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ON STOCHASTIC DISTRIBUTIONS AND CURRENTS

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Vincenzo Capasso and Franco Flandoli

Dedicated to Lucio Russo, on the occasion of his 70th birthday

In many applications, it is of great importance to handle random closed sets of different (even though integer) Hausdorff dimensions, including local information about initial conditions and growth parameters. Following a standard approach in geometric measure theory, such sets may be described in terms of suitable measures. For a random closed set of lower dimension with respect to the environment space, the relevant measures induced by its realizations are singular with respect to the Lebesgue measure, and so their usual Radon–Nikodym derivatives are zero almost everywhere. In this paper, how to cope with these difficulties has been suggested by introducing random generalized densities (distributions) á la Dirac–Schwarz, for both the deterministic case and the stochastic case. For the last one, mean generalized densities are analyzed, and they have been related to densities of the expected values of the relevant measures. Actually, distributions are a subclass of the larger class of currents; in the usual Euclidean space of dimension d, currents of any order $k \in \{0, 1, \dots, d\}$ or k-currents may be introduced. In this paper, the cases of 0-currents (distributions), 1-currents, and their stochastic counterparts are analyzed. Of particular interest in applications is the case in which a 1-current is associated with a path (curve). The existence of mean values has been discussed for currents too. In the case of 1-currents associated with random paths, two cases are of interest: when the path is differentiable, and also when it is the path of a Brownian motion or (more generally) of a diffusion. Differences between the two cases have been discussed, and nontrivial problems are mentioned which arise in the case of diffusions. Two significant applications to real problems have been presented too: tumor driven angiogenesis, and turbulence.

1. Introduction: preliminaries and notation

Many real phenomena may be modeled as random closed sets in $\mathbb{R}^d$, and in several situations as evolving random closed sets. Application areas include crystallization processes: Figures 1 and 2 (see [Capasso 2003; Capasso and Micheletti 2006], and
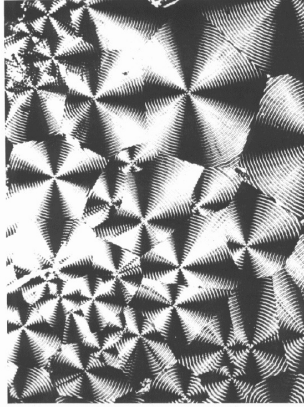
Figure 1. The final tessellation in a real experiment of a crystal-
lization process of a polymer [MONTELL-Italy].

references therein; see also [Ubukata 2003] for the crystallization processes on sea
shells, and [Callister Jr. 2007, p. 92; Hochrainer et al. 2007; Kassner et al. 2000]
for dislocations: Figure 3); tumor growth [Anderson 2003]; angiogenesis: Figure 6
[Carmeliet and Jain 2000]; patterns in biology: Figure 4; the spread of a pollutant
in an environment; etc.

A rigorous definition of the relevant geometric quantities in a stochastic setting
of the above systems (fibers for angiogenesis, dislocations for crystalline materials,
etc.) is very important for statistical applications (see, e.g., [Ambrosio et al. 2009;
Camerlenghi et al. 2014]), and in mean field approximations (see, e.g., [Bonilla
et al. 2017; Hochrainer et al. 2007; Bessaih et al. 2017]).

A presentation of an angiogenesis model will be offered later in Section 4.1.

For definitions and basic properties of Hausdorff measure and Hausdorff dimen-
sion see, e.g., [Ambrosio et al. 2000; Falconer 1986; Federer 1996; Morgan 1998].

We remind here the concepts and results of current literature which are relevant
for our analysis. Actually, the subject of stochastic geometry, considered here in
the direction of geometric measure theory, does have a nontrivial intersection with
the literature on convex geometry for which the reader may refer to [Baddeley et al.
2007] and references therein.
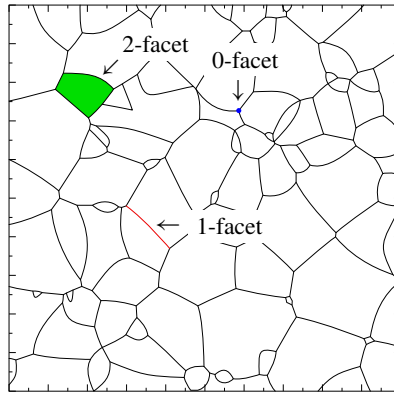
Figure 2. The final tessellation in a simulated experiment of a crystallization process of a polymer. This picture, together with the real one, shows the relevance of components at all integer Hausdorff dimensions for describing the final morphology [Burger et al. 2002].
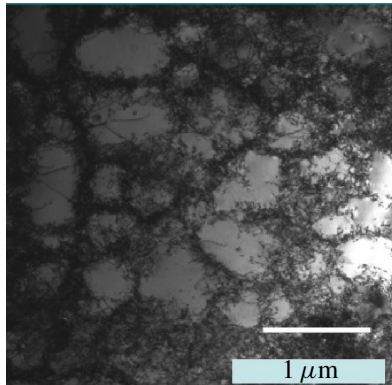

Figure 3. Dislocations in copper crystals [M. Kassner; private collection].

Figure 4. Pattern formation in a lichen colony [V. Capasso; private collection].

Figure 5.   Angiogenesis on a rat cornea [Dejana, personal collection].
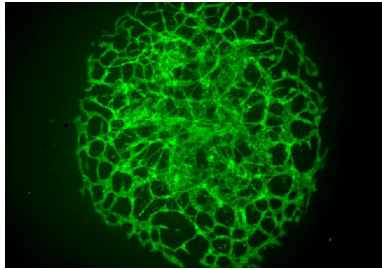

Figure 6.   Vascularization of an allantoid. [Dejana, personal collection]. An important example of a fiber process of Hausdorff dimension 1 in a 3D space.

Let us consider the space $\mathbb{R}^d$ for $d > 1$, and denote by ν^d the usual d-dimensional Lebesgue measure, and by $\mathcal{B}_{\mathbb{R}^d}$ the Borel σ-algebra of $\mathbb{R}^d$.

We know that every positive Radon measure μ on $\mathbb{R}^d$ can be represented as

$$\mu = \mu_{\ll} + \mu_{\perp},$$

where $\mu_{\ll}$ and $\mu_{\perp}$ are the absolutely continuous part with respect to ν^d and the singular part of μ, respectively. We shall denote by $B_r(x)$ the d-dimensional closed ball centered in x with radius r.

Let us denote by $\mathcal{H}^s$ the s-dimensional Hausdorff measure.

We will consider a class of subsets of $\mathbb{R}^d$ with integer Hausdorff dimension.

Definition 1. Given an integer $n \in [0, d]$, we say that a closed subset A of $\mathbb{R}^d$ is *n-regular* if it satisfies the following conditions:

(i) $\mathcal{H}^n(A \cap B_R(0)) < \infty$ for any $R > 0$;

(ii) $\displaystyle\lim_{r\to 0} \frac{\mathcal{H}^n(A \cap B_r(x))}{b_n r^n} = 1$ for $\mathcal{H}^n$-a.e. $x \in A$.

We recall that when n is integer, then $b(n) = b_n$, which is the volume of the unit ball in $\mathbb{R}^n$.

Note that condition (ii) is related to a characterization of the $\mathcal{H}^n$-rectifiability of the set A [Falconer 1986, p.256, 267; Ambrosio et al. 2000, p.83].

Remark 2. We may observe that if Θ is an n-regular closed set in $\mathbb{R}^d$, we have

$$\lim_{r\to 0} \frac{\mathcal{H}^n(\Theta \cap B_r(x))}{b_n r^n} = \begin{cases} 1 & \mathcal{H}^n\text{-a.e. } x \in \Theta, \\ 0 & \text{for } x \notin \Theta. \end{cases} \tag{1-1}$$

In fact, since Θ^C is open, for all $x \notin \Theta$ there exists $r_0 > 0$ such that for all $r \le r_0$, we have $B_r(x) \subset \Theta^C$, that is $\mathcal{H}^n(\Theta \cap B_r(x)) = 0$ for all $r \le r_0$; thus the limit equals 0 for all $x \in \Theta^C$.

For a general set A, problems about "$\mathcal{H}^n$-a.e." and "for all" may arise when we consider a point $x \in \partial A$ where the boundary is not a regular manifold. For example, if A is a closed square in $\mathbb{R}^2$, for all points x on the edges,

$$\lim_{r\to 0} \frac{\mathcal{H}^2(A \cap B_r(x))}{b_2 r^2} = \frac{1}{2},$$

while for each of the four vertices the limit equals $\frac{1}{4}$.

Observe that in both cases the set of such points has $\mathcal{H}^2$-measure zero.

From now on we shall consider n-regular closed sets Θ in $\mathbb{R}^d$, with $0 \le n \le d$. As a consequence, for $n < d$ (by assuming $0 \cdot \infty = 0$) by (1-1) we also have

$$\lim_{r\to 0} \frac{\mathcal{H}^n(\Theta \cap B_r(x))}{b_d r^d} = \lim_{r\to 0} \frac{\mathcal{H}^n(\Theta \cap B_r(x))}{b_n r^n} \frac{b_n r^n}{b_d r^d} = \begin{cases} \infty & \mathcal{H}^n\text{-a.e. } x \in \Theta, \\ 0 & \text{for } x \notin \Theta. \end{cases}$$

Note that in the particular case $n = 0$, with $\Theta = X_0$ point in $\mathbb{R}^d$ (X_0 is indeed a 0-regular closed set),

$$\lim_{r\to 0} \frac{\mathcal{H}^0(X_0 \cap B_r(x))}{b_d r^d} = \begin{cases} \infty & \text{if } x = X_0, \\ 0 & \text{if } x \ne X_0. \end{cases}$$

Note that if Θ is an n-regular closed set in $\mathbb{R}^d$ with $n < d$, then the Radon measure

$$\mu_\Theta(\cdot) := \mathcal{H}^n(\Theta \cap \cdot)$$

is a singular measure with respect to ν^d.

It is then clear that the quantity

$$\delta_\Theta(x) := \lim_{r\to 0} \frac{\mathcal{H}^n(\Theta \cap B_r(x))}{b_d r^d}, \quad x \in \mathbb{R}^d,$$

associated with Θ cannot be considered as a classical function. But, analogously to the Dirac delta function $\delta_{X_0}(x)$ associated with a point $X_0 \in \mathbb{R}^d$, we may refer to it as a *generalized density* (or the *generalized Radon–Nikodym derivative* of the measure μ_Θ with respect to ν_d), or better as a *distribution á la Schwarz* (see Section 2).

