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REDUCIBLE AND IRREDUCIBLE FORMS OF STABILISED GRADIENT ELASTICITY IN DYNAMICS

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The continualisation of discrete particle models has been a popular tool to formulate higher-order gradient elasticity models. However, a straightforward continualisation leads to unstable continuum models. Padé approximations can be used to stabilise the model, but the resulting formulation depends on the particular equation that is transformed with the Padé approximation. In this contribution, we study two different stabilised gradient elasticity models; one is an irreducible form with displacement degrees of freedom only, and the other is a reducible form where the primary unknowns are not only displacements but also the Cauchy stresses — this turns out to be Eringen's theory of gradient elasticity. Although they are derived from the same discrete model, there are significant differences in variationally consistent boundary conditions and resulting finite element implementations, with implications for the capability (or otherwise) to suppress crack tip singularities.

1. Introduction

Gradient elasticity is a methodology to enrich the continuum equations of elasticity with additional higher-order spatial (and occasionally temporal) derivatives of certain state variables. There are different versions of gradient elasticity, such as those equipped with strain gradients, stress gradients and acceleration gradients; see for instance [Askes and Aifantis 2011] for a recent (but by no means complete) review.

Certain formats of gradient elasticity bear a close relationship with discrete lattice models of materials with microstructure; indeed, it is often possible to derive gradient elasticity theories by continualising the response of a discrete model, for instance using Taylor series approximations [Chang and Gao 1995; Mühlhaus and Oka 1996; Suiker et al. 2001a; Suiker et al. 2001b; Ioannidou et al. 2001; Askes and Metrikine 2005]. However, such models often suffer from intrinsic deficiencies, such as loss of stability in dynamics and loss of uniqueness in statics [Askes et al. 2002]. This can be amended by applying Padé approximations or similar

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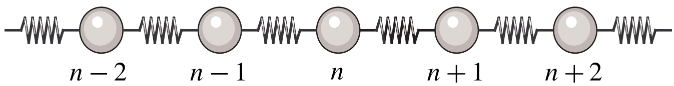

Figure 1. One-dimensional chain of masses connected by springs.

techniques, as has for instance been demonstrated in [Rosenau 1984; Rubin et al. 1995; Chen and Fish 2001; Andrianov 2002; Andrianov et al. 2003; Charlotte and Truskinovsky 2008]. Thus, stabilised gradient elasticity theories can be formulated that maintain their close link with discrete lattice models, thereby facilitating simple identification of the higher-order constitutive parameters (usually known as "intrinsic length scales" or "microstructural length scales").

In this paper, we compare two versions of stabilised gradient elasticity. Both can be derived from the response of a discrete lattice model, which is shown for the one-dimensional case. Variational formulations are presented for the multidimensional extensions. Throughout, a distinction is made between the so-called *irreducible form* where the only unknowns are the displacements and the *reducible form* where the unknowns are the displacements as well as the Cauchy stresses. The difference between these two forms has important consequences for the variationally consistent boundary conditions and finite element implementations. A numerical example will show the ability (or otherwise) of the two formulations to suppress singularities — this has historically been an important motivation for using gradient elasticity theories, and certain formats have been demonstrated to remove singularities even under restrictive conditions such as anisotropic material behaviour and bimaterial interface cracks [Kwong and Gitman 2012]. We also discuss the relation of the reducible form with Eringen's [1983] differential theory of nonlocal elasticity.

2. Continualisation of the response of a discrete chain

To illustrate the concepts of continualisation (this section) and stabilisation via Padé approximations (Section 3), the one-dimensional chain of particles and springs in Figure 1 is studied. All particles have mass M, and all springs have stiffness K. Furthermore, the interparticle distance is denoted by d. The equation of motion of particle n thus reads

$$M\ddot{u}_n = K(u_{n+1} - 2u_n + u_{n-1}) \tag{1}$$

where u_i is the displacement of particle i. A continuum approximation is obtained by replacing u_n with $u(x)$ and $u_{n\pm1}$ with $u(x \pm d)$. Taylor series expansions are applied according to

$$u(x \pm d) = u(x) \pm d\frac{\partial u}{\partial x} + \frac{1}{2}d^2\frac{\partial^2 u}{\partial x^2} \pm \frac{1}{6}d^3\frac{\partial^3 u}{\partial x^3} + \frac{1}{24}d^4\frac{\partial^4 u}{\partial x^4} \pm \cdots \tag{2}$$

so that (1) can be rewritten as

$$\rho\ddot{u} = E\left(\frac{\partial^2 u}{\partial x^2} + \frac{1}{12}d^2\frac{\partial^4 u}{\partial x^4} + \cdots\right) \tag{3}$$

where the mass density $\rho = M/Ad$ and the Young's modulus $E = Kd/A$, with A the (unit) cross-sectional area of the system. Multidimensional formulations in the spirit of (3) have been derived by Chang and Gao [1995], Mühlhaus and Oka [1996] and Suiker et al. [2001a; Suiker et al. [2001b], among others.

