x^2-1\right)+\sum_{s=0}^{\infty} a_s \sum_{r=0}^{\infty} c_{sr} T_r\left(\tfrac{1}{2}x^2-1\right)=1,$$

where $c_{sr}=b_{sr}/\pi$. Then

$$\sum_{r\in J} a_r T_r\left(\tfrac{1}{2}x^2-1\right)+\sum_{r\notin J} a_r T_r\left(\tfrac{1}{2}x^2-1\right)$$
$$+\sum_{s=0}^{\infty} a_s\left(\sum_{r\in J} c_{sr} T_r\left(\tfrac{1}{2}x^2-1\right)+\sum_{r\notin J} c_{sr} T_r\left(\tfrac{1}{2}x^2-1\right)\right)=1.$$

By (1),

$$\frac{1}{2}\sum_{r=1}^{\infty} a_r L_r(x) + \sum_{r\notin J} a_r T_r\left(\tfrac{1}{2}x^2-1\right) + \frac{1}{2}\sum_{s=0}^{\infty} a_s \sum_{r=1}^{\infty} c_{sr} L_r(x) + \sum_{s=0}^{\infty} a_s \sum_{r\notin J} c_{sr} T_r\left(\tfrac{1}{2}x^2-1\right) = 1$$

By Proposition 1, we note that solving (17), a subset of first $n+1$ rows and columns selected to perform Gaussian elimination, is due to Lucas–Lehmer polynomials. They not only cannot by themselves guarantee the convergence to the solution, but also their contributions can be neglected. In fact, by above reasoning, since

$$f(x) = \sum_{r\notin J} a_r T_r\left(\tfrac{1}{2}x^2-1\right) + \frac{1}{2}\sum_{r\in J} a_r L_r(x),$$

we have

$$\left| f(x) - \sum_{r\notin J} a_r T_r\left(\tfrac{1}{2}x^2-1\right)\right| \le \frac{1}{2}\sum_{r\in J}|a_r| = \frac{1}{2}\sum_{r=1}^{\infty}|a_{2^{r-1}}|.$$

Accordingly, when the term on the right hand side can be considered "small" with respect to other contributions, a convenient method of solving (17) should be to select the first $n+1$ rows and columns, perform Gaussian elimination and back-substitution for the last few coefficients — a_n, a_{n-1}, a_{n-2}, say — and delete terms due to Lucas–Lehmer polynomials.

3. An alternative approach: nonharmonic Fourier series

The capacitance of a circular parallel plate capacitor can be calculated by expanding the solution of the Love's integral equation in terms of a Fourier cosine series. In previous literature, this kind of expansion was carried out numerically, leading to accuracy problems at small plate separations. Norgren and Jonsson [2009] calculated analytically all expansion integrals in terms of the sine and cosine integrals. Hence, they approximated the kernel using considerably large matrices, resulting in improved numerical accuracy for the capacitance. Previously, G. T. Carlson and B. L. Illman [1994], solved the Love's equation through an expansion of the kernel into a Fourier cosine series. To calculate the expansion coefficients of the kernel, they use numerical integration. Hence, as noted in [Norgren and Jonsson 2009], their method is limited by a combination of the accuracy of the integration and the large number of terms needed. The accumulated errors effectively limit the expansion to about 100 terms, which is insufficient for the convergence at very small separations. Let us observe that both the methods recalled here make use of orthogonal expansions.

In this section we will use some basic facts about nonharmonic Fourier series, and we recall them below.

It is well known that the family of exponentials $\{e^{int}\}_{n\in\mathbb{Z}}$ forms an orthonormal basis in $L^2(-\pi,\pi)$. The natural question that arises is: what happens if we replace it by a classical system of exponentials $\{e^{i\lambda_n t}\}_{n\in\mathbb{Z}}$; these bases are very useful for the study of the so-called almost periodic functions. See, for example, [Andres et al. 2006; Besicovitch 1932].

The celebrated work of Paley and Wiener [1934] kicked off studies on classical systems of exponentials $\{e^{i\lambda_n t}\}_{n\in\mathbb{Z}}$ in $L^2(0,T)$, where $T>0$. They proved that if $\lambda_n\in\mathbb{R}$, $n\in\mathbb{Z}$, and

$$|\lambda_n - n| \le L < \pi^{-2}, \quad n\in\mathbb{Z}$$

then the system $\{e^{i\lambda_n t}\}_{n\in\mathbb{Z}}$ forms a Riesz basis in $L^2[-\pi,\pi]$, i.e., a family of the form $\{Ue_k\}_{k=1}^{\infty}$, where $\{e_k\}_{k=1}^{\infty}$ is an orthonormal basis for a separable infinite-dimensional Hilbert space $\mathcal{H}$ and $U:\mathcal{H}\to\mathcal{H}$ is a bounded bijective operator.