We may notice that in the case $\Theta = X_0$, the generalized density $\delta_{X_0}(x)$ coincides with the well-known Dirac delta function at a point X_0 that is the (generalized) density of the singular Dirac measure ε_{X_0} [Kolmogorov and Fomin 1970].

The usefulness of introducing these *generalized functions* associated with sets of any dimension $n \in \{0, \ldots, d\}$, in particular in the stochastic case, has been discussed in various papers [Matheron 1965; Ambrosio et al. 2009; Burger et al. 2002; Capasso and Villa 2006; 2007; Vladimirov 1979].

The previous ideas extend to another framework, the one of *currents*. First, let us restrict ourselves to the so-called 1-currents, which heuristically are distributional generalizations of the concept of vector fields. In the smooth case, a 1-current in $\mathbb{R}^d$ is simply a smooth vector field $\xi : \mathbb{R}^d \to \mathbb{R}^d$. A smooth vector field $\xi(x)$ acts on test functions (above we have introduced objects using the language of measures, but here it is more natural to use directly the language of test functions) as

$$(\xi, \theta) = \int_{\mathbb{R}^d} \langle \xi(x), \theta(x) \rangle \, \mathrm{d}x,$$

for all smooth test vector fields $\theta : \mathbb{R}^d \to \mathbb{R}^d$. But, similarly to the case of measures concentrated on lower-dimensional subsets Θ of $\mathbb{R}^d$, we may consider vector fields concentrated on lower-dimensional subsets. Let us restrict further our discussion of 1-currents to the case of currents associated with curves. Given a smooth curve $\gamma : [a, b] \to \mathbb{R}^d$, the concept of current associates to it a distribution vector field concentrated along the curve, and having the direction tangent to the curve. The action on smooth test vector fields $\theta : \mathbb{R}^d \to \mathbb{R}^d$ is

$$(\xi, \theta) = \int_a^b \langle \theta(\gamma(t)), \gamma'(t) \rangle \, \mathrm{d}t,$$

or, in the more intuitive language that we shall explain in the sequel,

$$\xi(x) = \int_a^b \delta_{\gamma(t)}(x) \gamma'(t) \, \mathrm{d}t,$$

which emphasizes the fact that the vector field is localized on the curve, with the direction of the tangent.

In Section 2 we present densities of measures as linear functionals for both the deterministic and the stochastic case; special emphasis is given to random functionals associated with random closed sets and their mean values. Section 3 is devoted

to 1-currents with their stochastic counterparts. Finally Section 4 presents examples of significant applications of real interest, such as tumor driven angiogenesis and turbulence.

2. Densities as linear functionals

2.1. *The deterministic case.* We know that the Dirac delta δ_{X_0} at a point $X_0 \in \mathbb{R}^d$ can be defined as a linear functional associated with a finite Borel measure, the well-known Dirac measure ε_{X_0}, concentrated at X_0; as such it is the (generalized) density of ε_{X_0}. In fact, we recall that, according to the Riesz representation theorem (see, e.g., [Folland 1999, p. 212]), Radon measures in $\mathbb{R}^d$ (i.e., nonnegative and σ-additive set functions defined on the Borel σ-algebra $\mathcal{B}_{\mathbb{R}^d}$ that are finite on bounded sets) can be canonically identified with linear and order preserving functionals on $C_c(\mathbb{R}^d, \mathbb{R})$, the space of continuous functions with compact support in $\mathbb{R}^d$. The identification is provided by the integral operator, i.e.,

$$(\mu, f) = \int_{\mathbb{R}^d} f \, \mathrm{d}\mu \quad \forall f \in C_c(\mathbb{R}^d, \mathbb{R}).$$

If $\mu \ll \nu^d$, it admits (as a Radon–Nikodym density) a classical function δ_μ defined almost everywhere in $\mathbb{R}^d$, so that

$$(\mu, f) = \int_{\mathbb{R}^d} f(x)\delta_\mu(x) \, \mathrm{d}x \quad \forall f \in C_c(\mathbb{R}^d, \mathbb{R}),$$

in the usual sense of a Lebesgue integral.

If $\mu \perp \nu^d$, we may speak of a density δ_μ only in the sense of distributions (formally, it is almost everywhere trivial, but it is ∞ on a set of ν^d-measure zero). In this case, the symbol

$$\int_{\mathbb{R}^d} f(x)\delta_\mu(x) \, \mathrm{d}x := (\mu, f)$$

can still be adopted, provided the integral on the left-hand side is understood in a generalized sense, and not as a Lebesgue integral.

In either case, from now on, we may denote by (δ_μ, f) the quantity (μ, f). Accordingly, we say that a sequence of measures μ_n *weakly* converges* to a Radon measure μ if (δ_{μ_n}, f) converges to (δ_μ, f) for any $f \in C_c(\mathbb{R}^d, \mathbb{R})$. A classical criterion (see, for instance, [Evans and Gariepy 1992, p. 54; Ambrosio et al. 2000]) states that μ_n weakly* converge to μ if and only if $\mu_n(A) \to \mu(A)$ for any bounded open set A with $\mu(\partial A) = 0$.

By the common integral representation for generalized functions,

$$\int_A \delta_{X_0}(x) \, \mathrm{d}x := \varepsilon_{X_0}(A) = \mathcal{H}^0(X_0 \cap A),$$

we have

$$(\delta_{X_0}, f) = \int_{\mathbb{R}^d} \delta_{X_0}(x) f(x) \, dx = f(X_0), \quad f \in C_c(\mathbb{R}^d, \mathbb{R}).$$

Now we are ready to introduce the delta function of an n-regular set Θ as the linear functional (the generalized function) δ_Θ in a similar way.

Consider the measure defined on the Borel σ-algebra of $\mathbb{R}^d$, as follows:

$$\mu_\Theta(A) := \mathcal{H}^n(\Theta \cap A), \quad A \in \mathcal{B}_{\mathbb{R}^d}. \tag{2-1}$$

The linear functional associated with it is

$$(\mu_\Theta, f) := \int_{\mathbb{R}^d} f(x) \mu_\Theta \, dx \tag{2-2}$$

for $f \in C_c(\mathbb{R}^d, \mathbb{R})$.

In accordance with what we have said in the Introduction, the following holds.

Proposition 3. *If n is an integer strictly less than dp the measure μ_Θ is a singular measure with respect to the usual Lebesgue measure on $\mathbb{R}^d$.*

In accordance with the usual representation of distributions in the theory of generalized functions, we formally write

$$\int_{\mathbb{R}^d} f(x) \mu_\Theta \, dx = \int_{\mathbb{R}^d} f(x) \delta_\Theta(x) \, dx =: (\delta_\Theta, f). \tag{2-3}$$

Remark 4. We may notice that the classical Dirac delta $\delta_{X_0}(x)$ associated to a point X_0 is now a particular case corresponding to $n = 0$. If Θ is a piecewise smooth surface S in $\mathbb{R}^d$ (and so 2-regular), then by the definition in (2-2), it follows that for any test function f,

$$(\delta_S, f) = \int_S f(x) \, dS,$$

which is the definition of δ_S in [Vladimirov 1979, p. 33].

In terms of the above arguments, we may state that $\delta_\Theta(x)$ is the *(generalized) density* of the measure μ_Θ, defined by (2-1), with respect to the usual Lebesgue measure ν^d on $\mathbb{R}^d$. Note that if $n = d$, then μ_Θ is absolutely continuous with respect to ν^d, so that δ_Θ is its classical Radon–Nikodym derivative [Kolmogorov 1956].

2.1.1. *Paths: 1-dimensional sets.*

Definition 5. A path (or curve or line) in $\mathbb{R}^d$ is a continuous mapping $\gamma : [a, b] \subset \mathbb{R} \to \mathbb{R}^d$. The point $\gamma(a)$ is called the initial point, and the point $\gamma(b)$ is called the final point. The image of the path $\gamma([a, b])$ is called the arc or the support of γ. A path γ is closed if its end points coincide; it is simple if it has no multiple points (apart from possibly the end points for a closed path).

If we denote by $e_j : j = 1, \ldots, d$ the canonical basis of $\mathbb{R}^d$, we may introduce the components of the vector function γ as

$$\gamma_j = \langle e_j, \gamma \rangle : [a, b] \to \mathbb{R}.$$

For $t \in (a, b)$, we put

$$\gamma'(t) := \lim_{h \to t; h \in [a,b]} \frac{\gamma(h) - \gamma(t)}{h - t}$$

whenever the limit exists. This is equivalent to state that all derivatives

$$\gamma_j'(t) := \lim_{h \to t; h \in [a,b]} \frac{\gamma_j(h) - \gamma_j(t)}{h - t}, \quad j = 1, \ldots, d,$$

exist in t.

Definition 6. A path $\gamma : [a, b] \to \mathbb{R}^d$ is said to be smooth if it is a function of class $C^1([a, b])$ and $\gamma'(t) \neq 0$ for any $t \in (a, b)$.

An additional piece of information related to the regularity of a path regards the evaluation of its arc length.

Definition 7. Let $\gamma : [a, b] \to \mathbb{R}^d$ be a path and let Π the set of all finite partitions $\pi := \{a = t_1 < \cdots < t_k = b\}$ of the interval $[a, b]$. Denote by

$$L(\gamma, \pi) := \sum_{i=1}^{k-1} \|\gamma(t_{i+1}) - \gamma(t_i)\|.$$

The path γ is said rectifiable or of bounded variation if

$$L(\gamma) := \sup_{\pi \in \Pi} L(\gamma, \pi) < +\infty;$$

this quantity is referred to as the length of γ.

The following theorem holds [Galbis and Maestre 2012, p. 24].

Theorem 8. *If γ is a smooth curve whose domain is an interval $[a, b] \subset \mathbb{R}$, then it is rectifiable and its length is given by*

$$L(\gamma) = \int_a^b \|\gamma'(t)\| \, dt.$$

Remark 9. The above theorem can be easily extended to a piecewise smooth path [Galbis and Maestre 2012, p. 26].

Example 10. The following are examples of smooth simple curves:

1. A segment in $\mathbb{R}^d$ with end points x_{ini} and $x_{\text{fin}} \in \mathbb{R}^d$:

$$\Gamma = \left\{ x_{\text{ini}} \cdot t + x_{\text{fin}} \cdot (1 - t) : t \in [0, 1] \right\}.$$

2. A circle in $\mathbb{R}^2$:
$$\Gamma = \big\{(\cos t, \sin t) : t \in [0, 2\pi]\big\}.$$

These sets are 1-regular, according to Definition 1.

