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AROUND TWO THEOREMS AND A LEMMA BY LUCIO RUSSO

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We describe two directions of study following early work of Lucio Russo. The first direction follows the famous Russo–Seymour–Welsh (RSW) theorem. We describe an RSW-type conjecture by the first author which, if true, would imply a coarse version of conformal invariance for critical planar percolation. The second direction is the study of "Russo's lemma" and "Russo's 0–1 law" for threshold behavior of Boolean functions. We mention results by Friedgut, Bourgain, and Hatami, and present a conjecture by Jeff Kahn and the second author, which may allow applications for finding critical probabilities.

1. Introduction

We have not met Lucio Russo in person but his mathematical work has greatly influenced our own and his wide horizons and interests in physics, mathematics, philosophy, and history have greatly inspired us. We describe here two directions of study following early work of Russo. The first section follows the famous Russo–Seymour–Welsh theorem regarding critical planar percolation. The second section follows the basic "Russo's lemma" and the deep "Russo's 0–1 law". In each direction we present one central conjecture.

2. Planar percolation

Consider $\frac{1}{2}$-Bernoulli bond percolation on a square lattice. Russo [1978] and Seymour and Welsh [1978] proved the *RSW theorem* relating the probability of having an open crossing in a $n \times cn$ rectangle to that of crossing a square. In particular, their results imply that

> *the probability of critical Bernoulli percolation crossing a long rectangle is bounded away from zero and depends only on the aspect ratio.*

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This fundamental fact was crucial in Kesten's proof [1980] that the critical probability for planar percolation is $\frac{1}{2}$, and has been used and extended to a variety of models using clever proofs. But until recently all proofs have depended on rotational symmetry. Vincent Tassion [2016] recently proved the RSW statement under various sets of weaker assumptions, and this has been the key to solving several known problems. On a personal note, we mention that the RSW lemma was essential in controlling the influence of a fixed edge on the crossing event, allowing us to establish, jointly with Oded Schramm, noise sensitivity of critical percolation; see [Benjamini et al. 1999; Garban and Steif 2015].

What about an RSW-type result for more general planar graphs going beyond Euclidean lattices and tessellations?

In what follows we suggest a conjectural extension of the RSW theorem to general planar triangulations. The motivation comes from conformal uniformization; see [Benjamini 2015].

There are strong ties between critical planar percolation and conformal geometry. In [Smirnov 2001] the scaling limit of critical Bernoulli site percolation on the triangular lattice was proved to be conformally invariant. Benjamini [2015] gave a far-reaching conjecture relating percolation and conformal uniformization and derived it from the conjectural extension of the RSW theorem.

A generalized RSW conjecture. Tile the unit square with (possibly infinitely many) squares of varying sizes so that at most three squares meet at corners. That is, the dual graph is a triangulation. Color each square black or white with equal probability independently.

Conjecture 2.1. *There is a universal $c > 0$ such that the probability of a black left-right crossing is bigger than c.*

At the moment we do not have a proof of the conjecture even when the squares are colored black with probability $\frac{2}{3}$. Behind the conjecture is a coarse version of conformal invariance. That is, the crossing probability is bounded away from zero and one if the tile shapes are uniformly close to circles (rotation invariance), and the squares can be of different sizes (dilation invariance). If true, the same should hold for a tiling or a packing of a triangulation, with a set of shapes that are of bounded Hausdorff distance to circles.

If the answer to Conjecture 2.1 is affirmative, this will imply (see [Benjamini 2015]) the following: Let G be the 1-skeleton of a bounded degree triangulation of an open disk. Assume G is transient for the simple random walk; then $\frac{1}{2}$-Bernoulli site percolation on G admits infinitely many infinite clusters almost surely. We do not know this even for any p-Bernoulli percolation with $1 > p > \frac{1}{2}$. In [Benjamini and Schramm 1996b] it is shown that such triangulations result in square tilings as in the conjecture. The proof there is an analogue of the RSW phenomenon for a simple random walk on the triangulation. We speculate that $\frac{1}{2}$-Bernoulli site percolation on G admits infinitely many infinite clusters almost surely if and only if G is transient.

How does the *influence* of a square in the tiling on the crossing probability at $p = \frac{1}{2}$ relate to its area? Establishing a *high-dimensional* version of the RSW lemma is a well-known and very important open problem. Dan Asimov and Dylan Thurston (private communication) worked out a $2k$-dimensional model with duality but not yet with RSW. Informally look at the critical p for a full infinite surface and prove RSW for plaquettes in cubes. That is (for $d = 3$, say), if the probability of no open path from top to bottom in an $n \times n \times n$ box is at least $\frac{1}{2}$, then there is no open path from top to bottom in a cube $2n \times 2n \times n$ with probability bounded away from 0 independently of n.