M. I. Kadec [1964] extended this result to the case $L<\frac{1}{4}$. This is the so-called *Kadec-$\frac{1}{4}$ theorem*, which over the following 50 years has been extensively generalized; see, for example [Avdonin 1974; Bailey 2010; Pavlov 1979; Sedletskii 2003; Sun and Zhou 1999; Vellucci 2015]. Let us recall Kadec's original result:

Theorem 7. *If $\{\lambda_n\}_{n\in\mathbb{Z}}$ is a sequence of real numbers for which*

$$\sup_n |\lambda_n - n| < \tfrac{1}{4}, \quad n = 0, \pm 1, \dots$$

then the system $\{e^{i\lambda_n t}\}_{n\in\mathbb{Z}}$ is a Riesz basis for $L^2[-\pi,\pi]$.

Therefore, if $L=\sup_n|\lambda_n-n|<\frac{1}{4}$, then the sine system $\{\sin\lambda_n t\}_1^{\infty}$ as well as the cosine system $1\cup\{\cos\lambda_n t\}_1^{\infty}$ is a Riesz basis in $L^2(0,\pi)$.

We now approach the problem described in [Norgren and Jonsson 2009]. The circular parallel plate capacitor is depicted in Figure 1.

The distance between the circular plates is here put equal to their common radius. Accordingly, the normalized separation between the plates, a constant k, is set for

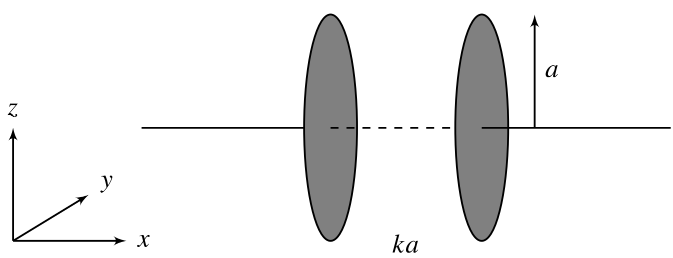

Figure 1. A circular parallel plate capacitor can be viewed as a cylindrical volume whose bases are the capacitor's plates.

the sake of simplicity equal to 1. The model is idealized in the sense that the plates have zero thickness.

The capacitance of the parallel plate capacitor is [Carlos and Illman 1994]

$$C = 4\varepsilon_0 a \int_0^1 f(s)\,ds, \tag{18}$$

where a is the radius of the circular plate and the function $f(s)$ is the solution of the modified Love's integral equation

$$f(s) - \int_0^1 K(s,t) f(t)\,dt = 1, \quad 0 \le s \le 1, \tag{19}$$

with kernel

$$K(s,t) = \frac{1}{\pi}\left(\frac{1}{1+(s-t)^2} + \frac{1}{1+(s+t)^2}\right). \tag{20}$$

To solve (19) numerically, we follow the approach in [loc. cit.] and expand the kernel and the unknown function into the (nonharmonic) Fourier cosine expansion in terms of the functions

$$\tilde{\psi}_n(s) = \sqrt{2-\delta_{n,0}}\cos(\lambda_n s), \quad n = 0, 1, \dots$$

which in our study have been orthonormalized to fulfill the orthogonality relation

$$\int_0^\pi \psi_n(s)\,\psi_m(s)\,ds = \delta_{m,n}$$

and satisfy Kadec's assumption on $L = \sup_n |\lambda_n - n| < \frac{1}{4}$. Here, $\delta_{m,n}$ denotes the Kronecker delta function.

This orthonormalization process is shown in the following

Theorem 8 (orthonormalization process). *Consider $L^2(-\pi,\pi)$ and a sequence $\{\lambda_n\}_{n\in\mathbb{Z}} \subset \mathbb{R}$ which satisfies Kadec's assumption. Let $P = (I-S)^{-1} = \sum_{m=0}^\infty S^m$, where*

$$S(f)(x) = \sum_{n=-\infty}^{\infty} \hat{f}(n)(e^{inx} - e^{i\lambda_n x})$$

and $\{\hat{f}(n)\}$ are the Fourier coefficients of f. Then $P(e^{i\lambda_n x}) = e^{inx}$ for each $n \in \mathbb{Z}$.