Apart from the lowest-order, standard terms, (3) also contains higher-order terms proportional to d^2, d^4, etc. These additional terms capture the microstructural effects that are present in the discrete model of (1) but that are absent in standard continuum theories as retrieved by taking $d = 0$ in (3). The simplest continuum model that incorporates microstructural effects is obtained by truncating the series in (3) after the term that is proportional to d^2; unfortunately, such a model is unstable and its solutions in a boundary-value problem may lack uniqueness [Askes et al. 2002]. Although stability and uniqueness can be restored by incorporating the next term, i.e., truncating after the d^4 term, the numerical implementation of such a model is complicated [Askes et al. 2002]; thus, alternative solution strategies are explored here.

3. Stabilising the continuum equations

Unstable gradient theories can be turned into stable gradient theories by means of Padé approximations, as has been explored in [Andrianov et al. 2003; Andrianov and Awrejcewicz 2008; Andrianov et al. 2010]. However, there are various ways to do this, and the format of the resulting equations depends on which equations are transformed by the Padé approximation.

3.1. *Irreducible form.* Firstly, (3) is truncated after the first nonstandard term. The various spatial derivatives are factorised as

$$\rho\ddot{u} = \left(1 + \frac{1}{12}d^2\frac{\partial^2}{\partial x^2}\right)E\frac{\partial^2 u}{\partial x^2}. \tag{4}$$

A [0, 1]-Padé approximation is used according to

$$1 + a \approx \frac{1}{1-a} \quad \text{for } a \ll 1. \tag{5}$$

For a in (5), we will substitute the operator $\frac{1}{12}d^2\,\partial^2/\partial x^2$, which allows us to rewrite (4) as

$$\left(1 - \frac{1}{12}d^2\frac{\partial^2}{\partial x^2}\right)\rho\ddot{u} = E\frac{\partial^2 u}{\partial x^2}. \tag{6}$$

The higher-order gradient term now appears on the inertia side of the equation, and for this reason, it has been called microinertia, internal inertia or higher-order inertia in the literature [Vardoulakis and Aifantis 1994; Wang and Sun 2002; Bennett et al. 2007]. Equation (6), or slight variations thereof, has also been obtained by various other researchers using asymptotic series equivalence; see for instance the work of Rubin et al. [1995], Chen and Fish [2001] and Pichugin et al. [2008].

Note that the only unknown appearing in (6) is the displacement; for this reason, this format is denoted as *irreducible*. Although at first sight it may appear that the micromechanical background of the higher-order terms is lost through the Padé approximation, an alternative interpretation of the microinertia contribution in terms of long-range interactions has been provided in [Askes and Gitman 2014].

3.2. ***Reducible form.*** It is also possible to extract a (one-dimensional) relation between stress σ and strain ε from (3) such that

$$\rho\ddot{u} = \frac{\partial \sigma}{\partial x} \quad \text{and} \quad \varepsilon = \frac{\partial u}{\partial x}. \tag{7}$$

The stress-strain relation then follows as

$$\sigma = E\left(\varepsilon + \frac{1}{12}d^2\frac{\partial^2 \varepsilon}{\partial x^2}\right) = E\left(1 + \frac{1}{12}d^2\frac{\partial^2}{\partial x^2}\right)\varepsilon \tag{8}$$

where series have again been truncated after the first nonstandard term. Applying the [0, 1]-Padé approximation to (8) yields

$$\left(1 - \frac{1}{12}d^2\frac{\partial^2}{\partial x^2}\right)\sigma = E\varepsilon. \tag{9}$$

Equations (7) and (9) can be combined into a system of coupled equations,

$$\rho\ddot{u} = \frac{\partial \sigma}{\partial x} \tag{10a}$$

together with

$$\sigma - \frac{1}{12}d^2\frac{\partial^2 \sigma}{\partial x^2} = E\frac{\partial u}{\partial x} \tag{10b}$$

where the unknowns are the displacement u as well as the stress σ. In contrast to the single fourth-order equation (6), (10) is a set of two second-order equations. They are termed *reducible* because it is possible to eliminate one of the unknowns, namely the stress σ. To do this, the second-order spatial derivative of (10a) must be taken and, multiplied with $\frac{1}{12}d^2$, subtracted from the original expression (10a):

$$\rho\left(\ddot{u} - \frac{1}{12}d^2\frac{\partial^2 \ddot{u}}{\partial x^2}\right) = \frac{\partial}{\partial x}\left(\sigma - \frac{1}{12}d^2\frac{\partial^2 \sigma}{\partial x^2}\right). \tag{11}$$

If (10b) is substituted into the right-hand side of (11), the stress will disappear from the expressions and thus it is possible to retrieve (6). This reduction of the number of unknowns, and its consequences, will be discussed in more depth below in Section 4.2.