If $\gamma : [a, b] \to \mathbb{R}^d$ is a smooth path, denote by $\Gamma := \gamma([a, b]) \subset \mathbb{R}^d$ its support; we may associate to Γ the Radon measure defined as

$$\mu_\Gamma : A \in \mathcal{B}_{\mathbb{R}^d} \mapsto \mu_\Gamma(A) := \mathcal{H}^1(\Gamma \cap A). \qquad (2\text{-}4)$$

The following theorem shows the action of the measure μ_Γ on test functions $f \in C_c(\mathbb{R}^d, \mathbb{R}^d)$ (see, e.g., [Evans and Gariepy 1992]).

Theorem 11. *If $\gamma : [a, b] \to \mathbb{R}^d$ is a $C^1([a, b])$ simple curve such that $\Gamma = \gamma([a, b])$, then for all $f \in C_c(\mathbb{R}^d, \mathbb{R}^d)$,*

$$\int f(x)\mu_\Gamma \, \mathrm{d}x = \int_a^b f(\gamma(t))\|\gamma'(t)\| \, \mathrm{d}t. \qquad (2\text{-}5)$$

Similarly, for every $A \in \mathcal{B}(\mathbb{R}^d)$,

$$\mu_\Gamma(A) = \int_a^b \epsilon_{\gamma(t)}(A)\|\gamma'(t)\| \, \mathrm{d}t. \qquad (2\text{-}6)$$

2.2. *The stochastic case.*

2.2.1. *Random Radon measures.* Consider the space $C_c(\mathbb{R}^d, \mathbb{R})$ of continuous functions with compact support. This space is the union of the separable Banach spaces $C(K, \mathbb{R})$ over all compact sets $K \in \mathbb{R}^d$; let us write $\| \cdot \|_K$ for the norm on $C(K)$:

$$\|f\|_K = \sup_{x \in K}|f(x)|.$$

The space $C_c(\mathbb{R}^d, \mathbb{R})$ can be endowed with the direct limit topology associated to a family $C(K_n, \mathbb{R})$ with K_n increasing to $\mathbb{R}^d$; we do not need the details of this definition but we need to know the following useful fact: a linear functional $I : C_c(\mathbb{R}^d, \mathbb{R}) \to \mathbb{R}$ is continuous if and only if for every compact $K \in \mathbb{R}^d$ there is $C_K > 0$ such that

$$|I(f)| \le C_K \|f\|_K \quad \text{for every } f \in C(K, \mathbb{R}).$$

Moreover, the dual of $C_c(\mathbb{R}^d, \mathbb{R})$ is the space of signed Radon measures, which we shall denote by $\mathcal{M}$.

Definition 12. Given a probability space $(\Omega, \mathcal{F}, P)$, a random Radon measure is a map $\mu : \Omega \to \mathcal{M}$ such that for every $f \in C_c(\mathbb{R}^d, \mathbb{R})$, the function $(\mu, f) := \int_{\mathbb{R}^d} f(x)\mu \, \mathrm{d}x : \Omega \to \mathbb{R}$ is measurable. We shall write (μ_ω, f) to emphasize the dependence on ω when μ is random.

Proposition 13 (expected value of a random Radon measure). *Let $\mu : \Omega \to \mathcal{M}$ be a random Radon measure. Assume that for every compact $K \in \mathbb{R}^d$ there is $C_K > 0$ such that*

$$\int_\Omega |(\mu_\omega, f)| P \, d\omega \le C_K \|f\|_K,$$

for every $f \in C(K, \mathbb{R})$. Then there exists an element $\hat{\mu} \in \mathcal{M}$ such that

$$(\hat{\mu}, f) = \int_\Omega (\mu_\omega, f) P \, d\omega = \mathbb{E}[(\mu, f)],$$

for every $f \in C_c(\mathbb{R}^d, \mathbb{R})$.

Proof. Define the number $(\hat{\mu}, f)$ by the previous identity; for every $f \in C_c(\mathbb{R}^d, \mathbb{R})$, since $f \in C(K, \mathbb{R})$ for some compact set K, we have $\int_\Omega |(\mu_\omega, f)| P \, d\omega < \infty$ by assumption, hence $\int_\Omega (\mu_\omega, f) P \, d\omega$ is well-defined.

Denote by $\Phi : f \in C_c(\mathbb{R}^d, \mathbb{R}) \mapsto \Phi(f) := (\mu, f) \in L^1(\Omega)$. We have to show that, as a function of f, the number $(\hat{\mu}, f)$ is linear continuous on $C_c(\mathbb{R}^d)$. Taking $f, g \in C_c(\mathbb{R}^d, \mathbb{R})$ and $\alpha, \beta \in \mathbb{R}$, since Φ is linear, we have $\Phi_\omega(\alpha f + \beta g) = \alpha \Phi_\omega(f) + \beta \Phi_\omega(g)$, hence

$$\int_\Omega \Phi_\omega(\alpha f + \beta g) P \, d\omega = \alpha \int_\Omega \Phi_\omega(f) P \, d\omega + \beta \int_\Omega \Phi_\omega(g) P \, d\omega,$$

which implies the linearity. For the continuity, for every compact set K and $f \in C(K, \mathbb{R})$, we have

$$|(\hat{\mu}, f)| \le \int_\Omega |\Phi_\omega(f)(\omega)| P \, d\omega \le C_K \|f\|_K$$

by assumption, hence $\hat{\mu}$ is a continuous functional. $\qquad\square$

Under the assumptions of Proposition 13, the measure $\hat{\mu} \in \mathcal{M}$, which satisfies the equality

$$(\hat{\mu}, f) = \mathbb{E}[(\mu, f)] \tag{2-7}$$

for every $f \in C_c(\mathbb{R}^d, \mathbb{R})$ will be called the expected value of the measure μ and will be denoted by $\mathbb{E}[\mu]$.

2.2.2. *Random linear functionals.* Let S be a separable linear metric space of test functions, for instance, $C_c(\mathbb{R}^d, \mathbb{R})$ or $C_c^\infty(\mathbb{R}^d, \mathbb{R})$ or $L^2(0, T)$. Denote by S' the space of continuous linear functionals on S. Let $(\Omega, \mathcal{F}, P)$ be a probability space. The following definition generalizes Definition 12.

Definition 14. We call random functional in the strict sense any map $\phi : \Omega \to S'$ such that (ϕ, f) is measurable for every $f \in S$.

Remark 15. The definition above is analogous to the well-known definition for Banach valued random variables (see, e.g., [Araujo and Giné 1980; Bharucha-Reid 1972; Bosq 2000]).

Later we will consider random functionals in a broad sense.

2.3. *Random functionals associated with random closed sets.* We recall that a *random closed set* Ξ in $\mathbb{R}^d$ is a measurable map

$$\Xi : (\Omega, \mathcal{F}, \mathbb{P}) \longrightarrow (\mathbb{F}, \sigma_{\mathbb{F}}),$$

where $\mathbb{F}$ denotes the class of the closed subsets in $\mathbb{R}^d$, and $\sigma_{\mathbb{F}}$ is the σ-algebra generated by the so-called *hit-or-miss topology* (see [Matheron 1975]).

Definition 16. Given an integer n with $0 \leq n \leq d$, we say that a random closed set Θ in $\mathbb{R}^d$ is *n-regular* if it satisfies the following conditions:

 (i) for almost all $\omega \in \Omega$, the set $\Theta(\omega)$ is an n-regular closed set in $\mathbb{R}^d$; and

 (ii) $\mathbb{E}[\mathcal{H}^n(\Theta \cap B_R(0))] < \infty$ for any $R > 0$.

Suppose now that Θ is a random n-regular closed set in $\mathbb{R}^d$. Thanks to the assumptions on the random set Θ, it can be shown that the random measure μ_Θ is P-a.s. a Radon measure on $\mathcal{B}_{\mathbb{R}^d}$; it can be further shown that, as a map $\mu_\Theta : \omega \in \Omega \mapsto \mu_{\Theta(\omega)} \in \mathcal{M}$, it is a random Radon measure according to Definition 12 (see, e.g., [Baddeley and Molchanov 1997; Matheron 1965; Zähle 1982]).

In this case it makes sense to define, for any $\omega \in \Omega$, the linear functional $\delta_{\Theta(\omega)}$ such that, for any $f \in C_c(\mathbb{R}^d, \mathbb{R})$:

$$(\delta_\Theta, f) := \int_{\mathbb{R}^d} f(x)\mu_\Theta(x)\,\mathrm{d}x, \quad P\text{-a.s.} \tag{2-8}$$

According to Definition 14, δ_Θ is then a random linear functional in the strict sense.

By recollecting all of the above, we may finally state the following.

Proposition 17. *Let Θ be a random closed n-regular set in $\mathbb{R}^d$. Then the associated random Radon measure μ_Θ satisfies the assumption of Proposition 13, and therefore the expected value $\mathbb{E}[\mu_\Theta]$ is well-defined. We therefore define the expected value of the generalized density δ_Θ by the following identities*:

$$
\begin{aligned}
(\mathbb{E}[\delta_\Theta], f) &= \int_{\mathbb{R}^d} f(x)\mathbb{E}[\delta_\Theta](x)\,\mathrm{d}x \\
&:= \int_{\mathbb{R}^d} f(x)\mathbb{E}[\mu_\Theta]\,\mathrm{d}x = \mathbb{E}\left[\int_{\mathbb{R}^d} f(x)\mu_\Theta(x)\,\mathrm{d}x\right] \\
&= \mathbb{E}\left[\int_{\mathbb{R}^d} f(x)\delta_\Theta\,\mathrm{d}x\right] = \mathbb{E}[(\delta_\Theta, f)], \tag{2-9}
\end{aligned}
$$

for any $f \in C_c(\mathbb{R}^d, \mathbb{R})$.