A comment on large graphs and percolation. In the category of planar graphs, in view of (discrete) conformal uniformization, transience (equivalently *conformal hyperbolicity*) is a natural notion of largeness. In the context of Cayley graphs, *nonamenability* serves as a notion of large Cayley graphs. Thus the still open conjecture [Benjamini and Schramm 1996a] that there is a nonempty interval of p's such that p-Bernoulli percolation admits infinitely many infinite clusters if and only if the group is nonamenable shares some flavor with Conjecture 2.1: both suggest that a graph is large provided there is a phase with infinitely many infinite clusters.

3. Isoperimetric inequalities and Russo's 0–1 law

We endow the discrete cube $\Omega_n = \{-1, 1\}^n$ with the product probability measure μ_p, where the probability for each bit to be 1 is p. A Boolean function f is a function from Ω_n to $\{-1, 1\}$, and f is monotone if changing the value of a variable from -1 to 1 does not change the value of f from 1 to -1. The influence of the k-th variable on f, denoted by $I_k^p(f)$, is the probability that changing the k-th variable will change the value of f. The total influence is $I^p(f) = \sum_{k=1}^n I_k^p(f)$. We denote $\mu_p(f) = \mu_p\{x : f(x) = 1\}$, and write $\mathrm{Var}_p(f) = 4\mu_p(f)(1 - \mu_p(f))$. (If $p = \frac{1}{2}$ we omit the superscript/subscript p.)

A basic result in extremal and probabilistic combinatorics going back to Harper (and others) is the isoperimetric inequality. For the measure μ_p the isoperimetric relation takes the form (see, e.g., [Kahn and Kalai 2007; Kalai 2016]):

Theorem 3.1. $$pI^p(f) \geq \mu_p(f) \log_p(1/\mu_p(f)).$$

If f is monotone then $\mu_p(f)$ is a monotone function of p. Fixing a small $\epsilon > 0$, the threshold interval of f is the interval $[p, q]$ where $\mu_p(f) = \epsilon$ and $\mu_q(f) = 1 - \epsilon$. A fundamental lemma by Russo [1982] and Margulis [1974] asserts that for a monotone Boolean function f,

$$d\mu_p(f)/dp = I^p(f).$$

The deep Russo's 0–1 law [1982] asserts informally that the threshold interval of a Boolean function is of size $o(1)$ if all variables have $o(1)$-influence. In view of the Russo–Margulis lemma, understanding the total influence is crucial for understanding the threshold window of a Boolean function. Sharp form of the Russo 0–1 theorem and various related results were proved in the last two decades, and Fourier methods played an important role in these developments. We mention especially the paper by Kahn, Kalai, and Linial [Kahn et al. 1988] and the subsequent papers [Bourgain et al. 1992; Talagrand 1994; Friedgut and Kalai 1996; Friedgut 1998] and the books [Garban and Steif 2015; O'Donnell 2014]. To a large extent, this study is centered around the following problem.

Problem. Understand the structure of Boolean functions of n variables for which

$$I^p(f) \le K\frac{1}{p}\mu_p(f)\log_p(1/\mu_p(f)).$$

We will quickly describe some main avenues of research and central results regarding this problem. For a more detailed recent survey, see [Kalai 2016].

(1) For the case where both p and $\mu_p(f)$ are bounded away from zero and one (or even when $\log(1/p)/\log n \to 0$) and K is bounded, Friedgut [1998] proved that such functions are approximately "juntas"; namely, they are determined (with high probability) by their values on a fixed bounded set of variables. This result can be seen as a sharp form of Russo's 0–1 law and it has a wide range of applications.

(2) For the case where K is bounded, $\mu_p(f)$ is bounded away from zero and one, but $\log(1/p)/\log n$ is bounded away from zero, there are important theorems by Friedgut [1999] and Bourgain [1999] (see below) and Hatami [2012]. These results have important applications for proving sharp threshold theorems. Hatami's work is based on the important, if mysterious, notion of pseudojuntas.

(3) The case where K is bounded and $\mu_p(f)$ is small is wide open. This case is important on its own and may have some applications for finding the critical probability; see Conjecture 3.3.