4. Energy functionals for the multidimensional case

Above, the governing equations have been derived from simple mechanical and mathematical arguments in a one-dimensional context. Next, we will show how the analogous multidimensional equations can be derived from variational principles. Hamilton's action S is defined as

$$S = \int_{t_0}^{t_1} L \, \mathrm{d}t. \tag{12}$$

The governing equations of the models can be derived by requiring stationarity of S, that is, $\delta S = 0$. The energy functional (or Lagrangian function) L is defined individually for the two different models below, but we will assume that L depends on the displacements u_i and their spatial and temporal derivatives, as well as on the stresses σ_{ij} and their spatial derivatives:

$$L = L(u_i; u_{i,j}; \dot{u}_i; \dot{u}_{i,j}; \sigma_{ij}; \sigma_{ij,k}). \tag{13}$$

Substituting (13) into (12) and requiring $\delta S = 0$ yields

$$\begin{aligned}
\int_{t_0}^{t_1} \delta L \, \mathrm{d}t
&= \int_{t_0}^{t_1} \left(\delta u_i \frac{\partial L}{\partial u_i} + \delta u_{i,j} \frac{\partial L}{\partial u_{i,j}} + \delta \dot{u}_i \frac{\partial L}{\partial \dot{u}_i} + \delta \dot{u}_{i,j} \frac{\partial L}{\partial \dot{u}_{i,j}} + \delta \sigma_{ij} \frac{\partial L}{\partial \sigma_{ij}} + \delta \sigma_{ij,k} \frac{\partial L}{\partial \sigma_{ij,k}} \right) \mathrm{d}t \\
&= 0,
\end{aligned} \tag{14}$$

which, as usual, can be rewritten as

$$\begin{aligned}
&\int_{t_0}^{t_1} \delta u_i \left(\frac{\partial L}{\partial u_i} - \frac{\partial}{\partial x_j} \frac{\partial L}{\partial u_{i,j}} - \frac{\partial}{\partial t} \frac{\partial L}{\partial \dot{u}_i} + \frac{\partial^2}{\partial x_j \partial t} \frac{\partial L}{\partial \dot{u}_{i,j}} \right) \mathrm{d}t \\
&\quad + \int_{t_0}^{t_1} \frac{\partial}{\partial x_j} \left(\delta u_i \frac{\partial L}{\partial u_{i,j}} - \delta u_i \frac{\partial}{\partial t} \frac{\partial L}{\partial \dot{u}_{i,j}} \right) \mathrm{d}t + \int_{t_0}^{t_1} \frac{\partial}{\partial t} \left(\delta u_i \frac{\partial L}{\partial \dot{u}_i} + \delta u_{i,j} \frac{\partial L}{\partial \dot{u}_{i,j}} \right) \mathrm{d}t \\
&\qquad + \int_{t_0}^{t_1} \delta \sigma_{ij} \left(\frac{\partial L}{\partial \sigma_{ij}} - \frac{\partial}{\partial x_k} \frac{\partial L}{\partial \sigma_{ij,k}} \right) \mathrm{d}t + \int_{t_0}^{t_1} \frac{\partial}{\partial x_k} \left(\delta \sigma_{ij} \frac{\partial L}{\partial \sigma_{ij,k}} \right) \mathrm{d}t = 0.
\end{aligned} \tag{15}$$

The third integral cancels through the requirement that $\delta u_i = 0$ and $\delta u_{i,j} = 0$ for $t = t_0$ and for $t = t_1$. The first and fourth integrals will lead to field equations, whereas the second and fifth will contribute to the natural boundary conditions.

4.1. ***Irreducible form.*** The Lagrangian function of the irreducible form can be written as

$$L_{\text{irred}} = \int_\Omega \tfrac{1}{2}\rho(\dot{u}_i\dot{u}_i + \ell^2\dot{u}_{i,j}\dot{u}_{i,j})\,\mathrm{d}V - \int_\Omega \tfrac{1}{2}u_{i,j}C_{ijkl}u_{k,l}\,\mathrm{d}V + \int_\Omega u_i b_i\,\mathrm{d}V + \int_{\Gamma_n} u_i t_i\,\mathrm{d}S \quad (16)$$

where the first integral is the kinetic energy, the second integral is the stored strain energy and the last two terms represent the work of the external forces. Thus, for this model, the Lagrangian takes the usual format of "kinetic energy minus potential energy", whereby the nonstandard contributions are included in the kinetic energy only [Lazar and Anastassiadis 2007; Polizzotto 2012]. Note that for the internal length scale we have now used the generic notation ℓ rather than the notation d that was used in the previous section in relation to the discrete model.

Substituting (16) into (15) and noting that $\delta u_i = 0$ on Γ_e leads to

$$\int_{t_0}^{t_1}\int_\Omega \delta u_i(b_i + C_{ijkl}u_{k,jl} - \rho\ddot{u}_i + \rho\ell^2\ddot{u}_{i,jj})\,\mathrm{d}V\,\mathrm{d}t + \int_{t_0}^{t_1}\int_{\Gamma_n} \delta u_i(t_i - n_j(C_{ijkl}u_{k,l} + \rho\ell^2\ddot{u}_{i,j}))\,\mathrm{d}S\,\mathrm{d}t = 0, \quad (17)$$

where, as usual, the boundary Γ of the domain Ω is decomposed into parts Γ_n and Γ_e associated with natural and essential boundary conditions: $\Gamma = \Gamma_n \cup \Gamma_e$ and $\varnothing = \Gamma_n \cap \Gamma_e$.