Proof. Let K be a compact in $\mathbb{R}^d$ and $f \in C_c(K, \mathbb{R})$. We have

$$\int_\Omega |(\mu_{\Theta(\omega)}, f)| P \, d\omega = \int_\Omega \left| \int_{\mathbb{R}^d} f(x) \mu_{\Theta(\omega)} \, dx \right| P \, d\omega$$

$$\leq \int_\Omega \int_{\mathbb{R}^d} |f(x)| \mu_{\Theta(\omega)} \, dx \, P \, d\omega$$

$$= \int_\Omega \int_K |f(x)| \mu_{\Theta(\omega)} \, dx \, P \, d\omega,$$

because f has support in K, and

$$\int_\Omega |(\mu_{\Theta(\omega)}, f)| P \, d\omega \leq \|f\|_K \int_\Omega \int_K \mu_{\Theta(\omega)} \, dx \, P \, d\omega$$

$$= \|f\|_K \int_\Omega \mu_{\Theta(\omega)}(K) P \, d\omega$$

$$= \|f\|_K \mathbb{E}[\mathcal{H}^n(\Theta \cap K)]$$

$$\leq C_K \|f\|_K,$$

by (ii) of the previous definition (the set K is included in a ball $B_R(0)$). Hence we may apply Proposition 13 for the existence of the expected value $\mathbb{E}[\mu_\Theta]$. $\square$

2.3.1. *Absolutely continuous (in mean) random sets.*

Remark 18. When $n = d$, the integral and expectation in (2-9) can be exchanged by Fubini's theorem, since in this case both μ_Θ and $\mathbb{E}[\mu_\Theta]$ are absolutely continuous with respect to ν^d and $\delta_\Theta(x) = \mathbb{1}_\Theta(x)$, ν^d-a.s.

In particular, $\delta_\Theta(x) = \mathbb{1}_\Theta(x)$, ν^d-a.s. implies that

$$\mathbb{E}[\delta_\Theta](x) = \mathbb{P}(x \in \Theta), \quad \nu^d\text{-a.s.},$$

and the following chain of equalities are well-known (according to our definition of $\mathbb{E}[\delta_\Theta]$ [Kolmogorov 1956, p. 46]):

$$\mathbb{E}[\nu^d(\Theta \cap A)] = \mathbb{E}\left(\int_{\mathbb{R}^d} \mathbb{1}_{\Theta \cap A}(x) \, dx \right) = \mathbb{E}\left(\int_A \mathbb{1}_\Theta(x) \, dx \right)$$

$$= \int_A \mathbb{E}(\mathbb{1}_\Theta(x)) \, dx \qquad = \int_A \mathbb{P}(x \in \Theta) \, dx. \qquad (2\text{-}10)$$

In materials science, the density

$$\rho(x) := \mathbb{E}[\delta_\Theta](x) = \mathbb{P}(x \in \Theta)$$

is known as the *(degree of) crystallinity.*

If Θ is not a pathological set, i.e., if $\mathcal{H}^n(\Theta)(\omega) > 0$ with $\mathbb{P}$-a.e. for $\omega \in \Omega$ ($n < d$), we may notice that, even though for a.e. realization Θ of Θ the measure μ_Θ is

positive and singular (and so it is not absolutely continuous), and the expected measure $\mathbb{E}[\mu_\Theta]$ may be absolutely continuous with respect to ν^d.

Example. Consider the case $n = 0$. Let $\Theta = X_0$ be a random point in $\mathbb{R}^d$; in this case, $\mathcal{H}^0(X_0 \cap A) = \mathbb{1}_A(X_0)$, and so

$$\mathbb{E}[\mathcal{H}^0(X_0 \cap A)] = \mathbb{P}(X_0 \in A).$$

If X_0 is a continuous random point with the pdf p_{X_0}, then $\mathbb{E}[\mathcal{H}^0(X_0 \cap \cdot)]$ is absolutely continuous and, in this case, $\mathbb{E}[\delta_{X_0}](x)$ is just the probability density function $p_{X_0}(x)$, so $\int_A \mathbb{E}[\delta_{X_0}](x)\nu^d \, dx$ is the usual Lebesgue integral. Note that we formally have

$$\mathbb{E}[\delta_{X_0}](x) = \int_{\mathbb{R}^d} \delta_y(x) p_{X_0}(y) \nu^d \, dy = \int_{\mathbb{R}^d} \delta_x(y) p_{X_0}(y) \nu^d \, dy = p_{X_0}(x);$$

and in accordance with Proposition 17,

$$\int_A \mathbb{E}[\delta_{X_0}](x)\nu^d \, dx = \int_A p_{X_0}(x)\nu^d \, dx = \mathbb{P}(X_0 \in A)$$

$$= \mathbb{E}[\mathcal{H}^0(X_0 \cap A)] = \mathbb{E}\left[\int_A \delta_{X_0}(x)\nu^d \, dx\right]. \quad (2\text{-}11)$$

If X_0 is discrete, i.e., $X_0 = x_i$ with probability p_i, only for an at most countable set of points $x_i \in \mathbb{R}^d$, then $\mathbb{E}[\mathcal{H}^0(X_0 \cap \cdot)]$ is singular and, as in the previous case, we have that $\mathbb{E}[\delta_{X_0}](x)$ coincides with the probability distribution p_{X_0} of X_0.

In fact, in this case $p_{X_0}(x) = \sum_i p_i \delta_{x_i}(x)$, and by computing the expectation of δ_{X_0}, we formally obtain

$$\mathbb{E}[\delta_{X_0}](x) = \delta_{x_1}(x)p_1 + \delta_{x_2}(x)p_2 + \cdots = \sum_i p_i \delta_{x_i}(x) = p_{X_0}(x).$$

Remark 19. By Remark 18 and the considerations in the above example, we may claim that, in the cases $n = d$ and $n = 0$ with X_0 being continuous, the expected linear functionals $\mathbb{E}[\delta_\Theta]$ and $\mathbb{E}[\delta_{X_0}]$ are defined by the function $\rho(x) := \mathbb{P}(x \in \Theta)$ and by the pdf p_{X_0} of X_0, respectively, in the following way:

$$(\mathbb{E}[\delta_\Theta], f) := \int_{\mathbb{R}^d} f(x)\rho(x) \, dx$$

and

$$(\mathbb{E}[\delta_{X_0}], f) := \int_{\mathbb{R}^d} f(x)p_{X_0}(x) \, dx.$$

In fact, let us consider the random point X_0; in accordance with Proposition 17,

$$(\mathbb{E}[\delta_{X_0}], f) := \int_{\mathbb{R}^d} f(x)p_{X_0}(x) \, dx = \mathbb{E}[f(X_0)] = \mathbb{E}[(\delta_{X_0}, f)].$$

Of particular interest is the case of fiber processes, i.e., 1-dimensional random sets that occur in medicine as models for vessels, nerves, etc., and in materials science as models for line dislocations (defects in a crystalline material) [Hochrainer et al. 2007], etc. In these cases, an empirical definition of line density is given as the total length of dislocation lines per unit volume (see, e.g., [Callister Jr. 2007, p. 177]). This has given a strong motivation for a rigorous definition of the mean density of random sets of any Hausdorff dimension; for a discussion about continuity and absolute continuity of random closed sets, we refer to [Capasso and Villa 2006; 2007].

2.3.2. *Example: an absolutely continuous* 1-*regular random set.* Let (X_t, V_t) be the solution of the stochastic differential system in $\mathbb{R}^d$:

$$\mathrm{d}X_t = V_t \, \mathrm{d}t,$$

$$\mathrm{d}V_t = b(X_t, V_t) \, \mathrm{d}t + \mathrm{d}W_t,$$

where b is Lipschitz continuous and W_t is a Brownian motion in $\mathbb{R}^d$. If the initial condition (X_0, V_0) of the above system has a smooth density $p_0(x, v)$ with respect to the usual Lebesgue measure on $\mathbb{R}^d$, thanks to hypoellipticity, (X_t, V_t) has (for every $t > 0$) a smooth density $p_t(x, v)$ with respect to the Lebesgue measure; it is a solution to the Fokker–Planck equation:

$$\frac{\partial p}{\partial t} = \Delta_v p - v \cdot \nabla_x p - \mathrm{div}_v(bp),$$

$$p|_{t=0} = p_0.$$

Given $T > 0$, consider the random path in $\mathbb{R}^d$ given by $t \in [0, T] \mapsto X_t \in \mathbb{R}^d$ (not necessarily simple); in accordance with Theorem 11, the associated random Radon measure μ_X is such that

$$\int f(x)\mu_X \, \mathrm{d}x = \int_0^T f(X_t)\|V_t\| \, \mathrm{d}t, \quad f \in C_c(\mathbb{R}^d, \mathbb{R}).$$

This formula defines a random generalized density $\delta_X(x)$ by the identity

$$(\delta_X, f) = \int_0^T f(X_t)\|V_t\| \, \mathrm{d}t, \quad f \in C_c(\mathbb{R}^d, \mathbb{R}).$$

It can be checked that we have

$$\delta_X(x) = \int_0^T \delta_{X_t}(x)\|V_t\| \, \mathrm{d}t = \int_0^T \delta_{X_t}(x)\|X_t'\| \, \mathrm{d}t.$$

Proposition 20. *Assume* $\mathbb{E}[\|X_0\| + \|V_0\|] < \infty$. *Then the random generalized density* δ_X, *which is a.s. concentrated on the random curve* X, *has a smooth average*

density $\mathbb{E}[\delta_X](x)$ *given by*

$$\mathbb{E}[\delta_X](x) = \int_0^T \int_{\mathbb{R}^d} \|v\| p_t(x, v) \, dv \, dt.$$

Proof. We only sketch the proof. We have, with Θ_1 equal to the support of the curve X,

$$\mathbb{E}\big[\mathcal{H}^1(\Theta_1 \cap B_R(0))\big] = \mathbb{E}\bigg[\int_0^T 1_{B_R(0)}(X_t) \|V_t\| \, dt\bigg]$$
$$\leq \mathbb{E}\bigg[\int_0^T \|V_t\| \, dt\bigg] = \int_0^T \mathbb{E}[\|V_t\|] \, dt.$$

We continue the proof under the additional assumption that b is bounded, otherwise it is sufficient to use a Gronwall-type argument (it is here that we use $\mathbb{E}[\|X_0\|] < \infty$). We have

$$V_t = V_0 + \int_0^t b(X_s, V_s) \, ds + W_t,$$

hence

$$\|V_t\| \leq \|V_0\| + T \|b\|_\infty + \|W_t\|,$$

which implies $\sup_{t \in [0,T]} \mathbb{E}[\|V_t\|] < \infty$ (because $\mathbb{E}[\|V_0\|] < \infty$), hence

$$\int_0^T \mathbb{E}[\|V_t\|] \, dt < \infty$$

and therefore $\mathbb{E}[\mathcal{H}^1(\Theta_1 \cap B_R(0))] < \infty$. By the general criterion above, this implies that $\mathbb{E}[\delta_X]$ exists and

$$(\mathbb{E}[\delta_X], f) = \int_0^T \mathbb{E}[f(X_t) \|V_t\|] \, dt.$$

It follows that

$$(\mathbb{E}[\delta_X], f) = \int_0^T \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(x) \|v\| p_t(x, v) \, dx \, dv \, dt$$
$$= \int_{\mathbb{R}^d} \bigg(\int_0^T \int_{\mathbb{R}^d} \|v\| p_t(x, v) \, dv \, dt\bigg) f(x) \, dx.$$

The arbitrariness of f gives us the existence of the average density $\mathbb{E}[\delta_X](x)$ and its formula. $\qquad\square$

Hence we may claim that the path X is a 1-dimensional absolutely continuous (in mean) random set in $\mathbb{R}^d$.

