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ON THE WELL-POSEDNESS OF THE GREEN–LINDSAY MODEL

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The present paper is devoted to an investigation of a nonclassical model for inhomogeneous anisotropic thermoelastic bodies with two constant relaxation times originally presented by Green and Lindsay. A variational formulation of the initial-boundary value problem corresponding to the linear dynamical three-dimensional Green–Lindsay model is applied. The corresponding spaces of vector-valued distributions with respect to the time variable with values in Sobolev spaces are defined and the existence and uniqueness of the solution in these spaces as well as continuous dependence of the solution on the given data is shown.

1. Introduction

The physically unrealistic feature of classical thermoelasticity, which is based on Fourier's law of heat conduction, and according to which heat spreads infinitely fast, was refuted by several experimental studies, where it was shown that heat propagates as a thermal wave at finite speed at low temperatures [Ackerman and Overton 1969; Caviglia et al. 1992; Coleman and Newman 1988; McNelly et al. 1970; Narayanamurti and Dynes 1972]. In various modern engineering constructions, such as high-speed aircraft, nuclear reactors, and recently developed ultrafast pulsed lasers, temperatures and temperature gradients are extremely high and the operation time periods are of the order of picoseconds. This results in thermal shocks and cannot be successfully described by the classical theory of thermoelasticity [Abdallah 2009; Dreyer and Struchtrup 1993; Wang and Xu 2002; Zhu et al. 1999]. Furthermore, mathematical models of propagation of heat as a thermal wave are used in order to describe various processes involving heat transfer, such as during chemotaxis [Dolak and Hillen 2003], in food technology [Saidane et al. 2005], in biological tissues [Afrin et al. 2011], in one of Saturn's moons [Bargmann et al. 2008], and in nanofluids [Vadasz et al. 2005].

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One of the theories devoted to eliminating the shortcomings of classical thermoelasticity was presented by Müller [1971] without making any assumptions regarding the form of the heat conduction law, and admitted a finite speed of propagation of thermal waves. By applying a similar approach and by using the entropy production inequality of Green and Laws [1972], a simpler and more explicit version of nonclassical thermoelasticity was presented by Green and Lindsay [1972], which also allows for finite speeds of thermal waves. In this theory, the classical forms of the entropy flux and entropy source are preserved and, as in Müller's theory, the temperature rate is included among the constitutive variables. Note that in the Green–Lindsay theory for materials with a center of symmetry at each point the classical Fourier law is satisfied.

In the nonclassical theory of thermoelasticity developed by Green and Lindsay, the constitutive relations for the stress tensor and the entropy are generalized by introducing two different relaxation times. A uniqueness theorem for the Green–Lindsay model in the case of a thermoelastic body consisting of a homogeneous material with an initial center of symmetry requiring only the usual symmetry properties of the elastic stiffness tensor was obtained by Green [1972]. The continuous dependence of the classical solution on given data and the existence of a generalized solution for initial-boundary value problems corresponding to the Green–Lindsay model (under the assumption that there is a classical solution of the problem) were proved for homogeneous isotropic thermoelastic bodies by Bem [1983]. By applying the method of potentials and the theory of integral equations, the problems of stable and pseudo-oscillations for the Green–Lindsay nonclassical model were studied by Burchuladze and Gegelia [1985]. For the Green–Lindsay nonclassical model, the problem of propagation of a thermoelastic wave was studied, and domain of influence results were obtained for a thermoelastic body consisting of homogeneous material with an initial center of symmetry by Carbonaro and Ignaczak [1987] in classical spaces of twice continuously differentiable functions.

Existence, uniqueness, and continuous dependence of the solution of the initial-boundary value problem corresponding to the Green–Lindsay model with Dirichlet boundary conditions for a temperature vanishing on the entire boundary in suitable function spaces were proved in [Karakostas and Massalas 1991]. For the Green–Lindsay nonclassical model, problems of wave propagation, methods of solution of the corresponding initial and initial-boundary value problems, and applications of the obtained results and related topics have been considered by many researchers (see [Chandrasekharaiah 1986; 1998; Hetnarski and Ignaczak 2000; Joseph and Preziosi 1989; Ignaczak and Ostoja-Starzewski 2010; Straughan 2011]).