A symmetric Hookean stress $\tau_{ij}^{\mathrm{H}} = C_{ijkl}u_{k,l}$ can be identified in terms of which the field equations and natural boundary conditions can be written as

$$\rho(\ddot{u}_i - \ell^2\ddot{u}_{i,jj}) = \tau_{ij,j}^{\mathrm{H}} + b_i \quad \text{in } \Omega, \quad (18a)$$

$$n_j(\tau_{ij}^{\mathrm{H}} + \rho\ell^2\ddot{u}_{i,j}) = t_i \quad \text{on } \Gamma_n. \quad (18b)$$

In our opinion, Hookean stress is appropriate terminology for τ_{ij}^{H}, not Cauchy stress, since the equations of motion and the natural boundary conditions contain additional gradients of the acceleration that are not included in the definition of τ_{ij}^{H}. In Appendix A this particular terminology is motivated.

Remark. A nonsymmetric stress tensor τ_{ij}^{B} can be identified as (see [Lazar and Anastassiadis 2007])

$$\tau_{ij}^{\mathrm{B}} = C_{ijkl}u_{k,l} + \rho\ell^2\ddot{u}_{i,j} \quad (19)$$

This would enable one to write the equations of motion and natural boundary conditions in terms of a stress tensor that is similar in role to a standard Cauchy stress as explained in Appendix A. However, since τ_{ij}^{B} is nonsymmetric, using the term Cauchy stress for this tensor is not obvious. This issue of nomenclature is left for future debate and discussion.

4.2. ***Reducible form.*** For the reducible form, the Lagrangian function adopts a less common appearance, which, to the authors' best knowledge, is novel:

$$L_{\text{red}} = \int_\Omega \tfrac{1}{2}\rho \dot{u}_i \dot{u}_i \,\mathrm{d}V - \int_\Omega u_{i,j}\sigma_{ij}\,\mathrm{d}V + \int_\Omega \tfrac{1}{2}(\sigma_{ij}S_{ijkl}\sigma_{kl} + \ell^2 \sigma_{ij,m}S_{ijkl}\sigma_{kl,m})\,\mathrm{d}V + \int_\Omega u_i b_i \,\mathrm{d}V + \int_{\Gamma_n} u_i t_i \,\mathrm{d}S, \tag{20}$$

where S_{ijkl} is the elastic compliance tensor. The first integral is again the kinetic energy, whilst the last two integrals contain the external work. The third integral contains the stored complementary energy with a positive rather than negative sign, but the effects of the lower-order part are offset by the effects of the second integral, which couples the effects of the two sets of unknowns, namely displacements and stresses. In the reducible form, the displacement derivative $u_{i,j}$ is no longer energy-conjugated to the (symmetric) stress σ_{ij}, unless $\ell = 0$. Therefore, the second integrand does *not* have the meaning of internal work. Expression (20) can also be rewritten as a Hellinger–Reissner functional whereby the displacements act as Lagrange multipliers to enforce balance of momentum in Ω and on Γ [Askes and Gutiérrez 2006; Polizzotto 2015].

Again making use of $\delta u_i = 0$ on Γ_e, substitution of (20) into (15) yields

$$\int_{t_0}^{t_1}\int_\Omega \delta u_i(b_i + \sigma_{ij,j} - \rho\ddot{u}_i)\,\mathrm{d}V\,\mathrm{d}t + \int_{t_0}^{t_1}\int_{\Gamma_n} \delta u_i(t_i - n_j\sigma_{ij})\,\mathrm{d}S\,\mathrm{d}t + \int_{t_0}^{t_1}\int_\Omega \delta\sigma_{ij}(-u_{i,j} + S_{ijkl}\sigma_{kl} - \ell^2 S_{ijkl}\sigma_{kl,mm})\,\mathrm{d}V\,\mathrm{d}t + \int_{t_0}^{t_1}\oint_\Gamma \delta\sigma_{ij}n_m S_{ijkl}\sigma_{kl,m}\,\mathrm{d}S\,\mathrm{d}t = 0 \tag{21}$$

so that the following set of coupled governing equations can be identified:

$$\rho\ddot{u}_i = \sigma_{ij,j} + b_i \quad \text{in } \Omega, \tag{22a}$$

$$n_j\sigma_{ij} = t_i \quad \text{on } \Gamma_n, \tag{22b}$$

$$S_{ijkl}(\sigma_{kl} - \ell^2\sigma_{kl,mm}) = \tfrac{1}{2}(u_{i,j} + u_{j,i}) \quad \text{in } \Omega, \tag{22c}$$

$$n_m \ell^2 S_{ijkl}\sigma_{kl,m} = 0 \quad \text{on } \Gamma. \tag{22d}$$

From the format of (22a) and (22b), it is clear that the meaning of σ_{ij} in the reducible model is that of the Cauchy stress. Equations (22) have also been derived, using different arguments, by Eringen [1983]; see Appendix B for a discussion.