2.4. *More on random functionals.* Given a probability space $(\Omega, \mathcal{F}, P)$, we denote by $L^0(\Omega)$ the space of P-equivalence classes of random variables $X : \Omega \to \mathbb{R}$, endowed with the Ky Fan topology, i.e., the topology of the convergence in probability.

In the previous sections, we defined as random functionals all measurable mappings from Ω to the dual space $C_c(\mathbb{R}^d, \mathbb{R})'$ (we continue to restrict ourselves to functionals on $C_c(\mathbb{R}^d, \mathbb{R})$). However, there exist examples of random objects which are natural to call random functionals, but do not enter in the previous definition. Let us then introduce a broader concept.

Definition 21. We use random functional in the broad sense of any continuous linear map $\Phi : C_c(\mathbb{R}^d, \mathbb{R}) \to L^0(\Omega)$. When there exists a random functional in the strict sense $\phi : \Omega \to C_c(\mathbb{R}^d, \mathbb{R})'$ such that

$$\Phi(f)(\omega) = (\phi_\omega, f)$$

for all $f \in C_c(\mathbb{R}^d, \mathbb{R})$ and for a.a. $\omega \in \Omega$, we say that ϕ_ω is a pathwise realization of Φ.

Example 22. Let W be a Brownian motion on $(\Omega, \mathcal{F}, P)$ and let $d = 1$. To any function $f \in C_c(\mathbb{R}, \mathbb{R})$, we associate the random variable

$$\Phi(f) = \int_0^T f(W_t) \, dW_t.$$

It is known that there is no simple pathwise meaning for this integral: we cannot fix $\omega \in \Omega$ and consider it as a map from $C_c(\mathbb{R}, \mathbb{R})$ to $\mathbb{R}$. This is an example of a random functional in the broad sense which has no pathwise realization.

Example 23. Referring to the above example, a pathwise realization exists if we restrict ourselves to more regular test functions f. Indeed, if $f \in C_c^1(\mathbb{R}, \mathbb{R})$, and $F' = f$, by Itô's formula we have

$$F(W_T) = F(W_0) + \int_0^T f(W_t) \, dW_t + \frac{1}{2} \int_0^T f'(W_t) \, dt,$$

and therefore the random functional in the strict sense ϕ_ω, defined as

$$(\phi_\omega, f) = F(W_T(\omega)) - F(W_0(\omega)) - \frac{1}{2} \int_0^T f'(W_t(\omega)) \, dt$$

(these expressions are all well-defined path by path) satisfies

$$(\phi_\cdot, f) = \int_0^T f(W_t) \, dW_t = \Phi(f), \quad P\text{-a.s.},$$

hence ϕ_ω is a pathwise realization of Φ; but only on $f \in C_c^1(\mathbb{R}, \mathbb{R})$.

Example 24. If we allow ourselves to use $L^2(0, T)$ as a set of test functions in place of $C_c(\mathbb{R}^d, \mathbb{R})$, the classical Wiener integral

$$\Phi(f) = \int_0^T f_t \, dW_t$$

is another example of random functional in the broad sense which has no pathwise realization. Again, if we change the set of test functions and we take $f \in W^{1,2}(0, T)$, the pathwise realization exists, given by

$$(\phi_\omega, f) = f_T W_T - \int_0^T f_t' W_t \, dt.$$

The previous examples have mainly an academic character, so we do not insist on this notion here. However, in the context of random currents, there are examples of great potential interest for applications, so we shall enter in more detail.

3. Currents

In this section we wish to extend our analysis to vector fields in $\mathbb{R}^d$, hence to random currents. Motivations for the study of random currents are the same as anticipated in Section 2.3 for 1-dimensional random sets, i.e., fiber processes.

We will start by defining line integrals of differential forms (see, e.g., [Giaquinta et al. 1998; Giaquinta and Modica 2009; Galbis and Maestre 2012, p. 21; Buck 1956]).

3.1. *Differential forms.* Let $g : U \subset \mathbb{R}^d \to \mathbb{R}$ be a real function of class C^1 in the open set U, i.e., it has all partial derivatives of the first order continuous in U. The differential of g at a point $x \in U$ is the linear function $\mathrm{d}g(x) : \mathbb{R}^d \to \mathbb{R}$ such that

$$h = (h_1, \ldots, h_d) \in \mathbb{R}^d \mapsto \mathrm{d}g(x)(h) = \sum_{j=1}^d \frac{\partial g}{\partial x_j}(x) h_j. \qquad (3\text{-}1)$$

It is usually suggestive to denote the partial increments $h_j = \langle e_j, h \rangle$ by $\mathrm{d}x_j$, so that the expression (3-1) is usually written as

$$\mathrm{d}x = (\mathrm{d}x_1, \ldots, \mathrm{d}x_d) \in \mathbb{R}^d \mapsto \mathrm{d}g(x)\,(\mathrm{d}x) = \sum_{j=1}^d \frac{\partial g}{\partial x_j}(x)\,\mathrm{d}x_j. \qquad (3\text{-}2)$$

Expressions (3-1) or (3-2) are a particular case of a more general mathematical entity called differential form of degree one, or simply 1-form, defined below.

Definition 25. Let $U \subset \mathbb{R}^d$ be an open set. A differential form of degree one or simply a 1-form on U is a mapping

$$\omega : U \to \mathcal{L}(\mathbb{R}^d, \mathbb{R}) = (\mathbb{R}^d)',$$

where $\mathcal{L}(\mathbb{R}^d, \mathbb{R})$ denotes the space of all real valued linear functions on $\mathbb{R}^d$, i.e., the dual space $(\mathbb{R}^d)'$ of $\mathbb{R}^d$.

Given a 1-form ω, a point $x \in U$, and an integer $j = 1, \dots, d$, we will denote the scalar $\omega(x)(e_j) \in \mathbb{R}$ by $f_j(x)$. Evidently $f_j : U \to \mathbb{R}$; by the linearity of $\omega(x) \in (\mathbb{R}^d)'$, for any $h = (h_1, \dots, h_d) \in \mathbb{R}^d$ we have

$$\omega(x)(h) = \sum_{j=1}^{d} f_j(x) h_j, \tag{3-3}$$

or

$$\omega(x)(\mathrm{d}x) = \sum_{j=1}^{d} f_j(x) \, \mathrm{d}x_j. \tag{3-4}$$

This expression is usually abbreviated into

$$\omega = \sum_{j=1}^{d} f_j \, \mathrm{d}x_j. \tag{3-5}$$

The functions f_j, where $j = 1, \dots d$, are called the component functions of the differential form ω. We will usually assume that the component functions of a 1-form are continuous; in such a case we say that the form is continuous.

Remark 26. The above discussion shows that the study of a 1-form is essentially equivalent to the study of a vector field, say

$$F(x) = (f_1(x), \dots, f_d(x)); \quad x \in U.$$

3.2. *Line integrals of differential forms.* The line integral of a differential form was originally motivated by the calculation of the work done by a force field along a path.

Consider a smooth path $\gamma : [a, b] \to \mathbb{R}^3$ all contained in an open set $U \subset \mathbb{R}^3$; and let $F : U \to \mathbb{R}^3$ be a force field acting on a point object. The work done by the force field F along the path γ can be obtained by taking into account two facts:

1. The work done by a force field along a path depends only upon the component of the force along the tangent direction of the path at each point.

2. The work done by a constant field F_0 to move an object along a line segment in its direction is given by the product of $\|F_0\|$ and the length of the line segment.

Consider a very fine partition $\pi = \{a = t_1 < \cdots < t_k = b\}$ of the interval $[a, b]$, so that the arc length of $\gamma|_{[t_j, t_{j+1}]}$, according to Theorem 8, can be approximated by

$$\|\gamma'(t_j)\|(t_{j+1} - t_j).$$

We know that the unit vector which is tangent to a smooth path at $\gamma(t)$, $t \in [a, b]$ is given by

$$T(t) = \frac{\gamma'(t)}{\|\gamma'(t)\|},$$

so that we may assume, at a good approximation, that along the path $\gamma|_{[t_j, t_{j+1}]}$ the force field has a constant value $F(\gamma(t_j))$. Consequently, the work done by the force field F along $\gamma|_{[t_j, t_{j+1}]}$ can be approximated by

$$\langle F(\gamma(t_j)), T(t_j) \rangle \|\gamma'(t_j)\|(t_{j+1} - t_j) = \langle F(\gamma(t_j)), \gamma'(t_j) \rangle (t_{j+1} - t_j),$$

where $j = 1, \ldots, k-1$. Summing up a good approximation of the work done in moving the object along γ is given by

$$\sum_{k=1}^{k-1} \langle F(\gamma(t_j)), \gamma'(t_j) \rangle (t_{j+1} - t_j);$$

the usual theorem of existence of the Riemann integral of a continuous function along a finite interval leads to the following proposition.

Proposition 27. *The work done by a continuous force field $F : U \to \mathbb{R}^3$ along a piecewise smooth path $\gamma : [a, b] \to \mathbb{R}^3$ all contained in an open set $U \subset \mathbb{R}^3$ is given by*

$$\int_\gamma F = \int_a^b \langle F(\gamma(t)), \gamma'(t) \rangle \, \mathrm{d}t. \tag{3-6}$$

The following definition is then meaningful.

Definition 28. Let $F : U \to \mathbb{R}^d$ be a continuous vector field along a piecewise smooth path $\gamma : [a, b] \to \mathbb{R}^d$ all contained in an open set $U \subset \mathbb{R}^d$. The line integral of F along γ is given by

$$\int_\gamma F := \int_a^b \langle F(\gamma(t)), \gamma'(t) \rangle \, \mathrm{d}t. \tag{3-7}$$

It is clear that the line integral of a vector field depends upon the orientation of the path, i.e.,

$$\int_{\gamma[a,b]} F := - \int_{\gamma[b,a]} F. \tag{3-8}$$

Thanks to the correspondence between 1-forms and vector fields, the following definition is appropriate.