(4) Cases where $K = 1+\epsilon$ are of different nature and are also of much interest. See [Ellis 2011], for example; work in progress of Ellis and N. Lifshitz is also relevant.

(5) There are few results regarding the case where K is unbounded and especially when K grows quicker than $\log n$. (One such result is by Bourgain and

Kalai [1997] for functions with various forms of symmetry.) This is of great interest already when both p and $\mu_p(f)$ are bounded away from zero and one.

We turn to a theorem of Bourgain and a related and far-reaching conjecture.

Theorem 3.2 [Bourgain 1999]. *There exists $\epsilon > 0$ with the following property: For every C there is $K(C)$ such that if $I^p(f) < pC$, then there exists a subset R of variables $|R| \leq K(C)$ such that*

$$\mu_p(x : f(x) = 1 \mid x_i = 1, i \in S) > (1+\epsilon)\mu_p(f).$$

Conjecture 3.3 [Kahn and Kalai 2007, Conjecture 6.1(a)]. *There exists $\epsilon > 0$ with the following property: For every C there is $K(C)$ such that if $I^p(f) < pC\mu_p(f)\log(1/\mu_p(f))$ then there exists a subset R of variables $|R| \leq K(C)$ $\log(1/\mu_p(f))$ such that*

$$\mu_p(x : f(x) = 1 \mid x_i = 1, i \in S) > (1+\epsilon)\mu_p(f).$$

Several attempted stronger conjectures (such as Conjectures 6.1(b), 6.1(c) in [Kahn and Kalai 2007]) turned out to be incorrect. Conjecture 3.3 was motivated by a far reaching conjecture from [Kahn and Kalai 2007] relating two notions of a threshold for random graphs. Related questions were raised in [Talagrand 2010]. We conclude with another approach for understanding Boolean functions with small influence. The first step is the important Fourier–Walsh expansion. Every Boolean function f can be written as a square free polynomial $f = \sum \hat{f}(S)x_S$, where $x_S = \prod\{x_i : i \in S\}$. (The coefficients $\hat{f}(S)$ are called the Fourier coefficients of f.) It is easy to verify that $\sum \hat{f}^2(S) = 1$ and that $\sum \hat{f}^2(S)|S| = I(f)$. Therefore:

Proposition 3.4. *For every $\epsilon > 0$, a Boolean function f can be $\epsilon \cdot \mathrm{Var}(f)$-approximated by the sign of a degree-d polynomial where $d = (1/\epsilon)I(f)$.*

However, we note that Boolean functions described as signs of low-degree polynomials may have large total influence. Our next step is to consider the representation of Boolean functions via Boolean circuits. Circuits allow us to build complicated Boolean functions from simple ones, and they have crucial importance in computational complexity. Starting with n variables $x_1, x_2, \ldots, x_n$, a *literal* is a variable x_i or its negation $-x_i$. Every Boolean function can be written as a formula in conjunctive normal form, namely as ANDs of ORs of literals. A *circuit* of depth d is defined inductively as follows: A circuit of depth zero is a literal. A circuit of depth one consists of an OR or AND *gate* applied to a set of literals. A circuit of depth k consists of an OR or AND gate applied to the outputs of circuits of depth $k-1$. (We can assume that gates in the odd levels are all OR gates and that the gates of the even levels are all AND gates.) The size of a circuit is the number of gates. The famous NP $\neq$ P conjecture (in a slightly stronger form) asserts that the Boolean function described by the graph property of containing a Hamiltonian cycle cannot be described by a polynomial-size circuit.

A theorem by Boppana [1984] (the monotone case) and Håstad [1989] (the general case) asserts that if f is described by a Boolean circuit of depth d and size M then $I(f) \le C(\log M)^{d-1}$. We conjecture that functions with low influence can be approximated by low-depth small-size circuits. A function g δ-approximates a function f if $|\mathbb{E}(f-g)^2| \le \epsilon$.

The next conjecture is slightly extended from one in [Benjamini et al. 1999].

Conjecture 3.5 (Benjamini, Kalai, and Schramm). *For some absolute constant C the following holds: For every $\epsilon > 0$ a Boolean function f can be $\epsilon \cdot \mathrm{Var}(f)$-approximated by a circuit of depth d and size M, where*

$$(\log M)^{Cd}\mathrm{Var}(f) \le I(f).$$

Conclusion

Congratulations Lucio on your remarkable career and contributions and best wishes for the future. It is time for us to meet!

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