Equations (22) form a set of coupled equations with independent unknowns u_i and σ_{ij}, but they are reducible in the sense that it is possible to eliminate the stresses σ_{ij}. To do so, firstly the Laplacian of (22a) is taken and multiplied with ℓ^2, after which the result is subtracted from the original expression (22a). This gives

$$\rho(\ddot{u}_i - \ell^2\ddot{u}_{i,jj}) = \sigma_{ij,j} - \ell^2\sigma_{ij,kk} + b_i - \ell^2 b_{i,jj}. \tag{23}$$

Next, (22c) is premultiplied with the elastic stiffness tensor C_{ijkl} and substituted into (23), leading to

$$\rho(\ddot{u}_i - \ell^2 \ddot{u}_{i,jj}) = C_{ijkl} u_{k,jl} + b_i - \ell^2 b_{i,jj}, \tag{24}$$

which is equivalent to (18a) except for the presence of the Laplacian of the body forces $b_{i,jj}$ and a mismatch in the associated variationally consistent boundary conditions. Note that the effect of the higher-order gradients disappears altogether in statics in the case $b_{i,jj} = 0$.

Remark. From (22c) it is clear that the gradient enrichment affects the constitutive part of the field equations, and therefore the term "gradient elasticity" seems appropriate for what is here denoted as the reducible form. In contrast, it could be argued that using the term "gradient elasticity" is less suitable for the irreducible format represented in (24), because the gradient enrichment operates on the accelerations, not stresses or strains — i.e., the elasticity part of the irreducible form retains its classical format. However, we still prefer to refer to the irreducible form as a particular variant of gradient elasticity, because of the close relation between the reducible form and the irreducible form. Due to the coupling between the equations of motion and the constitutive equations, the gradient enrichment of the accelerations will affect the stresses and strains, albeit indirectly.

5. Finite element equations

In order to obtain solutions of the relevant partial differential equations for domains of arbitrary geometry, a numerical solution strategy is required. Here, the finite element method will be used for the spatial discretisation, whereas the Newmark time integrator will be adopted to progress the solution in the time domain. The finite element equations of the irreducible form are well established and need not be revisited here — the interested reader is referred to [Fish et al. 2002a; 2002b; Askes and Aifantis 2011].

For the reducible form, we write $\underline{u} = \boldsymbol{N}_u \boldsymbol{d}$ and $\underline{\sigma} = \boldsymbol{N}_\sigma \boldsymbol{s}$ where $\underline{u}$ and $\underline{\sigma}$ are column vectors containing the relevant components of the displacements and Cauchy stresses, respectively. Furthermore, the matrices $\boldsymbol{N}_u$ and $\boldsymbol{N}_\sigma$ contain the shape functions for displacements and Cauchy stresses whereas $\boldsymbol{d}$ and $\boldsymbol{s}$ are the nodal displacements and nodal Cauchy stresses. The spatial discretisation of (20) can thus be written as

$$\begin{aligned} L_{\text{red}}^{\text{FE}} &= \int_\Omega \tfrac{1}{2}\rho \dot{\boldsymbol{d}}^T \boldsymbol{N}_u^T \boldsymbol{N}_u \dot{\boldsymbol{d}} \,\mathrm{d}V - \int_\Omega \boldsymbol{d}^T \boldsymbol{B}_u^T \boldsymbol{N}_\sigma \boldsymbol{s} \,\mathrm{d}V \\ &+ \int_\Omega \tfrac{1}{2}\boldsymbol{s}^T \Big(\boldsymbol{N}_\sigma^T \boldsymbol{S} \boldsymbol{N}_\sigma + \sum_{i=1}^{3} \ell^2 \frac{\partial \boldsymbol{N}_\sigma^T}{\partial x_i} \boldsymbol{S} \frac{\partial \boldsymbol{N}_\sigma}{\partial x_i} \Big) \boldsymbol{s} \,\mathrm{d}V + \int_\Omega \boldsymbol{d}^T \boldsymbol{N}_u^T \underline{b} \,\mathrm{d}V + \int_{\Gamma_n} \boldsymbol{d}^T \boldsymbol{N}_u^T \underline{t} \,\mathrm{d}S \end{aligned} \tag{25}$$

where $\underline{b}$ and $\underline{t}$ contain the components of the distributed body and surface forces, respectively. Furthermore, $\boldsymbol{B}_u$ is the standard strain-displacement matrix with derivatives of the displacement shape functions $\boldsymbol{N}_u$ and $\boldsymbol{S}$ is the matrix counterpart of the compliance tensor S_{ijkl}.