Definition 29. Let $U \subset \mathbb{R}^d$ be an open set, let ω be a continuous 1-form on U, and let $\gamma : [a, b] \to \mathbb{R}^d$ be a piecewise smooth path, all contained in U. If $F(x) =$

$(f_1(x), \ldots, f_d(x))$, for $x \in U$, is the vector field associated with ω, the line integral of ω along γ is given by

$$\int_\gamma \omega := \int_a^b \omega(\gamma(t))\gamma'(t) \, dt = \int_a^b \langle F(\gamma(t)), \gamma'(t) \rangle \, dt$$

$$= \sum_{j=1}^d \int_a^b f_j(\gamma(t))\gamma_j'(t) \, dt. \tag{3-9}$$

3.3. 1-currents.
In the sequel, a 1-*current* will be a continuous linear functional on $C_c(\mathbb{R}^d, \mathbb{R}^d)$ (more generally the literature considers a continuous linear functional on $C_c^\infty(\mathbb{R}^d, \mathbb{R}^d)$).

Typical examples of 1-currents are those induced by regular curves in $\mathbb{R}^d$. Let $\gamma : [a, b] \subset \mathbb{R} \to \mathbb{R}^d$ be a smooth curve in $\mathbb{R}^d$; the linear functional ξ defined by

$$\theta \in C_c(\mathbb{R}^d, \mathbb{R}^d) \mapsto (\xi, \theta) = \int_a^b \langle \theta(\gamma(t)), \gamma'(t) \rangle \, dt$$

$$= \sum_{j=1}^d \int_a^b \theta_j(\gamma(t))\gamma_j'(t) \, dt \in \mathbb{R} \tag{3-10}$$

is a 1-current.

Another case is one in which the 1-current is induced by a vector field. By an abuse of notation, let $\xi : \mathbb{R}^d \to \mathbb{R}^d$ be a vector field; the associated current is

$$\theta \in C_c(\mathbb{R}^d, \mathbb{R}^d) \mapsto (\xi, \theta) = \int_{\mathbb{R}^d} \langle \theta(x), \xi(x) \rangle \, dx = \sum_{j=1}^d \int_{\mathbb{R}^d} \theta_j(x)\xi_j(x) \, dx. \tag{3-11}$$

We may recover the case (3-10) as a particular case of (3-11) by the localization on the regular path $\{\gamma(t), \ t \in [a, b]\}$ made by a usual Dirac delta distribution

$$\xi(x) = \int_a^b \delta_{\gamma(t)}(x)\gamma'(t) \, dt, \quad x \in \mathbb{R}^d; \tag{3-12}$$

i.e.,

$$\xi = \int_a^b \delta_{\gamma(t)}(\cdot)\gamma'(t) \, dt. \tag{3-13}$$

3.4. *Random* 1-*currents and their mean densities.*
We have said above that a deterministic 1-current is a linear continuous functional on $C_c^\infty(\mathbb{R}^d, \mathbb{R}^d)$. Since it is sufficient for our purposes, in order to stress the analogy with Radon measures, we consider afterwards only 1-currents which are linear continuous functionals on $C_c(\mathbb{R}^d, \mathbb{R}^d)$. There are two definitions of *random* 1-currents, as in the case of random functionals (see Section 2.2.2).

The most natural one is the following [Flandoli et al. 2009; Bessaih et al. 2017].

Definition 30. Given a probability space $(\Omega, \mathcal{F}, P)$, consider the space $C_c(\mathbb{R}^d, \mathbb{R}^d)$ of compact support continuous vector fields endowed with the topology of convergence on compact sets and let $C_c(\mathbb{R}^d, \mathbb{R}^d)'$ be the space of 1-currents on $\mathbb{R}^d$, endowed with the dual topology, and the corresponding Borel sigma algebra. We call a random 1-current in the strict sense any measurable map

$$\xi : \omega \in \Omega \mapsto \xi_\omega \in C_c(\mathbb{R}^d, \mathbb{R}^d)'. \tag{3-14}$$

The second, a weaker one, relates to the real value obtained when applying the current to a test vector field.

Definition 31. Given a probability space $(\Omega, \mathcal{F}, P)$, consider the space $L^0(\Omega)$ of real-valued random variables endowed with the Ky Fan topology, i.e., the topology of the convergence in probability, and the corresponding Borel sigma algebra. We call a random 1-current in the broad sense any continuous linear map $\Xi : C_c(\mathbb{R}^d, \mathbb{R}^d) \to L^0(\Omega)$. When there exists a random 1-current in the strict sense $\xi : \Omega \to C_c(\mathbb{R}^d, \mathbb{R})'$ such that

$$\Xi(f)(\omega) = (\xi_\omega, f)$$

for all $f \in C_c(\mathbb{R}^d, \mathbb{R}^d)$ and a.a. $\omega \in \Omega$, we say ξ_ω is a pathwise realization of Ξ.

One can show, as in the case of random functionals, that strict sense implies broad sense, but the opposite is not true. Below we provide a typical example for each category and compute the corresponding mean densities.

3.4.1. *The mean of a 1-current.* Consider a random 1-current in the strict sense $\xi_\omega \in C_c(\mathbb{R}^d, \mathbb{R}^d)'$, and $\omega \in \Omega$. We say that it admits a mean value if a 1-current $\hat{\xi} \in C_c(\mathbb{R}^d, \mathbb{R}^d)'$ exists such that, for any $\theta \in C_c(\mathbb{R}^d, \mathbb{R}^d)$,

$$(\hat{\xi}, \theta) = \mathbb{E}[(\xi, \theta)]. \tag{3-15}$$

Whenever this happens, we will identify the current $\mathbb{E}[\xi] := \hat{\xi}$ as the *mean value of the current* ξ.

We will say that the mean current $\mathbb{E}[\xi]$ is induced by a vector field if a locally integrable vector field $u : \mathbb{R}^d \to \mathbb{R}^d$ exists such that

$$(\mathbb{E}[\xi], \theta) = \int_{\mathbb{R}^d} \langle \theta(x), u(x) \rangle \, dx. \tag{3-16}$$

3.5. *Example of a random 1-current in the strict sense and its expectation.* Let (X_t, V_t) be the solution of the stochastic equation in $\mathbb{R}^d \times \mathbb{R}^d$:

$$dX_t = V_t \, dt,$$
$$dV_t = b(t, X_t, V_t) \, dt + dW_t,$$

where b is Lipschitz continuous and W_t is a Brownian motion in $\mathbb{R}^d$. Assume that (X_0, V_0) has a smooth density $p_0(x, v)$ with respect to Lebesgue measure. Thanks to hypoellipticity, (X_t, V_t) has, for every $t > 0$, a smooth density $p_t(x, v)$ with respect to Lebesgue measure; it is a solution to the Fokker–Planck equation:

$$\frac{\partial p}{\partial t} = \Delta_v p - v \cdot \nabla_x p - \operatorname{div}_v(bp),$$

$$p|_{t=0} = p_0.$$

Given $T > 0$, consider the random curve in $\mathbb{R}^d$ given by $t \in [0, T] \mapsto X_t \in \mathbb{R}^d$ (not necessarily simple) and consider the associated random 1-current ξ_X formally defined as

$$\xi_X(x) = \int_0^T \delta_{X_t}(x) V_t \, dt, \quad x \in \mathbb{R}^d; \tag{3-17}$$

namely

$$(\xi_X, \theta) = \int_0^T \langle \theta(X_t), V_t \rangle \, dt, \quad \theta \in C_c(\mathbb{R}^d, \mathbb{R}^d).$$

One can easily recognize that ξ_X is a 1-current in the strict sense. We have

$$\mathbb{E}[(\xi_X, \theta)] = \mathbb{E}\left[\int_0^T \langle \theta(X_t), V_t \rangle \, dt\right] = \int_0^T \mathbb{E}[\langle \theta(X_t), V_t \rangle] \, dt$$

$$= \int_0^T \left(\int_{\mathbb{R}^d \times \mathbb{R}^d} \langle \theta(x), v \rangle p_t(x, v) \, dx \, dv\right) dt.$$

We may also proceed as in Proposition 20, and see directly from (3-17) that, for any $x \in \mathbb{R}^d$, we do have formally

$$\mathbb{E}[\xi_X(x)] = \int_0^T \mathbb{E}[\delta_{X_t}(x) V_t] \, dt$$

$$= \int_0^T dt \int_{\mathbb{R}^d \times \mathbb{R}^d} \delta_y(x) v p_t(y, v) \, dy \, dv$$

$$= \int_0^T dt \int_{\mathbb{R}^d \times \mathbb{R}^d} \delta_x(y) v p_t(y, v) \, dy \, dv$$

$$= \int_0^T dt \int_{\mathbb{R}^d} v p_t(x, v) \, dv. \tag{3-18}$$

In accordance with the above, we have proven the following.

Proposition 32. *The 1-current ξ_X admits a mean current $E[\xi_X]$ on $\mathbb{R}^d$, induced by the vector field*

$$u(x) = \int_0^T \int_{\mathbb{R}^d} v p_t(x, v) \, dv \, dt, \quad x \in \mathbb{R}^d. \tag{3-19}$$

3.6. *Example of a random **1**-current in the broad sense and its expectation.* Let
X_t be the solution of the stochastic equation in $\mathbb{R}^d$:

$$dX_t = b(t, X_t)\, dt + dW_t,$$

where b is Lipschitz continuous and W_t is a Brownian motion in $\mathbb{R}^d$. Assume that
X_0 has a smooth density $p_0(x)$ with respect to Lebesgue measure. Also X_t has,
for every $t > 0$, a smooth density $p_t(x)$ with respect to Lebesgue measure; it is a
solution to the Fokker–Planck equation:

$$\frac{\partial p}{\partial t} = \Delta_x p - \mathrm{div}_x(bp),$$

$$p|_{t=0} = p_0.$$

Given $T > 0$, consider again the random curve X_t in $\mathbb{R}^d$ (not necessarily simple)
and consider the associated random 1-current ξ_X formally defined as

$$\xi_X(x) = \int_0^T \delta_{X_t}(x)\, dX_t, \quad x \in \mathbb{R}^d;$$

namely,

$$(\xi_X, \theta) = \int_0^T \langle \theta(X_t), dX_t \rangle, \quad \theta \in C_c(\mathbb{R}^d, \mathbb{R}^d),$$

where now the integral is understood as a stochastic integral. There are two main
choices: Itô and Stratonovich integrals. Let us discuss only the Itô case, but the
other one is also not difficult.