It should be pointed out that three-dimensional initial-boundary value problems with general mixed boundary conditions for displacement and temperature corresponding to the linear Green–Lindsay dynamical model for an inhomogeneous anisotropic thermoelastic body have not been investigated yet. The well-posedness results are mainly obtained for the case of purely Dirichlet or Neumann types of boundary conditions. The initial-boundary value problem with mixed boundary conditions corresponding to the Green–Lindsay linear model for a homogeneous isotropic thermoelastic plate was investigated in first-order Sobolev spaces in the paper [Avalishvili et al. 2010] by applying a variational approach. In the present paper, we investigate the well-posedness of the linear three-dimensional initial-boundary value problem corresponding to the Green–Lindsay model with general mixed boundary conditions, provided that on certain parts of the boundary of the space domain surface force and heat flux along the outward normal vector are prescribed and on the remaining parts displacement and temperature vanish. We obtain new existence, uniqueness, and continuous dependence results in the corresponding Sobolev spaces.

In Section 2, we consider a differential formulation of the initial-boundary value problem corresponding to the Green–Lindsay linear dynamical three-dimensional model for an inhomogeneous anisotropic thermoelastic body and obtain integral equations that are equivalent to the original problem in spaces of sufficiently smooth functions. On the basis of these integral equations, we present a variational formulation of the three-dimensional problem in corresponding spaces of vector-valued distributions with respect to the time variable with values in Sobolev spaces. Furthermore, we formulate results regarding the existence and uniqueness of the solution of the three-dimensional initial-boundary value problem, and regard the continuous dependence of the solution on given data in suitable function spaces.

2. Well-posedness of the Green–Lindsay model

In this paper we denote for each real $s \geq 0$ by $H^s(\Omega)$ and $H^s(\check{\Gamma})$ the Sobolev spaces of real-valued functions based on $H^0(\Omega) = L^2(\Omega)$ and $H^0(\check{\Gamma}) = L^2(\check{\Gamma})$, respectively, where $\Omega \subset \mathbb{R}^n$, $n \in \mathbb{N}$, is a bounded Lipschitz domain and $\check{\Gamma}$ is an element of a Lipschitz dissection of the boundary $\Gamma = \partial\Omega$ [McLean 2000]. We refer to the corresponding spaces of vector-valued functions by

$$\boldsymbol{H}^s(\Omega) = [H^s(\Omega)]^3, \quad \boldsymbol{H}^s(\check{\Gamma}) = [H^s(\check{\Gamma})]^3 \ (s \geq 0), \quad \boldsymbol{L}^{s_1}(\check{\Gamma}) = [L^{s_1}(\check{\Gamma})]^3 \ (s_1 \geq 1)$$

and by $\mathbf{tr} : \boldsymbol{H}^1(\Omega) \to \boldsymbol{H}^{1/2}(\Gamma)$ and $\mathrm{tr} : H^1(\Omega) \to H^{1/2}(\Gamma)$ the trace operators. $C^{0,1}(\bar{\Omega})$ characterizes the space of Lipschitz continuous function on $\bar{\Omega}$. For any measurable set D, $(\cdot,\cdot)_{\boldsymbol{L}^2(D)}$ and $(\cdot,\cdot)_{L^2(D)}$ are the classical scalar products in $\boldsymbol{L}^2(D)$ and $L^2(D)$, respectively. For a Banach space X, $C([0, T]; X)$ is the space of continuous vector functions on $[0, T]$ with values in X. $L^m(0, T; X)$, $1 \leq m \leq \infty$, is the space of such measurable vector functions $g : (0, T) \to X$ that $\|g\|_X \in L^m(0, T)$, and the generalized derivative of g we denote by $g' = \mathrm{d}g/\mathrm{d}t \in$

$\mathfrak{D}'(0, T; X)$ [Dautray and Lions 1992]. If $g \in L^1(0, T; X)$ and X is a space of functions of variable $x \in \Omega$, then we identify g with a function $g(x, t)$ and $g(t)$ denotes the function $g(t) : x \to g(x, t)$, for almost all $t \in (0, T)$. The distributional derivative $\mathrm{d}g/\mathrm{d}t$ we identify with the derivative $\partial g/\partial t$ of g in $\mathfrak{D}'(\Omega \times (0, T))$.