Requiring $\delta L^{\mathrm{FE}}_{\mathrm{red}} = 0$ leads to a system of finite element equations according to

$$\begin{bmatrix} \boldsymbol{M}_{uu} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \ddot{\boldsymbol{d}} \\ \ddot{\boldsymbol{s}} \end{bmatrix} + \begin{bmatrix} \boldsymbol{0} & \boldsymbol{K}_{u\sigma} \\ \boldsymbol{K}_{\sigma u} & \boldsymbol{K}_{\sigma\sigma} \end{bmatrix} \begin{bmatrix} \boldsymbol{d} \\ \boldsymbol{s} \end{bmatrix} = \begin{bmatrix} \boldsymbol{f} \\ \boldsymbol{0} \end{bmatrix}, \tag{26}$$

where

$$\boldsymbol{M}_{uu} = \int_\Omega \rho \boldsymbol{N}_u^T \boldsymbol{N}_u \,\mathrm{d}V, \tag{27a}$$

$$\boldsymbol{K}_{u\sigma} = \boldsymbol{K}_{\sigma u}^T = \int_\Omega \boldsymbol{B}_u^T \boldsymbol{N}_\sigma \,\mathrm{d}V, \tag{27b}$$

$$\boldsymbol{K}_{\sigma\sigma} = -\int_\Omega \left(\boldsymbol{N}_\sigma^T \boldsymbol{S} \boldsymbol{N}_\sigma + \sum_{i=1}^{3} \ell^2 \frac{\partial \boldsymbol{N}_\sigma^T}{\partial x_i} \boldsymbol{S} \frac{\partial \boldsymbol{N}_\sigma}{\partial x_i} \right) \mathrm{d}V. \tag{27c}$$

Finite-element implementation of (26) was carried out using the recommendations of the statics theory given in [Askes and Gutiérrez 2006], in particular the use of quadratic shape functions for $\boldsymbol{s}$ and linear shape functions for $\boldsymbol{d}$. This particular choice of shape functions avoids oscillations in the displacement field, although a formal investigation of the inf-sup condition may require further refinement of the two sets of interpolations.

6. Numerical example

Although the reducible form can be transformed into the irreducible form as shown in (23) and (24), the associated change in variationally consistent boundary conditions has implications when it comes to the simulation of crack tip stresses. This will be demonstrated by means of the numerical example shown in Figure 2.

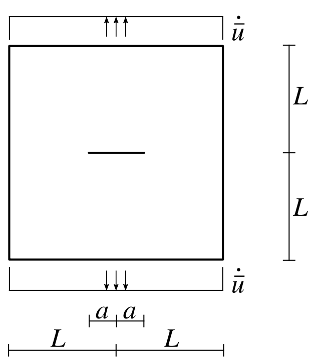

Figure 2. Strip with central crack: geometry and loading conditions.

A square strip with dimension $2L = 2\,\mathrm{m}$ has a central crack of length $2a = 0.5\,\mathrm{m}$. The material properties are mass density $\rho = 1\,\mathrm{kg/m^3}$, Young's modulus $E = 100\,\mathrm{N/m^2}$ and Poisson's ratio $\nu = \frac{1}{4}$, whilst a plane stress assumption has been made. Furthermore, the gradient elasticity length scale $\ell = 0.1\,\mathrm{m}$. The strip is subjected to outward vertical velocities $\dot{\bar{u}} = 10\,\mathrm{m/s}$ imposed on the top and bottom edges, as indicated, which leads to stress waves propagating towards the centre of the strip. Away from the crack, the stress waves will have the shape of a block wave due to the nature of the loading conditions, but the presence of the crack will disturb this pattern, and indeed in a classical elasticity setting, this will lead to singular stresses and strains at the tips of the crack. It is the aim of this example to verify whether these singularities can be avoided in the reducible and irreducible formulations of gradient elasticity discussed above. For reasons of symmetry, only the top quarter of the strip is modelled.

The irreducible format of gradient elasticity is implemented with four-noded quadrilateral elements for the displacements. The reducible format is implemented with eight-noded elements for the stresses and four-noded quadrilateral elements for the displacements — see [Askes and Gutiérrez 2006] for details on this particular choice. Structured finite element meshes consisting of square elements are used, and a sequence of uniformly refined meshes is taken to monitor the behaviour of the stresses at the crack tip. Since in the irreducible format the stresses are postprocessed from linear displacements whereas in the reducible format the stresses are primary unknowns interpolated with quadratic shape functions, there is an obvious mismatch in stress resolution between the two formats. To address this mismatch, the meshes used range from 16×16 to 128×128 elements for the irreducible format, whereas they range from 8×8 to 64×64 for the reducible format.

Regarding the imposition of traction boundary conditions, it must be realised that the stresses are primary variables in the reducible formulation, whereas they are derived quantities in the irreducible formulation. In the reducible formulation, traction boundary conditions are thus essential boundary conditions and are imposed by assigning prescribed values to the relevant stress components (e.g., $\sigma_{yy} = 0$ on the crack face). On the other hand, traction boundary conditions are natural boundary conditions in the irreducible formulation; applying zero tractions on the crack face means that the left-hand-side of (18b) is set equal to zero, which is handled straightforwardly in a finite element context. Finally, and for the sake of completeness, it is noted that displacement (and velocity) boundary conditions have been implemented using Lagrange Multipliers in the reducible formulation.