Given $\theta \in C_c(\mathbb{R}^d, \mathbb{R}^d)$, (ξ, θ) is a well defined random variable, because the
process $\theta(X_t)$ is adapted, $E[\int_0^T \|\theta(X_t)\|^2\, dt] < \infty$, and $\int_0^T \langle \theta(X_t), dX_t \rangle$ is given by

$$\int_0^T \langle \theta(X_t), dX_t \rangle = \int_0^T \langle \theta(X_t), b(t, X_t) \rangle\, dt + \int_0^T \langle \theta(X_t), dW_t \rangle.$$

So ξ_X is a current in the broad sense. In general, it is not clear if it is a current
also in the strict sense because we cannot "fix ω" and consider the map $\omega \in \Omega \mapsto$
$\int_0^T \langle \theta(X_t(\omega)), dX_t(\omega) \rangle$ (the Itô integral is an equivalence class and its pointwise
evaluation at ω is not a well-defined concept).

We have

$$\mathbb{E}\left[\int_0^T \langle \theta(X_t), dX_t \rangle\right] = \int_0^T \mathbb{E}[\langle \theta(X_t), b(t, X_t) \rangle]\, dt + \mathbb{E}\left[\int_0^T \langle \theta(X_t), dW_t \rangle\right].$$

Since for an Itô integral $E[\int_0^T \langle \theta(X_t), dW_t \rangle] = 0$, we have

$$\int_0^T \mathbb{E}[\langle \theta(X_t), b(t, X_t) \rangle]\, dt = \int_0^T \left(\int_{\mathbb{R}^d} \langle \theta(x), b(t, x) \rangle\, p_t(x)\, dx\right) dt. \qquad (3\text{-}20)$$

Hence we have proved the following.

Proposition 33. *The 1-current ξ_X admits a mean current $\mathbb{E}[\xi_X]$ on $\mathbb{R}^d$, induced by the vector field*

$$u(x) = \int_0^T b(t, x) p_t(x) \, \mathrm{d}t.$$

4. Advanced applications and open problems

4.1. *A mathematical model for tumor induced angiogenesis.* The main features of the process of formation of a tumor-driven vessel network are (see [Chaplain and Stuart 1993; Plank and Sleeman 2004; Bonilla et al. 2017])

 (i) vessel branching;

 (ii) vessel extension;

 (iii) chemotaxis in response to a generic tumor angiogenic factor (TAF), released by tumor cells;

 (iv) haptotactic migration in response to fibronectin gradient, emerging from the extracellular matrix and through degradation and production by endothelial cells themselves; and

 (v) anastomosis, the coalescence of a capillary tip with an existing vessel.

We will limit ourselves to describe the dynamics of tip cells at the front of growing vessels, as a consequence of chemotaxis in response to a generic TAF released by tumor cells.

The i-th tip cell is characterized by its position and velocity $(X_t^i, V_t^i) \in \mathbb{R}^{2d}$ for $t \geq 0$; also its history

$$(X_s^i, V_s^i)_{s \in [T^i, \Theta^i \wedge t)}$$

plays a role; the random variables T^i and Θ^i are respectively the birth (branching) and death (anastomosis) times of the i-th tip cell. All random variables and processes are defined on a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$.

The number of tip cells changes in time, due to proliferation and death. We denote this random number by N_t for $t \geq 0$.

The growth factor is a random function $C : \Omega \times [0, \infty) \times \mathbb{R}^d \to \mathbb{R}$, that we write as $C(t, x)$.

Tip cells and growth factor satisfy the system

$$\begin{aligned}
\mathrm{d}X_t^i &= V_t^i, \\
\mathrm{d}V_t^i &= \left[-k_1 V_t^i + f(C(t, X_t^i))\nabla C(t, X_t^i)\right] \mathrm{d}t + \sigma \, \mathrm{d}W_t^i, \\
\frac{\partial}{\partial t} C(t, x) &= k_2 \delta_A(x) + d_1 \Delta C(t, x) - \eta(t, x, S.) C(t, x),
\end{aligned} \tag{4-1}$$

where $k_1, k_2, \sigma, d_1 > 0$ and $f : \mathbb{R} \to \mathbb{R}$ are given, W_t^i for $i \in \mathbb{N}$ are independent Brownian motions, A is a Borel set of $\mathbb{R}^d$ representing the tumoral region acting as a source of the TAF; initial conditions X_0^i, V_0^i and $C(0, x)$ are also given.

Let us describe the term $\eta(t, x, S.)$. For every $t \geq 0$, we introduce the measure

$$S_t := \sum_{i=1}^{N_t} 1_{t \in [T^i, \Theta^i)} \epsilon_{(X_t^i, V_t^i)},$$

where ϵ denotes the usual Dirac measure. With these notations we may assume that, for every $t \geq 0$, the function $\eta(t, \cdot, \cdot)$ maps $\mathbb{R}^d \times C([0, t]; \mathcal{M}(\mathbb{R}^d))$ into $\mathbb{R}$:

$$\eta(t, \cdot, \cdot) : \mathbb{R}^d \times C([0, t]; \mathcal{M}(\mathbb{R}^d)) \to \mathbb{R}.$$

With the notation $\eta(t, x, S.)$, we understand $\eta(t, x, \{S_s\}_{s \in [0,t]})$.

We may leave the function η unspecified, with suitable assumptions. However, to help with intuition, we may assume the following structure:

$$\eta(t, x, S.) = \int_0^t e^{-(t-s)/\tau} \left(\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} K(x - x') |v'| S_s(dx', dv') \right) ds$$

for a suitable function $K : \mathbb{R}^d \to \mathbb{R}$.

In the SDE system (4-1), besides the friction force, there is a force due to the underlying TAF field $C(t, \mathbf{x})$; from the relevant literature we take [Plank and Sleeman 2004; Stéphanou et al. 2006]

$$f(C) = \frac{d_1}{(1 + \gamma_1 C)^q}. \tag{4-2}$$

The capillary network of endothelial cells $X(t)$ consists of the union of all random trajectories representing the extension of individual capillary tips from the random time of birth (branching) T^i to the random time of death (anastomosis) Θ^i,

$$X(t) = \bigcup_{i=1}^{N_t} \{X^i(s), T^i \leq s \leq \min\{t, \Theta^i\}\}, \tag{4-3}$$

giving rise to a stochastic network. Thanks to the choice of a Langevin model for the vessel's extension, we may assume that the trajectories are sufficiently regular and have integer Hausdorff dimension 1.

Hence the random measure [Capasso and Villa 2008]

$$A \in \mathcal{B}_{\mathbb{R}^d} \mapsto \mathcal{H}^1(X(t) \cap A) \in \mathbb{R}_+ \tag{4-4}$$

may admit a random generalized density $\delta_{X(t)}(x)$ with respect to the usual Lebesgue measure on $\mathbb{R}^d$ such that, for any $A \in \mathcal{B}_{\mathbb{R}^d}$,

$$\mathcal{H}^1(X(t) \cap A) = \int_A \delta_{X(t)}(x)\,\mathrm{d}x. \tag{4-5}$$

By Theorem 11, we may then state that

$$\mathcal{H}^1(X(t) \cap A) = \int_0^t \sum_{i=1}^{N_s} \epsilon_{X^i(s)}(A) \Big| \frac{\mathrm{d}}{\mathrm{d}s} X^i(s) \Big| \mathbb{1}_{s \in [T^i, \Theta^i)}\,\mathrm{d}s.$$

Hence,

$$\delta_{X(t)} = \int_0^t \sum_{i=1}^{N_s} \delta_{X^i(s)} \Big| \frac{\mathrm{d}}{\mathrm{d}s} X^i(s) \Big| \mathbb{1}_{s \in [T^i, \Theta^i)}\,\mathrm{d}s.$$

4.1.1. *Vessel branching.* Two kinds of branching have been identified; either from a tip or from a vessel.

The birth process of new tips can be described in terms of a marked point process (see, e.g., [Brémaud 1981]), by means of the random measure Φ on $\mathcal{B}_{\mathbb{R}^+ \times \mathbb{R}^d \times \mathbb{R}^d}$ such that, for any $t \geq 0$ and any $B \in \mathcal{B}_{\mathbb{R}^d \times \mathbb{R}^d}$,

$$\Phi((0, t] \times B) := \int_0^t \int_B \Phi(\mathrm{d}s \times \mathrm{d}x \times \mathrm{d}v), \tag{4-6}$$

where $\Phi(\mathrm{d}s \times \mathrm{d}x \times \mathrm{d}v)$ is the random variable that counts those tips born either from an existing tip, or from an existing vessel, during times in $(s, s + \mathrm{d}s]$, with positions in $(x, x + \mathrm{d}x]$, and velocities in $(v, v + \mathrm{d}v]$.

Given the history $\mathcal{F}_{t-}$ of the whole process up to time t^-, we claim that the compensator of the random measure $\Phi(\mathrm{d}s \times \mathrm{d}x \times \mathrm{d}v)$ is

$$\alpha(C(s, x))G_{v_0}(v)S_s(\mathrm{d}(x, v))\,\mathrm{d}s$$
$$+ \beta(C(s, x))G_{v_0}(v)(K_1 * \delta_{X(t)})(x)\,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}s, \quad (4\text{-}7)$$

where $\alpha(C)$ and $\beta(C)$ are nonnegative functions; for example, we may take

$$\alpha(C) = \beta(C) = \alpha_1 \frac{C}{C_R + C}, \tag{4-8}$$

where C_R is a reference density parameter [Capasso and Morale 2009]; $K_1 : \mathbb{R}^d \to \mathbb{R}$ is a suitable mollifying kernel.

As a technical simplification, we will further assume that the initial value of the state of a new tip is $(T^{N(t)+1}, X^{N(t)+1}, v^{N(t)+1})$, where $T^{N(t)+1}$ is the random time of branching, $X^{N(t)+1}$ is the random point of branching, and $v^{N(t)+1}$ is a random velocity, selected out of a normal distribution G_{v_0} with mean v_0, and some variance.

4.1.2. *Anastomosis.* When a vessel tip meets an existing vessel, it joins at that point and time and it stops moving. This process is called tip-vessel anastomosis.

As in the case of the branching process, we may model this process via a marked counting process; anastomosis is modeled as a "death" process.