Let us consider a thermoelastic body with initial configuration $\overline{\Omega}$ consisting of general inhomogeneous anisotropic thermoelastic material, which is described by the Green–Lindsay linear dynamical three-dimensional model [Green and Lindsay 1972], and whose thermal and elastic properties are characterized by the following consistently spatially dependent thermoelastic parameters:

(a) an elasticity tensor $\mu_{ijpq}(x),\ x \in \Omega\ \ (i, j, p, q = 1, 2, 3)$, which satisfies the symmetry and positive definiteness conditions

$$\mu_{ijpq}(x) = \mu_{pqij}(x) = \mu_{jipq}(x) \quad \forall x \in \Omega, \tag{1}$$

$$\sum_{i,j,p,q=1}^{3} \mu_{ijpq}(x)\varepsilon_{pq}\varepsilon_{ij} \geq c_\mu \sum_{i,j=1}^{3} (\varepsilon_{ij})^2 \quad \forall \varepsilon_{ij} \in \mathbb{R},\ x \in \Omega, \tag{2}$$

where c_μ is a constant > 0 and $\varepsilon_{ij} = \varepsilon_{ji}$;

(b) a mass density $\rho(x),\ x \in \Omega$;

(c) a thermal conductivity tensor $\lambda_{pq}(x),\ x \in \Omega$ $(p, q = 1, 2, 3)$, which satisfies the following symmetry and positive definite conditions:

$$\lambda_{pq}(x) = \lambda_{qp}(x) \quad \forall x \in \Omega, \tag{3}$$

$$\sum_{p,q=1}^{3} \lambda_{pq}(x)\varepsilon_p\varepsilon_q \geq c_\lambda \sum_{p=1}^{3} (\varepsilon_p)^2 \quad \forall \varepsilon_p \in \mathbb{R},\ x \in \Omega, \tag{4}$$

where c_λ is a constant > 0;

(d) a thermal capacity $\varkappa(x),\ x \in \Omega$;

(e) a stress-temperature tensor $\eta_{pq}(x)$, and thermal coefficients $\beta_p(x),\ x \in \Omega$ $(p, q = 1, 2, 3)$, such that

$$\eta_{pq}(x) = \eta_{qp}(x) \quad \forall x \in \Omega; \tag{5}$$

(f) relaxation times $\tau_0 = \text{const} > 0$ and $\tau_1 = \text{const} > 0$;

(g) temperature of thermoelastic body in natural state $\Theta_0 = \text{const} > 0$, which is considered as a reference temperature.

It should be noted that the constraint of constant relaxation times will be removed in a forthcoming paper [Avalishvili et al. 2017].

We consider mixed boundary conditions on the boundary $\Gamma = \partial\Omega$ of the thermoelastic body, such that on certain parts of the boundary the displacement or the temperature vanishes, and on the remaining parts the stress vector or the heat flux along the outward normal of the boundary are given. We assume that the body is clamped along a part $\Gamma_0 \subset \Gamma$ and that the temperature θ vanishes along a part $\Gamma_0^\theta \subset \Gamma$. The body is subjected to:

(i) an applied body force with density $\boldsymbol{f} = (f_i) : \Omega \times (0, T) \to \mathbb{R}^3$;

(ii) an applied surface force with density $\boldsymbol{g} = (g_i) : \Gamma_1 \times (0, T) \to \mathbb{R}^3$ is given along the part $\Gamma_1 = \Gamma \backslash \bar{\Gamma}_0$ of the boundary of Ω, where $\partial\Omega = \Gamma_0 \cup \Gamma_{01} \cup \Gamma_1$, $\Gamma_0 \cap \Gamma_1 = \varnothing$, is a Lipschitz dissection of $\partial\Omega$ [McLean 2000];

(iii) a heat source with density $f^\theta : \Omega \times (0, T) \to \mathbb{R}$;

(iv) a heat flux with density $g^\theta : \Gamma_1^\theta \times (0, T) \to \mathbb{R}$ along the outward normal vector of Γ, which is given on $\Gamma_1^\theta = \Gamma \backslash \overline{\Gamma_1^\theta}$, where $\partial\Omega = \Gamma_0^\theta \cup \Gamma_{01}^\theta \cup \Gamma_1^\theta$, $\Gamma_0^\theta \cap \Gamma_1^\theta = \varnothing$, is a Lipschitz dissection of $\partial\Omega$.