The Newmark constant average acceleration scheme is used for the time integration. This scheme is unconditionally stable; therefore, the only criterion for selecting the time step is accuracy. Following the recommendations given in [Askes et al. 2008; Bennett and Askes 2009], the time step is chosen such that waves

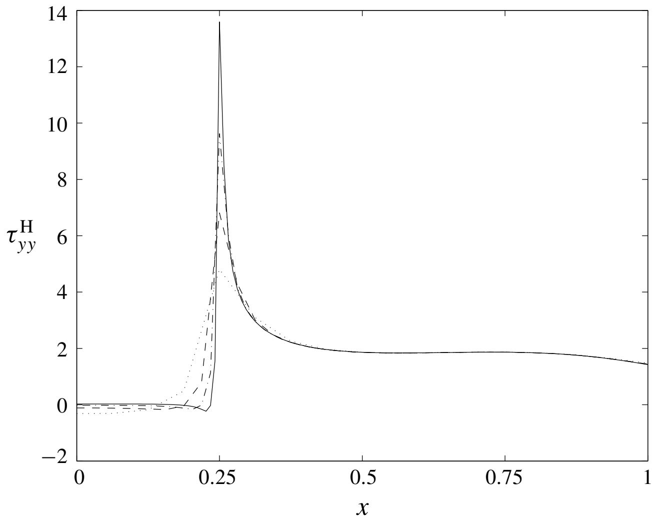

Figure 3. Vertical normal stress τ^{H}_{yy} ($\mathrm{N/m^2}$) versus x (m) for the irreducible format — 16×16 elements (dotted), 32×32 elements (dashed), 64×64 elements (dot-dashed) and 128×128 elements (solid).

propagate approximately half an element per time step. Time domain simulations were carried out from time $t = 0$ s to $t = 0.2$ s.

Figures 3 and 4 show the profiles of the vertical normal stress for both formats and the indicated range of finite element meshes, where the origin of the coordinate system is chosen at the centre of the crack. For the irreducible format (Figure 3), we have plotted the Hookean stress τ^{H}_{yy} (see Section 4.1) whilst for the reducible format the Cauchy stress σ_{yy} is plotted (Figure 4).

The stress profiles for the irreducible formulation appear to converge towards a unique solution, except for the crack tip value. At the crack tip, the stress increases significantly for every refinement of the mesh. This is an indication that a stress singularity is present at the crack tip. To analyse this in more depth, Richardson extrapolations have been carried out for the crack tip stresses. Table 1 reports the

mesh	τ^{H}_{yy}	extrapolation
16×16	4.8077	
32×32	6.8091	8.8105
64×64	9.6223	13.6438
128×128	13.5923	20.0751

Table 1. Crack tip stress and Richardson extrapolation in $\mathrm{N/m^2}$ for irreducible form.

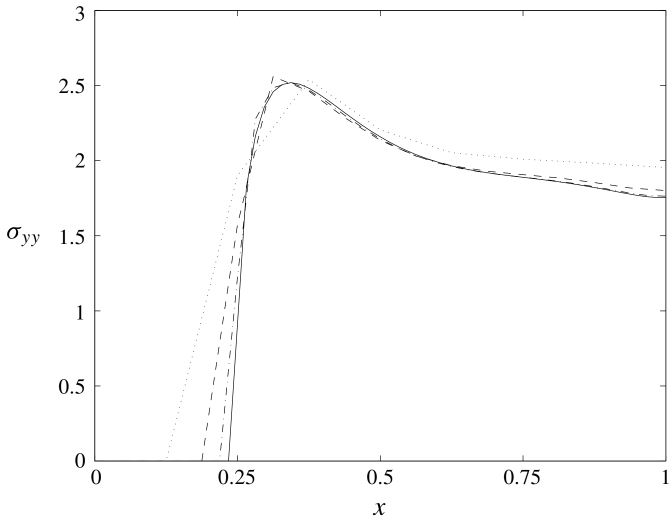

Figure 4. Vertical normal stress σ_{yy} (N/m^2) versus x (m) for the reducible format — 8×8 elements (dotted), 16×16 elements (dashed), 32×32 elements (dot-dashed) and 64×64 elements (solid).

values of the crack tip stress and their extrapolations. (The first extrapolation is a two-point extrapolation based on the coarsest two meshes, the second is a three-point extrapolation based on the coarsest three meshes, and mutatis mutandis for the last extrapolation.) The numerical results confirm that the crack tip stress grows in a seemingly unbounded manner, whereas the difference between numerical stress and extrapolated stress increases with refinement of the mesh. This confirms the suggestion that a singularity is present. Thus, it must be concluded that the irreducible format is not capable of avoiding stress singularities. This is reported for the Hookean stress τ^{H}_{yy} but will carry over to the pseudo Cauchy stress τ^{B}_{yy} since the latter quantity includes the former.

On the other hand, the results of the reducible format clearly converge towards a unique, nonsingular solution, and the singularities that plague classical elasticity formulations are avoided. However, it must be noted that the maximum stress occurs not at the crack tip but further inside the material. This is in line with the analysis and results reported in [Simone et al. 2004].