Let Ψ denote the random measure on $\mathcal{B}_{\mathbb{R}^+ \times \mathbb{R}^d \times \mathbb{R}^d}$ such that, for any $t \geq 0$ and any $B \in \mathcal{B}_{\mathbb{R}^d \times \mathbb{R}^d}$,

$$\Psi((0, t] \times B) := \int_0^t \int_B \Psi(\mathrm{d}s \times \mathrm{d}x \times \mathrm{d}v), \tag{4-9}$$

where $\Psi(\mathrm{d}s \times \mathrm{d}x \times \mathrm{d}v)$ is the random variable counting those tips that are absorbed by the existing vessel network during time $(s, s + \mathrm{d}s]$, with position in $(x, x + \mathrm{d}x]$, and velocity in $(v, v + \mathrm{d}v]$.

We assume that the compensator of the random measure $\Psi(\mathrm{d}s \times \mathrm{d}x \times \mathrm{d}v)$ is

$$\gamma (K_1 * \delta_{X(s)})(x) S_s(\mathrm{d}(x, v)) \, \mathrm{d}s, \tag{4-10}$$

where γ is a suitable constant.

4.2. *Turbulence.* The topic of turbulence is too wide and deep to be recalled here; let us mention a general reference [Frisch 1995] and one example to attempt to develop fragments of a rigorous theory based on the stochastic Navier–Stokes equations and their invariant measures [Flandoli et al. 2008]. Among the several ideas to approach turbulence, which is to a wide extent a statistical theory, there is one of Alexander Chorin [1994] which is particularly attractive. Based on some evidence that turbulent fluids are "made" (this sentence has to be taken in a very intuitive sense) of vortex structures, in particular filament-like structures (see [Frisch 1995] for a discussion), Chorin had the idea to describe such "vortex filaments" by means of paths of stochastic processes, and relate the statistical properties of these processes to the statistical properties of turbulent fluids. In particular, Chorin considered the so called self-avoiding walk and tried to connect its Flory exponents to the Kolmogorov exponents of K41 theory of turbulence — we cannot enter in further detail here, see [Chorin 1994]. Unfortunately, as Chorin admitted in his book, this connection is not clearly identified, so it was only a research suggestion; taken by some authors, like Gallavotti [2002], Lions and Majda [2000] and Flandoli and Gubinelli [2002] who tried to develop part of such arguments using Brownian motion instead of self-avoiding walk. The problem remains essentially open after these contributions.

The concept of a random 1-current described in the present work is strongly related to — in fact, it was strongly motivated by — stochastic vortex filaments. It does not solve in itself the question of connection with K41 or other statistical theories of turbulence, but it gives a precise language to approach it. Before we

continue, let us say again that vortex filaments are an intuitive concept, since classical continuum mechanics allows us only to define the vorticity field of a fluid, and the identification of "structures" in it is not rigorous, at least at present. Having said this, the intuitive idea of a vortex filament is a curve, a vortex line (an integral curve of the vorticity field) over which the vorticity field is supported — maybe with the generalization of a finite number of such lines, not only one. This directly leads to the concept of a 1-current, a distributional vector field, concentrated along curves. Since a statistical theory should be developed, random 1-currents are the right objects. If we denote by X the process whose paths represent the vortex filaments, the random distributional vorticity field is given by

$$\xi_X = \int_0^T \delta_{X_s} dX_s.$$

The two main open questions are about statistical properties and about the realism of these random vortex structures. Concerning the realism, we mean the connection between these structures and more classical objects of fluid dynamics, typically the partial differential equations of continuum mechanics. The only partial result until now in this direction is the mean field result [Bessaih et al. 2017], where a smoothed version of 3D Euler equations is obtained as the mean field of interacting vortex filaments — also with a smoothed Biot–Savart kernel.

Concerning the statistical properties, first attempts have been made in [Chorin 1994; Lions and Majda 2000; Flandoli and Gubinelli 2002] by using Gibbs measures. The problem, not yet solved, is to identify a stochastic process X such that the random distributional vorticity field ξ_X written above has statistical properties like those of K41 theory. This means that the associated random velocity field $u_X = K * \xi_X$ has to be introduced (K denotes here the Biot–Savart kernel) and expected values like

$$\mathbb{E}[|u(x+re) - u(x)|^2],$$

where r is a real number and e is a unitary vector, have to be computed and compared with the prescription of statistical theories of turbulence (K41 prescribes a behavior of the form $Cr^{2/3}$ for small r).

A more modest but still quite open problem, very related to the machinery developed in the present work, is the following one: given a vorticity field $\xi(x)$ — generic, or typical of turbulent fluids — find a stochastic process X such that

$$\xi(x) = \mathbb{E}[\xi_X](x).$$

Solving this problem may increase some understanding of the statistical problem above and provide relevant initial conditions for the mean-field approach of [Bessaih et al. 2017].

4.3. *Pathwise stochastic analysis.* In recent years, a new direction in stochastic analysis grew up thanks to a number of outstanding contributions. The beginning of this direction is often traced back to a paper by Hans Föllmer [1981] where he derived an Itô formula for the composition $f(x_t)$ of a C^2 function $f : \mathbb{R}^d \to \mathbb{R}$ with a *single deterministic path* $x : [0, T] \to \mathbb{R}^d$, which possesses quadratic variation along a sequence of partitions (π_n) of $[0, T]$, in the sense that

$$[x_{\cdot}^{(i)}]_t^{(\pi_n)} := \lim_{n \to \infty} \sum_{t_i \in \pi_n} (x_{t_{i+1} \wedge t}^{(i)} - x_{t_i \wedge t}^{(i)})^2$$

exists finite and continuous in $t \in [0, T]$, for each $i = 1, \dots, d$, where $x_t^{(i)}$ denotes the i-coordinate of x_t. The stochastic integral $\int_0^t \langle \nabla f(x_s), \mathrm{d}x_s \rangle$ required in the formula,

$$f(x_t) - f(x_0) - \frac{1}{2} \int_0^t \sum_{i,j=1}^d \partial_i \partial_j f(x_s) \, \mathrm{d}[x_{\cdot}^{(i)}, x_{\cdot}^{(j)}]_s^{(\pi_n)} = \int_0^t \langle \nabla f(x_s), \mathrm{d}x_s \rangle, \quad (4\text{-}11)$$

is not a priori defined, but it exists from the formula itself: the limit or Riemann sums

$$\lim_{n \to \infty} \sum_{t_i \in \pi_n} \langle \nabla f(x_{t_i \wedge t}), x_{t_{i+1} \wedge t} - x_{t_i \wedge t} \rangle$$

exists since the sum of the other terms in the formula have a limit, thanks to the existence of the quadratic variation (the so-called joint quadratic variation between two coordinates $[x_{\cdot}^{(i)}, x_{\cdot}^{(j)}]_t^{(\pi_n)}$ exists by polarization and it is a bounded variation function, hence the integral on the left-hand side of (4-11) exists as limit of Riemann–Stieltjes sums). Probability is not totally excluded by this approach, but its role is *localized*: probability provides the existence of the quadratic variation $[x_{\cdot}^{(i)}]_t^{(\pi_n)}$ for almost every path of relevant stochastic processes, like Brownian motion and more generally continuous semimartingales; it is otherwise very difficult to construct a deterministic function having *nonzero* quadratic variation $[x_{\cdot}^{(i)}]_t^{(\pi_n)}$ (if it is zero, the formula above is just the usual chain rule). In a sense, a main topic in stochastic analysis, which previously has been treated by probability from the first to the last step (namely Itô's formula from the viewpoint of Itô's approach), is now *decoupled*: one half of the story is based on probability — the existence of the quadratic variation — and the other half is purely deterministic.

The question arisen by that paper is: to what extent can one develop a similar approach for other pieces of stochastic analysis? A breakthrough is rough path calculus developed by Terry Lyons [1998], which introduces new classes of paths, defines stochastic integrals for them and solves stochastic differential equations. Again, the theory is fully deterministic but at the foundation there is the concept of rough path, a sort of path x enriched by its *Lévy areas* $\int_0^t x_s^{(i)} \mathrm{d}x_s^{(j)}$ (we refer here to

a subclass for the general theory); the latter object exists for almost every path of Brownian motion and semimartingales using probability, namely using Itô's theory of stochastic integration.

The theory of rough paths has been revisited with a novel approach by Massimiliano Gubinelli [2004] and, in a sense based on this new viewpoint, a sort of multidimensional analog (namely for functions $x : \mathbb{R}^d \to \mathbb{R}$ or more precisely for distributions) has been developed in the outstanding work of Martin Hairer [2014], see also the alternative theory by Gubinelli et al. [2015]. Again in these theories, probability guarantees the existence of fundamental objects, after which the procedure to define other objects and solve (ordinary or partial) differential equations is fully deterministic. Let us also mention this kind of two-step approach in other recent directions of stochastic analysis, like stochastic homogenization [Gloria and Otto 2015].

After reviewing this introduction to the subject of "stochastic calculus without probability", let us mention an open research direction related to the topics of the present paper. Probability allows us to define stochastic integrals of the form

$$\int_0^t \langle \theta(X_s), \mathrm{d}X_s \rangle$$

for several stochastic processes X and functions θ. The map $\theta \mapsto \int_0^t \langle \theta(X_s), \mathrm{d}X_s \rangle$ is a random 1-current in the broad sense. In many cases, thanks to probabilistic estimates, there is a random 1-current in the strict sense $\xi_{X,t}(\omega)$ associated to it, given by

$$\xi_{X,t} = \int_0^t \delta_{X_s} \mathrm{d}X_s.$$

This is, for a given ω, a deterministic 1-current associated to the deterministic path $X_{\cdot}(\omega)$. Conceptually, this is similar to the quadratic variation or the Lévy area associated to $X_{\cdot}(\omega)$: concepts which are well-defined by probability, and would be extremely difficult to define without. The question then is: starting from the deterministic pair

$$(X_{\cdot}(\omega), \xi_{X,\cdot}(\omega)),$$

is it possible to develop, by purely deterministic methods, some pieces of stochastic calculus, as it was done by the theories recalled above starting from the pairs (path, quadratic variation), (path, Lévy area) and so on? For instance, is it possible to formulate and solve differential equations driven by single paths of Brownian motions? At present, this program has not been developed. Only the regularity, in terms of distributions, of $\xi_{X,\cdot}(\omega)$ has been partially understood, see [Flandoli et al. 2005; 2009].

Acknowledgements

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