The dynamical linear three-dimensional model for the stress-strain state of a thermoelastic body Ω obtained by Green and Lindsay [1972] is given by the following initial-boundary value problem in differential form:

$$\rho \frac{\partial^2 u_i}{\partial t^2} = \sum_{j=1}^{3} \frac{\partial}{\partial x_j} \left(\sum_{p,q=1}^{3} \mu_{ijpq} e_{pq}(\boldsymbol{u}) + \eta_{ij}\theta + \eta_{ij}\tau_1 \frac{\partial \theta}{\partial t} \right) + f_i \text{ in } \Omega \times (0, T), \tag{6}$$

$$\varkappa \left(\frac{\partial \theta}{\partial t} + \tau_0 \frac{\partial^2 \theta}{\partial t^2} \right) - \sum_{p=1}^{3} \beta_p \frac{\partial^2 \theta}{\partial t \partial x_p} = \sum_{p,q=1}^{3} \frac{\partial}{\partial x_p} \left(\lambda_{pq} \frac{\partial \theta}{\partial x_q} \right) + \sum_{p=1}^{3} \frac{\partial}{\partial x_p} \left(\beta_p \frac{\partial \theta}{\partial t} \right) + \Theta_0 \sum_{p,q=1}^{3} \eta_{pq} e_{pq} \frac{\partial \boldsymbol{u}}{\partial t} + f^\theta \text{ in } \Omega \times (0, T), \tag{7}$$

$$\begin{aligned} \boldsymbol{u}(x, 0) &= \boldsymbol{u}_0(x), & \frac{\partial \boldsymbol{u}}{\partial t}(x, 0) &= \boldsymbol{u}_1(x), \\ \theta(x, 0) &= \theta_0(x), & \frac{\partial \theta}{\partial t}(x, 0) &= \theta_1(x) \text{ in } \Omega, \end{aligned} \tag{8}$$

$$\begin{gathered} \boldsymbol{u} = \boldsymbol{0} \text{ on } \Gamma_0 \times (0, T), \\ \sum_{j=1}^{3} \left(\sum_{p,q=1}^{3} \mu_{ijpq} e_{pq}(\boldsymbol{u}) + \eta_{ij}\theta + \eta_{ij}\tau_1 \frac{\partial \theta}{\partial t} \right) \nu_j = g_i \text{ on } \Gamma_1 \times (0, T), \end{gathered} \tag{9}$$

$$\begin{gathered} \theta = 0 \text{ on } \Gamma_0^\theta \times (0, T), \\ -\sum_{p=1}^{3} \left(\sum_{q=1}^{3} \lambda_{pq} \frac{\partial \theta}{\partial x_q} + \beta_p \frac{\partial \theta}{\partial t} \right) \nu_p = g^\theta \text{ on } \Gamma_1^\theta \times (0, T), \end{gathered} \tag{10}$$

where $e_{ij}(\boldsymbol{v}) = \frac{1}{2}\left(\partial_i v_j + \partial_j v_i\right)$ $(i, j = 1, 2, 3)$, $\boldsymbol{\nu} = (\nu_i)$ is the outward unit normal to Γ, $\boldsymbol{u} = (u_i) : \Omega \times (0, T) \to \mathbb{R}^3$ is the displacement vector-function of the thermoelastic body, $\theta : \Omega \times (0, T) \to \mathbb{R}$ is the temperature distribution, $\boldsymbol{u}_0 = (u_{0i})_{i=1}^3$

and $\boldsymbol{u}_1 = (u_{1i})_{i=1}^3$ are the initial displacement and velocity vector-functions, and θ_0 is the initial distribution of temperature.

Remark. If the thermoelastic body consists of a material that initially has a center of symmetry at each point, then parameters β_p $(p = 1, 2, 3)$ vanish.

By multiplying (6) by arbitrary continuously differentiable functions $v_i : \overline{\Omega} \to \mathbb{R}$ $(i = 1, 2, 3)$, which vanish on Γ_0, and (7) by a continuously differentiable function $\varphi : \overline{\Omega} \to \mathbb{R}$, such that $\varphi = 0$ on Γ_0^θ, by using Green's formula, the symmetry properties of the tensors μ_{ijpq}, η_{ij} and $e_{pq}(\boldsymbol{v})$, and the boundary conditions (9) and (10), we obtain the following integral equations:

$$\sum_{i=1}^3 \int_\Omega \rho \frac{\partial^2 u_i}{\partial t^2} v_i \,\mathrm{d}x + \sum_{i,j=1}^3 \int_\Omega \sum_{p,q=1}^3 \mu_{ijpq} e_{pq}(\boldsymbol{u}) e_{ij}(\boldsymbol{v}) \,\mathrm{d}x \\ + \sum_{i,j=1}^3 \int_\Omega \left(\eta_{ij}\theta + \eta_{ij}\tau_1 \frac{\partial \theta}{\partial t} \right) e_{ij}(\boldsymbol{v}) \,\mathrm{d}x = \sum_{i=1}^3 \int_\Omega f_i v_i \,\mathrm{d}x + \sum_{i=1}^3 \int_{\Gamma_1} g_i v_i \,\mathrm{d}\Gamma, \tag{11}$$

$$\int_\Omega \varkappa \left(\frac{\partial \theta}{\partial t} + \tau_0 \frac{\partial^2 \theta}{\partial t^2} \right) \varphi \,\mathrm{d}x - \int_\Omega \sum_{p=1}^3 \beta_p \frac{\partial^2 \theta}{\partial t \partial x_p} \varphi \,\mathrm{d}x + \sum_{p,q=1}^3 \int_\Omega \lambda_{pq} \frac{\partial \theta}{\partial x_q} \frac{\partial \varphi}{\partial x_p} \,\mathrm{d}x \\ + \sum_{p=1}^3 \int_\Omega \beta_p \frac{\partial \theta}{\partial t} \frac{\partial \varphi}{\partial x_p} \,\mathrm{d}x - \Theta_0 \int_\Omega \sum_{p,q=1}^3 \eta_{pq} e_{pq}\left(\frac{\partial \boldsymbol{u}}{\partial t} \right) \varphi \,\mathrm{d}x \\ = \int_\Omega f^\theta \varphi \,\mathrm{d}x - \int_{\Gamma_1^\theta} g^\theta \varphi \,\mathrm{d}\Gamma. \tag{12}$$

Therefore, if $\boldsymbol{u} = (u_i)_{i=1}^3$ and θ are solutions to (6) and (7) and satisfy the boundary conditions (9) and (10), then $\boldsymbol{u} = (u_i)_{i=1}^3$ and θ are solutions to (11) and (12). Conversely, if $\boldsymbol{u} = (u_i)_{i=1}^3$ and θ are twice continuously differentiable solutions of the integral equations (11) and (12), then we use Green's formula to obtain

$$\sum_{i=1}^3 \int_\Omega \left(\rho \frac{\partial^2 u_i}{\partial t^2} - \sum_{j=1}^3 \frac{\partial}{\partial x_j} \left(\sum_{p,q=1}^3 \mu_{ijpq} e_{pq}(\boldsymbol{u}) + \eta_{ij}\theta + \eta_{ij}\tau_1 \frac{\partial \theta}{\partial t} \right) \right) v_i \,\mathrm{d}x \\ + \sum_{i,j=1}^3 \int_{\Gamma_1} \left(\sum_{p,q=1}^3 \mu_{ijpq} e_{pq}(\boldsymbol{u}) + \eta_{ij}\theta + \eta_{ij}\tau_1 \frac{\partial \theta}{\partial t} \right) v_i \nu_j \,\mathrm{d}\Gamma \\ = \sum_{i=1}^3 \int_\Omega f_i v_i \,\mathrm{d}x + \sum_{i=1}^3 \int_{\Gamma_1} g_i v_i \,\mathrm{d}\Gamma, \tag{13}$$

$$\int_\Omega\Big(\varkappa\Big(\frac{\partial\theta}{\partial t}+\tau_0\frac{\partial^2\theta}{\partial t^2}\Big)-\sum_{p=1}^{3}\beta_p\frac{\partial^2\theta}{\partial t\partial x_p}-\sum_{p,q=1}^{3}\frac{\partial}{\partial x_p}\Big(\lambda_{pq}\frac{\partial\theta}{\partial x_q}\Big)$$
$$-\sum_{p=1}^{3}\frac{\partial}{\partial x_p}\Big(\beta_p\frac{\partial\theta}{\partial t}\Big)-\Theta_0\sum_{p,q=1}^{3}\eta_{pq}e_{pq}\Big(\frac{\partial \boldsymbol{u}}{\partial t}\Big)\Big)\varphi\,\mathrm{d}x$$
$$+\sum_{p=1}^{3}\int_{\Gamma_1^\theta}\Big(\sum_{q=1}^{3}\lambda_{pq}\frac{\partial\theta}{\partial x_q}+\beta_p\frac{\partial\theta}{\partial t}\Big)\varphi\nu_p\,\mathrm{d}\Gamma$$
$$=\int_\Omega f^\theta\varphi\,\mathrm{d}x-\int_{\Gamma_1^\theta}g^\theta\varphi\,\mathrm{d}\Gamma, \tag{14}$$