Proof. By Kadec's theorem, we have that $\|S\| < 1$. Hence, $P = (I-S)^{-1} = \sum_{m=0}^\infty S^m$. To show that $P(e^{i\lambda_n x}) = e^{inx}$, we write

$$e^{i\lambda_n x} = (I-S)e^{inx} = e^{inx} - \sum_k c_k(e^{ikx} - e^{i\lambda_k x})$$

where $c_k = \langle e^{inx}, e^{ikx}\rangle$. Thus

$$e^{inx} - e^{i\lambda_n x} = \sum_k \delta_{n,k}(e^{ikx} - e^{i\lambda_k x}). \qquad \square$$

In this way we have orthonormalized the Riesz basis $\{e^{i\lambda_n x}\}$, in an easy way. Further results on the orthonormalization of more complex Riesz bases, such as $\{\phi(t-n)\}_{n\in\mathbb{Z}}$, applied for example to the study of a "digital filter," can be found in [Meyer 1989]. For our purposes it is sufficient to consider the basis introduced at the beginning of Section 3 and used in Theorem 7.

Carrying out the expansions of $f(s)$ and $K(s,t)$ in terms of $\{\psi_n\}$, we obtain

$$f(s)=\sum_{m=0}^{\infty} f_m\psi_m(s), \quad \text{where } f_m=\int_0^{\pi} f(s)\psi_m(s)\,ds \tag{21}$$

and

$$K(s,t)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} K_{mn}\psi_m(s)\psi_n(t), \tag{22}$$

where

$$K_{mn}=\int_0^{\pi}\int_0^{\pi} K(s,t)\psi_n(t)\psi_m(s)\,ds\,dt.$$

These equations yield the infinite linear system of equations for the coefficients $\{f_n\}_{n=0}^{\infty}$:

$$\sum_{n=0}^{\infty}(\delta_{m,n}-K_{mn})f_n=\delta_{m,0}, \quad m=0,1,\dots. \tag{23}$$

From (18), (21), and from the orthonormalization process described in Theorem 8 and guaranteed by Kadec's assumption, which allows to us expand the kernel and the unknown function into the (nonharmonic) Fourier cosine expansion in terms of the functions $\{\cos(\lambda_n s)\}$, the capacitance reduces to $C=4\varepsilon_0 a f_0$, where f_0 is simply the $(0,0)$-element in the inverse of the matrix with elements $\delta_{m,n}-K_{mn}$, as obtained in [Norgren and Jonsson 2009].

Furthermore, Norgren and Jonsson derive analytical expressions for the expansion of the kernel $K(s,t)$. Proceeding as in [loc. cit.], it is easy to prove that, in the general case when $m\neq n$ and $m,n>0$,

$$K_{mn}=\tfrac{2}{\pi}\tilde{I}_3(n\pi,m\pi), \tag{24}$$

where $\tilde{I}_3(n\pi,m\pi)=P\,I_3(\lambda_n\pi,\lambda_m\pi)$, with P as in Theorem 8 and I_3 as defined in [loc. cit.]. The application of the operator P denotes here the orthonormalization process performed on the set of functions $\{\cos(\lambda_n s)\}_{n\in\mathbb{Z}}$.

We have extended the results of [Carlos and Illman 1994; Norgren and Jonsson 2009] to a (nonharmonic) Fourier cosine expansion in terms of the set of functions $\{\cos(\lambda_n s)\}_{n\in\mathbb{Z}}$, employing a simple procedure, due to Theorem 8, to orthonormalize the Riesz basis $\{e^{i\lambda_n x}\}$ under Kadec's assumption. Therefore, we have found a further expansion of the solution that it is not in terms of orthogonal polynomials, but in terms of nonharmonic functions $\cos(\lambda_n s)$, $s\in\mathbb{R}$.

4. Conclusion and perspectives

Orthogonal functions, other classes of polynomials, and Riesz bases have shown to be very powerful for the search of solutions of several problems in disparate fields, from physics to engineering, from economics to biology, and so on. In this paper we applied a new class of orthogonal polynomials, called Lucas–Lehmer polynomials [Vellucci and Bersani 2016] and the tool of Riesz bases [Paley and Wiener 1934] in order to reinvestigate a classical problem, due to Love [1949], obtaining a further expansion of the solution that it is not in terms of orthogonal polynomials, but in terms of nonharmonic functions $\cos(\lambda_n s)$, $s \in \mathbb{R}$, suitably orthonormalized, thanks to Theorem 8 which uses the celebrated result due to Kadec [1964]. Many other applications can be investigated in the future, mainly in the field of mechanics. In particular, great attention has been recently paid to peridynamics and fracture mechanics, which are approached in terms of integral equations. In this framework our researches will be addressed to apply our techniques to the study of some integral equations of the type introduced by Piola in 1848, recently rediscovered in the framework of peridynamics and fracture dynamics, and reported in the paper [dell'Isola et al. 2015].

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