7. Conclusions

We have reviewed and systematically compared two formats of gradient elasticity. Both formats can be derived by continualising a one-dimensional discrete model and stabilising the resulting equations, but the models differ in respect of which particular equation is stabilised — either the field equation (leading to what is denoted as the "irreducible format") or the constitutive equation (leading to the "reducible format"). The multidimensional case, including the associated boundary conditions, has been derived from a variational principle. It is noted that the field equations of the irreducible format can be retrieved from those of the reducible format (assuming that the Laplacian of the body forces vanishes), but the variationally consistent boundary conditions are different for the two models.

This has implications for the solution of initial-boundary-value problems. We have presented a crack problem, and it was demonstrated that the irreducible format is not capable of avoiding singularities in the stress field. On the other hand, no singularities were found when the reducible format was used. Thus, for the dynamic analysis of stresses around sharp cracks, the reducible format is to be preferred.

Appendix A: Nomenclature in gradient elasticity: Cauchy stress

In the literature, there is a lack of consistency in which quantity is denoted as the Cauchy stress in gradient elasticity theories. Some eminent authors have used this term to indicate the derivative of the strain energy density with respect to the strain — see for instance [Mindlin 1964, p. 57] or [Shu et al. 1999, p. 375]. However, we have followed the arguments set out by Borino and Polizzotto [2003, Remark 3], who state that the term Cauchy stress should be used for the total stress quantity as it appears in the equilibrium equations; conversely, we have used the term Hookean stress for the derivative of the strain energy density with respect to the strain. We believe the former is in line with the conceptualisation of Cauchy himself, who discussed stresses as forming equilibrium (or indeed accelerating) systems by acting on surfaces, rather than as derivatives of energy functionals — see for instance [Cauchy 1823; 1827; 1843].

However, it is also noted that extending the concept of Cauchy stress as "force divided by area" to gradient-enriched continua leads, in general, to much more complicated expressions. This is illustrated by the format of the natural boundary conditions in Mindlin's [1964, pp. 67–68] theory of gradient elasticity. Askes and Metrikine [2005] as well as Froiio et al. [2010] have provided physical interpretations of the nonstandard boundary conditions.

Appendix B: Eringen's 1983 differential theory of nonlocal elasticity

The reducible format presented in Section 4 has been derived earlier in [Eringen 1983] from an integral formulation. Because the coupled nature of the governing equations of Eringen's theory is not always appreciated, it is worthwhile to summarise Eringen's theory. Adopting his notation unless stated otherwise, the equations of motion are given by [Eringen 1983, (2.1)] as

$$t_{kl,k} + \rho(f_l - \ddot{u}_l) = 0 \tag{28}$$

where t_{kl} is the Cauchy stress tensor and f_l is the body force density. With the restriction to isotropic linear elasticity, a Hookean stress σ^0_{kl} is defined via [Eringen 1983, (2.3) and (2.4)] as

$$\sigma^0_{kl} = \lambda \delta_{kl} u_{j,j} + \mu u_{k,l} + \mu u_{l,k} \tag{29}$$

where a superscript 0 is included in σ^0 to avoid confusion with the Cauchy stress of the reducible theory discussed in Section 4.2. Furthermore, λ and μ are the Lamé constants and δ_{kl} is the Kronecker delta.

The field equations are completed by a differential relation between the Cauchy stress t_{kl} and the Hookean stress σ^0_{kl}. The particular relation that seems to have attracted most interest in the literature is given in [Eringen 1983, (3.19)] as

$$t_{kl} - \ell^2 t_{kl,jj} = \sigma^0_{kl} \tag{30}$$

where the higher-order coefficient is simply indicated by ℓ^2 (Eringen uses a more intricate notation with multiple symbols, which are not required in the present discussion).

Eringen [1983, pp. 4704–4705] also discusses the elimination of the stress t_{kl} from the system of equations. Combining (3.13) and (3.18), he arrives at the irreducible form

$$\sigma^0_{kl,k} + (1 - \ell^2 \nabla^2)(\rho f_l - \rho \ddot{u}_l) = 0. \tag{31}$$

Next, he notes that the particular case of statics with vanishing body forces leads to

$$\sigma^0_{kl,k} = 0. \tag{32}$$

However, regarding natural boundary conditions, Eringen [1983, p. 4704] explicitly states that "[b]oundary conditions involving tractions [are] based on the stress tensor t_{kl}, *not* on σ^0_{kl}", while Eringen [2002, p. 100] also emphasises that "the real stress is not σ^0_{kl} but t_{kl}" — in both quotations we have added the superscript 0 to σ as explained above. This means that (32) cannot be used in isolation to solve *general* boundary-value problems involving prescribed tractions.

In summary, in our opinion, a divergence-free Hookean stress σ^0 should not be considered as a fundamental equation of the Eringen theory because, firstly, it can only be retrieved by making the assumptions of zero body force and zero acceleration and, secondly, it cannot be used to solve general equilibrium problems due to a lack of associated traction boundary conditions. In this respect, we disagree with Lazar and Polyzos [2015], who suggest that (32) is an equilibrium equation in its own right — although these authors do confirm that the correct natural boundary conditions are in terms of t_{kl} rather than σ^0_{kl}.

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