for all continuously differentiable functions $\boldsymbol{v}=(v_i)_{i=1}^3$ and φ vanishing on Γ_0 and Γ_0^θ, respectively. By letting $\varphi\in C_0^1(\Omega)=\{\psi\in C^1(\Omega)\mid\psi=0 \text{ on } \Gamma\}$ and $\boldsymbol{v}=(v_i)_{i=1}^3\in(C_0^1(\Omega))^3$, and by taking into account the density of $C_0^1(\Omega)$ in $L^2(\Omega)$ from (13) and (14) we obtain that $\boldsymbol{u}$ and θ satisfy (6) and (7). Now, if we assume that $\boldsymbol{v}=(v_i)_{i=1}^3\in(C_0^1(\Gamma_1))^3=\{\boldsymbol{v}=(v_i)_{i=1}^3\in(C^1(\Gamma_1))^3\mid\boldsymbol{v}=\mathbb{0}\text{ on }\Gamma_{01}\}$ and $\varphi\in C_0^1(\Gamma_1^\theta)=\{\varphi\in C^1(\Gamma_1^\theta)\mid\varphi=0\text{ on }\Gamma_{01}^\theta\}$ are arbitrary continuous functions, then by applying (6) and (7), and the density of $C_0(\Gamma_1)$ and $C_0(\Gamma_1^\theta)$ being in $L^2(\Gamma_1)$ and $L^2(\Gamma_1^\theta)$, respectively, we infer that $\boldsymbol{u}$ and θ satisfy the boundary conditions (9) and (10).

Hence the initial-boundary value problem (6)–(10) corresponding to the Green–Lindsay dynamical three-dimensional model is equivalent to the integral equations (11) and (12), together with the initial conditions (8) in the spaces of twice continuously differentiable functions. On the basis of these equations we present the so-called weak or variational formulation of the initial-boundary value problem (6)–(10), and investigate the existence and uniqueness of a weak solution in suitable spaces of vector-valued distributions with values in the corresponding Sobolev spaces.

Let us introduce the following function spaces, which are used in the variational formulation of the initial-boundary value problem (6)–(10):

$$\boldsymbol{V}(\Omega)=\{\boldsymbol{v}=(v_i)_{i=1}^3\in\boldsymbol{H}^1(\Omega);\quad \mathbf{tr}(\boldsymbol{v})=\mathbb{0}\text{ on }\Gamma_0\},$$
$$V^\theta(\Omega)=\{\varphi\in H^1(\Omega);\quad \mathrm{tr}(\varphi)=0\text{ on }\Gamma_0^\theta\}.$$

Note that $\boldsymbol{V}(\Omega)$ and $V^\theta(\Omega)$ are Hilbert spaces equipped with the norms $\|\cdot\|_{\boldsymbol{H}^1(\Omega)}$ and $\|\cdot\|_{H^1(\Omega)}$, respectively.

We identify the unknown vector function $\boldsymbol{u}$ and the function θ with vector-functions defined on $[0,T]$ with values in suitable spaces of functions defined on Ω. By applying the integral equations (11) and (12), we consider the following variational formulation of problem (6)–(10) in the spaces of vector-valued distributions: find $\boldsymbol{u}\in C([0,T];\boldsymbol{V}(\Omega))$, $\boldsymbol{u}'\in L^\infty(0,T;\boldsymbol{V}(\Omega))$, $\boldsymbol{u}''\in L^\infty(0,T;\boldsymbol{L}^2(\Omega))$,

$\theta \in C([0,T]; V^\theta(\Omega))$, $\theta' \in L^\infty(0,T; V^\theta(\Omega))$, $\theta'' \in L^\infty(0,T; L^2(\Omega))$, which satisfy the following equations in the sense of distributions on $(0,T)$:

$$(\rho \boldsymbol{u}'', \boldsymbol{v})_{\boldsymbol{L}^2(\Omega)} + a(\boldsymbol{u}, \boldsymbol{v}) + b(\theta, \boldsymbol{v}) + \tau_1 b(\theta', \boldsymbol{v}) \\ = (\boldsymbol{f}, \boldsymbol{v})_{\boldsymbol{L}^2(\Omega)} + (\boldsymbol{g}, \boldsymbol{v})_{\boldsymbol{L}^2(\Gamma_1)} \quad \forall \boldsymbol{v} \in \boldsymbol{V}(\Omega), \tag{15}$$

$$(\varkappa\theta', \varphi)_{L^2(\Omega)} + \tau_0(\varkappa\theta'', \varphi)_{L^2(\Omega)} - b^\theta(\theta', \varphi) + a^\theta(\theta, \varphi) \\ + b^\theta(\varphi, \theta') - \Theta_0 b(\varphi, \boldsymbol{u}') = (f^\theta, \varphi)_{L^2(\Omega)} - (g^\theta, \varphi)_{L^2(\Gamma_1^\theta)} \quad \forall \varphi \in V^\theta(\Omega), \tag{16}$$

together with the initial conditions

$$\boldsymbol{u}(0) = \boldsymbol{u}_0, \quad \boldsymbol{u}'(0) = \boldsymbol{u}_1, \quad \theta(0) = \theta_0, \quad \theta'(0) = \theta_1, \tag{17}$$

where

$$a(\hat{\boldsymbol{v}}, \boldsymbol{v}) = \int_\Omega \sum_{i,j,p,q=1}^{3} \mu_{ijpq} e_{pq}(\hat{\boldsymbol{v}}) e_{ij}(\boldsymbol{v}) \, \mathrm{d}x \quad \forall \boldsymbol{v}, \hat{\boldsymbol{v}} \in \boldsymbol{H}^1(\Omega),$$

$$a^\theta(\hat{\varphi}, \varphi) = \int_\Omega \sum_{p,q=1}^{3} \lambda_{pq} \frac{\partial \hat{\varphi}}{\partial x_q} \frac{\partial \varphi}{\partial x_p} \, \mathrm{d}x \quad \forall \varphi, \hat{\varphi} \in H^1(\Omega),$$

$$b(\tilde{\varphi}, \boldsymbol{v}) = \int_\Omega \sum_{i,j=1}^{3} \eta_{ij} \tilde{\varphi} e_{ij}(\boldsymbol{v}) \, \mathrm{d}x,$$

$$b^\theta(\varphi, \tilde{\varphi}) = \int_\Omega \sum_{p=1}^{3} \beta_p \frac{\partial \varphi}{\partial x_p} \tilde{\varphi} \, \mathrm{d}x \quad \forall \tilde{\varphi} \in L^2(\Omega),\ \varphi \in H^1(\Omega),\ \boldsymbol{v} \in \boldsymbol{H}^1(\Omega).$$

Note that since $\boldsymbol{u}$ and θ are continuous with respect to the time variable t, the equations in (17) for $\boldsymbol{u}(0)$ and $\theta(0)$ are understood in the sense of the spaces $\boldsymbol{V}(\Omega)$ and $V^\theta(\Omega)$, respectively. From the embedding theorem [Dautray and Lions 1992] it follows that $\boldsymbol{u}' \in C([0,T]; \boldsymbol{L}^2(\Omega))$, $\theta' \in C([0,T]; L^2(\Omega))$, and, consequently, the equations in (17) for $\boldsymbol{u}'(0)$ and $\theta'(0)$ are understood in the sense of the spaces $\boldsymbol{L}^2(\Omega)$ and $L^2(\Omega)$, respectively.

For the problem (15)–(17), which is equivalent to the initial-boundary value problem (6)–(10) in the spaces of classical smooth enough functions, the following existence, uniqueness, and continuous dependence theorem is valid.

Theorem 2.1. *Suppose that parameters characterizing thermal and elastic properties of thermoelastic body satisfy conditions* (1)–(5), $\tau_1 > 0$, $\tau_0 > 0$ *and*

$$\rho(x) > c_\rho = \text{const} > 0, \quad \varkappa(x) > c_\varkappa = \text{const} > 0 \quad \forall x \in \Omega,$$

and $\mu_{ijpq}, \lambda_{pq}, \eta_{ij}, \beta_p \in C^{0,1}(\bar{\Omega})$ $(i, j, p, q = 1, 2, 3)$, $\rho, \varkappa \in L^\infty(\Omega)$. *If the densities of body and surface forces, heat source, and heat flux are such that*

$$\begin{aligned} &\boldsymbol{f}, \boldsymbol{f}', \boldsymbol{f}'' \in L^2(0, T; \boldsymbol{L}^{6/5}(\Omega)), \qquad \boldsymbol{g}, \boldsymbol{g}', \boldsymbol{g}'' \in L^2(0, T; \boldsymbol{L}^{4/3}(\Gamma_1)), \\ &f^\theta, f^{\theta\prime}, f^{\theta\prime\prime} \in L^2(0, T; L^{6/5}(\Omega)), \quad g^\theta, g^{\theta\prime}, g^{\theta\prime\prime} \in L^2(0, T; L^{4/3}(\Gamma_1^\theta)), \end{aligned}$$

and initial conditions $\boldsymbol{u}_0 \in \boldsymbol{H}^2(\Omega) \cap \boldsymbol{V}(\Omega)$, $\boldsymbol{u}_1 \in \boldsymbol{V}(\Omega)$, $\theta_0 \in H^2(\Omega) \cap V^\theta(\Omega)$, $\theta_1 \in V^\theta(\Omega)$ *satisfy the following compatibility conditions*:

$$g_i(0) = \sum_{j=1}^{3} \left(\sum_{p,q=1}^{3} \mu_{ijpq} e_{pq}(\boldsymbol{u}_0) + \eta_{ij}\theta_0 + \eta_{ij}\tau_1\theta_1 \right) \nu_j \Big|_{\Gamma_1},$$

$$g^\theta(0) = -\sum_{p=1}^{3} \left(\sum_{q=1}^{3} \left(\lambda_{pq} \frac{\partial \theta_0}{\partial x_q} + \beta_p \theta_1 \right) \nu_p \right) \Big|_{\Gamma_1^\theta},$$

where $i = 1, 2, 3$, *then the initial-boundary value problem* (15)–(17) *possesses a unique solution, which continuously depends on the given data, i.e., the mapping* $(\boldsymbol{u}_0, \boldsymbol{u}_1, \theta_0, \theta_1, \boldsymbol{f}, \boldsymbol{f}', \boldsymbol{g}, \boldsymbol{g}', f^\theta, f^{\theta\prime}, g^\theta, g^{\theta\prime}) \to (\boldsymbol{u}, \boldsymbol{u}', \theta, \theta')$ *is linear and continuous from space*

$$\begin{aligned} \boldsymbol{V}(\Omega) &\times \boldsymbol{L}^2(\Omega) \times V^\theta(\Omega) \times L^2(\Omega) \times L^2(0, T; \boldsymbol{L}^{6/5}(\Omega)) \times L^2(0, T; \boldsymbol{L}^{6/5}(\Omega)) \\ &\times L^2(0, T; \boldsymbol{L}^{4/3}(\Gamma_1)) \times L^2(0, T; \boldsymbol{L}^{4/3}(\Gamma_1)) \times L^2(0, T; L^{6/5}(\Omega)) \\ &\quad \times L^2(0, T; L^{6/5}(\Omega)) \times L^2(0, T; L^{4/3}(\Gamma_1^\theta)) \times L^2(0, T; L^{4/3}(\Gamma_1^\theta)) \end{aligned}$$

to space

$$C([0, T]; \boldsymbol{V}(\Omega)) \times C([0, T]; \boldsymbol{L}^2(\Omega)) \times C([0, T]; V^\theta(\Omega)) \times C([0, T]; L^2(\Omega)).$$

Further details and extensions will be presented in [Avalishvili et al. 2017].

3. Conclusions

We studied an initial-boundary value problem with general mixed boundary conditions for displacement and temperature corresponding to the Green–Lindsay linear dynamical three-dimensional model for an inhomogeneous anisotropic thermoelastic body. We obtained a variational formulation of the three-dimensional problem in the corresponding spaces of vector-valued distributions with respect to the time variable with values in Sobolev spaces, which is equivalent to the original differential formulation in spaces of sufficiently smooth functions. We formulated a new theorem on the existence and uniqueness of the solution of the three-dimensional initial-boundary value problem, and the continuous dependence of the solution on given data in suitable function spaces.

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